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QUANTUM OF BANACH ALGEBRA

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Abstract. A variety of Banach algebras is a non-empty class of Banach algebras, for which there exists a family of laws such that its elements satisfy all of the laws. Each variety has a unique core (see [3]) which is generated by it. Each Banach algebra is not a core but, in this paper, we show that for each Banach algebra there exists a cardinal number (quantum of that Banach algebra) which shows the elevation of that Banach algebra for bearing a core. The class of all cores has interesting properties. Also, in this paper, we shall show that each core of a variety is generated by essential elements and each algebraic law of essential elements permeates to all of the elements of all of the Banach algebras belonging to that variety, which shows the existence of considerable structures in the cores.

1. Introduction

Throughout this paper by a polynomial we mean a polynomial in several non-commuting variables without constant term and also we shall identify Banach algebras if they are isometrically isomorphic. For each Banach algebra A and polynomial P, we define

$$\|P\|_A = \sup\{\|P(x_1, \ldots, x_n)\| : x_i \in A, \|x_i\| \leq 1\}.$$

We shall denote the class of all polynomials by L.

Definition 1.1. By a law we mean a formal expression

$$\|P\| \leq K,$$

where P is a polynomial and $K \in \mathbb{R}$. If A is a Banach algebra, then we say that A satisfies the above law if

$$\|P\|_A \leq K,$$

and it is a homogeneous law, if P is a homogeneous polynomial.

Definition 1.2. Let V be a family of Banach algebras. We say that V is a variety of Banach algebras, or simply a variety, if it is closed under taking (i) direct sums, (ii) closed subalgebras, (iii) quotients (by closed ideals), and (iv) isometric isomorphisms. Equivalently, V is a variety, if there exists a family of laws such that each element of V satisfies all of the laws (see [3]).

Definition 1.3. Let V be a variety. For each polynomial P, we define

$$|P|_V = \inf\{K_P : \text{ every element of } V \text{ satisfies the law } \|P\| \leq K_P\}.$$

Throughout this paper, for each Banach algebra A, A_1 will be the unit ball of A.

2. The class of cores

In [4], the author has proved that each variety V can be obtained by means of the family of laws

$$\{\|P\| \leq |P|_V\}_{P \in L},$$

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and it has a worthy property: varieties can be compared by them. Also, for each variety V there exists a Banach algebras $A \in V$ (a generator of V) such that for all polynomials P, $\|P\|_A = |P|_V$, i.e., V is singly-generated (each element of V is a quotient of a closed subalgebra of a direct sum of some copies of A up to isometric isomorphic). Each variety has got many generators, but just one of them has a maximum property, namely the core of that variety.

Definition 2.1. Let V be a variety. We say that $A \in V$ is a core of the variety V, if there is a sequence $\{a_i\}_{i=1}^{\infty}$ of members of A_1, such that the subalgebra generated by $\{a_1, a_2, \ldots\}$ is dense in A and

$$|P|_V = \|P(a_1, \ldots, a_n)\|$$

for all polynomials $P = P(X_1, \ldots, X_n)$.

Definition 2.2. Let A be a normed algebra. We say that A is elevated, if there is a sequence $\{a_i\}_{i=1}^{\infty}$ of members of A_1, such that for each polynomial $P = P(X_1, \ldots, X_n)$,

$$\sup\{\|P(x_1, \ldots, x_n)\| : x_i \in A_1\} = \|P(a_1, \ldots, a_n)\|.$$

The Lemma 2.3 and Theorem 2.4 are consequences of the text of [2].

Lemma 2.3. *Let A be a Banach algebra. Then*

(i) *if A is a core of some variety V, then A is elevated,*

(ii) *if A is elevated, then there exists a closed subalgebra of A which is a core of $V(A)$.*

Proof. (i) Let A be a core of some variety V. Then there is a sequence $\{a_i\}_{i=1}^{\infty}$ of elements of A_1, such that for all polynomials $P = P(X_1, \ldots, X_n)$,

$$|P|_V = \|P(a_1, \ldots, a_n)\|.$$

So we have

$$\|P\|_A \le |P|_V = \|P(a_1, \ldots, a_n)\| \le \|P\|_A.$$

Thus $\|P\|_A = \|P(a_1, \ldots, a_n)\|$.

(ii) If A is elevated, then there exists a sequence $\{a_i\}_{i=1}^{\infty}$ of elements of A_1, such that for all polynomials P, $\|P\|_A = \|P(a_1, \ldots, a_n)\|$. We have

$$|P|_{V(A)} = \|P\|_A = \|P(a_1, \ldots, a_n)\|.$$

Let A_0 be the normed subalgebra of A generated by $\{a_1, a_2, \ldots\}$. Let $\overline{A_0}$ be the closure of A_0 , then $\overline{A_0} \in V(A)$, and for all polynomials P, we have

$$\|P(a_1, \ldots, a_n)\|_{\overline{A_0}} = \|P(a_1, \ldots, a_n)\|_A = |p|_{V(A)},$$

so $\overline{A_0}$ is a core of $V(A)$. $\square$

Theorem 2.4. *Each variety has a unique core (up to isometric isomorphic).*

Proof. For each variety V, there exist $A \in V$ and a sequence $\{a_i\}_{i=1}^{\infty}$ in A_1 such that, for each polynomial $P(X_1, \ldots, X_n)$,

$$\|P(a_1, \ldots, a_n)\| \le \|P\|_A \le \sup\{\|P\|_A : A \in V\} = \|P(a_1, \ldots, a_n)\|,$$

so A is elevated (see [2]). Hence by the Lemma 2.3, it has a core. Now, suppose that V is a variety with two cores A and B. Hence there are sequence

$$\{a_i\}_{i=1}^{\infty}, \quad \{b_i\}_{i=1}^{\infty}$$

of members of A_1 and B_1, respectively, such that for each polynomial $P = P(X_1, \ldots, X_n)$, we have

$$|P|_V = \|P(a_1, \ldots, a_n)\|$$

and

$$|P|_V = \|P(b_1, \ldots, b_n)\|.$$

Let A_0, B_0 be the normed subalgebra determined by $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$, *respectively.* Let $Q : A_0 \to B_0$ be defined by

$$Q(P(a_1, \dots, a_n)) = P(b_1, \dots, b_n)$$

where $P(X_1, \dots, X_n)$ is a polynomial. The mapping Q is well-defined, because, if $P(a_1, \dots, a_n) = 0$, then $\|P(b_1, \dots, b_n)\| = 0$. so $P(b_1, \dots, b_n) = 0$. The mapping Q is a homomorphism of A_0 onto B_0, and also it is clear that Q is isometric. Thus A_0, B_0 are isometrically isomorphic. By definition 2.1, $\overline{A_0} = A$ and $\overline{B_0} = B$, hence A is isometrically isomorphic with B. □

The class of cores has wonderful properties and its study is worthy.

Definition 2.5. Let A be the core of a variety V. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements of A_1 such that the algebra generated by $\{a_1, a_2, \dots\}$ is dense in A and for all polynomials $P = P(X_1, \dots, X_n)$,

$$|P|_V = \|P(a_1, \dots, a_n)\|.$$

The sequence $\{a_n\}_{n=1}^{\infty}$ will be called an essential sequence of A and for each $i \in \mathbb{N}, a_i$ will be called an essential element of A .

Theorem 2.6. *Let V be a variety with the core A, and $\{a_n\}_{n=1}^{\infty}$ be an essential sequence of A. Let $B \in V$, and $n, k \in \mathbb{N}$. Then for each $x \in B_1$ and each essential elements $a_{m_1}, \dots, a_{m_n}$, we have*

(i) $\|x^n\| \le \|a_{m_1} \dots a_{m_n}\|$,
(ii) $\|x^k\| \le \|a^k + P(a_{m_1}, \dots a_{m_n})\|$ *where $P = P(X_1, \dots, X_n)$ is a polynomial and a is an essential element of A,*
(iii) *if V has a unital Banach algebra, then for each polynomial $P = P(X_1, \dots X_n)$,* $\|P(1)\| \le \|P(a_{m_1}, \dots, a_{m_n})\|$.

Proof. (i) Consider the polynomial $P = X_1 \dots X_n$ where some of the X_i may repeat.

(ii) If for each $1 \le i \le n,\ a \ne a_{m_i}$, then consider the polynomial $Q(Y, X_1, \dots, X_n) = Y^k + P(X_1, \dots, X_n)$ where for each $i \ge 1,\ Y$ is different from X_i. Since

$$\sup_{y, x_1, \dots, x_n \in B_1} \|Q(y, x_1, \dots, x_n)\| \le \|Q(a^k, a_{m_1}, \dots, a_{m_n})\|,$$

by substituting $x_1 = \dots = x_n = 0$, we obtain

$$\|x^k\| \le \|a^k + P(a_{m_1}, \dots, a_{m_n})\|.$$

If for some $1 \le i \le n,\ a = a_{m_i}$, then consider the polynomial

$$Q(X_1, \dots, X_n) = X_i^k + P(X_1, \dots, X_n).$$

(iii) It is straightforward. □

Corollary 2.7. *Let V be a variety with the core A and $\{a_n\}_{n=1}^{\infty}$ be an essential sequence of A. Then*

(i) *if $B \in V$, then for each $x \in B_1$ and each essential element a, $r(x) \le r(a)$,*
(ii) *for each essential elements a and b, $r(a) = r(b)$.*

Theorem 2.8. *Let V be a variety with the core A. Let $\{a_n\}_{n=1}^{\infty}$ be an essential sequence of A. Then*

(i) *if 1 is an essential element, then A is commutative,*
(ii) *if some of the essential elements of A satisfy an algebraic law, then all of the members of all Banach algebras in V also satisfy the law.*

Proof. (i) Let $x, y \in A$ and let a be an essential element. Then

$$\|xy - yx\| \le \|XY - YX\|_A = \|a1 - 1a\| = 0.$$

So $xy = yx$, which proves the theorem.

(ii) Let $P = P(X_1, \ldots, X_n)$ be a polynomial. Without lose of generality we can suppose that P is a homogeneous polynomial (see [3]). Let for some essential elements $a_{m_1}, \ldots, a_{m_n}, P(a_{m_1}, \ldots, a_{m_n}) = 0$. Let $B \in V$ and $x_1, \ldots, x_n \in B_1$. Then

$$\|P(x_1, \ldots, x_n)\| \leq |P|_V = \|P(a_{m_1}, \ldots, a_{m_n})\| = 0.$$

So $P(x_1, \ldots, x_n) = 0$. Now, let $y_1, \ldots, y_n \in B$. Then, there is $k > 1$ such that for each $1 \leq i \leq n$, $\|y_i\|/k < 1$, so $P(y_1, \ldots, y_n) = 0$, and the theorem is proved. □

Corollary 2.9. *(i) If two essential elements of the core of a variety are commutative, then all of the Banach algebras in that variety are commutative,*

(ii) for a variety V, $V = N_n$, if and only if there exist essential elements $a_{m_1}, \ldots, a_{m_n}$ such that

$$a_{m_1} \ldots a_{m_n} = 0.$$

Lemma 2.10. *Let V be a variety with the core A. Then the essential elements are linearly independent.*

Proof. Let $a_{m_1}, \ldots, a_{m_n}$ be essential elements of A such that

$$\lambda_1 a_{m_1} + \cdots + \lambda_n a_{m_n} = 0.$$

Let $\lambda_1 \neq 0$. Then by the Theorem (2.8) for each essential element a, we have

$$\lambda_1 a + \lambda_2 a_{m_2} + \cdots + \lambda_n a_{m_n} = 0.$$

Thus $a_{m_1} = a$, which implies that $a_{m_1} = 0$. But zero can not be an essential element. So the lemma is proved. □

3. Elevated Banach algebras

If for some X, card(X) is the cardinal number of X, then for each Banach algebra A, the set of all card(X) where the core of $V(A)$ is a closed subalgebra of A^X is not empty (see [2]). Since the class of all cardinal numbers is well-ordered, it has a least element, which we shall denote by Q_A and call it "the quantum of A"(see [6]). It is clear that A is elevated if and only if $Q_A = 1$.

The quantum of a Banach algebra shows its degree of elevation, moreover, the condition of elevation of a Banach algebra is not simple, it seems just a few number of Banach algebras can have it. But, indeed, they are uncountable. Now, we shall show some properties of quantum of a Banach algebra.

Lemma 3.1. *Let $\{A_\alpha\}_{\alpha \in I}$ be a non-empty family of Banach algebras such that for each $\alpha \in I, Q_{A_\alpha} = 1$. Then*

$$Q_{\oplus A_\alpha} = 1.$$

Proof. Let for each $\alpha \in I, \{a_n^\alpha\}_{n=1}^\infty$ be a sequence of members of $(A_\alpha)_1$ such that for each polynomial $P = P(X_1, \ldots, X_n)$,

$$\|P\|_{A_\alpha} = \|P(a_1^\alpha, \ldots, a_n^\alpha)\|.$$

Let

$$P = \sum_{i=1}^n c_i X_i + \sum_{1 \leq i_1, i_2 \leq n} c_{i_1 i_2} X_{i_1} X_{i_2} + \cdots + \sum_{1 \leq i_1, \ldots, i_k \leq n} c_{i_1 \ldots i_k} X_{i_1} \ldots X_{i_k}.$$

Then we have

$$\begin{aligned}
&\{P(a_1^\alpha, \ldots, a_n^\alpha)\}_\alpha \\
&= \left\{ \sum_{i=1}^n c_i a_i^\alpha + \sum_{1 \leq i_1, i_2 \leq n} c_{i_1 i_2} a_{i_1}^\alpha a_{i_2}^\alpha + \cdots + \sum_{1 \leq i_1, \ldots, i_k \leq n} c_{i_1 \ldots i_k} a_{i_1}^\alpha \ldots a_{i_k}^\alpha \right\}_\alpha \\
&= \sum_{i=1}^n c_i \{a_i^\alpha\}_\alpha + \sum_{1 \leq i_1, i_2 \leq n} c_{i_1 i_2} \{a_{i_1}^\alpha\}_\alpha \{a_{i_2}^\alpha\}_\alpha + \cdots + \sum_{1 \leq i_1, \ldots, i_k \leq n} c_{i_1 \ldots i_k} \{a_{i_1}^\alpha\}_\alpha \ldots \{a_{i_k}^\alpha\}_\alpha.
\end{aligned}$$

Thus,

$$\begin{aligned}\|P\|_{\oplus A_\alpha} &= \sup_{\alpha\in I}\|P\|_{A_\alpha} = \sup_{\alpha\in I}\|P(a_1^\alpha,\dots,a_n^\alpha)\| \\ &= \|\{P(a_1^\alpha,\dots,a_n^\alpha)\}_{\alpha\in I}\| = \|P(\{a_1^\alpha\}_{\alpha\in I},\dots,\{a_n^\alpha\}_{\alpha\in I})\|.\end{aligned}$$

But for each $n \in N, \{a_n^\alpha\}_{\alpha\in I} \in (\oplus_{\alpha\in I}A_\alpha)_1$. So the lemma is proved. □

Lemma 3.2. *Let $\{A_\alpha\}_{\alpha\in I}$ be a non-empty family of Banach algebras. Let there exist $\alpha_1 \in I$ such that A_{α_1} is elevated, and for each polynomial $P = P(X_1,\dots,X_n)$,*

$$\|P\|_{A_\beta} \le \|P\|_{A_{\alpha_1}}, \quad \beta \in I.$$

Then $Q_{\oplus A_\alpha} = 1$.

Proof. Let $P = P(X_1,\dots,X_n)$ be a polynomial. Let $\{a_n\}_{n=1}^\infty$ be a sequence of members of $(A_\alpha)_1$ such that $\|P\|_{A_{\alpha_1}} = \|P(a_1,\dots,a_n)\|$. We have

$$\|P\|_{\oplus A_\beta} = \sup_{\beta\in I}\|P\|_{A_\beta} = \|P\|_{A_{\alpha_1}} = \|P(a_1,\dots,a_n)\| = \|P(b_1,\dots,b_n)\|,$$

where $b_i = (b_{i,\alpha})_{\alpha\in I}$, with

$$b_{i,\alpha} = \begin{cases} a_i & \text{if } \alpha = \alpha_1, \\ 0 & \text{if } \alpha \ne \alpha_1. \end{cases}$$

Since $\|b_i\| = \|a_i\| \le 1$, the lemma is proved. □

Lemma 3.3. *Let A be a Banach algebra and let Y be a non-empty set. Then*

$$Q_{A^Y} \le Q_A \le \operatorname{card} Y . Q_{A^Y}.$$

Proof. Let Z be a set such that $\operatorname{card} Z = Q_{A^Y}$. For each $f \in A^{Y\times Z}$, define $f_z(y) = f(y,z)$, then $f_z \in A^Y$. Let $e_f(z) = f_z$, then $e_f \in (A^Y)^Z$. Hence $f \to e_f$ is an isometric isomorphism of $A^{Y\times Z}$ onto $(A^Y)^Z$. So the core of $V(A^Y)$ is a closed subalgebra of $(A^Y)^Z$. Therefore, the core of $V(A)$ is a closed subalgebra of $A^{Y\times Z}$. Hence,

$$Q_A \le \operatorname{card}(Y\times Z) = \operatorname{card} Y . Q_{A^Y}.$$

Now, suppose that the core of $V(A)$ is a closed subalgebra of A^X, where $\operatorname{card} X = Q_A$. Since A^X is a closed subalgebra of $(A^Y)^X$, hence $Q_A = \operatorname{card} X \ge Q_{A^Y}$, and the lemma is proved. □

Let A be a Banach algebra and let X be a set such that $\operatorname{card} X = Q_A$. We shall denote A^X by A^{Q_A}.

Theorem 3.4. *Let A be a Banach algebra. Then*

$$Q_{A^m} = Q_A \quad (m \in \mathbb{N}).$$

Proof. Let $Q_{A^m} = 1$, and let $\{(a_i^1,\dots,a_i^m)\}_{i=1}^\infty$ be a sequence of members of $(A^m)_1$ such that for all polynomials $P = P(X_1,\dots,X_n)$,

$$\|P\|_{A^m} = \|P((a_1^1,\dots,a_1^m),\dots,(a_n^1,\dots,a_n^m))\|.$$

Then we have,

$$\begin{aligned}\|P\|_A = \|P\|_{A^m} &= \|(P(a_1^1,\dots,a_n^1),\dots,P(a_1^m,\dots,a_n^m))\| \\ &= \|(P(a_1^1,\dots,a_n^1)),\dots,(P(a_1^m,\dots,a_n^m))\| \\ &= \sup\{\|P(a_1^1,\dots,a_n^1)\|,\dots,\|P(a_1^m,\dots,a_n^m)\|\}.\end{aligned}$$

Now, let $c : \mathbb{N} \to \{a_i^k : 1 \le k \le m, 1 \le i\}$ be a surjective mapping. Then $\{c_i\}_1^\infty$ is a sequence of members of $(A^m)_1$ such that for some $c_{i_1},\dots,c_{i_n}$,

$$\|P\|_A = \|P(c_{i_1},\dots,c_{i_n})\|.$$

Hence $Q_A = 1$. Since $Q_{(A^m)^{Q_{A^m}}} = 1$, $Q_{(A^{Q_{A^m}})^m} = 1$. Thus, $Q_{A^{Q_{A^m}}} = 1$. Hence, $Q_A \le Q_{A^m}$. By the Lemma (3.3), $Q_{A^m} \le Q_A$, and the theorem is proved. □

Let for each $n \in N, n = \operatorname{card}\{1,\dots,n\}$.

Lemma 3.5. *Let A be a Banach algebra. If $Q_A > 1$, then $Q_A \geq \aleph$.*

Proof. Let for some $n \in N$, $Q_A \leq n$. Then A^{Q_A} is elevated. So A^n is elevated. Thus $Q_{A^n} = 1$, so by the Theorem (3.3), $Q_A = 1$, which is an contradiction. $\square$

Lemma 3.6. *Let A be a Banach algebra. Then*

$$Q_A \leq (\operatorname{card} A)^c,$$

where $c = \operatorname{card} \mathbb{R}$.

Proof. Let M be the core of $V(A)$. Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of members of $(M)_1$ such that for each polynomial $P = P(X_1, \ldots, X_n), \|P\|_A = \|P(a_1, \ldots, a_n)\|$, and let M_0 be the algebra generated by $\{a_1, \ldots\}$. Then M is a closed subalgebra of $(A)^{A_1^{M_0}}$. Since

$$M_0 = \{P(a_{m_1}, \ldots, a_{m_n}) : P(X_1, \ldots, X_n) \text{ is a polynomial}\}.$$

So $\operatorname{card} M_0 \leq c$. Therefore,

$$Q_A \leq \operatorname{card} A_1^{M_0} \leq (\operatorname{card} A)^{\operatorname{card} M_0} \leq (\operatorname{card} A)^c.$$

$\square$

Lemma 3.7. *Let $c = \operatorname{card} \mathbb{R}$ .Then*

$$\aleph \leq Q_{\mathbb{C}} \leq c^c.$$

Proof. Since $\|X + Y\|_{\mathbb{C}} = \|X - Y\|_{\mathbb{C}} = 2$, by a simple calculating we see that $\mathbb{C}$ is not elevated. Hence, $Q_{\mathbb{C}} > 1$. Therefore, $Q_{\mathbb{C}} \geq \aleph$, and by the Lemma (3.3) , $Q_{\mathbb{C}} \leq c^c$. So the theorem is proved. $\square$

Lemma 3.8. *Let A, B be two Banach algebras. Then*

$$Q_{A \oplus B} \leq Q_A Q_B.$$

Proof. Let $\operatorname{card} X = Q_A$ and $\operatorname{card} Y = Q_B$. Let

$$\psi : (A \oplus B)^{X \times Y} \longrightarrow (A^X)^Y \oplus (B^Y)^X$$

be defined by

$$\psi(f) = (e_{P_A \circ f}, e'_{P_B \circ f}), \quad f \in (A \oplus B)^{X \times Y},$$

where P_A, P_B are the projections onto A and B, respectively, and the maps

$$e_{P_A \circ f} : Y \longrightarrow A^X, \quad e'_{P_B \circ f} : X \longrightarrow B^Y$$

are defined by

$$e_{P_A \circ f}(y) = P_A \circ f_y, \quad e'_{P_B \circ f}(x) = P_B \circ f^x$$

where the maps $f_y : X \longrightarrow A \oplus B$ and $f^x : Y \longrightarrow A \oplus B$ are defined by

$$f_y(x) = f(x, y), \quad f^x(y) = f(x, y).$$

It is easily verified that ψ is an isometric isomorphism. Since A^X and B^Y are elevated, by Lemma 3.1, $(A^X)^Y$ and $(B^Y)^X$ are elevated. Therefore, $(A^X)^Y \oplus (B^Y)^X$ is elevated. Hence $(A \oplus B)^{X \times Y}$ is an elevated element of the variety $V(A \oplus B)$, so $Q_{A \oplus B} \leq \operatorname{card} X \times Y = Q_A Q_B$. $\square$

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