Methods of Functional Analysis and Topology
Vol. 12 (2006), no. 1, pp. 1–31

THE COMPLEX MOMENT PROBLEM AND DIRECT AND INVERSE SPECTRAL PROBLEMS FOR THE BLOCK JACOBI TYPE BOUNDED NORMAL MATRICES

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Abstract. We continue to generalize the connection between the classical power moment problem and the spectral theory of selfadjoint Jacobi matrices. In this article we propose an analog of the Jacobi matrix related to the complex moment problem and to a system of polynomials orthogonal with respect to some probability measure on the complex plane. Such a matrix has a block three-diagonal structure and gives rise to a normal operator acting on a space of l_2 type. Using this connection we prove existence of a one-to-one correspondence between probability measures defined on the complex plane and block three-diagonal Jacobi type normal matrices. For simplicity, we investigate in this article only bounded normal operators. From the point of view of the complex moment problem, this restriction means that the measure in the moment representation (or the measure, connected with the orthonormal polynomials) has compact support.

1. Introduction

This article consists of two parts: in the first part (Sections 2 and 3) we present some results about one dimensional complex moment problem. These results will be used in the second part of the article. It is necessary to note that this part is contained in our article [9], which was devoted to the infinite dimensional complex moment problem and was presented to Mathematische Nachrichten on September, 2004, however, without this part it would be impossible to give a further account.

The second part of the article (Sections 4, 5 and 6) is a presentation of the direct and the inverse spectral problems. This is a generalization of the classical problems for Jacobi matrices and orthogonal polynomials on the real axis $\mathbb{R}$ to the case of block Jacobi type normal matrices and the corresponding orthogonal polynomials on the complex plane $\mathbb{C}$. This part continues our previous article [11] in which we have investigated unitary matrices and orthogonal polynomials on the unit circle $\mathbb{T} \subset \mathbb{C}$.

A few words are due about first part of the article. An investigation of the one dimensional complex moment problem is carried out in many works, we mention here the following [24, 1, 2, 12, 39, 13, 40, 41]. But our proof of this complex moment representation (here and in [9, 10]) is based on the generalized eigenfunction expansion of the corresponding normal operator and will be described in detail in Sections 2 and 3. Here we only note that this method goes back to the old works of M. G. Krein [25, 26].

To understand the second part of this article, it is necessary to recall at first the situation with the direct and the inverse spectral problems for the classical Jacobi matrices and orthogonal polynomials on the axis $\mathbb{R}$ (see, for example, [1, 4, 44]). In this classical theory one studies, on the space l_2 of sequences $f = (f_n)_{n=0}^{\infty}$, the Hermitian Jacobi

2000 *Mathematics Subject Classification.* Primary 44A60, 47A57, 47A70.

Key words and phrases. Classical and complex moment problems, block three-diagonal matrix, eigenfunction expansion, generalized eigenvector.

matrix

$$(1)\qquad J=\begin{bmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad b_n\in\mathbb{R}, \quad a_n>0, \quad n\in\mathbb{N}_0=\{0,1,2,\ldots\}.$$

This matrix gives rise, on finite sequences $f\in l_{\mathrm{fin}}\subset l_2$, to an operator J on l_2 that is Hermitian with equal defect numbers and therefore has a selfadjoint extensions on l_2. Under some conditions on J (for example, $\sum\limits_{n=0}^{\infty} a_n^{-1}=\infty$) the closure $\tilde J$ of J is selfadjoint.

The direct spectral problem, i.e., the eigenfunction expansion for $\tilde J$ (for simplicity, we will assume that $\tilde J$ is selfadjoint), is constructed in the following way. We introduce $\forall\lambda\in\mathbb{R}$ a sequence of polynomials $P(\lambda)=(P_n(\lambda))_{n=0}^{\infty}$ as a solution of the equation $JP(\lambda)=\lambda P(\lambda), P_0(\lambda)=1$, i.e., $\forall n\in\mathbb{N}_0$

$$(2)\qquad a_{n-1}P_{n-1}(\lambda)+b_nP_n(\lambda)+a_nP_{n+1}(\lambda)=\lambda P_n(\lambda), \quad P_0(\lambda)=1, \quad P_{-1}(\lambda)=0.$$

This recurrence relation has a solution; it is only necessary to proceed step by step, starting with $P_0(\lambda)$. This is possible, since all $a_n>0$.

The sequence $P(\lambda)$ of polynomials belongs $\forall\lambda$ to $l=\mathbb{C}^\infty$ (more exactly, to the real part of l) and is a generalized eigenvector for $\tilde J$ corresponding to the eigenvalue λ (we use a certain rigging of l_2). The corresponding Fourier transform $\widehat{\ }$ in generalized eigenfunctions of $\tilde J$ is

$$(3)\qquad l_2\supset l_{\mathrm{fin}}\ni f=(f_n)_{n=0}^{\infty}\longmapsto \hat f(\lambda)=\sum_{n=0}^{\infty} f_nP_n(\lambda)\in L^2(\mathbb{R},d\rho(\lambda))=L^2.$$

It is an unitary operator (after taking the closure) between the spaces l_2 and L^2. The image of $\tilde J$ is the operator of multiplication by λ on the space L^2. The polynomials $P_n(\lambda)$ are orthonormal w.r.t. $d\rho(\lambda)$.

The inverse problem in this classical case is the following. Let us have a probability Borel measure $d\rho(\lambda)$ on $\mathbb{R}$ which has all its moments,

$$(4)\qquad s_n=\int\limits_{\mathbb{R}}\lambda^n\,d\rho(\lambda), \quad n\in\mathbb{N}_0$$

(and the support of $d\rho(\lambda)$ contains an infinite set in a finite interval). The question is whether it is possible to recover the corresponding Jacobi matrix J such that the initial $d\rho(\lambda)$ would be equal to the spectral measure for $\tilde J$, and how to obtain such a reconstruction?

The answer is very simple, — it is necessary to take the following sequence of functions from L^2 (according to (4)):

$$(5)\qquad 1,\lambda,\lambda^2,\ldots$$

(which are linearly independent due to the condition on the support of $d\rho(\lambda)$) and apply to it the classical Schmidt orthogonalization procedure. As a result, we get a sequence of orthonormal polynomial basis in L^2,

$$(6)\qquad P_0(\lambda)=1,P_1(\lambda),P_2(\lambda),\ldots .$$

Then the matrix J is reconstructed by the formulas

$$(7)\qquad a_n=\int\limits_{\mathbb{R}}\lambda P_n(\lambda)P_{n+1}(\lambda)\,d\rho(\lambda), \quad b_n=\int\limits_{\mathbb{R}}\lambda(P_n(\lambda))^2\,d\rho(\lambda), \quad n\in\mathbb{N}_0.$$

The above mentioned connections between Jacobi matrices, the classical moment problem, and orthogonal polynomials is very fruitful for studying these objects. Many mathematicians worked in this direction, but it is necessary to single out the corresponding results of M. G. Krein [26, 28, 23] and N. I. Achiezer [1].

The main question treated in the second part of this articles is following: in what way it is necessary to proceed for obtaining a generalization of the above mentioned classical theory to orthonormal polynomials on the complex plane $\mathbb{C}$ (or on some subset of $\mathbb{C}$, for example, on the unit circle $\mathbb{T} \subset \mathbb{C}$)? Roughly speaking, it is necessary to replace selfadjoint operators on l_2 with normal (or unitary) operators that act on some space similar to l_2.

More exactly, instead of the space $l_2 = \mathbb{C} \oplus \mathbb{C} \oplus \cdots$, it is necessary to take the space

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \text{where} \quad \mathcal{H}_n = \mathbb{C}^{n+1}, \tag{8}$$

and to replace the scalar matrix (1) with the following Jacobi type block matrix with the elements a_n, b_n, and c_n that are finite dimensional operators (matrices) acting between the corresponding spaces $\mathcal{H}_n$ in (8), namely,

$$J = \begin{bmatrix} b_0 & c_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & c_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & c_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad \begin{aligned} a_n &: \mathcal{H}_n \longrightarrow \mathcal{H}_{n+1}, \\ b_n &: \mathcal{H}_n \longrightarrow \mathcal{H}_n, \\ c_n &: \mathcal{H}_{n+1} \longrightarrow \mathcal{H}_n, \quad n \in \mathbb{N}_0. \end{aligned} \tag{9}$$

Such a matrix (9), on finite vectors $\mathbf{l}_{\text{fin}} \subset \mathbf{l}_2$, in a natural way induces an operator J on $\mathbf{l}_2$. For simplicity, *we will demand everywhere in the sequel that the norms of all the matrices a_n, b_n, and c_n are uniformly bounded and, therefore, the operator J is bounded on $\mathbf{l}_2$.*

The essential conditions $a_n > 0$ in (1) now have the form

$$a_n = \underbrace{\left.\begin{bmatrix} a_{n;0,0} & * & * & \dots & * \\ 0 & a_{n;1,1} & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n;n,n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}\right\}}_{n+1} n+2,$$

$$c_n = \underbrace{\left.\begin{bmatrix} c_{n;0,0} & c_{n;0,1} & 0 & \dots & 0 & 0 \\ * & * & c_{n;1,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \dots & c_{n;n-1,n} & 0 \\ * & * & * & \dots & * & c_{n;n,n+1} \end{bmatrix}\right\}}_{n+2} n+1, \tag{10}$$

$$a_{n;0,0}, a_{n;1,1}, \dots, a_{n;n,n} > 0, \quad c_{n;0,1}, c_{n;1,2}, \dots, c_{n;n,n+1} > 0, \quad n \in \mathbb{N}_0$$

(it is convenient to denote a vector $x \in \mathcal{H}_n = \mathbb{C}^{n+1}$ by $x = (x_0, x_1, \dots, x_n)$).

Under some simple conditions on a_n, b_n, and c_n, $n \in \mathbb{N}_n$ (see Section 6), the matrix J is formally normal, i.e., $JJ^+ = J^+J$, where J^+ denotes the matrix adjoint to J. Therefore (due to the boundedness of J) the closure $\tilde{J}$ is a bounded normal operator on $\mathbf{l}_2$.

Let $z \in \mathbb{C}$ belong to the spectrum of $\tilde{J}$ and $P(z) = (P_n(z))_{n=0}^{\infty}$ be the corresponding generalized eigenvector of $\tilde{J}$. Here $P_n(z) \in \mathcal{H}_n$ is a vector-valued polynomial with respect to $z, \bar{z}$ of degree n, i.e., its $n+1$ coordinates are some linear combinations of $z^j \bar{z}^k$,

$0 \le j+k \le n$. According to the generalized eigenvectors expansion theorem, it is some solution of two equations of type (2) (but with matrix coefficients),

$$JP(z) = zP(z), \quad J^+P(z) = \bar{z}P(z). \tag{11}$$

The corresponding Fourier transform $\widehat{\ }$ for the operator $\tilde{J}$ has the form

$$\mathbf{l}_2 \supset \mathbf{l}_{\text{fin}} \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(z) = \sum_{n=0}^{\infty} (f_n, P_n(z))_{\mathcal{H}_n} \in L^2(\mathbb{C}, d\rho(z)) = L^2, \tag{12}$$

where $d\rho(z)$ is the spectral measure of $\tilde{J}$ and has compact support. Operator (12) is a unitary operator (after taking the closure) between $\mathbf{l}_2$ and L_2. The polynomials $P_n(z)$ are orthonormal with respect to $d\rho(z)$ and form a basis in the space L^2. Note that these results are formulated in Theorem 6, but for us it is convenient to denote here these polynomials by

$$(\overline{Q_{n;0}(z)}, \overline{Q_{n;1}(z)}, \ldots, \overline{Q_{n;n}(z)}) = (P_{n;0}(z), P_{n;1}(z), \ldots, P_{n;n}(z)) = P_n(z).$$

So, the results described above make a direct spectral problem for J of type (9) and (10).

The inverse spectral problem now is formulated in the following way. Let us have a probability Borel measure $d\rho(z)$ on $\mathbb{C}$ with a compact support; assume that all the complex moments

$$s_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \, d\rho(z), \quad m, n \in \mathbb{N}_0, \tag{13}$$

exist and the support of $d\rho(z)$ must be such that all the functions $z^j \bar{z}^k$, $j, k \in \mathbb{N}_0$ (belonging to L^2, see (13)), are linearly independent in this space (for example, the support contains some open subset of $\mathbb{C}$). It is necessary to construct a Jacobi type block matrix (9) satisfying (10) in such a way that for the normal operator $\tilde{J}$, its spectral measure be equal to the initial measure.

As in the classical case, it is necessary to apply the standard Schmidt orthogonalization procedure to the sequence of functions in L^2,

$$(z^j \bar{z}^k)_{j,k=0}^{\infty} \tag{14}$$

(instead of (5)). But the sequence (14) has two indices and, therefore, it is necessary to choose a convenient global (linear) order for (14). We order it in the following way:

$$z^0\bar{z}^0 = 1; \quad z^1\bar{z}^0, z^0\bar{z}^1; \quad z^2\bar{z}^0, z^1\bar{z}^1, z^0\bar{z}^2; \quad \ldots\ ; \quad z^n\bar{z}^0, z^{n-1}\bar{z}^1, \ldots, z^0\bar{z}^n; \quad \ldots \tag{15}$$

(see Figure 1 and (49)).[1]

After such a orthogonalization, we get s sequence of polynomials,

$$P_n(z) = (P_{n;0}(z), P_{n;1}(z), \ldots, P_{n;n}(z)), \quad n \in \mathbb{N}_0,$$

and the matrix (9) and (10) is reconstructed by using the formulas of type (7).

The above mentioned results are presented in Sections 4 and 5; it is convenient to start with the orthogonalization (Section 4). All necessary references connected with the projection spectral theorem will be given in Sections 2 and 5. Note that we give now references on some results concerning the related topics that are useful for the presented theory, [19, 35, 14]. Note also that the theory of block Jacobi matrices that are either

[1] *A proofreading remark.* This order of orthogonalization is not new (see [45], Ch. 12, [46]). For the case under consideration, it is necessary to take into account that, e.g., for a bounded operator A to be normal, its parts, $\operatorname{Re} \mathrm{A} = 1/2(\mathrm{A} + \mathrm{A}^*)$ and $\operatorname{Im} \mathrm{A} = 1/2\mathrm{i}(\mathrm{A} - \mathrm{A}^*)$, must be selfadjoint and commuting. Note that the books [45] and [46] contain many interesting facts connected with this article. Our Theorem 6, (82), and results in Section 6 provide answers to some questions formulated in [45], Ch. 12, Subsection 12.3.

Hermitian or selfadjoint operators acting on the spaces $l_2(\mathcal{H}) = \mathcal{H} \oplus \mathcal{H} \oplus \cdots$, where $\mathcal{H}$ is a Hilbert space, was investigated for the first time in [27] for the case $\dim\mathcal{H} < \infty$ and in [3, 4] for the case $\dim\mathcal{H} \leq \infty$. For a discussion of families of commuting selfadjoint operators that act on a symmetric Fock space, see [7]. Note that the Fock space has the form (8) with $\mathcal{H}_n$ that are, for $n > 0$, n-particle infinite-dimensional Hilbert space.

Remark 1. It is very interesting to develop the spectral theory of block Jacobi type normal matrices J (9) and (10) on the space $\mathbf{l}_2$ (8) in the case where the normal operator $\tilde{J}$ is unbounded. What are the conditions on the elements of the matrix J which would guarantee that the operator $\tilde{J}$ is normal? In what terms would it be possible to describe all normal extensions of J on $\mathbf{l}_2$, similarly to the classical Jacobi matrices?

Let us pass now to a comparison of the results contained in this and our previous article [11].

Roughly speaking, it is necessary to apply the above mentioned theory to the case where the matrix J (9) and (10) is a unitary operator. But here is an essential difference: the spectrum of $\tilde{J}$ lies in the unit circle $\mathbb{T} \subset \mathbb{C}$, but on this set the functions (15) are linearly dependent, because $\forall n \in \mathbb{N}_0$

$$z^{j+n}\bar{z}^{k+n} = z^j\bar{z}^k, \quad j,k \in \mathbb{N}_0, \quad z \in \mathbb{T}.$$

Therefore, it is necessary to take only such functions from (15) for which $j \cdot k = 0$, i.e., the functions

$$\begin{aligned}
&z^0\bar{z}^0 = 1; \\
&z^1\bar{z}^0 = z^1, \quad z^0\bar{z}^1 = \bar{z}^1; \\
&z^2\bar{z}^0 = z^2, \quad z^0\bar{z}^2 = \bar{z}^2; \qquad \ldots \\
&z^n\bar{z}^0 = z^n, \quad z^0\bar{z}^n = \bar{z}^n; \qquad \ldots, \qquad z \in \mathbb{T}.
\end{aligned} \tag{16}$$

We stress that, for the classical Jacobi matrices, the situation is similar; there $\bar{z} = z$ for $z \in \mathbb{R}$. The support of $d\rho(z)$ on $\mathbb{T}$ must be an infinite set, then the functions (16) are linearly independent in $L^2(\mathbb{T}, d\rho(z))$.

The global (linear) order for the sequence (16) must be the same as in (15) and Picture 1. But now every "diagonal" in the place $n = 1, 2, \ldots$ contains only 2 points (instead of $n+1$, as earlier). Therefore, it is convenient to consider our operator $\tilde{J}$ on some subspace $\mathbf{l}_{2,u}$ of the space $\mathbf{l}_2$, namely on

$$\mathbf{l}_{2,u} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \text{where} \quad \mathcal{H}_0 = \mathbb{C}, \mathcal{H}_1 = \mathcal{H}_2 = \cdots = \mathbb{C}^2. \tag{17}$$

The corresponding block matrix (9) acts on the space (17), and therefore, its blocks have the structure

$$\begin{aligned}
a_0 :& \quad \mathbb{C} \longrightarrow \mathbb{C}^2, \\
b_0 :& \quad \mathbb{C} \longrightarrow \mathbb{C}, \\
c_0 :& \quad \mathbb{C}^2 \longrightarrow \mathbb{C}, \\
a_n, b_n, c_n :& \quad \mathbb{C}^2 \longrightarrow \mathbb{C}^2, \qquad n \in \mathbb{N} = \{1, 2, \ldots\}.
\end{aligned} \tag{18}$$

The conditions of type $a_n > 0$ from (1) and (10) now have the following form:

$$\begin{gathered}
a_0 = \underbrace{\left.\begin{bmatrix} a_{0;0,0} \\ 0 \end{bmatrix}\right\}}_{1} 2, \quad b_0 = \underbrace{\left.\begin{bmatrix} b_{0;0,0} \end{bmatrix}\right\}}_{1} 1, \quad c_0 = \underbrace{\left.\begin{bmatrix} c_{0;0,0} & c_{0;0,1} \end{bmatrix}\right\}}_{2} 1; \\
a_n = \underbrace{\left.\begin{bmatrix} a_{n;0,0} & a_{n;0,1} \\ 0 & 0 \end{bmatrix}\right\}}_{2} 2, \quad c_n = \underbrace{\left.\begin{bmatrix} 0 & 0 \\ c_{n;1,0} & c_{n;1,1} \end{bmatrix}\right\}}_{2} 2; \\
a_{0;0,0}, c_{0;0,1}, a_{n;0,0}, c_{n;1,1} > 0, \quad n \in \mathbb{N}.
\end{gathered} \tag{19}$$

All previous results are preserved for unitary operator $\tilde{J}$ that acts on the space $\mathbf{l}_{2,u}$ (17) and is given by the block Jacobi type matrix (9) and (18) with condition (19). Namely, the direct spectral problem leads to the Fourier transform of type (12) between the spaces $\mathbf{l}_{2,u}$ and $L^2(\mathbb{T}, d\rho(z))$. The inverse problem has a solution similar to the one described above using the orthogonalization of the sequence (16). These problems are connected with the trigonometric moment problem.

Remark 2. It is necessary to stress that in this article we consider the Jacobi type normal bounded matrices J in "the general case" where the functions (15) are linearly independent in the space $L^2(\mathbb{C}, d\rho(z))$ that is constructed from the spectral measure $d\rho(z)$ of the bounded normal operator $N = \tilde{J}$. This condition in terms of the operator $N = \tilde{J}$ means that if for some coefficients $c_{j,k} \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{j,k=0}^{n} c_{j,k} N^j N^{*k} = 0, \tag{20}$$

then $c_{j,k} = 0$, $\forall j, k \in \{0, 1, \dots, n\}$.

It is easy to understand that the last condition is equivalent to the linear independence of (15). So, let $dE(z)$ be the resolution of identity of N. Then (20) can be written as follows: $\forall f \in \mathbf{l}_2$

$$\int\limits_{\mathbb{C}} \Big| \sum_{j,k=0}^{n} c_{j,k} z^j \bar{z}^k \Big|^2 d(E(z)f, f)_{\mathbf{l}_2} = 0.$$

Using boundedness of the support of $E(\alpha)$ we conclude that the last equality means that $\sum\limits_{j,k=0}^{n} c_{j,k} z^j \bar{z}^k \in L^2(\mathbb{C}, d\rho(z))$ and equals 0. By our assumption, all $c_{j,k} = 0$, i.e, functions (15) are linearly independent in $L^2(\mathbb{C}, d\rho(z))$. The converse assertion is also clear.

Remark 3. It follows from the last remark, that if the normal operator N satisfies a condition of type $NN^* = 1$, $N = N^*$ etc., then its matrix must be considered on the corresponding subspace of $\mathbf{l}_2$ spanned by the respective basis (constructed from (16) or $z^0 \bar{z}^0, z^1 \bar{z}^0, z^2 \bar{z}^0, \dots$, etc).

Note that a similar situation happens also for classical Jacobi matrices, — if the corresponding selfadjoint operator N on the space l_2 has the property that $\sum\limits_{j=0}^{n} c_j N^j = 0$ with some $c_j \in \mathbb{R}$, $(c_0, \dots, c_n) \neq 0$, then its matrix gives rise to an operator on a finite-dimensional subspace of l_2.

The previous article [11] is closely related to the new book [38] and numerous works on orthogonal polynomials on the unit circle; they are cited in [38]. Especially it is related to the articles [15, 16] where it was proposed for the first time to orthogonalize the system (16). But in all these works, the corresponding unitary matrix was investigated on ordinary space l_2 as a 5-diagonal scalar matrix.

This article, of course, is connected to a vast number of works devoted to orthogonal polynomials w.r.t. some measure on the complex plane. We will not consider here these relations and indicate only essential works that contain main results on such polynomials and the corresponding references [22, 43, 42].

It is necessary to stress that many proofs in [11] and in this article are similar. But we wanted to make reading of this article independent of [11].

2. Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space and N a normal operator defined on $\mathrm{Dom}(N)$ in $\mathcal{H}$; N^* be its adjoint, $\mathrm{Dom}(N^*) = \mathrm{Dom}(N)$. Consider a rigging of $\mathcal{H}$,

$$\mathcal{H}_- \supset \mathcal{H} \supset \mathcal{H}_+ \supset \mathcal{D}, \tag{21}$$

such that $\mathcal{H}_+$ is a Hilbert space topologically and quasinuclearly embedded into $\mathcal{H}$ (topologically means densely and continuously; quasinuclearly means that the inclusion operator is of the Hilbert-Schmidt type); $\mathcal{H}_-$ is the dual of $\mathcal{H}_+$ with respect to the space $\mathcal{H}$; $\mathcal{D}$ is a linear, topological space, topologically embedded into $\mathcal{H}_+$.

The operator N is called standardly connected with the chain (21) if $\mathcal{D} \subset \mathrm{Dom}(N)$ and the restrictions $N|\mathcal{D}$, $N^*|\mathcal{D}$ act from $\mathcal{D}$ into $\mathcal{H}_+$ continuously. We consider just this case.

Let us recall that a vector $\Omega \in \mathcal{D}$ is called a strong cyclic vector of operators N and N^* if for any $p, q \in \mathbb{N}$ we have $\Omega \in \mathrm{Dom}(N^p) \cap \mathrm{Dom}((N^*)^q)$, $N^p(N^*)^q\Omega \in \mathcal{D}$ and the set of all these vectors together with Ω, as $p, q = \mathbb{N}_0$, is total in the space $\mathcal{H}_+$ (and, hence, also in $\mathcal{H}$).

Assuming that the strong cyclic vector exists we formulate a short version of the projection spectral theorem (see [5], Ch. 3, Theorem 2.7, or [4] Ch. 5, [6], Ch. 15; [33]).

Theorem 1. *For a normal operator N with a strong cyclic vector in a separable Hilbert space $\mathcal{H}$, there exists a nonnegative finite Borel measure $d\rho(z)$ such that for ρ-almost every $z \in \mathbb{C}$ there exists a generalized joint eigenvector $\xi_z \in \mathcal{H}_-$, i.e.,*

$$(\xi_z, N^*f)_{\mathcal{H}} = z(\xi_z, f)_{\mathcal{H}}, \quad (\xi_z, Nf)_{\mathcal{H}} = \bar{z}(\xi_z, f)_{\mathcal{H}}, \quad z \in \mathbb{C}, \quad f \in \mathcal{D}, \quad \xi_z \neq 0. \tag{22}$$

The corresponding Fourier transform F given by

$$\mathcal{H} \supset \mathcal{H}_+ \ni f \mapsto (Ff)(z) = \hat{f}(z) = (f, \xi_z)_{\mathcal{H}} \in L_2(\mathbb{C}, d\rho(z)) \tag{23}$$

is a unitary operator (after taking the closure) acting from $\mathcal{H}$ into $L^2(\mathbb{C}, d\rho(z))$. The image of the operator N (N^*) under F is an operator of multiplication by z ($\bar{z}$) in $L^2(\mathbb{C}, d\rho(z))$.

Let us also recall that for a selfadjoint operator A defined on $\mathrm{Dom}(A)$ in $\mathcal{H}$, a vector $f \in \bigcap_{n=0}^{\infty} \mathrm{Dom}(A^n)$ is called quasianalytic [31, 32] if the class $\mathrm{C}\{m_n\}$ where, in our case $m_n = \sqrt{\|A^n f\|_{\mathcal{H}}}$, is quasianalytic (recall that this class of functions on $[a, b] \subset \mathbb{R}$ is defined by

$$\mathrm{C}\{m_n\} = \{f \in C^\infty([a, b]) \ \exists K = K_f > 0, \ |f^{(n)}(t)| \leq K^n m_n, \ t \in [a, b], \ n \in \mathbb{N}_0\},$$

or

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\|A^n f\|_{\mathcal{H}}}} = \infty. \tag{24}$$

The quasianalyticity is used in the criterion of selfadjointness and commutativity [4, 31, 32, 5, 6] (see also [37]). For us it will be essential to use following two theorems from [5], Ch. 5, §1, or [6], Ch. 13, §9, and from [30] (see also [34]).

Theorem 2. *A closed Hermitian operator A on a Hilbert space $\mathcal{H}$ is selfadjoint iff there exists a total in $\mathcal{H}$ set of quasianalytic vectors.*

Theorem 3. *Let A_1 and A_2 be two Hermitian operators defined on* $\mathrm{Dom}(A_1)$ *and* $\mathrm{Dom}(A_2)$ *in a Hilbert space $\mathcal{H}$, and let a dense in $\mathcal{H}$ linear set $\mathcal{D}$ be contained in the domains of the operators A_1, A_2, A_1^2, A_1A_2, A_2A_1, and A_2^2 so that $A_1A_2f = A_2A_1f$, for all $f \in \mathcal{D}$.*

If the restriction $A_1^2+A_2^2$ to $\mathcal{D}$ is essentially selfadjoint, then A_1 and A_2 are selfadjoint and commute in the strong resolvent sense.

3. The complex moment problem

The complex moment problem consists of finding a condition on the sequence $\{s_{m,n}\}$, $m,n \in \mathbb{N}_0$, of complex numbers that would imply existence of a nonnegative Borel measure $d\rho(z)$ on the complex plane $\mathbb{C}$ such that

$$s_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n d\rho(z), \quad m,n \in \mathbb{N}_0. \tag{25}$$

Theorem 4. *If a given sequence of complex numbers $\{s_{m,n}\}_{m,n=0}^{\infty}$ admits representation (25) then it is positive definite, i.e.,*

$$\sum_{j,k,m,n=0}^{\infty} f_{j,k}\bar{f}_{m,n}s_{j+n,k+m} \geq 0 \tag{26}$$

for all finite sequences of complex numbers, $(f_{j,k})_{j,k=0}^{\infty}$, $f_{j,k} \in \mathbb{C}$.

For a given sequence of complex numbers$\{s_{m,n}\}_{m,n=0}^{\infty}$, representation (25) exists and is unique if it is positive definite and

$$\sum_{p=1}^{\infty} \frac{1}{\sqrt[2p]{s_{2p,2p}}} = \infty. \tag{27}$$

It is easy to see that condition (26) is necessary for the sequence to have representation (25). It is shown in [41] that replacing (26) to a more complicated and hard to verify condition gives a necessary and sufficient condition for representation (25) to hold but the measure in (25) will not necessary be unique. Our result states that the conditions (26) and (27) give the representation (25) with a unique measure, namely, condition (27) gives this uniqueness. For a one parameter real sequence, another version of Theorem 4 is discussed also in [8].

Proof. Necessity of the condition (26) is obvious. Indeed, if the sequence $\{s_{m,n}\}_{m,n=0}^{\infty}$ has representation (25), then for an arbitrary finite sequence $f = (f_{m,n})_{m,n=0}^{\infty}$, $f_{m,n} \in \mathbb{C}$, we have

$$\sum_{j,k,m,n=0}^{\infty} f_{j,k}\bar{f}_{m,n}s_{j+n,k+m} = \int_{\mathbb{C}} \Big| \sum_{m,n=0}^{\infty} f_{m,n}z^m\bar{z}^n \Big|^2 d\rho(z) \geq 0. \tag{28}$$

Denote by l the linear space $\mathbb{C}^\infty$ of sequences $f = (f_{m,n})_{m,n=0}^{\infty}$, $f_{m,n} \in \mathbb{C}$, and by l_{fin} its linear subspace consisting of finite sequences $f = (f_{m,n})_{m,n=0}^{\infty}$, i.e., the sequences such that $f_{m,n} \neq 0$ for only a finite number of n and m. Let $\delta_{m,n}$, $m,n \in \mathbb{N}_0$, be the δ-sequence such that each $f \in l_{\mathrm{fin}}$ has the representation $f = \sum_{n,m=0}^{\infty} f_{m,n}\delta_{m,n}$.

Let us consider the linear operators on l_{fin},

$$J : (Jf)_{j,k} = f_{j,k-1}, \quad J^+ : (J^+f)_{j,k} = f_{j-1,k}, \quad j,k \in \mathbf{N}_0, \tag{29}$$

where always $f_{j,-1} = f_{-1,k} \equiv 0$. The operators J and J^+ are the "creation" type operators. For the δ-sequence we get

$$J\delta_{j,k} = \delta_{j,k+1}, \quad J^+\delta_{j,k} = \delta_{j+1,k}. \tag{30}$$

The operator J is formally adjoint to J^+ with respect to the (quasi)scalar product, consistent with (28),

$$(f,g)_S = \sum_{j,k,m,n=0}^{\infty} f_{j,k}\bar{g}_{m,n}s_{j+n,k+m}, \quad f,g \in l_{\mathrm{fin}}. \tag{31}$$

Indeed,

$$\begin{aligned}(Jf,g)_S &= \sum_{j,k,m,n=0}^{\infty} (Jf)_{j,k}\bar{g}_{m,n}s_{j+n,k+m}\\ &= \sum_{j,k,m,n=0}^{\infty} f_{j,k-1}\bar{g}_{m,n}s_{j+n,k+m} = \sum_{j,k,m,n=0}^{\infty} f_{j,k}\bar{g}_{m,n}s_{j+n,k+m+1}\\ &= \sum_{j,k,m,n=0}^{\infty} f_{j,k}\bar{g}_{m-1,n}s_{j+n,k+m} = \sum_{j,k,m,n=0}^{\infty} f_{j,k}\overline{(J^+g)}_{m,n}s_{j+n,k+m}\\ &= (f,J^+g)_S.\end{aligned}$$

The operator J commutes with J^+ on l_{fin},

$$(J^+Jf)_{j,k} = f_{j-1,k-1} = (JJ^+f)_{j,k}.$$

Hence, the operator J is formally normal and its adjoint is J^+.

Let S be the Hilbert space obtained as the completion of the factor space

$$\dot{l}_{\mathrm{fin}} = l_{\mathrm{fin}}/\{h \in l_{\mathrm{fin}} \mid (h,h)_S = 0\}.$$

The element f of S is a representative of the class $\dot{f}$ of equivalent elements of $\dot{l}_{\mathrm{fin}}$. Hence, the operators $\dot{J}$ and $\dot{J}^+$ are well defined on S. This fact in case of a selfadjoint operator is described in detail in [4], Ch. 8, §1, Subsect. 4 and [5], Ch. 5, §5, Subsect. 2. Similarly to this case, we get

$$\dot{J}\dot{f} = (Jf)^{\cdot}, \quad f \in \mathrm{Dom}(\dot{J}) = \dot{l}_{\mathrm{fin}}; \quad \dot{J}^+\dot{f} = (J^+f)^{\cdot}, \quad f \in \mathrm{Dom}(\dot{J}^*) = \dot{l}_{\mathrm{fin}}. \tag{32}$$

In the next considerations denote by N and N^+ the closure $\sim$ of $\dot{J}$ and $\dot{J}^+$ in S.

In the next step we use Theorem 3. For simplicity, we suppose that the given sequence $\{s_{m,n}\}$ is nondegenerate, i.e., if $(f,f)_S = 0$ for $f \in l_{\mathrm{fin}}$, then $f = 0$, and now $\dot{f} = f$ and $\tilde{\dot{J}} = N$. The investigation in the general case is more complicated, see in [4], Ch. 8, §1, Subsect. 4 and [5], Ch. 5, §5, Subsect. 1–3. We also assume for a moment that the operator N is normal. Later we will prove that N is normal under the condition (27). In general, conditions for normality of extensions of a formally normal operator connected with the complex moment problem are described in [24, 41, 40, 39] (see also [20, 21]).

Let us construct a rigging of spaces,

$$(l_2(p))_{-,S} \supset S \supset l_2(p) \supset l_{\mathrm{fin}}, \tag{33}$$

where $l_2(p)$ is a weighted l_2-space with the weight $p = (p_{m,n})_{m,n=0}^{\infty}$, $p_n \geq 1$. The norm in $l_2(p)$ is given by $\|f\|^2_{l_2(p)} = \sum_{m,n=0}^{\infty} |f_{m,n}|^2 p_{m,n}$; $(l_2(p))_{-,S} = \mathcal{H}_-$ is a negative space with respect to the positive space $l_2(p) = \mathcal{H}_+$ and the zero space $S = \mathcal{H}$. The space $l_{\mathrm{fin}} = \mathcal{D}$ is provided with the coordinate-wise uniform finite convergence.

Lemma 1. *If the sequence $p_{m,n}$ is sufficiently fast increasing, then there is an embedding $l_2(p) \hookrightarrow S$ and it is quasinuclear.*

Proof. The equality (26) means that the multimatrix $(K_{j,k;m,n})_{j,k,m,n=0}^{\infty}$, where $K_{j,k;m,n} = s_{j+n,k+m}$ is nonnegative definite and, therefore,

$$|s_{j+n,k+m}|^2 = |K_{j,k;m,n}|^2 \leq K_{j,k;j,k}K_{m,n;m,n} = s_{j+k,j+k}s_{m+n,m+n}, \quad j,k,m,n \in \mathbb{N}_0. \tag{34}$$

Let the weight $q = (q_{j,k})_{j,k=0}^{\infty}$, $q_{j,k} \geq 1$, be such that $\sum_{j,k=0}^{\infty} s_{j+k,j+k} q_{j,k}^{-1} < \infty$. Then from (31) and (34) it follows that

$$\|f\|_S^2 = \sum_{j,k,m,n=0}^{\infty} f_{j,k} \bar{f}_{m,n} s_{j+n,k+m} \leq \Big(\sum_{j,k=0}^{\infty} \frac{s_{j+k,j+k}}{q_{j,k}} \Big) \|f\|_{l_2(q)}^2, \quad f \in l_{\mathrm{fin}}.$$

Therefore, $l_2(q) \hookrightarrow S$ topologically. And if $\sum_{j,k=0}^{\infty} q_{j,k} p_{j,k}^{-1} < \infty$, then $l_2(p) \hookrightarrow l_2(q)$ quasinuclearly. The composition $l_2(p) \hookrightarrow S$ of the quasinuclear and topological embeddings is also quasinuclear. $\square$

In the next step we use the rigging (33) to construct generalized eigenvectors. The inner structure of the space $(l_2(p))_{-,S}$ is complicated, because of the complicated structure of S. This is a reason for introducing a new auxiliary rigging,

$$l = (l_{\mathrm{fin}})' \supset (l_2(p^{-1})) \supset l_2 \supset l_2(p) \supset l_{\mathrm{fin}}, \tag{35}$$

where $l_2(p^{-1})$, $p^{-1} = (p_{m,n}^{-1})_{m,n=0}^{\infty}$ is a space negative with respect to the positive space $l_2(p)$ and the zero space l_2. Chains (33) and (35) have the same positive space $l_2(p)$. The next general Lemma [8] establishes that the space $(l_2(p))_{-,S}$ is isometric to the space $l_2(p^{-1})$.

Lemma 2. *Suppose we have two riggings,*

$$\mathcal{K}_- \supset \mathcal{K} \supset \mathcal{K}_+, \quad \mathcal{F}_- \supset \mathcal{F} \supset \mathcal{F}_+ = \mathcal{K}_+, \tag{36}$$

with equal positive spaces. Then there exist a unitary operator $U : \mathcal{K}_- \to \mathcal{F}_-$, $U\mathcal{K}_- = \mathcal{F}_-$, such that

$$(U\xi, f)_{\mathcal{F}} = (\xi, f)_{\mathcal{K}}, \quad \xi \in \mathcal{K}_-, \quad f \in \mathcal{K}_+ = \mathcal{F}_+. \tag{37}$$

This operator can be given as follows: $U = \mathbb{I}_{\mathcal{F}}^{-1} \mathbb{I}_{\mathcal{K}}$, where $\mathbb{I}_{\mathcal{F}}$ and $\mathbb{I}_{\mathcal{K}}$ are standard two unitaries in the corresponding chains ($\mathbb{I}_{\mathcal{F}} \mathcal{F}_- = \mathcal{F}_+$, $\mathbb{I}_{\mathcal{K}} \mathcal{K}_- = \mathcal{K}_+$).

Instead of riggings (36), we consider (33) and (35).

Let $\xi_z \in (l_2(p))_{-,S}$ be a generalized eigenvector of the operator N in terms of the chain (33). So, in this case due to Theorem 3,

$$(\xi_z, N^* f)_S = z(\xi_z, f)_S, \quad (\xi_z, N f)_S = \bar{z}(\xi_z, f)_S, \quad z \in \mathbb{C}, \quad f \in l_{\mathrm{fin}}. \tag{38}$$

Denote $P(z) = U\xi_z \in l_2(p^{-1}) \subset l$, $P(z) = (P_{m,n}(z))_{m,n=0}^{\infty}$, $P_{m,n}(z) \in \mathbb{C}$. Using (37) we can rewrite (38) in the form

$$(P(z), N^* f)_{l_2} = z(P(z), f)_{l_2}, \quad (P(z), N f)_{l_2} = \bar{z}(P(z), f)_{l_2}, \quad z \in \mathbb{C}, \quad f \in l_{\mathrm{fin}}. \tag{39}$$

The corresponding Fourier transform has the form

$$S \supset l_{\mathrm{fin}} \ni f \to (Ff)(z) = \hat{f}(z) = (f, P(z))_{l_2} \in L^2(\mathbb{C}, d\rho(z)). \tag{40}$$

Let us calculate $P(z)$. The operator N^* is obtained from the second formula in (29) and, therefore, (39) gives $\forall f \in l_{\mathrm{fin}}$

$$\begin{aligned} \sum_{m,n=0}^{\infty} z P_{m,n}(z) \bar{f}_{m,n} &= z(P(z), f)_{l_2} = (P(z), N^* f)_{l_2} \\ &= (P(z), J^+ f)_{l_2} = \sum_{m,n=0}^{\infty} P_{m+1,n}(z) \bar{f}_{m,n}. \end{aligned} \tag{41}$$

Analogously, using (29) and (39), we have $\forall f \in l_{\text{fin}}$

$$\begin{aligned}\sum_{m,n=0}^{\infty} \bar{z} P_{m,n}(z) \bar{f}_{m,n} &= \bar{z}(P(z), f)_{l_2} = (P(z), Nf)_{l_2} \\ &= (P(z), Jf)_{l_2} = \sum_{m,n=0}^{\infty} P_{m,n+1}(z) \bar{f}_{m,n}.\end{aligned} \tag{42}$$

Hence we have

$$zP_{m,n}(z) = P_{m+1,n}(z), \quad \bar{z}P_{m,n}(z) = P_{m,n+1}(z), \quad m,n \in \mathbb{N}_0.$$

Without loss of generality, we can take $P_{0,0}(z) = 1,\ z \in \mathbb{C}$. Then last two equalities give

$$P_{m,n}(z) = z^m \bar{z}^n, \quad m,n \in \mathbb{N}_0. \tag{43}$$

Thus the Fourier transform (40) has the form

$$S \supset l_{\text{fin}} \ni f \to (Ff)(z) = \hat{f}(z) = \sum_{m,n=0}^{\infty} f_{m,n} z^m \bar{z}^n \in L^2(\mathbb{C}, d\rho(z)), \tag{44}$$

and

$$(f,g)_S = \int_{\mathbb{C}} \hat{f}(z)\overline{\hat{g}(z)} d\rho(z), \quad f,g \in l_{\text{fin}}. \tag{45}$$

To construct the Fourier transform in (40) and verify the formulas (41)—(45), it is still necessary to check that, for our operators N and N^*, the vector $\Omega = \delta_{0,0} \in l_{\text{fin}}$ is strong cyclic in the sense of the chain (33). But this is evidently true, since by (30), $N^p(N^*)^q\Omega = J^p(J^+)^q\delta_{0,0} = \delta_{q,p}$.

The Parseval equality (45) immediately leads to representation (25); according to (43) and (44), $\hat{\delta}_{m,n} = z^m \bar{z}^n$ and $\hat{\delta}_{0,0} = 1$; by (31) we get

$$s_{m,n} = (\delta_{m,n}, \delta_{0,0})_S = (\hat{\delta}_{m,n}, \hat{\delta}_{0,0})_{L_2(\mathbb{C}, d\rho(z))} = \int_{\mathbb{C}} z^m \bar{z}^n d\rho(z), \quad m,n \in \mathbb{N}_0, \quad z \in \mathbb{C}.$$

The uniqueness of representation (25) follows from normality of the operator N (compare with [4], Ch.8).

So, to finish the proof of Theorem 4 it is only necessary to check that condition (27) provides normality of N. For this reason, we introduce two closed Hermitian operators defined on a linear set $\mathcal{D} = l_{\text{fin}} = \text{span}\{\delta_{m,n} \mid m,n \in \mathbb{N}_0\}$, invariant with respect to the action of these operators,

$$A_1 = \frac{1}{2}(N + N^*), \quad A_2 = \frac{1}{2i}(N - N^*).$$

For normality of N, it is sufficient to show that the operators A_1 and A_2 are selfadjoint and commutative in the strong resolvent sense. But to do this (see Theorems 2 and 3) we must only check that the operator $A_1^2 + A_2^2$ has a total set $\mathcal{D}$ of quasianalytic vectors. (Let us recall that, instead of Theorem 2 and 3, it is also possible to directly use Lemma 4 from [40]).

Due to (29), the operator $\mathcal{A} = A_1^2 + A_2^2$ acts on $\delta_{m,n} \in \mathcal{D}$ as follows:

$$\mathcal{A}\delta_{m,n} = (A_1^2 + A_2^2)\delta_{m,n} = NN^+\delta_{m,n} = \delta_{m+1,n+1}. \tag{46}$$

Obviously, $\mathcal{A} \geq 0$. For $p \geq 1$,

$$\mathcal{A}^p\delta_{m,n} = \delta_{m+p,n+p}.$$

According to (31) we have the norm $\|f\|_S = \sqrt{(f,f)_S}$ in S. Hence $\forall \delta_{m,n} \in \mathcal{D}$ we obtain

$$\|\mathcal{A}^p\delta_{m,n}\|_S^2 = \|\delta_{m+p,n+p}\|_S^2 = s_{m+n+2p,m+n+2p}.$$

Since

$$\sum_{p=1}^{\infty}\frac{1}{\sqrt[p]{\|\mathcal{A}^p\delta_{m,n}\|}}=\sum_{p=1}^{\infty}\frac{1}{\sqrt[2p]{s_{m+n+2p,m+n+2p}}},$$

we conclude that quasianalyticity of the class $\mathrm{C}\{\|\mathcal{A}^p\delta_{m,n}\|\}$ is equivalent to quasianalyticity of the class $\mathrm{C}\{\sqrt{s_{m+n+2p,m+n+2p}}\}$ and, due to the quasianalyticity properties [17, 29], it is equivalent to quasianalyticity of the class $\mathrm{C}\{\sqrt{s_{2p,2p}}\}$. But this quasianalyticity is equivalent to the condition (27), taking into account that $s_{2p,2p}=\|\mathcal{A}^p\delta_{0,0}\|_S^2$. This completes the proof of the Theorem 4. $\square$

Remark 4. The condition (26) is only a necessary one in Theorem 4, i.e., it is not enough to have this condition for the representation (25).To make it more clear, we give a simple counterexample.

Let A_1 and A_2 be two selfadjoint operators commuting on a linear set $\mathcal{D}$ dense in a Hilbert space $\mathcal{H}$, where $\mathcal{D}$ is invariant under the action of A_1 and A_2. Suppose A_1 and A_2 are essentially selfadjoint on $\mathcal{D}$, i.e., $A_1=(A_1|\mathcal{D})^\sim$, $A_2=(A_2|\mathcal{D})^\sim$, but A_1 and A_2 do not commute in the strong resolvent sense. Existence of such sort operators is guaranteed by the Nelson's example [30] (see also [6], Ch. 13, §9).

For the next, we put $N=(A_1+iA_2)|\mathcal{D}$ and $N^+=(A_1-iA_2)|\mathcal{D}$. In this case, N is a formally normal operator which has no normal extensions [18, 36].

The sequence

$$s_{m,n}:=(N^m f_0,N^n f_0)_{\mathcal{H}},\quad f_0\in\operatorname{Dom}(N^n),\quad n\in\mathbb{N},$$

obviously satisfies the condition (26), but cannot admit the representation (25) (see analogous situation in [4], Ch. 8). Indeed, for a finite sequence $f=(f_{j,k})_{j,k=0}^{\infty}$, we have

$$\begin{aligned}\sum_{j,k,m,n=0}^{\infty}f_{j,k}\bar f_{m,n}s_{j+n,k+m}&=\sum_{j,k,m,n=0}^{\infty}f_{j,k}\bar f_{m,n}(N^{j+n}f_0,N^{k+m}f_0)_{\mathcal{H}}\\&=\Big(\sum_{j,k=0}^{\infty}f_{j,k}N^j(N^+)^k f_0,\sum_{m,n=0}^{\infty}f_{m,n}N^m(N^+)^n f_0\Big)_{\mathcal{H}}\\&=\Big\|\sum_{m,n=0}^{\infty}f_{m,n}N^m(N^+)^n f_0\Big\|_{\mathcal{H}}^2\geq 0.\end{aligned}$$

Remark 5. Additionally to condition (26), we obtain from (46) the second necessary condition, $\mathcal{A}\geq 0$. This gives an additional positive definiteness condition of the form

$$\sum_{j,k,m,n=0}^{\infty}f_{j,k}\bar f_{m,n}s_{j+n+1,k+m+1}\geq 0 \tag{47}$$

on the same sequences as in (26). Note that two conditions (26) and (27) together give (47).

4. The orthogonalization procedure and construction of a three-diagonal block matrix of bounded normal operators

Let $d\rho(z)$ be a probability Borel measure on $\mathbb{C}$ with compact support and $L^2=L^2(\mathbb{C},d\rho(z))$ the space of square integrable complex-valued functions defined on $\mathbb{C}$. We suppose that the support of this measure is an infinite set such that the functions $\mathbb{C}\ni z\longmapsto z^m\bar z^n$, $m,n\in\mathbb{N}_0$, are linearly independent in L^2.

In order to find an analog of the Jacobi matrix J, there is need to choose an order for the orthogonalization in L^2 applied to the following family of functions:

$$\{z^j\bar z^k\},\quad j,k\in\mathbb{N}_0. \tag{48}$$

Our assumption about compactness of $\operatorname{supp}(d\rho(z))$ implies that this family is total in L^2.

We use the following total (linear) order for the orthogonalization via the Schmidt procedure:

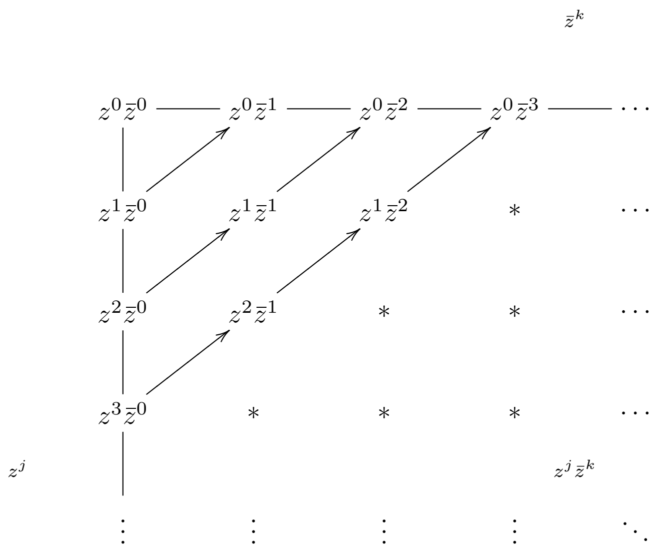

FIGURE 1. The order of the orthogonalization

According to Figure 1 we get

$$z^0\bar{z}^0; \quad z^1\bar{z}^0, z^0\bar{z}^1; \quad z^2\bar{z}^0, z^1\bar{z}^1, z^0\bar{z}^2; \quad \dots ; \quad z^n\bar{z}^0, z^{n-1}\bar{z}^1, \dots, z^0\bar{z}^n; \quad \dots \tag{49}$$

Applying the Schmidt orthogonalization procedure to (49) (see, for example [6] Ch. 7) we obtain an orthonormal system of polynomials indexed in the following way:

$$\begin{array}{lllll} P_{0;0}(z); & P_{1;0}(z), & P_{2;0}(z), & \dots; & P_{n;0}(z), \quad \dots \\ & P_{1;1}(z); & P_{2;1}(z), & & P_{n;1}(z), \\ & & P_{2;2}(z); & & P_{n;2}(z), \\ & & & & \quad\dots \\ & & & & P_{n;n}(z); \end{array} \tag{50}$$

where each polynomial has the form $P_{n;\alpha}(z) = k_{n;\alpha} z^{n-\alpha}\bar{z}^{\alpha} + \cdots$, $n \in \mathbb{N}_0$, $\alpha = 0, 1, \dots, n$, $k_{n;\alpha} > 0$; here $+\cdots$ denotes the preceding part of the corresponding polynomial; $P_{0;0}(z) = 1$. In such a way, $P_{n;\alpha}$ is some linear combination of

$$\{1;\ z^1\bar{z}^0, z^0\bar{z}^1;\ \dots;\ z^n\bar{z}^0, z^{n-1}\bar{z}^1, \dots, z^{n-\alpha}\bar{z}^{\alpha}\}. \tag{51}$$

Since the family (48) is total in the space L^2, the sequence (50) gives an orthonormal basis in this space.

Denote the subspace spanned by elements in (51) by $\mathcal{P}_{n;\alpha}$ $\forall n \in \mathbb{N} = \{1, 2, \dots\}$.

It is clear that $\forall n \in \mathbb{N}$ we have

$$\begin{aligned} &\mathcal{P}_{0;0} \subset \mathcal{P}_{1;0} \subset \mathcal{P}_{1;1} \subset \mathcal{P}_{2;0} \subset \mathcal{P}_{2;1} \subset \mathcal{P}_{2;2} \subset \cdots \subset \mathcal{P}_{n;0} \subset \mathcal{P}_{n;1} \subset \cdots \subset \mathcal{P}_{n;n} \subset \cdots, \\ &\mathcal{P}_{n;\alpha} = \{P_{0;0}(z)\} \oplus \{P_{1;0}(z)\} \oplus \{P_{1;1}(z)\} \oplus \{P_{2;0}(z)\} \oplus \{P_{2;1}(z)\} \oplus \{P_{2;2}(z)\} \oplus \cdots \\ &\qquad \oplus \{P_{n;0}(z)\} \oplus \{P_{n;1}(z)\} \oplus \cdots \oplus \{P_{n;\alpha}(z)\}, \end{aligned} \tag{52}$$

where $\{P_{n;\alpha}(z)\}$, $n \in \mathbb{N}$, $\alpha = 0, 1, \dots n$, denotes the one-dimensional space spanned by $P_{n;\alpha}(z)$; $\mathcal{P}_{0;0} = \mathbb{C}$.

For the next investigation we need, instead of the usual space l_2, the Hilbert space

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \mathcal{H}_n = \mathbb{C}^{n+1}, \quad n \in \mathbb{N}_0. \tag{53}$$

Each vector $f \in \mathbf{l}_2$ has the form $f = (f_n)_{n=0}^{\infty}$, $f_n \in \mathcal{H}_n$, and consequently $\forall f, g \in \mathbf{l}_2$

$$\|f\|_{\mathbf{l}_2}^2 = \sum_{n=0}^{\infty} \|f_n\|_{\mathcal{H}_n}^2 < \infty, \quad (f, g)_{\mathbf{l}_2} = \sum_{n=0}^{\infty} (f_n, g_n)_{\mathcal{H}_n}.$$

For $n \in \mathbb{N}_0$, the coordinates of the vector $f_n \in \mathcal{H}_n$ in some orthonormal basis $\{e_{n;0},\, e_{n;1},\, e_{n;2},\, \dots,\, e_{n;n}\}$ in the space $\mathbb{C}^{n+1}$ are denoted by $(f_{n;0}, f_{n;1}, f_{n;2}, \dots, f_{n;n})$ and, hence, we have $f_n = (f_{n;0}, f_{n;1}, f_{n;2}, \dots, f_{n;n})$. It is clear that the space $\mathbf{l}_2$ is isometric to $l_2 \times l_2$.

Using the orthonormal system (50) one can define a mapping of $\mathbf{l}_2$ into L^2. We put $\forall n \in \mathbb{N}_0$ and $\forall z \in \mathbb{C}$, $P_n(z) = (P_{n;0}, P_{n;1}(z), P_{n;2}(z), \dots, P_{n;n}) \in \mathcal{H}_n$,

$$\mathbf{l_2} \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(z) = \sum_{n=0}^{\infty} (f_n, P_n(z))_{\mathcal{H}_n} \in L^2. \tag{54}$$

Since for $n \in \mathbb{N}_0$ we get

$$(f_n, P_n(z))_{\mathcal{H}_n} = f_{n;0}\overline{P_{n;0}(z)} + f_{n;1}\overline{P_{n;1}(z)} + f_{n;2}\overline{P_{n;2}(z)} + \cdots + f_{n;n}\overline{P_{n;n}(z)}$$

and

$$\|f\|_{\mathbf{l}_2}^2 = \|(f_{0;0}, f_{1;0}, f_{1;1}, f_{2;0}, f_{2;1}, f_{2;2}, \dots, f_{n;0}, , f_{n;1}, \dots, f_{n;n}, \dots)\|_{l_2}^2,$$

we see that (54) is a mapping of the space $l_2 \times l_2$ into L^2, and the use of the orthonormal system (50) shows that this mapping is isometric. The image of $\mathbf{l}_2$ under the mapping (54) coincides with the space L^2, because under our assumption the system (50) is an orthonormal basis in L^2. Therefore the mapping (54) is a unitary transformation (denoted by I) that acts from $\mathbf{l}_2$ onto L^2.

Let A be a bounded linear operator defined on the space $\mathbf{l}_2$. It is possible to construct an operator matrix $(a_{j,k})_{j,k=0}^{\infty}$, where for each $j, k \in \mathbb{N}_0$ the element $a_{j,k}$ is an operator from $\mathcal{H}_k$ into $\mathcal{H}_j$, so that $\forall f, g \in \mathbf{l}_2$ we have

$$(Af)_j = \sum_{k=0}^{\infty} a_{j,k} f_k, \; j \in \mathbb{N}_0, \quad (Af, g)_{\mathbf{l}_2} = \sum_{j,k=0}^{\infty} (a_{j,k} f_k, g_j)_{\mathcal{H}_j}. \tag{55}$$

To prove (55), we only need to write the usual matrix of the operator A in the space $l_2 \times l_2$ using the basis

$$(e_{0;0};\; e_{1;0}, e_{0;1};\; e_{2;0}, e_{2;1}, e_{2;2};\; \dots;\; e_{n;0}, e_{n;1}, \dots, e_{n;n};\; \dots), \quad e_{0;0} = 1. \tag{56}$$

Then $a_{j,k}$ for each $j, k \in \mathbb{N}_0$ is an operator $\mathcal{H}_k \longrightarrow \mathcal{H}_j$ that has the matrix representation

$$a_{j,k;\alpha,\beta} = (Ae_{k;\beta}, e_{j;\alpha})_{\mathbf{l}_2}, \tag{57}$$

where $\alpha = 0, 1, \dots, j$ and $\beta = 0, 1, \dots, k$. We will write $a_{j,k} = (a_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{j,k}$ (including the cases $a_{0,k} = (a_{0,k;\alpha,\beta})_{\alpha,\beta=0}^{0,k}$, $a_{j,0} = (a_{j,0;\alpha,\beta})_{\alpha,\beta=0}^{j,0}$ and $a_{0,0} = (a_{0,0;\alpha,\beta})_{\alpha,\beta=0}^{0,0} = a_{0,0;0,0}$).

Note that the same representation (55) is also valid for a general operator A on the space $\mathbf{l}_2$ with the domain $\mathrm{Dom}(A) = \mathbf{l}_{\mathrm{fin}} \subset \mathbf{l}_2$, where $\mathbf{l}_{\mathrm{fin}}$ denotes the set of finite vectors from $\mathbf{l}_2$. In this case, the first formula in (55) takes place for $f \in \mathbf{l}_{\mathrm{fin}}$; in the second formula, $f \in \mathbf{l}_{\mathrm{fin}}$, $g \in \mathbf{l}_2$.

Let us consider the image $\hat{A} = IAI^{-1} : L^2 \longrightarrow L^2$ of the above bounded operator $A : \mathbf{l}_2 \longrightarrow \mathbf{l}_2$ defined by the mapping (54). Its matrix in the basis (50),

$$(P_{0;0}(z);\ P_{1;0}(z), P_{1;1}(z);\ P_{2;0}(z), P_{2,1}(z), P_{2,2}(z);\ \ldots;\\ P_{n;0}(z), P_{n;1}(z), \ldots, P_{n;n}(z); \ldots),$$

is equal to the usual matrix of operator A regarded as the operator: $l_2 \times l_2 \longrightarrow l_2 \times l_2$ in the corresponding basis (56). Using (57) and the above mentioned procedure, we get the operator matrix $(a_{j,k})_{j,k=0}^{\infty}$ of $A : l_2 \times l_2 \longrightarrow l_2 \times l_2$. *By the definition, this matrix is also the operator matrix of $\hat{A} : L^2 \longrightarrow L^2$.*

It is clear that $\hat{A}$ can be an arbitrary linear bounded operator in L^2.

Lemma 3. *For the polynomials $P_{n;\alpha}(z)$ and the subspaces $\mathcal{P}_{m,\beta}$, $n, m \in \mathbb{N}_0$, $\alpha = 0, 1, \ldots, n$, $\beta = 0, 1, \ldots, m$, the following relations hold:*

$$zP_{n;\alpha}(z) \in \mathcal{P}_{n+1;\alpha}, \quad \bar{z}P_{n;\alpha}(z) \in \mathcal{P}_{n+1;\alpha+1}. \tag{58}$$

Proof. According to (50), the polynomial $P_{n;\alpha}(z)$, $n \in \mathbb{N}_0$, is equal to some linear combination of $\{1;\ z^1\bar{z}^0,\ z^0\bar{z}^1;\ \ldots;\ z^n\bar{z}^0,\ z^{n-1}\bar{z}^1,\ \ldots,\ z^{n-\alpha}\bar{z}^{\alpha}\}$. Hence, multiplying by z we obtain a linear combination of $\{z;\ z^2\bar{z}^0,\ z^1\bar{z}^1;\ \ldots;\ z^{n+1}\bar{z}^0,\ z^n\bar{z}^1,\ \ldots,\ z^{n+1-\alpha}\bar{z}^{\alpha}\}$, and this linear combination belongs to $\mathcal{P}_{n+1;\alpha}$. Analogously multiplying by $\bar{z}$ we obtain a linear combination of $\{\bar{z}^1;\ z^1\bar{z}^1,\ z^0\bar{z}^2;\ \ldots;\ z^n\bar{z}^1,\ z^{n-1}\bar{z}^2,\ \ldots,\ z^{n-\alpha}\bar{z}^{\alpha+1}\}$, and this linear combination belongs to $\mathcal{P}_{n+1;\alpha+1}$, since $z^{n-\alpha}\bar{z}^{\alpha+1} \in \mathcal{P}_{n+1,\alpha+1}$. ☐

Lemma 4. *Let $\hat{A}$ be the normal bounded operator of multiplication by z in the space L^2:*

$$L^2 \ni \varphi(z) \longmapsto (\hat{A}\varphi)(z) = z\varphi(z) \in L^2.$$

The operator matrix $(a_{j,k})_{j,k=0}^{\infty}$ of $\hat{A}$ (i.e. of $A = I^{-1}\hat{A}I$) has a three-diagonal structure, $a_{j,k} = 0$ for $|j-k| > 1$.

Proof. Using (57) for $e_{n;\gamma} = I^{-1}P_{n;\gamma}(z)$, $n \in \mathbb{N}_0$; $\gamma = 0, 1, \ldots, n$, we have $\forall j, k \in \mathbb{N}_0$

$$a_{j,k;\alpha,\beta} = (Ae_{k;\beta}, e_{j;\alpha})_{l_2} = \int_{\mathbb{C}} zP_{k;\beta}(z)\overline{P_{j;\alpha}(z)}d\rho(z), \tag{59}$$

where $\alpha = 0, 1, \ldots, j$, $\beta = 0, 1, \ldots, k$. From (58) we have $zP_{k;\alpha}(z) \in \mathcal{P}_{k+1;\alpha}$. According to (52), the integral in (59) is equal to zero for $j > k+1$ and for each $\alpha = 0, 1, \ldots, j$.

On another hand, the integral in (59) has the form

$$a_{j,k;\alpha,\beta} = \overline{\int_{\mathbb{C}} \bar{z}P_{j;\alpha}(z)\overline{P_{k;\beta}(z)}\, d\rho(z)}. \tag{60}$$

From (58) we have now that $\bar{z}P_{j;\alpha}(z) \in \mathcal{P}_{j+1;\alpha+1}$. According to (52), the last integral is equal to zero for $k > j+1$ and each $\beta = 0, 1, \ldots, k$.

As a result, the integral in (60), i.e., the coefficients $a_{j,k;\alpha,\beta}$, $j, k \in \mathbb{N}_0$, are equal to zero for $|j-k| > 1$; $\alpha = 0, 1, \ldots, j$, $\beta = 0, 1, \ldots, k$. (In the previous considerations it was necessary to take into account that $e_{0;0} = I^{-1}P_{0;0}(z) = 1$). ☐

In such a way, the matrix $(a_{j,k})_{j,k=0}^{\infty}$ of the operator $\hat{A}$ has the three-diagonal block structure

$$\begin{bmatrix} a_{0,0} & a_{0,1} & 0 & 0 & 0 & \ldots \\ a_{1,0} & a_{1,1} & a_{1,2} & 0 & 0 & \ldots \\ 0 & a_{2,1} & a_{2,2} & a_{2,3} & 0 & \ldots \\ 0 & 0 & a_{3,2} & a_{3,3} & a_{3,4} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{61}$$

A more careful analysis of expressions (59) allows to find out which elements of the matrices $(a_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{j,k}$ are zero and which are not in each case for $|j-k|\leq 1$. We can also describe properties of the matrix with respect to permutation of the indexes j,k, and α,β.

Let us denote by $((a^*)_{j,k})_{j,k=0}^{\infty}$ the operator matrix of the operator $(\hat{A})^*$ that is adjoint to $\hat{A}$. Note that $(\hat{A})^*$ is an operator of multiplication by $\bar{z}$. Taking into account the expression (59) for $j,k\in\mathbb{N}_0$ we have

$$(a^*)_{j,k;\alpha,\beta}=\int_{\mathbb{C}}\bar{z}P_{k;\beta}(z)\overline{P_{j;\alpha}(z)}\,d\rho(z)=\overline{\int_{\mathbb{C}}zP_{j;\alpha}(z)\overline{P_{k;\beta}(z)}\,d\rho(z)}=\overline{a_{k,j;\beta,\alpha}}, \tag{62}$$

where $\alpha=0,1,\ldots,j$ and $\beta=0,1,\ldots,k$.

Lemma 5. *Let $(a_{j,k})_{j,k=0}^{\infty}$ be an operator matrix for the operator of multiplication by z in L^2, where $a_{j,k}:\mathcal{H}_k\longrightarrow\mathcal{H}_j$; $a_{j,k}=(a_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{j,k}$ are matrices of operators $a_{j,k}$ in the corresponding orthonormal basis. Then $\forall j\in\mathbb{N}_0$*

$$\begin{array}{ll}\forall\alpha=0,1,\ldots j-1 & a_{j,j+1;\alpha,\alpha+2}=a_{j,j+1;\alpha,\alpha+3}=\cdots=a_{j,j+1;\alpha,j+1}=0;\\ \forall\beta=0,1,\ldots j & a_{j+1,j;\beta+1,\beta}=a_{j+1,j;\beta+2,\beta}=\cdots=a_{j+1,j;j+1,\beta}=0.\end{array} \tag{63}$$

If we choose, inside of each diagonal $\{z^n\bar{z}^0,z^{n-1}\bar{z}^1,z^{n-2}\bar{z}^2,\ldots,z^0\bar{z}^n\}$ in Figure 1, another order (preserving the order of the diagonals), then Lemma 5 is not true, but it will still be possible to describe the zeros of the matrices $(a_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{j,k}$. Such matrices $(a_{j,k})_{j,k=0}^{\infty}$ also have a three-diagonal block structure and zeros although in another places.

Proof. According to (59) and (58), for $j\in\mathbb{N}_0$ we have $\forall\alpha=0,1,\ldots,j$ and $\forall\beta=0,1,\ldots,j+1$,

$$a_{j,j+1;\alpha,\beta}=\int_{\mathbb{C}}zP_{j+1,\beta}(z)\overline{P_{j;\alpha}(z)}\,d\rho(z)=\overline{\int_{\mathbb{C}}\bar{z}P_{j,\alpha}(z)\overline{P_{j+1;\beta}(z)}\,d\rho(z)},$$

where $\bar{z}P_{j;\alpha}(z)\in\mathcal{P}_{j+1;\alpha+1}$. But according to (52), $P_{j+1;\beta}(z)$ is orthogonal to $\mathcal{P}_{j+1;\alpha+1}$ for $\beta>\alpha+1$ and, hence, the last integral is equal to zero. This gives the first equality in (63).

Analogously from (59) and (58), for $j\in\mathbb{N}_0$ we have $\forall\alpha=0,1,\ldots,j+1$ and $\forall\beta=0,1,\ldots,j$ that

$$a_{j+1,j;\alpha,\beta}=\int_{\mathbb{C}}zP_{j,\beta}(z)\overline{P_{j+1;\alpha}(z)}\,d\rho(z),$$

where $zP_{j;\beta}(z)\in\mathcal{P}_{j+1;\beta}$. But according to (52), $P_{j+1;\alpha}(z)$ is orthogonal to $\mathcal{P}_{j+1;\beta}$ if $\alpha>\beta$ and, hence, the last integral is equal to zero. This gives the second identity in (63). ☐

The above shows that the $((j+1)\times(j+2))$ and $((j+2)\times(j+1))$-matrices in (61), $a_{j,j+1}$ and $a_{j+1,j}$, have all their entries above the second and below the first diagonal, respectively, zeros. Taking into account (61) we can conclude that the normal matrix of the operator of multiplication by z is a multi-diagonal usual scalar matrix in the usual basis of the space $l_2\times l_2$.

Lemma 6. *The following elements of the matrix $(a_{j,k})_{j,k=0}^{\infty}$ from Lemma 5 are positive:*

$$a_{0,1;0,1},a_{1,0;0,0};\quad a_{j,j+1;\alpha,\alpha+1},a_{j+1,j;\alpha,\alpha};\quad j\in\mathbb{N},\quad \alpha=0,1,\ldots,j. \tag{64}$$

Proof. We start with a study of $a_{1,0;0,0}$. Using (59) and denoting by $P'_{1;0}(z) = z-(z,1)_{L^2}$ the non normalized vector $P_{1;0}(z)$ we get

$$\begin{aligned} a_{1,0;0,0} &= \int_{\mathbb{C}} z\overline{P_{1;0}(z)}\,d\rho(z) = \|P'_{1;0}(z)\|^{-1}_{L^2}\int_{\mathbb{C}} z\overline{(z-(z,1)_{L^2})}\,d\rho(z) \\ &= \|P'_{1;0}(z)\|^{-1}_{L^2}(\|\bar{z}\|^2_{L^2} - |(\bar{z},1)_{L^2}|^2). \end{aligned} \tag{65}$$

The last difference is positive (see below, (67)), therefore $a_{1,0;0,0} > 0$.

Consider $a_{0,1;0,1}$. Denote, as before, by $P'_{1;1}(z)$ the non normalized vector $P_{1;1}(z)$. According to (49) and (50) we have

$$P'_{1;1}(z) = \bar{z} - (\bar{z}, P_{1;0}(z))_{L^2}P_{1;0}(z) - (\bar{z},1)_{L^2}.$$

Therefore using (59) we get

$$\begin{aligned} a_{0,1;0,1} &= \int_{\mathbb{C}} zP_{1;1}(z)\,d\rho(z) = \|P'_{1;1}(z)\|^{-1}_{L^2}\int_{\mathbb{C}} zP'_{1;1}(z)\,d\rho(z) \\ &= \|P'_{1;1}(z)\|^{-1}_{L^2}\int_{\mathbb{C}} z(\bar{z} - (\bar{z}, P_{1;0}(z))_{L^2}P_{1;0}(z) - (\bar{z},1)_{L^2})\,d\rho(z) \\ &= \|P'_{1;1}(z)\|^{-1}_{L^2}(\|\bar{z}\|^2_{L^2} - |(\bar{z}, P_{1;0}(z))_{L^2}|^2 - |(\bar{z},1)_{L^2}|^2). \end{aligned} \tag{66}$$

Also using (67) we conclude that the last expression is positive and, therefore, $a_{0,1;0,1} > 0$.

Positiveness in (65) and (66) follows from the Parseval equality applied to the decomposition of the function $\bar{z} \in L^2$ with respect to the orthonormal basis (50) in the space L^2,

$$|(\bar{z},1)_{L^2}|^2 + |(\bar{z}, P_{1;0}(z))_{L^2}|^2 + |(\bar{z}, P_{1;1}(z))_{L^2}|^2 + \cdots = \|\bar{z}\|^2_{L^2}, \quad (1 = P_{0;0}(z)). \tag{67}$$

Let us now pass to the proof of positiveness of $a_{j+1,j;\alpha,\alpha}$, where $j \in \mathbb{N}$, $\alpha = 0,1,\ldots,j$. From (59) we have

$$a_{j+1,j;\alpha,\alpha} = \int_{\mathbb{C}} zP_{j;\alpha}(z)\overline{P_{j+1;\alpha}(z)}\,d\rho(z). \tag{68}$$

According to (50) and (52),

$$P_{j;\alpha}(z) = k_{j;\alpha}z^{j-\alpha}\bar{z}^\alpha + R_{j;\alpha}(z), \tag{69}$$

where $R_{j;\alpha}(z)$ is some polynomial from $\mathcal{P}_{j;\alpha-1}$ if $\alpha > 0$, or from $\mathcal{P}_{j-1;j-1}$ if $\alpha = 0$. Therefore $zR_{j;\alpha}(z)$ is some polynomial from $\mathcal{P}_{j+1;\alpha-1}$ or from $\mathcal{P}_{j;j-1}$ (see (58) and (52)). Multiplying (69) by z we conclude that

$$zP_{j;\alpha}(z) = k_{j;\alpha}z^{j+1-\alpha}\bar{z}^\alpha + zR_{j;\alpha}(z); \quad zR_{j;\alpha}(z) \in \mathcal{P}_{j+1;\alpha-1} \quad \text{or to} \quad \mathcal{P}_{j;j-1} \subset \mathcal{P}_{j;j}. \tag{70}$$

On the other hand, equality (69) for $P_{j+1;\alpha}(z)$ gives

$$P_{j+1;\alpha}(z) = k_{j+1;\alpha}z^{j+1-\alpha}\bar{z}^\alpha + R_{j+1;\alpha}(z); \quad R_{j+1;\alpha}(z) \in \mathcal{P}_{j+1;\alpha-1} \quad \text{or to} \quad \mathcal{P}_{j;j}. \tag{71}$$

Find $z^{j+1-\alpha}\bar{z}^\alpha$ from (71) and substitute it into (70). We get

$$\begin{aligned} zP_{j;\alpha}(z) &= \frac{k_{j;\alpha}}{k_{j+1;\alpha}}(P_{j+1;\alpha}(z) - R_{j+1;\alpha}(z)) + zR_{j;\alpha}(z) \\ &= \frac{k_{j;\alpha}}{k_{j+1;\alpha}}P_{j+1;\alpha}(z) - \frac{k_{j;\alpha}}{k_{j+1;\alpha}}R_{j+1;\alpha}(z) + zR_{j;\alpha(z)}(z), \end{aligned} \tag{72}$$

where second two terms belong to $\mathcal{P}_{j+1;\alpha-1}$ or to $\mathcal{P}_{j;j}$ and are, in any case, orthogonal to $P_{j+1;\alpha}(z)$.

Therefore, after substituting the expression (72) into (68) we get that $a_{j+1,j;\alpha,\alpha} = \frac{k_{j;\alpha}}{k_{j+1;\alpha}} > 0$.

Consider at last the elements $a_{j,j+1;\alpha,\alpha+1}$, where $j \in \mathbb{N}$, $\alpha = 0, 1, \ldots, j$. From (59) we get

$$a_{j,j+1;\alpha,\alpha+1} = \int\limits_{\mathbb{C}} z P_{j+1,\alpha+1}(z) \overline{P_{j;\alpha}(z)}\, d\rho(z) = \overline{\int\limits_{\mathbb{C}} \bar{z} P_{j,\alpha}(z) \overline{P_{j+1;\alpha+1}(z)}\, d\rho(z)}. \tag{73}$$

For $P_{j;\alpha}(z)$ we have expression (69). Multiplying it by $\bar{z}$ we get similarly to (70) that

$$\bar{z} P_{j;\alpha}(z) = k_{j;\alpha} z^{j-\alpha} \bar{z}^{\alpha+1} + \bar{z} R_{j;\alpha}(z), \quad \bar{z} R_{j;\alpha}(z) \in \mathcal{P}_{j+1;\alpha} \quad \text{or to} \quad \mathcal{P}_{j;j}, \tag{74}$$

(but, at this point, it is necessary to use the second inclusion from (58) and of course (52)).

Now the equality (69) gives

$$P_{j+1;\alpha+1}(z) = k_{j+1;\alpha+1} z^{j-\alpha} \bar{z}^{\alpha+1} + R_{j+1;\alpha+1}(z), \quad R_{j+1;\alpha+1}(z) \in \mathcal{P}_{j+1;\alpha}. \tag{75}$$

Finding $z^{j-\alpha}\bar{z}^{\alpha+1}$ from (75) and substituting it into (74) gives

$$\begin{aligned} \bar{z} P_{j;\alpha}(z) &= \frac{k_{j;\alpha}}{k_{j+1;\alpha+1}} (P_{j+1;\alpha+1}(z) - R_{j+1;\alpha+1}(z)) + \bar{z} R_{j;\alpha}(z) \\ &= \frac{k_{j;\alpha}}{k_{j+1;\alpha+1}} P_{j+1;\alpha+1}(z) - \frac{k_{j;\alpha}}{k_{j+1;\alpha+1}} R_{j+1;\alpha+1}(z) + \bar{z} R_{j;\alpha}(z). \end{aligned} \tag{76}$$

As before, second two terms in (76) belong to $\mathcal{P}_{j+1;\alpha}$ or to $\mathcal{P}_{j;j}$ and are, in any case, orthogonal to $P_{j+1;\alpha+1}(z)$.

Therefore, substituting expression (76) into (73) gives $a_{j,j+1;\alpha,\alpha+1} = \frac{k_{j;\alpha}}{k_{j+1;\alpha+1}} > 0$. $\square$

In what follows we will use usual well known notations for the elements $a_{j,k}$ of the Jacobi matrix,

$$\begin{aligned} a_n = a_{n+1,n} &\ : \ \mathcal{H}_n \longrightarrow \mathcal{H}_{n+1}, \\ b_n = a_{n,n} &\ : \ \mathcal{H}_n \longrightarrow \mathcal{H}_n, \\ c_n = a_{n,n+1} &\ : \ \mathcal{H}_{n+1} \longrightarrow \mathcal{H}_n, \qquad n \in \mathbb{N}_0. \end{aligned} \tag{77}$$

All previous investigation are summarized in the following theorem.

Theorem 5. *The bounded normal operator $\hat{A}$ of multiplication by z in the space L^2, in the orthonormal basis (50) of polynomials, has the form of a three-diagonal block Jacobi type normal matrix $J = (a_{j,k})_{j,k=0}^{\infty}$ that acts on the space (53),*

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \mathcal{H}_n = \mathbb{C}^{n+1}, \quad n \in \mathbb{N}_0. \tag{78}$$

The norms of all the operators $a_{j,k} : \mathcal{H}_k \longrightarrow \mathcal{H}_j$ are uniformly bounded with respect to $j, k \in \mathbb{N}_0$. In notations (77), this matrix has the form

$$J = \begin{bmatrix} b_0 & c_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & c_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & c_2 & 0 & \cdots \\ 0 & 0 & a_2 & b_3 & c_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$(79) \qquad = \left[\begin{array}{c|cc|ccc|cccc|ccccc} * & * & + & & & & & & & & & & & & \\ \hline + & * & * & * & + & 0 & & & & & & & & & \\ 0 & * & * & * & * & + & & & & & & & 0 & & \\ \hline & + & * & * & * & * & * & + & 0 & 0 & & & & & \\ & 0 & + & * & * & * & * & * & + & 0 & & & & & \\ & 0 & 0 & * & * & * & * & * & * & + & & & & & \\ \hline & & & + & * & * & * & * & * & * & * & + & 0 & 0 & 0 \\ & & & 0 & + & * & * & * & * & * & * & * & + & 0 & 0 \\ & 0 & & 0 & 0 & + & * & * & * & * & * & * & * & + & 0 \\ & & & 0 & 0 & 0 & * & * & * & * & * & * & * & * & + \\ \hline & & & & \ddots & & & & \ddots & & & & & \ddots & \end{array}\right].$$

In (79) $\forall n \in \mathbb{N}_0$ b_n is an $((n+1)\times(n+1))$-matrix, $b_n = (b_{n;\alpha,\beta})_{\alpha,\beta=0}^{n,n}$, ($b_0 = b_{0;0,0}$ is a scalar); a_n is an $((n+2)\times(n+1))$-matrix, $a_n = (a_{n;\alpha,\beta})_{\alpha,\beta=0}^{n+1,n}$; c_n is an $((n+1)\times(n+2))$-matrix, $c_n = (c_{n;\alpha,\beta})_{\alpha,\beta=0}^{n,n+1}$. In these matrices a_n and c_n, some elements are always equal to zero, $\forall n \in \mathbb{N}$

$$(80) \qquad \begin{aligned} a_{0;1,0} = 0; \quad a_{n;\beta+1,\beta} = a_{n;\beta+2,\beta} = \cdots = a_{n;n+1,\beta} = 0, \quad & \beta = 0, 1, \ldots, n; \\ c_{n;\alpha,\alpha+2} = c_{n;\alpha,\alpha+3} = \cdots = c_{n;\alpha,n+1} = 0, \quad & \alpha = 0, 1, \ldots, n-1. \end{aligned}$$

Some other their elements are positive, namely $\forall n \in \mathbb{N}_0$

$$(81) \qquad a_{n;\alpha,\alpha}; c_{n;\alpha,\alpha+1} > 0, \quad \alpha = 0, 1, \ldots, n.$$

Thus, it is possible to say that $\forall n \in \mathbb{N}_0$ the matrices a_n (starting with the second diagonal and below) and the matrices c_n (starting with the third diagonal and above) have all their elements equal zero. All positive elements in (79) are denoted by $+$).

So, the matrix (79), in the scalar form, has the indicated multi-diagonal structure.

The adjoint operator $(\hat{A})^*$ in basis (50) has the form of a similar three-diagonal block Jacobi type matrix J^+.

These matrices J, J^+ act as follows: $\forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2$

$$(82) \qquad \begin{aligned} (Jf)_n &= a_{n-1} f_{n-1} + b_n f_n + c_n f_{n+1}, \\ (J^+ f)_n &= c_{n-1}^* f_{n-1} + b_n^* f_n + a_n^* f_{n+1}, \quad n \in \mathbb{N}_0, \quad f_{-1} = 0, \end{aligned}$$

(here $*$ denotes the usual matrix adjoint).

Let us indicate that the form of the coefficients in the expression for J^+ follows from (62) and (77).

5. The direct and inverse spectral problems for the three-diagonal block Jacobi type bounded normal operators

As it was mentioned above, the main result of the previous section is, actually, solving the inverse problem for the corresponding direct problem indicated in the title of this section.

We consider operators on the space $\mathbf{l}_2$ of the form (53). In addition to the space $\mathbf{l}_2$, we consider its rigging

$$(\mathbf{l}_{\rm fin})' \supset \mathbf{l}_2(p^{-1}) \supset \mathbf{l}_2 \supset \mathbf{l}_2(p) \supset \mathbf{l}_{\rm fin}, \tag{83}$$

where $\mathbf{l}_2(p)$ is a weighted $\mathbf{l}_2$ space with the weight $p = (p_n)_{n=0}^\infty$, $p_n \geq 1$, $(p^{-1} = (p_n^{-1})_{n=0}^\infty)$. In our case, $\mathbf{l}_2(p)$ is the Hilbert space of sequences $f = (f_n)_{n=0}^\infty$, $f_n \in \mathcal{H}_n$, such that

$$\|f\|^2_{\mathbf{l}_2(p)} = \sum_{n=0}^\infty \|f_n\|^2_{\mathcal{H}_n} p_n, \quad (f,g)_{\mathbf{l}_2(p)} = \sum_{n=0}^\infty (f_n, g_n)_{\mathcal{H}_n} p_n. \tag{84}$$

The space $\mathbf{l}_2(p^{-1})$ is defined analogously; recall that $\mathbf{l}_{\rm fin}$ is the space of finite sequences and $(\mathbf{l}_{\rm fin})'$ is the space conjugate to $\mathbf{l}_{\rm fin}$. It is easy to show that the embedding $\mathbf{l}_2(p) \hookrightarrow \mathbf{l}_2$ is quasinuclear if $\sum\limits_{n=0}^\infty np_n^{-1} < \infty$ (see, for example, [4] Ch. 7; [6] Ch. 15).

Let A be a normal operator standardly connected with the chain (83). According to the projection spectral theorem (see [5] Ch. 3, Theorem 2.7; [4] Ch. 5; [6], Ch. 15; [33]) such an operator has the representation

$$Af = \int\limits_{\mathbb{C}} z\Phi(z)\, d\sigma(z) f, \quad f \in \mathbf{l}_2, \tag{85}$$

where $\Phi(z) : \mathbf{l}_2(p) \longrightarrow \mathbf{l}_2(p^{-1})$ is the the generalized projection operator and $d\sigma(z)$ is a spectral measure. The operator adjoint to A, A^*, has the same representation (85), where $z\Phi(z)$ is replaced with $\bar{z}\Phi(z)$. For every $f \in \mathbf{l}_{\rm fin}$, the projection $\Phi(z)f \in \mathbf{l}_2(p^{-1})$ is a generalized eigenvector for the operators A and A^* with the corresponding eigenvalues z and $\bar{z}$. For all $f, g \in \mathbf{l}_{\rm fin}$ we have the Parseval equality

$$(f,g)_{\mathbf{l}_2} = \int\limits_{\mathbb{C}} (\Phi(z)f, g)_{\mathbf{l}_2} d\sigma(z); \tag{86}$$

and, after extending by continuity, the equality (86) takes place for $\forall f, g \in \mathbf{l}_2$.

Let us denote by π_n the operator of orthogonal projection in $\mathbf{l}_2$ on $\mathcal{H}_n$, $n \in \mathbb{N}_0$. Hence $\forall f = (f_n)_{n=0}^\infty \in \mathbf{l}_2$ we have $f_n = \pi_n f$. This operator acts analogously on the spaces $\mathbf{l}_2(p)$ and $\mathbf{l}_2(p^{-1})$ but possibly with the norm which is not equal to one.

Let us consider the operator matrix $(\Phi_{j,k}(z))_{j,k=0}^\infty$, where

$$\Phi_{j,k}(z) = \pi_j \Phi(z) \pi_k \ : \ \mathbf{l}_2 \longrightarrow \mathcal{H}_j, \quad (\text{ or } \quad \mathcal{H}_k \longrightarrow \mathcal{H}_j). \tag{87}$$

The Parseval equality (86) can be rewritten as follows: $\forall f, g \in \mathbf{l}_2$

$$\begin{aligned} (f,g)_{\mathbf{l}_2} &= \sum_{j,k=0}^\infty \int\limits_{\mathbb{C}} (\Phi(z)\pi_k f, \pi_j g)_{\mathbf{l}_2} d\sigma(z) = \sum_{j,k=0}^\infty \int\limits_{\mathbb{C}} (\pi_j \Phi(z) \pi_k f, g)_{\mathbf{l}_2} d\sigma(z) \\ &= \sum_{j,k=0}^\infty \int\limits_{\mathbb{C}} (\Phi_{j,k}(z) f_k, g_j)_{\mathbf{l}_2} d\sigma(z). \end{aligned} \tag{88}$$

Let us now pass to a study of a more special bounded operator A that acts on the space $\mathbf{l}_2$. Namely, let it be given by a matrix J which has a three-diagonal block structure of the form (79). So, this operator A is defined by the first expression in (82), the adjoint operator defined analogously by the second expression in (82). Recall that the norms of all elements a_n, b_n, and c_n are uniformly bounded with respect to $n \in \mathbb{N}_0$.

For the further investigations *we assume that conditions (80) and (81) are fulfilled and, additionally, the operator A given by (79) is bounded and normal on* $\mathbf{l}_2$. The conditions that would imply for the operator A to be bounded and normal will be investigated in Section 6.

At the next step we will rewrite the Parseval equality (88) in terms of generalized eigenvectors of the operator A. At first we prove the following lemma.

Lemma 7. *Let $\varphi(z)=(\varphi_n(z))_{n=0}^{\infty}$, $\varphi_n(z)\in\mathcal{H}_n$, $z\in\mathbb{C}$, be a generalized eigenvector from $(\mathbf{l}_{\mathrm{fin}})'$ of the operator A with an eigenvalue z and, as we have recalled above, it is also a generalized eigenvector of A^* with the eigenvalue $\bar z$. Multiplying $\varphi(z)$ by a scalar constant (depending on z) we can obtain that $\varphi_0(z)=\varphi_0$ is independent of z. Thus $\varphi(z)$ is a solution from $(\mathbf{l}_{\mathrm{fin}})'$ of the two difference equations (see (82))*

$$\begin{gathered}(J\varphi(z))_n=a_{n-1}\varphi_{n-1}(z)+b_n\varphi_n(z)+c_n\varphi_{n+1}(z)=z\varphi_n(z),\\ (J^+\varphi(z))_n=c^*_{n-1}\varphi_{n-1}(z)+b^*_n\varphi_n(z)+a^*_n\varphi_{n+1}(z)=\bar z\varphi_n(z),\\ n\in\mathbb{N}_0,\quad \varphi_{-1}(z)=0,\end{gathered}\tag{89}$$

with the initial condition $\varphi_0\in\mathbb{C}$.

We assert that this solution is the following: $\forall n\in\mathbb{N}$

$$\varphi_n(z)=Q_n(z)\varphi_0=(Q_{n;0},Q_{n;1},\dots,Q_{n;n},)\varphi_0.\tag{90}$$

Here $Q_{n;\alpha}$, $\alpha=0,1,\dots,n$, are polynomials in z and $\bar z$, and these polynomials have the form

$$Q_{n;\alpha}(z)=l_{n;\alpha}\bar z^{n-\alpha}z^{\alpha}+q_{n;\alpha}(z,\bar z),\quad \alpha=1,\dots,n,\tag{91}$$

where $l_{n;\alpha}>0$, and $q_{n;\alpha}(z)$ is some linear combinations of $\bar z^j z^k$, $0\le j+k\le n-1$ and $\bar z^{n-(\alpha-1)}z^{\alpha-1}$ (the last expressions are present in the case $\alpha=1,\dots n$).

Proof. For $n=0$ the system (89) has the form

$$\begin{aligned}b_0\varphi_0+c_0\varphi_1&=z\varphi_0,\\ b^*_0\varphi_0+a^*_0\varphi_1&=\bar z\varphi_0,\end{aligned}\quad\text{or}\quad\begin{aligned}\bar a_{0;0,0}\varphi_{1;0}+\bar a_{0;0,1}\varphi_{1;1}&=(\bar z-\bar b_{0;0,0})\varphi_0,\\ c_{0;0,0}\varphi_{1;0}+c_{0;0,1}\varphi_{1;1}&=(z-b_{0;0,0})\varphi_0.\end{aligned}\tag{92}$$

Here and in what follows we denote $\forall n\in\mathbb{N}$

$$\varphi_n(z)=(\varphi_{n;0}(z),\varphi_{n;1}(z),\dots,\varphi_{n;n}(z))\in\mathcal{H}_n;\quad \varphi_0=\varphi_{0;0}.$$

Using the assumptions (80) and (81) we rewrite the last two equalities in (92) in the form

$$\begin{gathered}\Delta_0\varphi_1(z)=((\bar z-\bar b_{0;0,0})\varphi_0,(z-b_{0;0,0})\varphi_0);\\ \Delta_0=\begin{pmatrix}a_{0;0,0}&0\\ c_{0;0,0}&c_{0;0,1}\end{pmatrix},\quad a_{0;0,0}>0,\quad c_{0;0,1}>0.\end{gathered}\tag{93}$$

Therefore,

$$\begin{aligned}\varphi_{1;0}(z)&=\frac{1}{a_{0;0,0}}(\bar z-\bar b_{0;0,0})\varphi_0=Q_{1;0}(z)\varphi_0,\\ \varphi_{1;1}(z)&=(r_1(\bar z-\bar b_{0;0,0})+r_2(z-b_{0;0,0})+r_3)\varphi_0=Q_{1;1}(z)\varphi_0,\end{aligned}\tag{94}$$

where $r_1>0$, r_2 and r_3 some constants. In another words, the solution $\varphi_n(z)$ of (89), for $n=1$, has the form (90) and (91).

Suppose, by induction, that for $n\in\mathbb{N}$ the coordinates $\varphi_{n-1}(z)$ and $\varphi(z)$ of our generalized eigenvector $\varphi(z)=(\varphi_n(z))_{n=0}^{\infty}$ have the form (90) and (91), and prove that $\varphi_{n+1}(z)$ is also of the form (90) and (91).

Our eigenvector $\varphi(z)$ satisfies the system (89) of two equations. But this system is overdetermined; it consists of $2(n+1)$ scalar equations from which it is necessary to find only $n+2$ unknowns $\varphi_{n+1;0}$, $\varphi_{n+1;1}$, …, $\varphi_{n+1;n+1}$ using, as the initial data, the previous $n+1$ values $\varphi_{n;0}$, $\varphi_{n;1}$, …, $\varphi_{n;n}$ of coordinates of the vector $\varphi_n(z)$.

We act in the following manner. According to Theorem 5, especially to (80) and (81), the $((n+1)\times(n+2))$-matrices a_n^* and c_n act on $\psi_{n+1}\in\mathcal{H}_n$ as follows:

$$
\begin{aligned}
a_n^*\psi_{n+1}(z) &= \begin{bmatrix} a_{n;0,0} & 0 & 0 & \dots & 0 & 0 \\ \bar a_{n;1,0} & a_{n;1,1} & 0 & \dots & 0 & 0 \\ \bar a_{n;2,0} & \bar a_{n;2,1} & a_{n;2,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar a_{n;n-1,0} & \bar a_{n;n-1,1} & \bar a_{n;n-1,2} & \dots & 0 & 0 \\ \bar a_{n;n,0} & \bar a_{n;n,1} & \bar a_{n;n,2} & \dots & a_{n;n,n} & 0 \end{bmatrix}\psi_{n+1}(z), \\
c_n\psi_{n+1}(z) &= \begin{bmatrix} c_{n;0,0} & c_{n;0,1} & 0 & \dots & 0 & 0 \\ c_{n;1,0} & c_{n;1,1} & c_{n;1,2} & \dots & 0 & 0 \\ c_{n;2,0} & c_{n;2,1} & c_{n;2,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n;n-1,0} & c_{n;n-1,1} & c_{n;n-1,2} & \dots & c_{n;n-1,n} & 0 \\ c_{n;n,0} & c_{n;n,1} & c_{n;n,2} & \dots & c_{n;n,n} & c_{n;n,n+1} \end{bmatrix}\psi_{n+1}(z),
\end{aligned} \tag{95}
$$

where $\psi_{n+1}(z)=(\psi_{n+1;0}(z),\psi_{n+1;1}(z),\dots,\psi_{n+1;n+1}(z))$.

Construct, similarly to (93), the following combination from the matrices (95), which is a $((n+2)\times(n+2))$-matrix:

$$
\Delta_n\psi_{n+1}(z) = \begin{bmatrix} a_{n;0,0} & 0 & 0 & \dots & 0 & 0 \\ c_{n;0,0} & c_{n;0,1} & 0 & \dots & 0 & 0 \\ c_{n;1,0} & c_{n;1,1} & c_{n;1,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n;n-1,0} & c_{n;n-1,1} & c_{n;n-1,2} & \dots & c_{n;n-1,n} & 0 \\ c_{n;n,0} & c_{n;n,1} & c_{n;n,2} & \dots & c_{n;n,n} & c_{n;n,n+1} \end{bmatrix}\psi_{n+1}(z), \tag{96}
$$

where $\psi_{n+1}(z)=(\psi_{n+1;0}(z),\psi_{n+1;1}(z),\dots,\psi_{n+1;n+1}(z))$.

The matrix (96) is invertible because its elements on the main diagonal are positive (see (81)). Rewrite the equalities (89) as follows:

$$
\begin{aligned}
a_n^*\varphi_{n+1}(z) &= \bar z\varphi_n(z) - c_{n-1}^*\varphi_{n-1}(z) - b_n^*\varphi_n(z), \\
c_n\varphi_{n+1}(z) &= z\varphi_n(z) - a_{n-1}\varphi_{n-1}(z) - b_n\varphi_n(z), \quad n\in\mathbb{N}.
\end{aligned} \tag{97}
$$

We see that the first $n+2$ scalar equations (from $2(n+1)$ scalar equations (97)) have the form

$$
\begin{aligned}
\Delta_n\varphi_{n+1}(z) = \Big(&\bar z Q_{n;0}(z) - (c_{n-1}^*Q_{n-1}(z) - (b^*Q_n(z))_{n;0}, \\
&zQ_{n;0}(z) - (a_{n-1}Q_{n-1}(z))_{n;0} - (b_nQ_n(z))_{n;0},\dots, \\
&zQ_{n;n}(z) - (a_{n-1}Q_{n-1}(z))_{n;n} - (b_nQ_n(z))_{n;n}\Big)\varphi_0.
\end{aligned} \tag{98}
$$

The construction of the matrix Δ_n and the form of the vector in the right-hand side of (98) and (90), (91) shows that

(99)

$$
\begin{aligned}
\varphi_{n+1;0}(z) = Q_{n+1;0}(z)\varphi_0 &= \frac{1}{a_{n;0,0}}(\bar z Q_{n;0}(z) - (c_{n-1}^*Q_{n-1}(z))_{n;0}) - (b_n^*Q_n(z))_{n;0})\varphi_0 \\
&= \frac{1}{a_{n;0,0}}(\bar z(l_{n;0}\bar z^n + q_{n;0}(z)) - (c_{n-1}^*Q_{n-1}(z))_{n;0} - (b_n^*Q_n(z))_{n;0})\varphi_0,
\end{aligned}
$$

i.e., the main summand in the right-hand side of (99) is equal to $\frac{l_{n;0}}{a_{n;0,0}}\bar z_{n+1}z^0$, so it has the form (91).

A similar calculation gives the same result for $\varphi_{n+1;1}(z)$, …, $\varphi_{n+1;n+1}(z)$. It is necessary to take into account that the next diagonal elements $c_{n;0,1}$, $c_{n;1,2}$, …, $c_{n;n,n+1}$ of the matrix Δ_n are positive due to (81). This completes the induction and (94) finishes the proof. $\square$

Remark 6. Note that we did not assert that a solution of the overdetermined system (89) exists for arbitrary initial data $\varphi_0 \in \mathbb{C}$; we have only proved that the generalized eigenvector from $(\mathbf{l}_{\text{fin}})'$ of the operator A is a solution of (89) and has the form (90) and (91).

In what follows, it will be convenient to look at $Q_n(z)$ with fixed z as a linear operator that acts from $\mathcal{H}_0$ into $\mathcal{H}_n$, i.e., $\mathcal{H}_0 \ni \varphi_0 \longmapsto Q_n(z)\varphi_0 \in \mathcal{H}_n$. We also regard $Q_n(z)$ as an operator-valued polynomial in $z, \bar{z} \in \mathbb{C}$; hence, for the adjoint operator we have $Q_n^*(z) = (Q_n(z))^* : \mathcal{H}_n \longrightarrow \mathcal{H}_0$. Using these polynomials $Q_n(z)$ we construct the following representation for $\Phi_{j,k}(z)$.

Lemma 8. *The operator $\Phi_{j,k}(z)$, $\forall z \in \mathbb{C}$ has the following representation:*

$$\Phi_{j,k}(z) = Q_j(z)\Phi_{0,0}(z)Q_k^*(z) : \mathcal{H}_k \longrightarrow \mathcal{H}_j, \quad j,k \in \mathbb{N}_0, \tag{100}$$

where $\Phi_{0,0}(z) \geq 0$ is a scalar.

Proof. For a fixed $k \in \mathbb{N}_0$, the vector $\varphi = \varphi(z) = (\varphi_j(z))_{j=0}^{\infty}$, where

$$\varphi_j(z) = \Phi_{j,k}(z) = \pi_j \Phi(z) \pi_k \in \mathcal{H}_j, \quad z \in \mathbb{C}, \tag{101}$$

is a generalized solution, in $(\mathbf{l}_{\text{fin}})'$, of the equation $J\varphi(z) = z\varphi(z)$, since $\Phi(z)$ is a projector onto generalized eigenvectors of the operator A with the corresponding generalized eigenvalues z. Therefore $\forall g \in \mathbf{l}_{\text{fin}}$ we have $(\varphi, J^+ g)_{\mathbf{l}_2} = z(\varphi, g)_{\mathbf{l}_2}$. Transfering the finite difference expression J^+ to φ we get $(J\varphi, g)_{\mathbf{l}_2} = z(\varphi, g)_{\mathbf{l}_2}$. Hence, it follows that $\varphi = \varphi(z) \in \mathbf{l}_2(p^{-1})$ exists as a usual solution of the equation $J\varphi = z\varphi$ with the initial condition $\varphi_0 = \pi_0 \Phi(z) \pi_k \in \mathcal{H}_0$.

Since $\forall f \in \mathbf{l}_{\text{fin}}$, the vector $\Phi(z) f \in \mathbf{l}_2(p^{-1})$ is also a generalized eigenvector of the operator A^* with the corresponding eigenvalue $\bar{z}$ (because A is normal), the same $\varphi = \varphi(z)$ in (101) is also a solution of the equation $J^+\varphi = \bar{z}\varphi$ with the same initial condition $\varphi_0 = \pi_0 \Phi(z) \pi_k$.

Using Lemma 7 and (90) we obtain

$$\Phi_{j,k}(z) = Q_j(z)(\Phi_{0,k}(z)), \quad j \in \mathbb{N}_0. \tag{102}$$

The operator $\Phi(z) : \mathbf{l}_2(p) \longrightarrow \mathbf{l}_2(p^{-1})$ is formally selfadjoint on $\mathbf{l}_2$, being the derivative of the resolution of identity of the operator A on $\mathbf{l}_2$ with respect to the spectral measure. Hence, according to (100) we get

$$(\Phi_{j,k}(z))^* = (\pi_j \Phi(z) \pi_k)^* = \pi_k \Phi(z) \pi_j = \Phi_{k,j}(z), \quad j,k \in \mathbb{N}_0. \tag{103}$$

For a fixed $j \in \mathbb{N}_0$ it follows from (103) and the previous discussion that the vector

$$\psi = \psi(z) = (\psi_k(z))_{k=0}^{\infty}, \quad \psi_k(z) = \Phi_{k,j}(z) = (\Phi_{j,k}(z))^*$$

is a usual solution of the equations $J\psi = z\psi$ and $J^+\psi = \bar{z}\psi$ with the initial condition $\psi_0 = \Phi_{0,j}(z) = (\Phi_{j,0}(z))^*$.

Again using Lemma 7 we obtain the representation of the type (102),

$$\Phi_{k,j}(z) = Q_k(z)(\Phi_{0,j}(z)), \quad k \in \mathbb{N}_0. \tag{104}$$

Taking into account (103) and (104) we get

$$\Phi_{0,k}(z) = (\Phi_{k,0}(z))^* = (Q_k(z)\Phi_{0,0}(z))^* = \Phi_{0,0}(z)(Q_k(z))^*, \quad k \in \mathbb{N}_0 \tag{105}$$

(here we used $\Phi_{0,0}(z) \geq 0$; this inequality follows from (86) and (87)). Substituting (105) into (102) we obtain (100). $\square$

Now it is possible to rewrite the Parseval equality (88) in a more concrete form. To this end, we substitute the expression (100) for $\Phi_{j,k}(z)$ into (88) and get that $\forall f,g\in\mathbf{l}_{\mathrm{fin}}$

(106)
$$\begin{aligned}(f,g)_{\mathbf{l}_2}&=\sum_{j,k=0}^{\infty}\int_{\mathbb{C}}(\Phi_{j,k}(z)f_k,g_j)_{\mathbf{l}_2}d\sigma(z)=\sum_{j,k=0}^{\infty}\int_{\mathbb{C}}(Q_j(z)\Phi_{0,0}(z)Q_k^*(z)f_k,g_j)_{\mathbf{l}_2}d\sigma(z)\\&=\sum_{j,k=0}^{\infty}\int_{\mathbb{C}}(Q_k^*(z)f_k,Q_j^*(z)g_j)_{\mathbf{l}_2}d\rho(z)=\int_{\mathbb{C}}\Big(\sum_{k=0}^{\infty}Q_k^*(z)f_k\Big)\overline{\Big(\sum_{j=0}^{\infty}Q_j^*(z)g_j\Big)}d\rho(z),\end{aligned}$$

$d\rho(z)=\Phi_{0,0}(z)\,d\sigma(z)$.

Introduce the Fourier transform $\widehat{\ }$ induced by the normal operator A on the space $\mathbf{l}_2$, $\forall f\in\mathbf{l}_{\mathrm{fin}}$,

$$\mathbf{l}_2\supset\mathbf{l}_{\mathrm{fin}}\ni f=(f_n)_{n=0}^{\infty}\longmapsto\hat f(z)=\sum_{n=0}^{\infty}Q_n^*(z)f_n\in L^2(\mathbb{C},d\rho(z)).\tag{107}$$

Hence, (106) gives the Parseval equality in a final form, $\forall f,g\in\mathbf{l}_{\mathrm{fin}}$

$$(f,g)_{\mathbf{l}_2}=\int_{\mathbb{C}}\hat f(z)\overline{\hat g(z)}\,d\rho(z).\tag{108}$$

Extending (108) by continuity, it becomes valid $\forall f,g\in\mathbf{l}_2$.

Orthogonality of the polynomials $Q_n^*(z)$ follows from (107) and (108). Namely, it is sufficient only to take $f=(0,\dots,0,f_k,0,\dots)$, $f_k\in\mathcal{H}_k$, $g=(0,\dots,0,g_j,0,\dots)$, $g_j\in\mathcal{H}_j$ in (107) and (108) . Then $\forall k,j\in\mathbb{N}_0$

$$\int_{\mathbb{C}}(Q_k^*(z)f_k)\overline{(Q_j^*(z)g_j)}\,d\rho(z)=\delta_{j,k}(f_j,g_j)_{\mathcal{H}_j}.\tag{109}$$

Using representation (90) for these polynomials we can rewrite the equality (109) in a more classical scalar form. To do this, we remark that $Q_0^*(z)=\bar Q_0(z)$ and for $n\in\mathbb{N}$, according to (90), $Q_n(z)=(Q_{n;0}(z),Q_{n;1}(z),\dots,,Q_{n;n}(z)):\mathcal{H}_0\longrightarrow\mathcal{H}_n$. Hence, for the adjoint operator $Q_n^*(z):\mathcal{H}_n\longrightarrow\mathcal{H}_0$ we have $\forall x\in\mathcal{H}_0$, $y=(y_0,y_1,\dots,y_n)\in\mathcal{H}_n$,

$$\begin{aligned}(Q_n(z)x,y)_{\mathcal{H}_n}&=((Q_{n;0}(z)x,Q_{n;1}(z)x,\dots,Q_{n;n}(z)x),(y_0,y_1,\dots,y_n))_{\mathcal{H}_n}\\&=Q_{n;0}(z)x\bar y_0+Q_{n;1}(z)x\bar y_1+\dots+Q_{n;n}(z)x\bar y_n\\&=x\overline{(\overline{Q_{n;0}(z)}y_0+\overline{Q_{n;1}(z)}y_1+\dots+\overline{Q_{n;n}(z)}y_n)}=(x,Q_n^*(z)y)_{\mathcal{H}_0},\end{aligned}$$

that is, $Q_n^*(z)y=\overline{Q_{n;0}(z)}y_0+\overline{Q_{n;1}(z)}y_1+\dots+\overline{Q_{n;n}(z)}y_n$.

Due to the last equality for $n\in\mathbb{N}$ and $f_n=(f_{n,0},f_{n,1},\dots,f_{n,n})\in\mathcal{H}_n$, $z\in\mathbb{C}$, we obtain

$$Q_n^*(z)f_n=\overline{Q_{n;0}(z)}f_{n;0}+\overline{Q_{n;1}(z)}f_{n;1}+\dots+\overline{Q_{n;n}(z)}f_{n;n},\quad Q_0^*(z)=1.\tag{110}$$

Therefore (109) has the form: $\forall f_{k;0},f_{k;1},\dots,f_{k;k},\ g_{j;0},g_{j;1},\dots,g_{j;j}\in\mathbb{C}$, $j,k\in\mathbb{N}_0$,

$$\int_{\mathbb{C}}\Big(\sum_{\alpha=0}^{k}\overline{Q_{k;\alpha}(z)}f_{k;\alpha}\Big)\overline{\Big(\sum_{\beta=0}^{j}\overline{Q_{j;\beta}(z)}f_{j;\beta}\Big)}d\rho(z)=\delta_{j,k}\sum_{\alpha=0}^{j}f_{j;\alpha}\bar g_{j;\alpha}.$$

This equality is equivalent to the following orthogonality relation in the usual classical form: $\forall j,k\in\mathbb{N}_0$, $\forall\alpha=0,1,\dots,j,\beta=0,1,\dots,k$,

$$\int_{\mathbb{C}}\overline{Q_{k;\beta}^*(z)}Q_{j;\alpha}d\rho(z)=\delta_{j,k}\delta_{\alpha,\beta}\quad(Q_{0;0}=Q_0(z)).\tag{111}$$

Let us remark that due to (110) the Fourier transform (107) can be rewritten as $\forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2$

$$\hat{f}(z) = \sum_{n=0}^{\infty} \sum_{\alpha=0}^{n} \overline{Q_{n;\alpha}(z)} f_{n;\alpha}, \quad z \in \mathbb{C}. \tag{112}$$

Using the stated above results of this section, we can formulate the following spectral theorem for our bounded normal operator A.

Theorem 6. *Consider the space (53),*

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus, \cdots, \quad \mathcal{H}_n = \mathbb{C}^{n+1}, \quad n \in \mathbb{N}_0, \tag{113}$$

and the linear operator A, which is defined on finite vectors $\mathbf{l}_{\mathrm{fin}}$ by a block three-diagonal Jacobi type matrix J of the form (79) with the help of the first expression in (82). We suppose that all its coefficients a_n, b_n and c_n, $n \in \mathbb{N}_0$, are uniformly bounded, some elements of these matrices are equal to zero or positive according to (80), (81) and the extension of A by continuity is a bounded normal operator on this space.

The eigenfunction expansion of the operator A has the following form. According to Lemma 7 we represent, using $\varphi_0 \in \mathbb{C}$, the solution $\varphi(z) = (\varphi_n(z))_{n=0}^{\infty}$, $\varphi_n(z) \in \mathcal{H}_n$, of equations (79) (which exists thanks to the projection spectral theorem) for $z \in \mathbb{C}$,

$$\varphi_n(z) = Q_n(z)\varphi_0 = (Q_{n;0}(z), Q_{n;1}(z), \cdots, Q_{n;n}(z))\varphi_0,$$

where $Q_{n;\alpha}(z)$, $\alpha = 0, 1, \ldots, n$, are polynomials in z and $\bar{z}$. Then the Fourier transform has the form

$$\mathbf{l}_2 \supset \mathbf{l}_{\mathrm{fin}} \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(z) = \sum_{n=0}^{\infty} Q_n^*(z) f_n = \sum_{n=0}^{\infty} \sum_{\alpha=0}^{n} \overline{Q_{n;\alpha}(z)} f_{n;\alpha} \in L^2(\mathbb{C}, d\rho(z)). \tag{114}$$

Here $Q_n^*(z) : \mathcal{H}_n \longrightarrow \mathcal{H}_0$ is the adjoint to the operator $Q_n(z) : \mathcal{H}_0 \longrightarrow \mathcal{H}_n$, $d\rho(z)$ is the probability spectral measure of A.

The Parseval equality has the following a form: $\forall f, g \in \mathbf{l}_{\mathrm{fin}}$

$$(f, g)_{\mathbf{l}_2} = \int_{\mathbb{C}} \hat{f}(z)\overline{\hat{g}(z)}\, d\rho(z); \quad (Jf, g)_{\mathbf{l}_2} = \int_{\mathbb{C}} z\hat{f}(z)\overline{\hat{g}(z)}\, d\rho(z). \tag{115}$$

Identities (114) and (115) are extended by continuity to $\forall f, g \in \mathbf{l}_2$ making the operator (114) unitary, mapping $\mathbf{l}_2$ onto the whole $L^2(\mathbb{C}, d\rho(z))$.

The polynomials $\overline{Q_{n;\alpha}(z)}$, $n \in \mathbb{N}$, $\alpha = 0, 1, \ldots, n$, and $Q_{0;0}(z) = 1$, form an orthonormal system in $L^2(\mathbb{C}, d\rho(z))$ in the sense of (111), and it is total in this space.

Proof. It is only necessary to show that the orthogonal polynomials $\overline{Q_{n;\alpha}(z)}$, $n \in \mathbb{N}$, $\alpha = 0, 1, \ldots, n$, and $Q_{0;0}(z) = 1$ form a total set in the space $L^2(\mathbb{C}, d\rho(z))$. For this reason we remark at first that due to the compactness of the support of the measure $d\rho(z)$ on $\mathbb{C}$, the elements $z^j \bar{z}^k$, $j, k \in \mathbb{N}_0$, form a total set in $L^2(\mathbb{C}, d\rho(z))$.

Let us suppose the contrary, i.e., that our system of polynomials is not total. Then there exists a non zero function $h(z) \in L^2(\mathbb{C}, d\rho(z))$ that is orthogonal to all these polynomials and hence, according to (91), to all $z^j \bar{z}^k$, $j, k \in \mathbb{N}_0$. Hence $h(z) = 0$. □

The last theorem solves the direct problem for the bounded normal operator A which is generated, on the space $\mathbf{l}_2$, by a matrix J of the form (79).

The inverse problem consists of constructing, from a given measure $d\rho(z)$ on $\mathbb{C}$ with compact support, a bounded normal matrix J of the form (79) that has its spectral measure equal to $d\rho(z)$. *This construction is conducted according to Theorem 5, with the use of the Schmidt orthogonalization procedure for the system (49). For a matrix J of* *the form (79), which is constructed from $d\rho(z)$, the spectral measure of the corresponding bounded normal operator A coincides with the starting measure.*

Proof. The claim holds true, since the system of orthogonal polynomials, connected with A, $\overline{Q_{n,\alpha}(z)}$, $\alpha = 0,1,\dots,n$, $n \in \mathbb{N}_0$, is orthonormal in $L^2(\mathbb{C}, d\rho(z))$ and, according to Lemma 7, is constructed from $\bar{z}^j z^k$, $z \in \mathbb{C}$, in the same way as system (50) was constructed from $z^j \bar{z}^k$, $j,k \in \mathbb{N}_0$. Hence, $\forall n \in \mathbb{N}$

$$Q_0(z) = 1 = P_0(z), \quad \overline{Q_{n,\alpha}(z)} = P_{n;\alpha}(z), \quad \alpha = 0,1,\dots,n, \tag{116}$$

Since both system of polynomials form a total set in $L^2(\mathbb{C}, d\rho(z))$, (116) shows that the spectral measure constructed from the operator and the given one coincide. $\square$

Let us remark that the expressions (59) (as it was known in the classical theory of Jacobi matrices) reestablish the initial matrix (79) from the spectral measure $d\rho(z)$ of the operator constructed from J on $\mathbf{l}_2$.

6. On a condition for normality of Jacobi type block matrices

We will find at first a condition that would guarantee the formal normality for the matrix J of type (9). The formal adjoint matrix J^+ has the form (compare with (62))

$$J^+ = \begin{bmatrix} b_0^* & a_0^* & 0 & 0 & \cdots \\ c_0^* & b_1^* & a_1^* & 0 & \cdots \\ 0 & c_1^* & b_2^* & a_2^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \begin{aligned} c_n^* &: \mathcal{H}_n \longrightarrow \mathcal{H}_{n+1}, \\ b_n^* &: \mathcal{H}_n \longrightarrow \mathcal{H}_n, \\ a_n^* &: \mathcal{H}_{n+1} \longrightarrow \mathcal{H}_n, \quad n \in \mathbb{N}_0. \end{aligned} \tag{117}$$

Multiplying matrices (9) and (117) we get

$$JJ^+ = \begin{bmatrix} b_0b_0^* + c_0c_0^* & b_0a_0^* + c_0b_1^* & c_0a_1^* & 0 & 0 & \cdots \\ a_0b_0^* + b_1c_0^* & a_0a_0^* + b_1b_1^* + c_1c_1^* & b_1a_1^* + c_1b_2^* & c_1a_2^* & 0 & \cdots \\ a_1c_0^* & a_1b_1^* + b_2c_1^* & a_1a_1^* + b_2b_2^* + c_2c_2^* & b_2a_2^* + c_2b_3^* & c_2a_3^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{118}$$

The expression for J^+J is analogous to (118) if a_n, b_n and c_n are replaced with c_n^*, b_n^* and a_n^*, respectively, and vice versa.

Comparing these expressions for JJ^+ and J^+J we conclude that the equality $JJ^+ = J^+J$ is equivalent to fulfillment of the following equalities (we take into account that b_0 is a scalar, $b_0^* = \bar{b}_0$):

$$\begin{aligned} &c_0c_0^* = a_0^*a_0; \\ &c_na_{n+1}^* = a_n^*c_{n+1}, \\ &b_na_n^* + c_nb_{n+1}^* = b_n^*c_n + a_n^*b_{n+1}, \\ &a_na_n^* + b_{n+1}b_{n+1}^* + c_{n+1}c_{n+1}^* = c_n^*c_n + b_{n+1}^*b_{n+1} + a_{n+1}^*a_{n+1}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{119}$$

Note that the necessary equalities

$$a_nb_n^* + b_{n+1}c_n^* = c_n^*b_n + b_{n+1}^*a_n, \quad a_{n+1}c_n^* = c_{n+1}^*a_n, \quad n \in \mathbb{N}_0,$$

follow from the third and the second equalities in (119) by taking the adjoints.

So, the conditions (119) are necessary and sufficient for the matrix equality $JJ^+ = J^+J$ to hold. If the norms of the operators a_n, b_n and c_n are uniformly bounded w.r.t. $n \in \mathbb{N}_0$, then the operator $\tilde{J}$ on $\mathbf{l}_2$ is bounded and conditions (119) gives the normality of the operator.

Taking the initial matrices a_0, b_0, c_0 and step by step finding from (119) a_1, b_1, c_1; a_2, b_2, c_2; ... etc. (in a non-unique manner) we can construct some normal matrix J. But for such a matrix, Theorem 6 in general is not valid, because it is necessary to find these matrices in such way that a_n and c_n be of the form (10) (i.e. (79)). Only in this case, according to Lemma 7 and (105), the functions (15) are linearly independent and Theorem 6 is applicable (or the condition of Remark 2 is fulfilled for $N = \tilde{J}$).

To find the matrices a_n, b_n and c_n, $n \in \mathbb{N}_0$, which solve equations (119) and such that a_n, c_n have the form (10) is a sufficiently complicated problem, and we here investigate only some special cases.

Namely, we assume in the first place that all the matrices b_n are selfadjoint, $b_n^* = b_n$, $n \in \mathbb{N}_0$. Then the conditions (119) can be written in the following form:

$$\begin{aligned}
&c_0 c_0^* = a_0^* a_0;\\
&c_n a_{n+1}^* = a_n^* c_{n+1},\\
&b_n(a_n^* - c_n) = (a_n^* - c_n) b_{n+1},\\
&a_n a_n^* + c_{n+1} c_{n+1}^* = c_n^* c_n + a_{n+1}^* a_{n+1}, \quad n \in \mathbb{N}_0.
\end{aligned} \tag{120}$$

Further, we assume that all the matrices a_n, c_n, $n \in \mathbb{N}_0$, have the form (10), where $a_{n;0,0}, a_{n;1,1}, \ldots, a_{n;n,n}$ and $c_{n;0,1}, c_{n;1,2}, \ldots, c_{n;n,n+1}$, $\forall n \in \mathbb{N}_0$, are positive and all other elements of these matrices are equal to zero. So, our matrices are the following:

(121)

$$a_n = \underbrace{\begin{bmatrix} a_{n;0,0} & 0 & \ldots & 0 \\ 0 & a_{n;1,1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{n;n,n} \\ 0 & 0 & \ldots & 0 \end{bmatrix}}_{n+1} \left.\vphantom{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}}\right\} n+2, \quad a_n^* = \underbrace{\begin{bmatrix} a_{n;0,0} & 0 & \ldots & 0 & 0 \\ 0 & a_{n;1,1} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & a_{n;n,n} & 0 \end{bmatrix}}_{n+2} \left.\vphantom{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\right\} n+1,$$

$$c_n = \underbrace{\begin{bmatrix} 0 & c_{n;0,1} & 0 & \ldots & 0 \\ 0 & 0 & c_{n;1,2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & c_{n;n,n+1} \end{bmatrix}}_{n+2} \left.\vphantom{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\right\} n+1, \quad c_n^* = \underbrace{\begin{bmatrix} 0 & 0 & \ldots & 0 \\ c_{n;0,1} & 0 & \ldots & 0 \\ 0 & c_{n;1,2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & c_{n;n,n+1} \end{bmatrix}}_{n+1} \left.\vphantom{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}}\right\} n+2,$$

$n \in \mathbb{N}_0$.

Multiplying the matrices of type (121) we can rewrite the first, the second, and the fourth equality in (120) in the form of the corresponding equalities for elements of these matrices, $c_{0;0,1} = a_{0;0,0}$ and $\forall n \in \mathbb{N}_0$

(122)

$$\begin{aligned}
&c_{n;0,1} a_{n+1;1,1} = a_{n;0,0} c_{n+1;0,1}, && a_{n;0,0}^2 + c_{n+1;0,1}^2 = a_{n+1;0,0}^2,\\
&c_{n;1,2} a_{n+1;2,2} = a_{n;1,1} c_{n+1;1,2}, && a_{n;1,1}^2 + c_{n+1;1,2}^2 = c_{n;0,1}^2 + a_{n+1;1,1}^2,\\
&c_{n;2,3} a_{n+1;3,3} = a_{n;2,2} c_{n+1;2,3}, && a_{n;2,2}^2 + c_{n+1;2,3}^2 = c_{n;1,2}^2 + a_{n+1;2,2}^2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots && \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&c_{n;n-1,n} a_{n+1;n,n} = a_{n;n-1,n-1} c_{n+1;n-1,n}, && a_{n;n,n}^2 + c_{n+1;n,n+1}^2 = c_{n;n-1,n}^2 + a_{n+1;n,n}^2,\\
&c_{n;n,n+1} a_{n+1;n+1,n+1} = a_{n;n,n} c_{n+1;n,n+1}; && c_{n+1;n+1,n+2}^2 = c_{n;n,n+1}^2 + a_{n+1;n+1,n+1}^2.
\end{aligned}$$

The system of equalities (122) is equivalent to system (120) for the case where all $b_n = 0$, $n \in \mathbb{N}_0$. At first we wish to find the solution $c_{n;\alpha,\beta}$, $a_{n;\gamma,\gamma}$ of this system step by step, taking $n \in \mathbb{N}_0$.

Let $n \in \mathbb{N}_0$ be fixed. Introduce the following notations for elements from (122):

$$\begin{aligned}
&c^2_{n;0,1} = x_0, && c^2_{n;1,2} = x_1, && \dots, && c^2_{n;n,n+1} = x_n;\\
&a^2_{n;n,n} = \xi_0, && a^2_{n;n-1,n-1} = \xi_1, && \dots, && a^2_{n;0,0} = \xi_n;\\
&c^2_{n+1;0,1} = y_0, && c^2_{n+1;1,2} = y_1, && \dots, && c^2_{n+1;n+1,n+2} = y_{n+1};\\
&a^2_{n+1;n+1,n+1} = \eta_0, && a^2_{n+1;n,n} = \eta_1, && \dots, && a^2_{n+1;0,0} = \eta_{n+1}.
\end{aligned} \tag{123}$$

Taking the square of the first $n+1$ equalities from (122), we can rewrite (122) in the form:

$$\begin{aligned}
&x_0\eta_n = \xi_n y_0, && \xi_n + y_0 = \eta_{n+1},\\
&x_1\eta_{n-1} = \xi_{n-1} y_1, && \xi_{n-1} + y_1 = x_0 + \eta_n,\\
&x_2\eta_{n-2} = \xi_{n-2} y_2, && \xi_{n-2} + y_2 = x_1 + \eta_{n-1},\\
&\dots\dots\dots\dots && \dots\dots\dots\dots\dots\dots\\
&x_{n-1}\eta_1 = \xi_1 y_{n-1}, && \xi_0 + y_n = x_{n-1} + \eta_1,\\
&x_n\eta_0 = \xi_0 y_n; && y_{n+1} = x_n + \eta_0.
\end{aligned} \tag{124}$$

Conditions (124) are regarded as a system of $2n+3$ linear equations w.r.t $2(n+2)$ positive unknowns $y_0, y_1, \dots, y_{n+1}; \eta_0, \eta_1, \dots, \eta_{n+1}$ with coefficients depending on $x_0, x_1, \dots,$ x_n; $\xi_0, \xi_1, \dots, \xi_n$.

We start with the initial data $c_{0;0,1} = a_{0;0,0} = \delta_0^{1/2}$, where $\delta_0 > 0$ is a fixed number, i.e., for $n = 0$ from $x_0 = \xi_0 = \delta_0$. Then (124) gives

$$x_0\eta_0 = \xi_0 y_0; \quad \xi_0 + y_0 = \eta_1, \quad y_1 = x_0 + \eta_0.$$

The general solution of this system is

$$y_0 = \eta_0 = \delta_1, \quad y_1 = \eta_1 = \delta_0 + \delta_1, \tag{125}$$

where $\delta_1 > 0$ is an arbitrary given number.

Consider the system (124) for $n = 1$ taking the coefficients $(x_0, x_1; \xi_0, \xi_1)$ equal to the solution $(y_0, y_1; \eta_0, \eta_1)$ (125) of the previous system. We get

$$\begin{aligned}
&x_0\eta_1 = \xi_1 y_0, && && \delta_1\eta_1 = (\delta_0+\delta_1)y_0,\\
&x_1\eta_0 = \xi_0 y_1; && && (\delta_0+\delta_1)\eta_0 = \delta_1 y_1;\\
&\xi_1 + y_0 = \eta_2, && \text{i.e.} && (\delta_0+\delta_1) + y_0 = \eta_2,\\
&\xi_0 + y_1 = x_0 + \eta_1, && && \delta_1 + y_1 = \delta_1 + \eta_1,\\
&y_2 = x_1 + \eta_0, && && y_2 = (\delta_0+\delta_1) + \eta_0.
\end{aligned} \tag{126}$$

The general solution of the system (126) is

$$y_0 = \eta_0 = \delta_2, \quad y_1 = \eta_1 = (\delta_0+\delta_1)\delta_2\delta_1^{-1}, \quad y_2 = \eta_2 = \delta_0 + \delta_1 + \delta_2, \tag{127}$$

where $\delta_2 > 0$ is arbitrary. We see that the solution is symmetric, $y_k = \eta_k$, $k = 0, 1, 2$, and therefore the system (124), for $n = 2$, has a simpler form, in this system, $x_k = \xi_k$, $k = 0, 1, 2$.

We assert that this symmetry takes place for arbitrary $n = 2, 3, \dots$.

Lemma 9. *Let the coefficients of the system (124) $x_k = \xi_k > 0$, $k = 0, 1, \dots, n$, $n = 2, 3, \dots$. Then its arbitrary solution $(y_0, y_1, \dots, y_{n+1}; \eta_0, \eta_1, \dots, \eta_{n+1})$ with the initial data $y_0 = \eta_0 = \delta_{n+1} > 0$ is symmetric, $y_k = \eta_k$, $k = 0, 1, \dots, n+1$.*

Proof. From the first and the last equations in the first column in (124), we conclude that

$$\eta_n = \xi_n y_0 x_0^{-1} = x_n \eta_0 \xi_0^{-1} = y_n.$$

This equality, together with the first and the last equations in the second column in (124), give

$$\eta_{n+1} = \xi_n + y_0 = x_n + \eta_0 = y_{n+1}.$$

The equality $\eta_n = y_n$ and the second and the fourth equation in the second column in (124) give:

$$\xi_{n-1} + y_1 = x_0 + \eta_n = x_0 + y_n = \xi_0 + y_n = x_{n-1} + \eta_1 = \xi_{n-1} + \eta_1,$$

from which we conclude that $\eta_1 = y_1$.

This equality and the second and the fourth equation in the first column in (124) give

$$\eta_{n-1} = \xi_{n-1} y_1 x_1^{-1} = x_{n-1} \eta_1 \xi_1^{-1} = y_{n-1}.$$

This equality, the third equation in the second column in (124) and the unwritten equation $\xi_1 + y_{n-1} = x_{n-2} + \eta_2$ in the second column give

$$\xi_{n-2} + y_2 = x_1 + \eta_{n-1} = x_1 + y_{n-1} = \xi_1 + y_{n-1} = x_{n-2} + \eta_2 = \xi_{n-2} + \eta_2,$$

from which we conclude that $\eta_2 = y_2$.

This equality and the third equation in the first column in (124) and the unwritten equation $x_{n-2}\eta_2 = \xi_2 y_{n-2}$ in this column give

$$\eta_{n-2} = \xi_{n-2} y_2 x_2^{-1} = x_{n-2} \eta_2 \xi_2^{-1} = y_{n-2}.$$

Repeating the last two steps, we obtain $\eta_3 = y_3$, $\eta_{n-3} = y_{n-3}$; $\eta_4 = y_4$, ... As a result, we have proved that $\eta_k = y_k$ for all $k = 0, 1, \dots, n+1$. $\square$

Using this lemma we can, instead of system (124), consider the more simple system $\forall n \in \mathbb{N}$:

(128)

$$\begin{array}{lll}
x_0 y_n = x_n y_0, & x_n + y_0 = y_{n+1}, & \\
x_1 y_{n-1} = x_{n-1} y_1, & x_{n-1} + y_1 = x_0 + y_n, & \\
x_2 y_{n-2} = x_{n-2} y_2, & x_{n-2} + y_2 = x_1 + y_{n-1}, & \\
\dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots & \\
x_{[n/2]} y_{[n/2]+1} = x_{[n/2]+1} y_{[n/2]}; & x_{[n/2]+1} + y_{[n/2]} = x_{[n/2]-1} + y_{[n/2]+2}, & \text{n odd}, \\
x_{[n/2]-1} y_{[n/2]+1} = x_{[n/2]+1} y_{[n/2]-1}; & x_{[n/2]} + y_{[n/2]} = x_{[n/2]-1} + y_{[n/2]+1}, & \text{n even}.
\end{array}$$

So, we get the following procedure for finding the matrices of the form (121) which satisfy the equality (120) where all $b_n = 0$, $n \in \mathbb{N}_0$.

Consider the system (128) of $n+1$ equations, where $(x_0, x_1, \dots, x_n)$ are positive coefficients and $(y_0, y_1, \dots, y_{n+1})$ are real unknowns.

For $n = 1$ we put $x_0 = \delta_1$, $x_1 = \delta_0 + \delta_1$ ($\delta_0, \delta_1 > 0$ are arbitrary initial data) and find its solution (127) with arbitrary $y_0 = \delta_2 > 0$.

Consider the system (128) for $n = 2$ with the coefficients (x_0, x_1, x_2) equal to the previous solution (y_0, y_1, y_2) and find the solution (y_0, y_1, y_2, y_3), where $y_0 = \delta_3 > 0$ is arbitrary.

Assume that $y_k > 0$, $k = 0, 1, 2, 3$, and consider (128) for $n = 3$ with the coefficients (x_0, x_1, x_2, x_3) equal to this (y_0, y_1, y_2, y_3) and the initial data $y_0 = \delta_4 > 0$ for a new solution. Continue this procedure, assuming at each step that the solution is positive (such positiveness depends on the choice of $\delta_0, \delta_1, \dots, > 0$). As a result, we find $(y_0, y_1, \dots, y_{n+1})$ for every $n \in \mathbb{N}_0$.

By means of formulas (123) (where $\xi_k = x_k$ and $\eta_k = y_k$) we determine elements of matrices (121). For this matrices and $b_n = 0, n \in \mathbb{N}_0$, the matrix J of type (9) is formally normal. If all y_k in the found solution $(y_0, y_1, \dots, y_{n+1})$ of (128) are uniformly bounded w.r.t. $n \in \mathbb{N}_0$, then the operator A, generated by J, is bounded and normal.

We have a similar situation in the more general case, where b_n may be not equal to zero. There it is necessary to find b_n, $n \in \mathbb{N}_0$, step by step, starting with initial $b_0 \in \mathbb{R}$ and finding $b_{n+1} = b_{n+1}^*$ from $b_n = b_n^*$ using the third equality in (120).

Example. Let us use the procedure described above for finding a solution of the system (128) in the case

$$\delta_n = q^n, \quad q > 0, \quad n \in \mathbb{N}_0. \tag{129}$$

It is easy to calculate that this solution, at the n-th step, has the form

$$y_0 = q^{n+1}, y_1 = q^n + q^{n+1}, y_2 = q^{n-1} + q^n + q^{n+1}, \dots, y_{n+1} = 1 + q + q^2 + \dots + q^{n+1}. \tag{130}$$

So, if $0 < q < 1$, the corresponding to the matrix J operator A is bounded, In this case, A is a normal operator. If $q \geq 1$, then J is unbounded and formally normal.

For the case (129), (130), it is easy to find the selfadjoint matrices b_n, $n \in \mathbb{N}_0$, for which the third equation in (120) is satisfied. As usual, we get a more general formally normal (or bounded normal) matrix of the type (9).

Acknowledgments. The authors are grateful to Professor L. B. Golinskii for stimulating discussions and O. A. Mokhonko for the help. The work is partially supported by the DFG 436 UKR 113/78/0-1 and the Ukrainian State Foundation for Fund. Studies, Grant № 01.07/027.

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Received 30/11/2005