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NEVANLINNA TYPE FAMILIES OF LINEAR RELATIONS AND THE DILATION THEOREM

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Abstract. Let $\mathcal{H}_1$ be a subspace in a Hilbert space $\mathcal{H}_0$ and let $\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ be the set of all closed linear relations from $\mathcal{H}_0$ to $\mathcal{H}_1$. We introduce a Nevanlinna type class $\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ of holomorphic functions with values in $\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ and investigate its properties. In particular we prove the existence of a dilation for every function $\tau_+(\cdot)\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$. In what follows these results will be used for the derivation of the Krein type formula for generalized resolvents of a symmetric operator with arbitrary (not necessarily equal) deficiency indices.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and let $\widetilde{\mathcal{C}}(\mathcal{H})$ be the set of all closed linear relations in $\mathcal{H}$. Recall that a holomorphic function (family of linear relations) $\tau(\cdot):\mathbb{C}_+\cup\mathbb{C}_-\to\widetilde{\mathcal{C}}(\mathcal{H})$ belongs to the Nevanlinna class $\widetilde{R}(\mathcal{H})$, if $\tau(\lambda)$ is a maximal dissipative linear relation for all $\lambda\in\mathbb{C}_+$ and $\tau^*(\lambda)=\tau(\overline{\lambda})$, $\lambda\in\mathbb{C}_+\cup\mathbb{C}_-$. It is well known that the class $\widetilde{R}(\mathcal{H})$ possesses a number of the interesting properties. In particular, for a function $\tau(\cdot)\in\widetilde{R}(\mathcal{H})$ there is a Hilbert space $\mathfrak{H}_1$ and a selfadjoint linear relation $\widetilde{\theta}\in\widetilde{\mathcal{C}}(\mathcal{H}\oplus\mathfrak{H}_1)$ (a dilation of the function $\tau(\cdot)$) such that the following equality holds [11, 6]

$$P_{\mathcal{H}}(\widetilde{\theta}-\lambda)^{-1}\upharpoonright\mathcal{H}=-(\tau(\lambda)+\lambda)^{-1},\quad\lambda\in\mathbb{C}_+\cup\mathbb{C}_-. \tag{1.1}$$

Here $P_{\mathcal{H}}$ is the orthoprojector in $\mathcal{H}\oplus\mathfrak{H}_1$ onto $\mathcal{H}$. Formula (1.1) is implied by the Naimark theorem [1, 12]. Note also that Nevanlinna families of linear relations naturally appear in the Krein-Naimark formula for generalized resolvents of a symmetric operator with equal defect numbers (see [8, 9, 2] and references therein).

Assume now that $\mathcal{H}_1$ is a subspace in a Hilbert space $\mathcal{H}_0$ and $\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ is the set of all closed linear relations from $\mathcal{H}_0$ to $\mathcal{H}_1$. In the present paper we introduce a Nevanlinna type class $\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ of holomorphic functions $\tau_+(\cdot):\mathbb{C}_+\to\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ and investigate its properties. The main result of the paper is Theorem 4.6 that may be considered as an analog of the dilation theorem in the form (1.1). Namely, we prove the existence of a dilation $\widetilde{\theta}$ (in a generalized form) for a function $\tau_+(\cdot)\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ and describe some properties of $\widetilde{\theta}$ in terms of $\tau_+(\cdot)$.

In the forthcoming paper we will present the main result of our investigations, the Krein-Naimark type formula for a symmetric operator A with arbitrary (not necessarily equal) defect numbers. The proof of this formula is based on Theorem 4.6. Moreover in this formula the class $\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ plays the role similar to that of the class $\widetilde{R}(\mathcal{H})$ in the Krein-Naimark formula. Namely, the functions $\tau_+(\cdot)\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ appear in this formula as parameters allowing to describe all generalized resolvents of A. Therefore the class $\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ may be considered as a natural generalization of the Navanlinna class to the case $\mathcal{H}_1\subset\mathcal{H}_0$, $\mathcal{H}_1\neq\mathcal{H}_0$.

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2. Preliminaries

2.1. **Notations.** The following notations will be used throughout the paper: $\mathfrak{H}$, $\mathcal{H}$ denote Hilbert spaces; $[\mathcal{H}_1, \mathcal{H}_2]$ is the set of all bounded linear operators defined on $\mathcal{H}_1$ with values in $\mathcal{H}_2$; $C(\mathcal{H}_1, \mathcal{H}_2)$ is the set of all operators $B \in [\mathcal{H}_1, \mathcal{H}_2]$ with $||B|| \le 1$; $A \upharpoonright L$ is the restriction of an operator A onto the linear manifold L; P_L is the orthogonal projector in $\mathfrak{H}$ onto the subspace $L \subset \mathfrak{H}$; $\mathbb{C}_+$ ($\mathbb{C}_-$) is the upper (lower) half-plain of the complex plain. We also let $[\mathcal{H}] := [\mathcal{H}, \mathcal{H}]$ and $C(\mathcal{H}) := C(\mathcal{H}, \mathcal{H})$.

For a Hilbert space $\mathfrak{H}$ we denote by $\dim \mathfrak{H}$ its dimension. Moreover we write $\dim \mathfrak{H} < \infty$, if $\mathfrak{H}$ is finite-dimensional and $\dim \mathfrak{H} = \infty$, if $\mathfrak{H}$ is an infinite-dimensional not necessarily separable Hilbert space.

Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be Hilbert spaces. A linear manifold $T \subset \mathcal{H}_0 \oplus \mathcal{H}_1$ is called a linear relation in $\mathcal{H}_0 \oplus \mathcal{H}_1$ (from $\mathcal{H}_0$ to $\mathcal{H}_1$). We denote by $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ ($\widetilde{\mathcal{C}}(\mathcal{H})$) the set of all closed linear relations(closed subspaces) in $\mathcal{H}_0 \oplus \mathcal{H}_1$ (in $\mathcal{H} \oplus \mathcal{H}$). For a linear relation $T \subset \mathcal{H}_0 \oplus \mathcal{H}_1$ we denote by $\mathcal{D}(T)$, $\mathcal{R}(T)$, $\operatorname{Ker} T$ and $T(0)$ the domain, the range, the kernel and the multivalued part of T respectively.

If T is a relation in $\mathcal{H}_0 \oplus \mathcal{H}_1$, then the inverse T^{-1} and adjoint T^* relations are defined as

$$T^{-1} = \{\{f', f\} : \{f, f'\} \in T\}, \quad T^{-1} \subset \mathcal{H}_1 \oplus \mathcal{H}_0, \tag{2.1}$$

$$T^* = \{\{g, g'\} \in \mathcal{H}_1 \oplus \mathcal{H}_0 : (f', g) = (f, g'), \quad \{f, f'\} \in T\}, \quad T^* \in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0). \tag{2.2}$$

A closed linear operator T from $\mathcal{H}_0$ to $\mathcal{H}_1$ is identified with its graph $\operatorname{gr} T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$.

For a linear relation $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we write $0 \in \rho(T)$ if $\operatorname{Ker} T = \{0\}$ and $\mathcal{R}(T) = \mathcal{H}_1$, which is equivalent to the condition $T^{-1} \in [\mathcal{H}_1, \mathcal{H}_0]$. Moreover we denote by $\rho(T) = \{\lambda \in \mathbb{C} : \ 0 \in \rho(T - \lambda)\}$ the resolvent set of a linear relation $T \in \widetilde{\mathcal{C}}(\mathcal{H})$.

2.2. **Linear relations and holomorphic functions.** Let H, $\mathcal{H}_0$ $\mathcal{H}_1$ be Hilbert spaces and let $K = (K_0 \ \ K_1)^\top \in [H, \mathcal{H}_0 \oplus \mathcal{H}_1]$. For a (not necessary closed) linear relation $\theta \subset \mathcal{H}_0 \oplus \mathcal{H}_1$ we write $\theta = \{K_0, K_1; H\}$ if $\operatorname{Ker} K = \{0\}$ (that is $\operatorname{Ker} K_0 \cap \operatorname{Ker} K_1 = \{0\}$) and

$$\theta = KH = \{\{K_0 h, K_1 h\} : h \in H\}.$$

Similarly let $C = (C_0 \ \ C_1) \in [\mathcal{H}_0 \oplus \mathcal{H}_1, H]$. For a linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we write $\theta = \{(C_0, C_1); H\}$ if $\mathcal{R}(C) = H$ and

$$\theta = \operatorname{Ker} C = \{\{h_0, h_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1 : C_0 h_0 + C_1 h_1 = 0\}.$$

It is clear that every linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ admits both representations $\theta = \{K_0, K_1; H\}$ and $\theta = \{(C_0, C_1); H'\}$. Moreover the equalities $\dim H = \dim \theta$, $\dim H' = \operatorname{codim} \theta$ are valid.

Lemma 2.1. *1) Let $K_0 \in [H, \mathcal{H}_0]$, $K_1 \in [H, \mathcal{H}_1]$ and let $B \in [\mathcal{H}_0, \mathcal{H}_1]$ be an operator such that $0 \in \rho(K_1 - BK_0)$. Then $\operatorname{Ker} K_0 \cap \operatorname{Ker} K_1 = \{0\}$ and the equality $\theta = \{K_0, K_1; H\}$ define a closed linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ such that $0 \in \rho(\theta - B)$. Moreover in this case*

$$(\theta - B)^{-1} = K_0 (K_1 - BK_0)^{-1}. \tag{2.3}$$

2) Assume that $\theta = \{K_0, K_1; H\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$, $\tau = \{N_1, N_0; H'\} \in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0)$ and there is an operator $B \in [\mathcal{H}_0, \mathcal{H}_1]$ such that $0 \in \rho(K_1 - BK_0) \cap \rho(N_0 - B^* N_1)$ (i.e., $0 \in \rho(\theta - B) \cap \rho(\tau - B^*)$). Then the following equivalence holds

$$\tau = \theta^* \iff N_0^* K_0 = N_1^* K_1. \tag{2.4}$$

Proof. 1) Since $\operatorname{Ker}(K_1 - BK_0) = \{0\}$, it follows that $\operatorname{Ker} K_0 \cap \operatorname{Ker} K_1 = \{0\}$. Let now $\theta = \{K_0, K_1; H\}$ be a linear relation in $\mathcal{H}_0 \oplus \mathcal{H}_1$. Then

$$\theta - B = \{K_0, K_1 - BK_0; H\} \tag{2.5}$$

and hence $\mathcal{R}(\theta - B) = \mathcal{R}(K_1 - BK_0) = \mathcal{H}_1$, $\operatorname{Ker}(\theta - B) = K_0 \operatorname{Ker}(K_1 - BK_0) = \{0\}$. Consequently $0 \in \rho(\theta - B)$ and therefore the relation θ is closed. Finally the equality (2.3) is implied by (2.5).

2) Since $(\theta - B)^{-1} = K_0(K_1 - BK_0)^{-1}$ and $(\tau - B^*)^{-1} = N_1(N_0 - B^*N_1)^{-1}$, the following chain of equivalences is valid

$$\tau = \theta^* \iff (\theta - B)^{-1} = (\tau - B^*)^{-1*} \iff K_0(K_1 - BK_0)^{-1} = (N_0^* - N_1^*B)^{-1}N_1^*$$
$$\iff (N_0^* - N_1^*B)K_0 = N_1^*(K_1 - BK_0).$$

This yields the equivalence (2.4). □

Let $\mathcal{D}$ be an open set in $\mathbb{C}$ and let $K_0(\cdot) : \mathcal{D} \to [H, \mathcal{H}_0]$, $K_1(\cdot) : \mathcal{D} \to [H, \mathcal{H}_1]$ be a pair of holomorphic operator functions. Such a pair will be called admissible if $\operatorname{Ker} K_0(\lambda) \cap \operatorname{Ker} K_1(\lambda) = \{0\}$, $\lambda \in \mathcal{D}$.

Definition 2.2. Let $\{K_0(\cdot), K_1(\cdot)\}$ and $\{K_0'(\cdot), K_1'(\cdot)\}$ be two admissible pairs of holomorphic operator functions, $K_j : \mathcal{D} \to [H, \mathcal{H}_j]$, $K_j' : \mathcal{D} \to [H', \mathcal{H}_j]$, $j \in \{0, 1\}$. Two such pairs are said to be equivalent if $K_0'(\lambda) = K_0(\lambda)\varphi(\lambda)$ and $K_1'(\lambda) = K_1(\lambda)\varphi(\lambda)$ for some operator function $\varphi(\cdot) : \mathcal{D} \to [H', H]$, which is holomorphic and invertible on $\mathcal{D}$.

Definition 2.3. (cf. [7]). A function $\tau(\cdot)$, defined on an open set $\mathcal{D} \subset \mathbb{C}$ with values in $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ is called holomorphic on $\mathcal{D}$ if there exist a Hilbert space H and an admissible pair of holomorphic operator functions $K_j(\cdot) : \mathcal{D} \to [H, \mathcal{H}_j]$, $j \in \{0, 1\}$ such that

$$\tau(\lambda) = \{K_0(\lambda), K_1(\lambda); H\} = \{\{K_0(\lambda)h, K_1(\lambda)h\} : h \in H\}, \quad \lambda \in \mathcal{D}. \tag{2.6}$$

It is clear that two pairs $\{K_0(\cdot), K_1(\cdot)\}$ and $\{K_0'(\cdot), K_1'(\cdot)\}$ define by (2.6) the same holomorphic function $\tau(\cdot)$, if and only if they are equivalent. Therefore we will identify (by means of (2.6)) a holomorphic $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$-valued function $\tau(\cdot)$ and the corresponding class of equivalent admissible pairs $\{K_0(\cdot), K_1(\cdot)\}$.

Proposition 2.4. (cf. [7]). *Let $\tau(\cdot) : \mathcal{D} \to \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ be a $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$-valued function and let $N(\cdot) : \mathcal{D} \to [\mathcal{H}_0, \mathcal{H}_1]$ be a holomorphic operator function such that $0 \in \rho(\tau(\lambda) - N(\lambda))$, $\lambda \in \mathcal{D}$. Then the function $\tau(\cdot)$ is holomorphic on $\mathcal{D}$ if and only if so is an operator function $(\tau(\lambda) - N(\lambda))^{-1}$, $\lambda \in \mathcal{D}$.*

Proof. Let $\tau(\cdot)$ be a holomorphic function with the corresponding representation (2.6). Then

$$\tau(\lambda) - N(\lambda) = \{\{K_0(\lambda)h, (K_1(\lambda) - N(\lambda)K_0(\lambda))h\} : h \in H\}$$

and the pair $\{K_0(\lambda), K_1(\lambda) - N(\lambda)K_0(\lambda)\}$ is admissible because so is the pair $\{K_0(\lambda), K_1(\lambda)\}$. Therefore $\tau(\lambda) - N(\lambda) = \{K_0(\lambda), K_1(\lambda) - N(\lambda)K_0(\lambda); H\}$, $\lambda \in \mathcal{D}$ and hens $(\tau(\lambda) - N(\lambda))^{-1} = K_0(\lambda)(K_1(\lambda) - N(\lambda)K_0(\lambda))^{-1}$. This imply that the function $(\tau(\lambda) - N(\lambda))^{-1}$ is holomorphic.

Conversely assume that $S(\lambda) := (\tau(\lambda) - N(\lambda))^{-1}$ is a holomorphic operator function. It is clear that $\tau(\lambda) = \{S(\lambda), I_{\mathcal{H}_1} + N(\lambda)S(\lambda); \mathcal{H}_1\}$, $\lambda \in \mathcal{D}$ and the pair $\{S(\lambda), I_{\mathcal{H}_1} + N(\lambda)S(\lambda)\}$ is admissible. Hence by Definition 2.3 $\tau(\cdot)$ is a holomorphic function. □

Corollary 2.5. *Suppose that $\tau(\cdot) : \mathcal{D} \to \widetilde{\mathcal{C}}(\mathcal{H})$ is a $\widetilde{\mathcal{C}}(\mathcal{H})$-valued function and there is a point $\mu \in \mathbb{C}$ such that $\mu \in \rho(\tau(\lambda))$ for every $\lambda \in \mathcal{D}$. Then $\tau(\cdot)$ is holomorphic on $\mathcal{D}$ if and only if so is the operator function $(\tau(\lambda) - \mu)^{-1}, \lambda \in \mathcal{D}$.*

3. Linear relations from a Hilbert space to its subspace

Let $\mathcal{H}_1$ be a subspace in a Hilbert space $\mathcal{H}_0$ and let $\mathcal{H}_2 = \mathcal{H}_0 \ominus \mathcal{H}_1$. Denote by P_i the orthoprojector in $\mathcal{H}_0$ onto $\mathcal{H}_i$, $i \in \{1,2\}$ and introduce the operators

(3.1)
$$J_{01}=\begin{pmatrix} P_2 & -iI_{\mathcal{H}_1} \\ iP_1 & 0 \end{pmatrix}:\mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1, \quad J_{10}=\begin{pmatrix} 0 & -iP_1 \\ iI_{\mathcal{H}_1} & P_2 \end{pmatrix}:\mathcal{H}_1 \oplus \mathcal{H}_0 \to \mathcal{H}_1 \oplus \mathcal{H}_0,$$

(3.2)
$$U_{01}=\begin{pmatrix} P_1 & 0 \\ iP_2 & I_{\mathcal{H}_1} \end{pmatrix}:\mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_0, \qquad U_{10}=\begin{pmatrix} I_{\mathcal{H}_1} & -iP_2 \\ 0 & P_1 \end{pmatrix}:\mathcal{H}_1 \oplus \mathcal{H}_0 \to \mathcal{H}_0 \oplus \mathcal{H}_1.$$

It is easily seen that J_{01} and J_{10} are signature operators, i.e., $J_{01} = (J_{01})^* = (J_{01})^{-1}$ and $J_{10} = (J_{10})^* = (J_{10})^{-1}$. Furthermore since $U_{10} = U_{01}^*$ and $U_{10}U_{01} = I_{\mathcal{H}_0\oplus\mathcal{H}_1}$, $U_{01}U_{10} = I_{\mathcal{H}_1\oplus\mathcal{H}_0}$, it follows that U_{01} and U_{10} are unitary operators and $U_{10} = (U_{01})^{-1}$.

For every linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we put

$$\theta^\times = J_{01}(\theta^\perp) = (J_{01}\theta)^\perp, \quad \theta^\times \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1). \tag{3.3}$$

It is clear that $\theta^\times$ is the set of all vectors $\hat{k} = \{k_0, k_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1$ such that

$$(k_1, h_0) - (k_0, h_1) + i(P_2k_0, P_2h_0) = 0, \quad \{h_0, h_1\} \in \theta. \tag{3.4}$$

If $\mathcal{H}_1 = \mathcal{H}_0 := \mathcal{H}$, then a linear relation $\theta^\times \in \widetilde{\mathcal{C}}(\mathcal{H})$ coincides with θ^*. In the next proposition we show that in the general case (i.e., if $\mathcal{H}_1 \subset \mathcal{H}_0$) $\theta^\times$ possesses a number of properties similar that of θ^*.

Proposition 3.1. *1) The linear relations $\theta^\times \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ and $\theta^* \in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0)$ are connected via $\theta^* = U_{01}\theta^\times$, $\theta^\times = U_{10}\theta^*$;*

2) $\theta^{\times\times} = \theta$;

3) $\theta_1 \subset \theta_2 \iff \theta_2^\times \subset \theta_1^\times$;

4) Let $\theta = \{K_0, K_1; H\}$, where $K_0 = (K_{01} \quad K_{02})^\top : H \to \mathcal{H}_1 \oplus \mathcal{H}_2$ and $K_1 \in [H, \mathcal{H}_1]$. Then $\theta^\times = \{(\widetilde{C}_0, \widetilde{C}_1); H\}$ where

$$\widetilde{C}_0 = (K_1^* \quad -iK_{02}^*) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to H, \quad \widetilde{C}_1 = -K_{01}^* \in [\mathcal{H}_1, H]. \tag{3.5}$$

Similarly let $\theta = \{(C_0, C_1); H\}$ where $C_0 = (C_{01} \quad C_{02}) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to H$ and $C_1 \in [\mathcal{H}_1, H]$. Then $\theta^\times = \{\widetilde{K}_0, \widetilde{K}_1; H\}$ where

$$\widetilde{K}_0 = (-C_1^* \quad -iC_{02}^*)^\top : H \to \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \widetilde{K}_1 = C_{01}^* \in [H, \mathcal{H}_1]. \tag{3.6}$$

If in particular B is a bounded operator from $\mathcal{H}_0$ to $\mathcal{H}_1$ and $B = (B_1 \quad B_2) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1$, then $B^\times$ is a linear relation defined by

$$B^\times = \{(I_{\mathcal{H}_1} \quad -iB_2^*)^\top, B_1^*; \mathcal{H}_1\}. \tag{3.7}$$

Proof. 1) Let $\{k_0, k_1\} \in \theta^\times$ and let $\{l_1, l_0\} = U_{01}\{k_0, k_1\}$, that is $l_1 = P_1k_0$ and $l_0 = iP_2k_0 + k_1$. Then by (3.4)

$$(l_0, h_0) - (l_1, h_1) = i(P_2k_0, P_2h_0) + (k_1, h_0) - (k_0, h_1) = 0, \quad \{h_0, h_1\} \in \theta.$$

Hence $\{l_1, l_0\} \in \theta^*$ and therefore $U_{01}\theta^\times \subset \theta^*$.

Let now $\{l_1, l_0\} \in \theta^*$ and let $\{k_0, k_1\} = U_{10}\{l_1, l_0\}$, so that $k_0 = l_1 - iP_2l_0$ and $k_1 = P_1l_0$. Then

$$(k_1, h_0) - (k_0, h_1) + i(P_2k_0, P_2h_0) = (l_0, h_0) - (l_1, h_1) = 0, \quad \{h_0, h_1\} \in \theta,$$

that is $\{k_0, k_1\} \in \theta^\times$. Hence $U_{10}\theta^* \subset \theta^\times$, which yields the desired statement.

2) It follows from (3.3) that

$$\theta^{\times\times} = J_{01}[(\theta^\times)^\perp] = J_{01}(J_{01}\theta) = J_{01}^2\theta = \theta.$$

3) If $\theta_1 \subset \theta_2$, then $\theta_2^\perp \subset \theta_1^\perp$ and by (3.3) $\theta_2^\times \subset \theta_1^\times$. Conversely if $\theta_2^\times \subset \theta_1^\times$, then $\theta_1 = \theta_1^{\times\times} \subset \theta_2^{\times\times} = \theta_2$.

4) Let $K = (K_0 \quad K_1)^\top \in [H, \mathcal{H}_0 \oplus \mathcal{H}_1]$. Then $\theta = KH$ and by (3.3)

$$\theta^\times = (J_{01}KH)^\perp = \operatorname{Ker}(J_{01}K)^* = \operatorname{Ker}(K^*J_{01}) = \operatorname{Ker}\widetilde{C},$$

where

$$\widetilde{C} = K^*J_{01} = (K_0^* \quad K_1^*)\begin{pmatrix} P_2 & -iI_{\mathcal{H}_1} \\ iP_1 & 0 \end{pmatrix} = (iK_1^*P_1 + K_{02}^*P_2 \quad -iK_{01}^*). \tag{3.8}$$

Moreover since $\operatorname{Ker} K = \{0\}$ and $\overline{\mathcal{R}(K)} = \mathcal{R}(K)(=\theta)$, one has $\mathcal{R}(K^*) = H$ and therefore $\mathcal{R}(\widetilde{C}) = H$. This and (3.8) yield the first part of the statement 4).

To prove the second part consider a linear relation $\widetilde{\theta} = \{\widetilde{K}_0, \widetilde{K}_1; H\}$, where $\widetilde{K}_0$ and $\widetilde{K}_1$ are given by (3.6). It follows from (3.5) that $\widetilde{\theta}^\times = \{(C_0, C_1); H\} = \theta$. Therefore $\theta^\times = \widetilde{\theta} = \{\widetilde{K}_0, \widetilde{K}_1; H\}$.

Finally formula (3.7) is implied by (3.6) and the obvious relation $B = \{(B, -I_{\mathcal{H}_1}); \mathcal{H}_1\}$. □

Let $\mathcal{H}_1$ be a subspace in a Hilbert space $\mathcal{H}_0$. For a linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we let

$$\varphi_\theta(\hat{h}) = 2\operatorname{Im}(h_1, h_0) + ||P_2h_0||^2, \quad \hat{h} = \{h_0, h_1\} \in \theta. \tag{3.9}$$

Definition 3.2. A linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ belongs to the class:

1) $\operatorname{Dis}_0(\mathcal{H}_0, \mathcal{H}_1)$ $(Ac_0(\mathcal{H}_0, \mathcal{H}_1))$, if $\varphi_\theta(\hat{h}) \geq 0$ $(\varphi_\theta(\hat{h}) \leq 0)$ for all $\hat{h} \in \theta$;
2) $\operatorname{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$, if $\theta \subset \theta^\times$;
3) $\operatorname{Self}(\mathcal{H}_0, \mathcal{H}_1)$, if $\theta = \theta^\times$.

It is easily seen that $\theta \in \operatorname{Sym}_0(\mathcal{H}_0, \mathcal{H}_1) \iff \varphi_\theta(\hat{h}) = 0,\ \hat{h} \in \theta$.

Definition 3.3. A linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ belongs to one of the classes $\operatorname{Dis}(\mathcal{H}_0, \mathcal{H}_1)$, $\operatorname{Ac}(\mathcal{H}_0, \mathcal{H}_1)$ or $\operatorname{Sym}(\mathcal{H}_0, \mathcal{H}_1)$ if it belongs to the class $\operatorname{Dis}_0(\mathcal{H}_0, \mathcal{H}_1)$, $\operatorname{Ac}_0(\mathcal{H}_0, \mathcal{H}_1)$ or $\operatorname{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$ respectively and there are not extensions $\widetilde{\theta} \supset \theta$, $\widetilde{\theta} \neq \theta$ in the corresponding class.

Note that in the case $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$ the classes $\operatorname{Dis}(\mathcal{H}, \mathcal{H})$, $\operatorname{Ac}(\mathcal{H}, \mathcal{H})$, $\operatorname{Sym}(\mathcal{H}, \mathcal{H})$ and $\operatorname{Self}(\mathcal{H}, \mathcal{H})$ coincide with the sets of all maximal dissipative, maximal accumulative, maximal symmetric and selfadjoint linear relations in $\mathcal{H}$ respectively.

In the next proposition we describe classes Dis, Ac, Sym and Self in the form convenient for applications.

Proposition 3.4. *1) Assume that θ is a (not necessary closed) linear relation in $\mathcal{H}_0 \oplus \mathcal{H}_1$ and $\theta = \{K_0, K_1; H\}$, where $K_0 = (K_{01} \quad K_{02})^\top : H \to \mathcal{H}_1 \oplus \mathcal{H}_2$, $K_1 \in [H, \mathcal{H}_1]$. Moreover let*

$$S_\theta := 2\operatorname{Im}(K_{01}^*K_1) + K_{02}^*K_{02}, \quad S_\theta \in [H]. \tag{3.10}$$

Then:

i) *$\theta \in \operatorname{Dis}(\mathcal{H}_0, \mathcal{H}_1)$ if and only if $S_\theta \geq 0$ and*

$$0 \in \rho(K_1 + \lambda K_0) \quad \text{for some (equivalently for every)} \quad \lambda \in \mathbb{C}_+; \tag{3.11}$$

ii) *$\theta \in \operatorname{Ac}(\mathcal{H}_0, \mathcal{H}_1)$ if and only if $S_\theta \leq 0$ and*

$$0 \in \rho(K_1 + \lambda K_{01}) \quad \text{for some (equivalently for every)} \quad \lambda \in \mathbb{C}_-; \tag{3.12}$$

iii) $\theta \in \mathrm{Sym}(\mathcal{H}_0, \mathcal{H}_1)$ $(\theta \in \mathrm{Self}(\mathcal{H}_0, \mathcal{H}_1))$ *if and only if* $S_\theta = 0$ *and at least one of the condition (respectively both the conditions)* (3.11), (3.12) *is fulfilled. Therefore* $\theta \in \mathrm{Self}(\mathcal{H}_0, \mathcal{H}_1)$ *if and only if* $\theta \in \mathrm{Dis}(\mathcal{H}_0, \mathcal{H}_1) \cap \mathrm{Ac}(\mathcal{H}_0, \mathcal{H}_1)$.

2) Let $\theta = \{(C_0, C_1); H\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$, *where* $C_0 = (C_{01} \quad C_{02}) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to H$, $C_1 \in [\mathcal{H}_1, H]$ *and let*

$$\widetilde{S}_\theta := 2\mathrm{Im}(C_1 C_{01}^*) - C_{02} C_{02}^*, \quad \widetilde{S}_\theta \in [H].$$

Then:

i) $\theta \in \mathrm{Dis}(\mathcal{H}_0, \mathcal{H}_1)$ *if and only if* $\widetilde{S}_\theta \geq 0$ *and*

$$0 \in \rho(C_{01} - \lambda C_1) \quad \text{for some (equivalently for every)} \quad \lambda \in \mathbb{C}_+; \tag{3.13}$$

ii) $\theta \in \mathrm{Ac}(\mathcal{H}_0, \mathcal{H}_1)$ *if and only if* $\widetilde{S}_\theta \leq 0$ *and*

$$0 \in \rho(C_0 - \lambda C_1 P_1) \quad \text{for some (equivalently for every)} \quad \lambda \in \mathbb{C}_-; \tag{3.14}$$

iii) $\theta \in \mathrm{Sym}(\mathcal{H}_0, \mathcal{H}_1)$ $(\theta \in \mathrm{Self}(\mathcal{H}_0, \mathcal{H}_1))$ *if and only if* $\widetilde{S}_\theta = 0$ *and at least one of the condition (respectively both the conditions)* (3.13), (3.14) *is fulfilled.*

Proof. 1) Let $\hat{h} = \{h_0, h_1\} \in \theta$, where $h_0 = K_0 h, \ h_1 = K_1 h, \ \ h \in H$. Then in view of (3.9)

$$\varphi_\theta(\hat{h}) = 2\mathrm{Im}(K_1 h, K_0 h) + ||P_2 K_0 h||^2 = ((2\mathrm{Im}(K_{01}^* K_1) + K_{02}^* K_{02})h, h) = (S_\theta h, h)$$

and, therefore, the following equivalences hold

$$\theta \in \mathrm{Dis}_0(\mathcal{H}_0, \mathcal{H}_1) \iff S_\theta \geq 0, \quad \theta \in \mathrm{Ac}_0(\mathcal{H}_0, \mathcal{H}_1) \iff S_\theta \leq 0, \\ \theta \in \mathrm{Sym}_0(\mathcal{H}_0, \mathcal{H}_1) \iff S_\theta = 0. \tag{3.15}$$

Let further $\lambda \in \mathbb{C}_-$ and let

$$Y_\lambda = \frac{1}{\sqrt{-2\mathrm{Im}\lambda}} \begin{pmatrix} -\lambda P_1 + \sqrt{-2\mathrm{Im}\lambda}\, P_2 & I_{\mathcal{H}_1} \\ -\overline{\lambda} P_1 & I_{\mathcal{H}_1} \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1, \tag{3.16}$$

$$Z_\lambda = \frac{1}{i\sqrt{-2\mathrm{Im}\lambda}} \begin{pmatrix} P_1 + i\sqrt{-2\mathrm{Im}\lambda}\, P_2 & -I_{\mathcal{H}_1} \\ \overline{\lambda} P_1 & -\lambda I_{\mathcal{H}_1} \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1. \tag{3.17}$$

The immediate checking shows that $Y_\lambda Z_\lambda = Z_\lambda Y_\lambda = I$. Hence $0 \in \rho(Y_\lambda)$ and $Z_\lambda = (Y_\lambda)^{-1}$. Furthermore introduce the invertible operator $F_\lambda \in [\mathcal{H}_0]$ by

$$F_\lambda = \begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & -\lambda^{-1}\sqrt{-2\mathrm{Im}\lambda} I_{\mathcal{H}_2} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Consider the (Cayley) transform

$$\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1) \ni \theta \to \tau = \tau(\theta) := Y_\lambda \theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1), \quad \lambda \in \mathbb{C}_- \tag{3.18}$$

Clearly, $\tau = \{N_0, N_1; H\}$ where

$$N_0 = (K_1 - \lambda K_{01} \quad \sqrt{-2\mathrm{Im}\lambda} K_{02})^\top = F_\lambda(K_1 - \lambda K_0), \quad N_1 = K_1 - \overline{\lambda} K_{01}. \tag{3.19}$$

Moreover since $0 \in \rho(Y_\lambda)$, the transform (3.18) bijectively maps the set $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ onto itself.

It follows from (3.19) that

$$N_1^* N_1 - N_0^* N_0 = 2\mathrm{Im}\lambda\, S_\theta.$$

Therefore in view of (3.15) the following assertions hold: 1) $\theta \in \mathrm{Dis}_0(\mathcal{H}_0, \mathcal{H}_1)$ iff τ is a closed contraction from $\mathcal{R}(N_0)$ to $\mathcal{H}_1$; 2) $\theta \in \mathrm{Ac}_0(\mathcal{H}_0, \mathcal{H}_1)$ iff τ^{-1} is a closed contraction from $\mathcal{R}(N_1)$ to $\mathcal{H}_0$; 3) $\theta\in \mathrm{Sym}_0(\mathcal{H}_0,\mathcal{H}_1)$ iff τ is an isometry from $\mathcal{R}(N_0)(=\overline{\mathcal{R}(N_0)})$ onto $\mathcal{R}(N_1)$. This and Definition 3.3 yield

$$\theta\in \mathrm{Dis}(\mathcal{H}_0,\mathcal{H}_1)\iff \tau\in C(\mathcal{H}_0,\mathcal{H}_1)\iff S_\theta\ge 0 \ \text{ and }\ \mathcal{R}(N_0)=\mathcal{H}_0, \tag{3.20}$$

$$\theta\in \mathrm{Ac}(\mathcal{H}_0,\mathcal{H}_1)\iff \tau^{-1}\in C(\mathcal{H}_1,\mathcal{H}_0)\iff S_\theta\le 0 \ \text{ and }\ \mathcal{R}(N_1)=\mathcal{H}_1, \tag{3.21}$$

$$\begin{aligned}\theta\in \mathrm{Sym}(\mathcal{H}_0,\mathcal{H}_1)\iff \tau \ \text{ is an isometry such that }\ \mathcal{D}(\tau)=\mathcal{H}_0\ \text{ or (and) }\ \mathcal{R}(\tau)=\mathcal{H}_1\\ \iff S_\theta=0\ \text{ and }\ \mathcal{R}(N_0)=\mathcal{H}_0\ \text{ or (and) }\ \mathcal{R}(N_1)=\mathcal{H}_1.\end{aligned}$$

Combining (3.20),(3.21) with (3.19) we arrive at the desired statements for the classes Dis, Ac and Sym.

Next we show that

$$\tau(\theta^\times)=(\tau(\theta))^{-1*},\quad \theta\in\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1). \tag{3.22}$$

Letting

$$J'=\begin{pmatrix} I_{\mathcal{H}_0} & 0\\ 0 & -I_{\mathcal{H}_1}\end{pmatrix}:\mathcal{H}_0\oplus\mathcal{H}_1\to\mathcal{H}_0\oplus\mathcal{H}_1$$

one obtains $\theta^{-1*}=J'\theta^\perp$, $\theta\in\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$. Moreover it easily seen that for every invertible operator $X\in[\mathcal{H}_0\oplus\mathcal{H}_1]$ the equality $(X\theta)^\perp=X^{-1*}\theta^\perp$ holds. This and (3.18) yield $(\tau(\theta))^{-1*}=J'(Y_\lambda\theta)^\perp=J'Y_\lambda^{-1*}\theta^\perp$. On the other hand in view of (3.3) one has $\tau(\theta^\times)=Y_\lambda\theta^\times=Y_\lambda J_{01}\theta^\perp$. Thus (3.22) is equivalent to the relation

$$Y_\lambda J_{01}\theta^\perp=J'Y_\lambda^{-1*}\theta^\perp$$

which in turn follows from the directly verified equality $Y_\lambda J_{01}Y_\lambda^*=J'$.

Now in view of (3.22) and (3.20), (3.21) one has

$$\begin{aligned}\theta=\theta^\times\iff \tau(\theta)=(\tau(\theta))^{-1*}\iff \tau(\theta)\in C(\mathcal{H}_0,\mathcal{H}_1)\ \text{ and }\ (\tau(\theta))^{-1}\in C(\mathcal{H}_1,\mathcal{H}_0)\\ \iff \theta\in \mathrm{Dis}(\mathcal{H}_0,\mathcal{H}_1)\cap \mathrm{Ac}(\mathcal{H}_0,\mathcal{H}_1).\end{aligned}$$

This yields the required statement for the class $\mathrm{Self}(\mathcal{H}_0,\mathcal{H}_1)$.

2) It follows from (3.22) that $\tau(\theta)\in C(\mathcal{H}_0,\mathcal{H}_1)\iff (\tau(\theta^\times))^{-1}\in C(\mathcal{H}_1,\mathcal{H}_0)$. Therefore by (3.20), (3.21) the following equivalence holds

$$\theta\in \mathrm{Dis}(\mathcal{H}_0,\mathcal{H}_1)\iff \theta^\times\in \mathrm{Ac}(\mathcal{H}_0,\mathcal{H}_1). \tag{3.23}$$

Let now $\theta=\{(C_0,C_1);H\}$. Then by Proposition 3.1,4) $\theta^\times=\{\widetilde{K}_0,\widetilde{K}_1;H\}$, where $\widetilde{K}_0$ and $\widetilde{K}_1$ are defined by (3.6). Hence,

$$S_{\theta^\times}=-2\mathrm{Im}(C_1C_{01}^*)+C_{02}C_{02}^*=-\widetilde{S}_\theta$$

and by (3.6)

$$\widetilde{K}_1+\lambda\widetilde{K}_0=\begin{pmatrix} C_{01}^*-\lambda C_1^*\\ -i\lambda C_{02}^*\end{pmatrix}=\Phi_\lambda\begin{pmatrix} C_{01}^*-\lambda C_1^*\\ C_{02}^*\end{pmatrix},\quad \widetilde{K}_1+\lambda\widetilde{K}_{01}=C_{01}^*-\lambda C_1^*,$$

where $\Phi_\lambda=\mathrm{diag}(I_{\mathcal{H}_1},-i\lambda I_{\mathcal{H}_2})\in[\mathcal{H}_1\oplus\mathcal{H}_2]$. Therefore

$$(\widetilde{K}_1+\lambda\widetilde{K}_0)^*=(C_{01}-\overline{\lambda}C_1\ \ C_{02})\Phi_\lambda^*=(C_0-\overline{\lambda}C_1P_1)\Phi_\lambda^*,\quad (\widetilde{K}_1+\lambda\widetilde{K}_{01})^*=C_{01}-\overline{\lambda}C_1,$$

which in view of the inclusion $0\in\rho(\Phi_\lambda^*)$ yields the equivalences

$$0\in\rho(\widetilde{K}_1+\lambda\widetilde{K}_0)\iff 0\in\rho(C_0-\overline{\lambda}C_1P_1),\quad \lambda\in\mathbb{C}_+,$$

$$0\in\rho(\widetilde{K}_1+\lambda\widetilde{K}_{01})\iff 0\in\rho(C_{01}-\overline{\lambda}C_1),\quad \lambda\in\mathbb{C}_-.$$

Now the desired statement is implied by (3.23) and the statement 1). □

It is known that a maximal accretive (dissipative, symmetric) linear relation $\theta\in\widetilde{\mathcal{C}}(\mathcal{H})$ admits the orthogonal decomposition in the operator and multivalued parts. In the next corollary we specify a similar result for the classes Ac, Dis, Sym and Self.

Corollary 3.5. *Suppose that a linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ belongs to one of the classes* Dis, Ac, Sym *or* Self *and let $\mathcal{H}_1' := \mathcal{H}_1 \ominus \theta(0)$, $\mathcal{H}_0' := \mathcal{H}_0 \ominus \theta(0)$ so that $\mathcal{H}_1' \subset \mathcal{H}_0'$ and $\mathcal{H}_0' \ominus \mathcal{H}_1' = \mathcal{H}_2 (= \mathcal{H}_0 \ominus \mathcal{H}_1)$. Then*

$$\theta = \theta_s \oplus \hat{\theta}(0), \quad \hat{\theta}(0) = \{0\} \oplus \theta(0), \tag{3.24}$$

where θ_s is an operator from $\mathcal{H}_0'$ to $\mathcal{H}_1'$ with $\mathcal{D}(\theta_s) = \mathcal{D}(\theta)$, which belongs to the same class (in $\mathcal{H}_0' \oplus \mathcal{H}_1'$) as θ.

Proof. We prove the corollary for $\theta \in \mathrm{Ac}(\mathcal{H}_0, \mathcal{H}_1)$, since for other classes the proof is similar. Put in (3.18) $\lambda = -i$ and consider the Cayley transform $\tau = Y_{-i}\theta$. It follows from (3.21) that $\tau = \{N, I_{\mathcal{H}_1}; \mathcal{H}_1\}$ where $N = \tau^{-1} \in [\mathcal{H}_1, \mathcal{H}_0]$ and $||N|| \leq 1$. Moreover $\theta = (Y_{-i})^{-1}\tau = Z_{-i}\tau$ where Z_{-i} is the operator (3.17). Hence θ has the representation $\theta = \{K_0, K_1; \mathcal{H}_1\}$ with operators $K_0 \in [\mathcal{H}_1, \mathcal{H}_0]$ and $K_1 \in [\mathcal{H}_1]$ given by

$$K_0 = \frac{1}{2i}[(P_1 + i\sqrt{2}P_2)N - I_{\mathcal{H}_1}], \quad K_1 = \frac{1}{2}(P_1 N + I_{\mathcal{H}_1}). \tag{3.25}$$

Let $\mathcal{H}_1'' = \{h_1 \in \mathcal{H}_1 : Nh_1 = h_1\}$, $\mathcal{H}_0' = \mathcal{H}_0 \ominus \mathcal{H}_1''$ and $\mathcal{H}_1' = \mathcal{H}_1 \ominus \mathcal{H}_1''$. Since N is a contraction, it follows that

$$N = \begin{pmatrix} N' & 0 \\ 0 & I \end{pmatrix} : \mathcal{H}_1' \oplus \mathcal{H}_1'' \to \mathcal{H}_0' \oplus \mathcal{H}_1''.$$

This and (3.25) imply that

$$K_0 = \begin{pmatrix} K_0' & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{H}_1' \oplus \mathcal{H}_1'' \to \mathcal{H}_0' \oplus \mathcal{H}_1'', \quad K_1 = \begin{pmatrix} K_1' & 0 \\ 0 & I \end{pmatrix} : \mathcal{H}_1' \oplus \mathcal{H}_1'' \to \mathcal{H}_1' \oplus \mathcal{H}_1''$$

where $K_0' \in [\mathcal{H}_1', \mathcal{H}_0']$ and $K_1' \in [\mathcal{H}_1']$ are some operators. Letting now $\theta_s := \{K_0', K_1'; \mathcal{H}_1'\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0', \mathcal{H}_1')$ and taking into account Proposition 3.4, 1) one obtains the desired statement. □

In the next proposition we show that a linear relation $\theta \in \mathrm{Self}(\mathcal{H}_0, \mathcal{H}_1)$ has the normalized representation. Note that for selfadjoint relations this result is well known (see for instance [10, 3]).

Proposition 3.6. *A linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ belongs to the class* $\mathrm{Self}(\mathcal{H}_0, \mathcal{H}_1)$ *if and only if there is a representation $\theta = \{K_0, K_1; \mathcal{H}_1\}$ with operators $K_0 = (K_{01} \;\; K_{02})^\top : \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ and $K_1 \in [\mathcal{H}_1]$ satisfying the relations*

$$K_{01}^* K_1 - K_1^* K_{01} + i\, K_{02}^* K_{02} = 0, \quad K_{01}^* K_{01} + K_1^* K_1 + K_{02}^* K_{02} = I_{\mathcal{H}_1}, \quad 2K_{02} K_{02}^* = I_{\mathcal{H}_2}, \tag{3.26}$$

$$K_{01} K_1^* - K_1 K_{01}^* = 0, \quad K_{01} K_{01}^* + K_1 K_1^* = I_{\mathcal{H}_1}, \quad (K_1 + i\, K_{01}) K_{02}^* = 0. \tag{3.27}$$

Proof. Let $\theta \in \mathrm{Self}(\mathcal{H}_0, \mathcal{H}_1)$. Put in (3.18) $\lambda = -i$ and consider the Cayley transform $\tau = Y_{-i}\theta$. It was shown under the proof of Proposition 3.4 that $\tau = \mathrm{gr}\, V = \{I_{\mathcal{H}_0}, V; \mathcal{H}_0\}$ where V is a unitary operator from $\mathcal{H}_0$ onto $\mathcal{H}_1$. Hence $\theta = Z_{-i}(\mathrm{gr}\, V)$ where Z_{-i} is the operator (3.17). This implies that the linear relation θ has a representation $\theta = \{K_0, K_1; \mathcal{H}_1\}$ with

$$K_0 = \frac{1}{2}(P_1 + i\sqrt{2}P_2 - V)V^* = \frac{1}{2}(P_1 V^* + i\sqrt{2}P_2 V^* - I_{\mathcal{H}_1}), \tag{3.28}$$

$$K_1 = \frac{i}{2}(P_1 + V)V^* = \frac{i}{2}(P_1 V^* + I_{\mathcal{H}_1}). \tag{3.29}$$

Let $V=(V_1\ \ V_2)(\in[\mathcal{H}_1\oplus\mathcal{H}_2,\mathcal{H}_1])$ and $V^*=(V_1^*\ \ V_2^*)^\top(\in[\mathcal{H}_1,\mathcal{H}_1\oplus\mathcal{H}_2])$ be block-matrix representations of the operators V and V^* respectively. Then by (3.28) and (3.29) one has

$$K_{01}(=P_1K_0)=\frac{1}{2}(V_1^*-I_{\mathcal{H}_1}),\quad K_{02}(=P_2K_0)=\frac{i}{\sqrt{2}}V_2^*,\quad K_1=\frac{i}{2}(V_1^*+I_{\mathcal{H}_1}). \tag{3.30}$$

Moreover since $V^*V=I_{\mathcal{H}_0}$ and $VV^*=I_{\mathcal{H}_1}$, it follows that

$$V_1^*V_1=I,\quad V_2^*V_2=I,\quad V_1^*V_2=0,\quad V_1V_1^*+V_2V_2^*=I.$$

Now the immediate calculations give the relations (3.26) and (3.27) for the operators (3.30).

Conversely, assume that the representation $\theta=\{K_0,K_1;\mathcal{H}_1\}$ satisfies (3.26), (3.27) and let S_θ be the operator (3.10). Then by the first equality in (3.26) one has $S_\theta=0$. Moreover, the relations (3.26) and (3.27) yield

$$(K_1^*+i\,K_{01}^*)(K_1-i\,K_{01})=(K_1-i\,K_{01})(K_1^*+i\,K_{01}^*)=I_{\mathcal{H}_1},$$

so that $0\in\rho(K_1-i\,K_{01})$.

Next consider the operators $X:=\operatorname{diag}(I_{\mathcal{H}_1},\sqrt{2}I_{\mathcal{H}_2})\in[\mathcal{H}_1\oplus\mathcal{H}_2]$ and

$$N:=X(K_1+i\,K_0)=\begin{pmatrix}K_1+i\,K_{01}\\ \sqrt{2}K_{02}\end{pmatrix}:\mathcal{H}_1\to\mathcal{H}_1\oplus\mathcal{H}_2.$$

Clearly, $N^*=(K_1^*-i\,K_{01}^*\ \ \sqrt{2}K_{02}^*)$ and in view of (3.26), (3.27) one has

$$N^*N=(K_1^*-i\,K_{01}^*)(K_1+i\,K_{01})+2K_{02}^*K_{02}=I_{\mathcal{H}_1},$$

$$NN^*=\begin{pmatrix}(K_1+i\,K_{01})(K_1^*-i\,K_{01}^*) & \sqrt{2}(K_1+i\,K_{01})K_{02}^*\\ \sqrt{2}K_{02}(K_1^*-i\,K_{01}^*) & 2K_{02}K_{02}^*\end{pmatrix}=\begin{pmatrix}I_{\mathcal{H}_1} & 0\\ 0 & I_{\mathcal{H}_2}\end{pmatrix}=I_{\mathcal{H}_0}.$$

Hence $0\in\rho(N)$ and therefore $0\in\rho(K_1+i\,K_0)$. Now Proposition 3.4 imply the inclusion $\theta\in\operatorname{Self}(\mathcal{H}_0,\mathcal{H}_1)$. □

Remark 3.7. i) If $\theta=\{K_0,K_1;H\}\in\operatorname{Dis}(\mathcal{H}_0,\mathcal{H}_1)$, then by (3.11) $\dim H=\dim\mathcal{H}_0$ and hence $\dim\theta=\dim\mathcal{H}_0$, $\operatorname{codim}\theta=\dim\mathcal{H}_1$. Therefore every linear relation $\theta\in\operatorname{Dis}(\mathcal{H}_0,\mathcal{H}_1)$ admits the representations $\theta=\{K_0,K_1;\mathcal{H}_0\}=\{(C_0,C_1);\mathcal{H}_1\}$. Similarly it follows from (3.12) that every $\theta\in\operatorname{Ac}(\mathcal{H}_0,\mathcal{H}_1)$ can be represented as $\theta=\{K_0,K_1;\mathcal{H}_1\}=\{(C_0,C_1);\mathcal{H}_0\}$. Observe also that for $\theta\in\operatorname{Self}(\mathcal{H}_0,\mathcal{H}_1)$ the equality $\dim\theta=\dim\mathcal{H}_0=\dim\mathcal{H}_1$ is valid. Therefore if $\mathcal{H}_1\neq\mathcal{H}_0$, then the set $\operatorname{Self}(\mathcal{H}_0,\mathcal{H}_1)$ is not empty if and only if $\dim\mathcal{H}_0=\dim\mathcal{H}_1=\infty$.

ii) If $\theta=\{K_0,K_1;\mathcal{H}_1\}\in\operatorname{Ac}(\mathcal{H}_0,\mathcal{H}_1)$, then by Proposition 3.4 $0\in\rho(K_1-i\,K_{01})$. Therefore for a linear relation $\theta\in\operatorname{Ac}(\mathcal{H}_0,\mathcal{H}_1)$ there is a unique representation $\theta=\{K_0',K_1';\mathcal{H}_1\}$ such that $K_1'-i\,K_{01}'=I_{\mathcal{H}_1}$.

4. The class $\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ and the dilation theorem

4.1. **The class $\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ and its properties.** Let $\mathcal{H}_1$ be a subspace in a Hilbert space $\mathcal{H}_0$, $\mathcal{H}_2=\mathcal{H}_0\ominus\mathcal{H}_1$ and let $\tau_+(\cdot):\mathbb{C}_+\to\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$, $\tau_-(\cdot):\mathbb{C}_-\to\widetilde{\mathcal{C}}(\mathcal{H}_1,\mathcal{H}_0)$ be holomorphic functions. Then by Definition 2.3

$$\tau_+(\lambda)=\{K_0(\lambda),K_1(\lambda);H_+\}=\{(K_{01}(\lambda)\ \ K_{02}(\lambda))^\top,K_1(\lambda);H_+\},\quad \lambda\in\mathbb{C}_+, \tag{4.1}$$

where H_+ is an auxiliary Hilbert space, $K_j(\cdot):\mathbb{C}_+\to[H_+,\mathcal{H}_j]$, $j\in\{0,1\}$ are holomorphic operator functions and

(4.2)

$$K_0(\lambda)=(K_{01}(\lambda)\ \ K_{02}(\lambda))^\top\ (\in[H_+,\mathcal{H}_1\oplus\mathcal{H}_2]),\quad K_{0j}(\cdot):\mathbb{C}_+\to[H_+,H_j],\quad j\in\{1,2\}$$

is the block-matrix representation of the operator function $K_0(\cdot)$.

Similarly the function $\tau_-(\cdot)$ admits the representation

$$\tau_-(z) = \{N_1(z), N_0(z); H_-\} = \{N_1(z), (N_{01}(z)\ \ N_{02}(z))^\top; H_-\}, \quad z \in \mathbb{C}_-, \tag{4.3}$$

where H_- is a Hilbert space, $N_j(\cdot) : \mathbb{C}_- \to [H_-, \mathcal{H}_j],\ j \in \{0, 1\}$ are holomorphic operator functions and

$$N_0(z) = (N_{01}(z)\ \ N_{02}(z))^\top\ (\in [H_-, \mathcal{H}_1 \oplus \mathcal{H}_2]), \quad N_{0j}(\cdot) : \mathbb{C}_- \to [H_-, H_j], \quad j \in \{1, 2\} \tag{4.4}$$

is the block-matrix representation of the operator function $N_0(\cdot)$.

Definition 4.1. A holomorphic $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$-valued function $\tau_+(\cdot) : \mathbb{C}_+ \to \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ belongs to the class $\widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$, if $-\tau_+(\lambda) \in \mathrm{Ac}(\mathcal{H}_0, \mathcal{H}_1)$ for every $\lambda \in \mathbb{C}_+$.

Definition 4.2. A pair of holomorphic functions $\tau_+(\cdot) : \mathbb{C}_+ \to \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ and $\tau_-(\cdot) : \mathbb{C}_- \to \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0)$ belongs to the class $\widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ if $\tau_+(\cdot) \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ and $\tau_-(\overline{\lambda}) = \tau_+^*(\lambda)$ for every $\lambda \in \mathbb{C}_+$. In what follows such a pair of functions $\tau_+(\cdot)$ and $\tau_-(\cdot)$ will be denoted by $\tau = \{\tau_+, \tau_-\}$.

A pair of functions $\tau = \{\tau_+(\cdot), \tau_-(\cdot)\} \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ is referred to the class $\widetilde{R}^0(\mathcal{H}_0, \mathcal{H}_1)$ if $\tau_+(\lambda) = \tau_+,\ \lambda \in \mathbb{C}_+;\ \ \tau_-(z) = \tau_-,\ z \in \mathbb{C}_-$ (1,e., the functions $\tau_+(\cdot)$ and $\tau_-(\cdot)$ are constant on their domains) and $-\tau_+ \in \mathrm{Self}(\mathcal{H}_0, \mathcal{H}_1)$.

In the case $\mathcal{H}_1 = \mathcal{H}_0 := \mathcal{H}$ we put $\widetilde{R}_+(\mathcal{H}) := \widetilde{R}_+(\mathcal{H}, \mathcal{H})$ and $\widetilde{R}(\mathcal{H}) := \widetilde{R}(\mathcal{H}, \mathcal{H})$.

Let $\tau = \{\tau_+, \tau_-\} \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ and let (4.1), (4.3) be representations of τ_+ and τ_-. Then in view of Remark 3.7 $\dim H_+ = \dim \mathcal{H}_1,\ \dim H_- = \dim \mathcal{H}_0$ and, therefore, there exist representations (4.1), (4.3) with $H_+ = \mathcal{H}_1$ and $H_- = \mathcal{H}_0$. At the same time the spaces H_+ and H_- can be chosen equal if and only if $\dim \mathcal{H}_1 = \dim \mathcal{H}_0$. In particular such a choice is possible if $\tau \in \widetilde{R}^0(\mathcal{H}_0, \mathcal{H}_1)$.

In the following proposition we describe classes $\widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ and $\widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ in terms of the corresponding pairs $\{K_0(\cdot), K_1(\cdot)\}$ and $\{N_1(\cdot), N_0(\cdot)\}$.

Proposition 4.3. *1) The equality* (4.1) *establishes a bijective correspondence between all functions $\tau_+(\cdot) \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ and all pairs $\{K_0(\cdot), K_1(\cdot)\}$ of holomorphic operator functions satisfying the relations*

$$2\mathrm{Im}(K_{01}^*(\lambda)K_1(\lambda)) - K_{02}^*(\lambda)K_{02}(\lambda) \geq 0, \quad 0 \in \rho(K_1(\lambda) + i\,K_{01}(\lambda)), \quad \lambda \in \mathbb{C}_+. \tag{4.5}$$

2) The equalities (4.1) *and* (4.3) *establish a bijective correspondence between all pairs $\tau = \{\tau_+, \tau_-\} \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ and all pairs $\{K_0(\cdot), K_1(\cdot)\}$, $\{N_1(\cdot), N_0(\cdot)\}$ of holomorphic operator functions satisfying* (4.5) *and the following relations*

$$2\mathrm{Im}(N_1^*(z)N_{01}(z)) - N_{02}^*(z)N_{02}(z) \leq 0, \quad 0 \in \rho(N_0(z)) - i\,N_1(z)), \quad z \in \mathbb{C}_-, \tag{4.6}$$

$$N_0^*(\overline{\lambda})K_0(\lambda) - N_1^*(\overline{\lambda})K_1(\lambda) = 0, \quad \lambda \in \mathbb{C}_+. \tag{4.7}$$

Moreover a pair $\tau = \{\tau_+, \tau_-\}$ belongs to the class $\widetilde{R}^0(\mathcal{H}_0, \mathcal{H}_1)$ if and only if it admits the representation (4.1), (4.3) *such that: i) $H_+ = H_-$; ii) $K_j(\lambda) = K_j,\ \lambda \in \mathbb{C}_+;\ N_j(z) = N_j,\ z \in \mathbb{C}_-,\ j \in \{0, 1\}$ (i.e., the functions $K_j(\cdot)$ and $N_j(\cdot)$ are constant on their domains); iii) operators K_j and N_j satisfy* (4.5)–(4.7) *and the following condition*

$$N_1 = K_{01}, \quad N_{01} = K_1, \quad N_{02} = -i\,K_{02}. \tag{4.8}$$

Proof. 1) Let a function $\tau_+(\cdot)$ belongs to the class $\widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$. Then according to Definition 2.3 it admits the representation (4.1), where $\{K_0(\cdot), K_1(\cdot)\}$ is an admissible pair. Consequently, $\mathrm{Ker}\, K_0(\lambda) \cap \mathrm{Ker}(-K_1(\lambda)) = \{0\}$ and therefore $-\tau_+(\lambda) = \{K_0(\lambda), -K_1(\lambda); H_+\}, \lambda \in \mathbb{C}_+$. Now (4.5) is implied by the inclusion $-\tau_+(\lambda) \in \mathrm{Ac}(\mathcal{H}_0, \mathcal{H}_1)$ and Proposition 3.4, 1).

Conversely, let a pair $\{K_0(\cdot), K_1(\cdot)\}$ of holomorphic operator-functions satisfies (4.5). Since $\mathrm{Ker}(K_1(\lambda)+i\,K_{01}(\lambda))=\{0\}$, this pair is admissible. Moreover in view of (4.5) and Proposition 3.4, 1) a linear relation $\tau'(\lambda):=\{K_0(\lambda),-K_1(\lambda);H_+\}$ belongs to the class $\mathrm{Ac}(\mathcal{H}_0,\mathcal{H}_1)$ (for every $\lambda\in\mathbb{C}_+$) and ,therefore, is closed. Consequently, the equality (4.1) defines a holomorphic $\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$-valued function $\tau_+(\lambda)=-\tau'(\lambda)$ and $-\tau_+(\lambda)=\tau'(\lambda)\in \mathrm{Ac}(\mathcal{H}_0,\mathcal{H}_1),\ \lambda\in\mathbb{C}_+$. Hence $\tau_+(\cdot)\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$.

2) Assume that $\tau=\{\tau_+,\tau_-\}\in\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ and (4.1), (4.3) is the representation of $\tau_+(\cdot)$ and $\tau_-(\cdot)$ respectively. The statement 1) imply that the operator functions $K_0(\cdot)$ and $K_1(\cdot)$ satisfy (4.5). Let U_{10} be the operator (3.2) and let $\widetilde{\tau}(z)=U_{10}(-\tau_-(z))$, so that $\widetilde{\tau}(z)\in\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ and

$$\widetilde{\tau}(z)=\{(N_1(z)\ \ i\,N_{02}(z))^\top,-N_{01}(z);H_-\},\quad z\in\mathbb{C}_-. \tag{4.9}$$

It follows from Proposition 3.1 ,1) that $\widetilde{\tau}(z)=U_{10}(-\tau_+(\overline{z}))^*=(-\tau_+(\overline{z}))^\times,\ z\in\mathbb{C}_-$. Therefore by (3.23) $\widetilde{\tau}(z)\in\mathrm{Dis}(\mathcal{H}_0,\mathcal{H}_1)$. Applying now Proposition 3.4, 1) to (4.9) we arrive at the relations (4.6). Furthermore, (4.7) immediately follows from the representations (4.1), (4.3) and the equality $\tau_-(\overline{\lambda})=\tau_+^*(\lambda)$.

Conversely, let pairs $\{K_0(\cdot),K_1(\cdot)\}$ and $\{N_1(\cdot),N_0(\cdot)\}$ satisfy (4.5) – (4.7). It follows from the statement 1) that (4.1) defines a function $\tau_+(\cdot)\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$. Let now $B=-i\,P_1\in[\mathcal{H}_0,\mathcal{H}_1]$, so that $B^*=i\,I_{\mathcal{H}_1}\in[\mathcal{H}_1,\mathcal{H}_0]$. Then by (4.5) and (4.6)

$$0\in\rho(K_1(\lambda)-BK_0(\lambda))\cap\rho(N_0(z)-B^*N_1(z)),\quad \lambda\in\mathbb{C}_+,\quad z\in\mathbb{C}_-.$$

This and Lemma 2.1, 1) show that for every $z\in\mathbb{C}_-$ a linear relation $\tau_-(z)=\{N_1(z), N_0(z),H_-\}$ is closed. Therefore the equality (4.3) defines a holomorphic $\widetilde{\mathcal{C}}(\mathcal{H}_1,\mathcal{H}_0)$-valued function $\tau_-(\cdot)$. Finally, applying Lemma 2.1, 2) to the linear relations $\tau_+(\lambda)=\{K_0(\lambda),K_1(\lambda;H_+)\}$ and $\tau_-(\overline{\lambda})=\{N_1(\overline{\lambda}),N_0(\overline{\lambda});H_-)\}$ and taking into account (4.7) we arrive at the equality $\tau_-(\overline{\lambda})=\tau_+^*(\lambda),\ \lambda\in\mathbb{C}_+$. Thus a pair $\tau:=\{\tau_+(\cdot),\tau_-(\cdot)\}$ belongs to the class $\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$.

It remains to describe the class $\widetilde{R}^0(\mathcal{H}_0,\mathcal{H}_1)$. Let $\tau_+\in\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ and $\tau_-\in\widetilde{\mathcal{C}}(\mathcal{H}_1,\mathcal{H}_0)$ be a pair of linear relations such that $\tau_-=\tau_+^*$. Then by Proposition 3.1,1) $-\tau_-=(-\tau_+)^*=U_{01}(-\tau_+)^\times$ where U_{01} is the operator (3.2). This implies the equivalence

$$-\tau_+\in\mathrm{Self}(\mathcal{H}_0,\mathcal{H}_1)\iff -\tau_-=U_{01}(-\tau_+). \tag{4.10}$$

Moreover if $\tau_+=\{(K_{01}\ \ K_{02})^\top,K_1;H_+\}$ then by (3.2)

$$U_{01}(-\tau_+)=\{K_{01},(-K_1\ \ i\,K_{02})^\top;H_+\}. \tag{4.11}$$

Now the desired statement for the class $\widetilde{R}^0(\mathcal{H}_0,\mathcal{H}_1)$ is implied by (4.10), (4.11) and the description of the class $\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$. □

Remark 4.4. i) It follows from Proposition 4.3 that for every pair $\tau=\{\tau_+,\tau_-\}\in\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ there exists a unique representation (4.1), (4.3) such that $H_+=\mathcal{H}_1, H_-=\mathcal{H}_0$ and $K_1(\lambda)+i\,K_{01}(\lambda)=I_{\mathcal{H}_1},\ \lambda\in\mathbb{C}_+;\ \ N_0(z)-i\,N_1(z)=I_{\mathcal{H}_0},\ z\in\mathbb{C}_-$.

ii) In the case $\mathcal{H}_1=\mathcal{H}_0:=\mathcal{H}$ the class $\widetilde{R}(\mathcal{H})(=\widetilde{R}(\mathcal{H},\mathcal{H}))$ coincides with the well known class of Nevanlinna functions with values in $\widetilde{\mathcal{C}}(\mathcal{H})$ [8, 3, 4, 5]. More precisely, the equality

$$\tau(\lambda)=\begin{cases}\tau_+(\lambda), & \lambda\in\mathbb{C}_+\\ \tau_-(\lambda), & \lambda\in\mathbb{C}_-\end{cases}$$

gives a bijective correspondence between all pairs $\tau=\{\tau_+(\cdot),\tau_-(\cdot)\}\in\widetilde{R}(\mathcal{H})$ and all Nevanlinna functions $\tau(\cdot):\mathbb{C}_+\cup\mathbb{C}_-\to\widetilde{\mathcal{C}}(\mathcal{H})$. Observe also that for the class $\widetilde{R}(\mathcal{H})$ Proposition 4.3 was obtained in [3].

In view of Proposition 4.3 we will identify a function $\tau_+(\cdot) \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ and the corresponding class of equivalent (in the sense of Definition 2.2) pairs of operator functions $\{K_0(\cdot), K_1(\cdot)\}$ satisfying (4.5). Similarly a pair of functions $\tau = \{\tau_+(\cdot), \tau_-(\cdot)\} \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ will be identified with two classes of equivalent pairs $\{K_0(\cdot), K_1(\cdot)\}$ and $\{N_1(\cdot), N_0(\cdot)\}$ which satisfy (4.5)–(4.7).

4.2. **The dilation theorem.** Assume that $\mathcal{H}_1$ is a subspace in a Hilbert space $\mathcal{H}_0$ and $\mathcal{H}_2 = \mathcal{H}_0 \ominus \mathcal{H}_1$. Denote by $\mathfrak{H}'$ a Hilbert space with $\dim \mathfrak{H}' = \infty$, if $\dim \mathcal{H}_2 < \infty$, and $\dim \mathfrak{H}' = \dim \mathcal{H}_2$, if $\dim \mathcal{H}_2 = \infty$. Since $\dim(\mathfrak{H}' \oplus \mathcal{H}_2) = \dim \mathfrak{H}'$, there exists a unitary operator $V = (V_1 \;\; V_2) : \mathfrak{H}' \oplus \mathcal{H}_2 \to \mathfrak{H}'$ from $\mathfrak{H}' \oplus \mathcal{H}_2$ onto $\mathfrak{H}'$. Moreover without loss of generality one may consider that $\operatorname{Ker}(V_1 - I_{\mathfrak{H}'}) = \{0\}$.

Let θ be a linear relation in $\mathcal{H}_0 \oplus \mathcal{H}_1$ with the representation $\theta = \{K_0, K_1; H\}$, $K_0 = (K_{01} \;\; K_{02})^\top \in [H, \mathcal{H}_1 \oplus \mathcal{H}_2]$, $K_1 \in [H, \mathcal{H}_1]$. Put $\mathcal{H}' := \mathfrak{H}' \oplus \mathcal{H}_2 \oplus \mathcal{H}_1$ and consider the operators

$$K_0' = \begin{pmatrix} -\frac{i}{2}(V_1 - I) & -\frac{i}{2}V_2 & 0 \\ 0 & \frac{i}{2}I_{\mathcal{H}_2} & -\frac{i}{\sqrt{2}}K_{02} \\ 0 & 0 & K_{01} \end{pmatrix} : \mathfrak{H}' \oplus \mathcal{H}_2 \oplus H \to \underbrace{\mathfrak{H}' \oplus \mathcal{H}_2 \oplus \mathcal{H}_1}_{\mathcal{H}'}, \tag{4.12}$$

$$K_1' = \begin{pmatrix} \frac{1}{2}(V_1 + I) & \frac{1}{2}V_2 & 0 \\ 0 & \frac{1}{2}I_{\mathcal{H}_2} & \frac{1}{\sqrt{2}}K_{02} \\ 0 & 0 & K_1 \end{pmatrix} : \mathfrak{H}' \oplus \mathcal{H}_2 \oplus H \to \underbrace{\mathfrak{H}' \oplus \mathcal{H}_2 \oplus \mathcal{H}_1}_{\mathcal{H}'}. \tag{4.13}$$

Denote by θ' a linear relation in $\mathcal{H}'$ given by $\theta' := \{K_0', K_1'; \mathcal{H}'\}$.

Lemma 4.5. *1) A linear relation θ belongs to one of the classes $\operatorname{Dis}(\mathcal{H}_0, \mathcal{H}_1)$, $\operatorname{Ac}(\mathcal{H}_0, \mathcal{H}_1)$, $\operatorname{Sym}(\mathcal{H}_0, \mathcal{H}_1)$ or $\operatorname{Self}(\mathcal{H}_0, \mathcal{H}_1)$ if and only if θ' is a maximal dissipative, maximal accumulative, maximal symmetric or selfadjoint linear relation in $\mathcal{H}'$ respectively.*

2) If $\theta \in \operatorname{Self}(\mathcal{H}_0, \mathcal{H}_1)$, then $\theta'(0) = \theta(0) (\subset \mathcal{H}_1)$.

Proof. 1) Since $V^*V = I_{\mathfrak{H}' \oplus \mathcal{H}_2}$ and $VV^* = I_{\mathfrak{H}'}$, it follows that

$$V_1^*V_1 = I, \quad V_2^*V_2 = I, \quad V_1^*V_2 = 0, \quad V_1V_1^* + V_2V_2^* = I.$$

Hence by the immediate calculation one obtains

$$2\operatorname{Im}(K_0^{'*}K_1') = \operatorname{diag}(0, 0, 2\operatorname{Im}(K_{01}^*K_1) + K_{02}^*K_{02}).$$

Moreover, $K_1' - i\,K_0' = \operatorname{diag}(I_{\mathfrak{H}'}, I_{\mathcal{H}_2}, K_1 - i\,K_{01})$ and

$$K_1' + i\,K_0' = \begin{pmatrix} V_1 & V_2 & 0 \\ 0 & 0 & \sqrt{2}K_{02} \\ 0 & 0 & K_1 + i\,K_{01} \end{pmatrix} : \mathfrak{H}' \oplus \mathcal{H}_2 \oplus H \to \mathfrak{H}' \oplus \mathcal{H}_2 \oplus \mathcal{H}_1.$$

Therefore $K_1' + i\,K_0' = V \oplus X(K_1 + i\,K_0)$, where $X = \operatorname{diag}(I_{\mathcal{H}_1}, \sqrt{2}I_{\mathcal{H}_2}) \in [\mathcal{H}_1 \oplus \mathcal{H}_2]$. This and Proposition 3.4, 1) yield the desired statement.

2) If $\theta \in \operatorname{Self}(\mathcal{H}_0, \mathcal{H}_1)$, then by Proposition 3.4, 1)

$$-2\operatorname{Im}(K_1^*K_{01}) + K_{02}^*K_{02} = 0.$$

Hence $\operatorname{Ker} K_{01} \subset \operatorname{Ker} K_{02}$ and, therefore, $\operatorname{Ker} K_{01} = \operatorname{Ker} K_0$. This and (4.12) show that $\operatorname{Ker} K_0' = \{\{0, 0, h\} \in \mathcal{H}' : h \in \operatorname{Ker} K_0\}$. Consequently,

$$\theta'(0) = K_1' \operatorname{Ker} K_0' = \{\{0, 0, K_1 h\} : h \in \operatorname{Ker} K_0\} = \theta(0).$$

□

Now we are ready to prove the main result of the paper – the dilation theorem.

Theorem 4.6. *Suppose that* $\mathcal{H}_0$ *and* $\mathfrak{H}_1$ *are Hilbert spaces,* $\mathcal{H}_1$ *is a subspace in* $\mathcal{H}_0$, $\mathcal{H}_2 = \mathcal{H}_0 \ominus \mathcal{H}_1$ *and* $\widetilde{\mathcal{H}}_0 := \mathcal{H}_0 \oplus \mathfrak{H}_1$, $\widetilde{\mathcal{H}}_1 := \mathcal{H}_1 \oplus \mathfrak{H}_1$ *, so that* $\widetilde{\mathcal{H}}_1 \subset \widetilde{\mathcal{H}}_0$ *and* $\widetilde{\mathcal{H}}_0 \ominus \widetilde{\mathcal{H}}_1 = \mathcal{H}_2$*. Then for every linear relation* $\widetilde{\theta} \in \mathrm{Self}(\widetilde{\mathcal{H}}_0, \widetilde{\mathcal{H}}_1)$ *there exists a representation* $\widetilde{\theta} = \{\widetilde{K}_0, \widetilde{K}_1; \widetilde{\mathcal{H}}_1\}$ *with the following properties:*

i) the operators $\widetilde{K}_0 \in [\widetilde{\mathcal{H}}_1, \widetilde{\mathcal{H}}_0]$ *and* $\widetilde{K}_1 \in [\widetilde{\mathcal{H}}_1]$ *have the block-matrix representations*

$$\widetilde{K}_0 = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} : \mathcal{H}_1 \oplus \mathfrak{H}_1 \to \mathcal{H}_0 \oplus \mathfrak{H}_1, \quad \widetilde{K}_1 = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} : \mathcal{H}_1 \oplus \mathfrak{H}_1 \to \mathcal{H}_1 \oplus \mathfrak{H}_1 \tag{4.14}$$

such that $0 \in \rho(N_4 - \lambda K_4), \ \lambda \in \mathbb{C}_+$*;*

ii) the equalities

$$K_0(\lambda) = -K_1 + K_2(N_4 - \lambda K_4)^{-1}(N_3 - \lambda K_3), \quad \lambda \in \mathbb{C}_+, \tag{4.15}$$

$$K_1(\lambda) = N_1 - N_2(N_4 - \lambda K_4)^{-1}(N_3 - \lambda K_3), \quad \lambda \in \mathbb{C}_+ \tag{4.16}$$

define holomorphic operator functions $K_0(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_1, \mathcal{H}_0]$ *and* $K_1(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_1]$ *such that the function* $\tau_+(\lambda) := \{K_0(\lambda), K_1(\lambda); \mathcal{H}_1\}, \ \lambda \in \mathbb{C}_+$ *belongs to the class* $\widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$.

Conversely, assume that $\mathcal{H}_1$ *is a subspace in a Hilbert space* $\mathcal{H}_0$*. Then every function* $\tau_+(\cdot) \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ *admits the representation* $\tau_+(\lambda) = \{K_0(\lambda), K_1(\lambda); \mathcal{H}_1\}, \ \lambda \in \mathbb{C}_+$ *with the following properties: there exist a Hilbert space* $\mathfrak{H}_1$ *and operators* $\widetilde{K}_0 \in [\mathcal{H}_1 \oplus \mathfrak{H}_1, \mathcal{H}_0 \oplus \mathfrak{H}_1]$, $\widetilde{K}_1 \in [\mathcal{H}_1 \oplus \mathfrak{H}_1]$ *with the block-matrix representations* (4.14) *such that a linear relation* $\widetilde{\theta} := \{\widetilde{K}_0, \widetilde{K}_1; \mathcal{H}_1 \oplus \mathfrak{H}_1\}$ *belongs to the class* $\mathrm{Self}(\mathcal{H}_0 \oplus \mathfrak{H}_1, \mathcal{H}_1 \oplus \mathfrak{H}_1)$ *and the operator functions* $K_0(\cdot)$, $K_1(\cdot)$ *satisfy the equalities* (4.15), (4.16).

Proof. 1) First we prove the theorem for the case $\mathcal{H}_0 = \mathcal{H}_1 := \mathcal{H}$. It is well known [11, 6] that for every selfadjoint linear relation $\widetilde{\theta} \in \widetilde{\mathcal{C}}(\mathcal{H} \oplus \mathfrak{H}_1)$ the equality

$$-(\tau_+(\lambda) + \lambda)^{-1} = P_{\mathcal{H}}(\widetilde{\theta} - \lambda)^{-1} \upharpoonright \mathcal{H}, \quad \lambda \in \mathbb{C}_+ \tag{4.17}$$

uniquely defines a function $\tau_+(\cdot) \in \widetilde{R}_+(\mathcal{H})$. Conversely, for every function $\tau_+(\cdot) \in \widetilde{R}_+(\mathcal{H})$ there are a Hilbert space $\mathfrak{H}_1$ and a selfadjoint linear relation $\widetilde{\theta} \in \widetilde{\mathcal{C}}(\mathcal{H} \oplus \mathfrak{H}_1)$ such that (4.17) holds.

Let now $\widetilde{\theta} = \widetilde{\theta}^* \in \widetilde{\mathcal{C}}(\mathcal{H} \oplus \mathfrak{H}_1)$. Then by Remark 3.7, ii) there is a representation $\widetilde{\theta} = \{\widetilde{K}_0, \widetilde{K}_1; \mathcal{H} \oplus \mathfrak{H}_1\}$ such that $\widetilde{K}_1 - i\,\widetilde{K}_0 = I$. Therefore the block-matrix representations (4.14) satisfy the relations

$$N_1 - i\,K_1 = I, \quad N_2 = i\,K_2, \quad N_3 = i\,K_3, \quad N_4 - i\,K_4 = I. \tag{4.18}$$

In view of (4.14) one has

$$\mathrm{Im}(\widetilde{K}_0^* \widetilde{K}_1) = \begin{pmatrix} * & * \\ * & \mathrm{Im}(K_2^* N_2) + \mathrm{Im}(K_4^* N_4) \end{pmatrix}. \tag{4.19}$$

Since $\mathrm{Im}(\widetilde{K}_0^* \widetilde{K}_1) = 0$, it follows from (4.19) and the second equality in (4.18) that

$$\mathrm{Im}(K_4^* N_4) = -\mathrm{Im}(K_2^* N_2) = -K_2^* K_2 \le 0.$$

Moreover by (4.18) $0 \in \rho(N_4 - i\,K_4)$. Consequently, $\theta_4 := \{K_4, N_4; \mathfrak{H}_1\}$ is a maximal accumulative linear relation in $\mathfrak{H}_1$, so that $0 \in \rho(N_4 - \lambda K_4), \ \lambda \in \mathbb{C}_+$.

Let $K_0(\lambda)$ and $K_1(\lambda)$ be operator functions (4.15), (4.16) and let $\tau_+(\lambda) = \{K_0(\lambda), K_1(\lambda); \mathcal{H}\}$. We show that

$$P_{\mathcal{H}} \widetilde{K}_0(\widetilde{K}_1 - \lambda \widetilde{K}_0)^{-1} \upharpoonright \mathcal{H} = -K_0(\lambda)(K_1(\lambda) + \lambda K_0(\lambda))^{-1}, \quad \lambda \in \mathbb{C}_+. \tag{4.20}$$

Using the Frobenius formula one derives

$$(\widetilde{K}_1-\lambda\widetilde{K}_0)^{-1}=\begin{pmatrix}N_1-\lambda K_1 & N_2-\lambda K_2\\ N_3-\lambda K_3 & N_4-\lambda K_4\end{pmatrix}^{-1}=\begin{pmatrix}S_1(\lambda) & *\\ S_2(\lambda) & *\end{pmatrix},$$

where

$$S_1(\lambda)=[N_1-\lambda K_1-(N_2-\lambda K_2)(N_4-\lambda K_4)^{-1}(N_3-\lambda K_3)]^{-1}=(K_1(\lambda)+\lambda K_0(\lambda))^{-1},$$
$$S_2(\lambda)=-(N_4-\lambda K_4)^{-1}(N_3-\lambda K_3)(K_1(\lambda)+\lambda K_0(\lambda))^{-1}.$$

Hence

$$\begin{aligned}P_{\mathcal{H}}\widetilde{K}_0(\widetilde{K}_1-\lambda\widetilde{K}_0)^{-1}\upharpoonright\mathcal{H}&=K_1S_1(\lambda)+K_2S_2(\lambda)\\&=\big(K_1-K_2(N_4-\lambda K_4)^{-1}(N_3-\lambda K_3)\big)(K_1(\lambda)+\lambda K_0(\lambda))^{-1}\\&=-K_0(\lambda)(K_1(\lambda)+\lambda K_0(\lambda))^{-1},\end{aligned}$$

which proves (4.20). Since $(\widetilde{\theta}-\lambda)^{-1}=\widetilde{K}_0(\widetilde{K}_1-\lambda\widetilde{K}_0)^{-1}$ and $-(\tau_+(\lambda)+\lambda)^{-1}=-K_0(\lambda)(K_1(\lambda)+\lambda K_0(\lambda))^{-1}$, it follows from (4.20) that the function $\tau_+(\lambda)$ satisfies (4.17). Therefore $\tau_+(\cdot)\in\widetilde{R}_+(\mathcal{H})$.

Conversely assume that $\tau_+(\cdot)\in\widetilde{R}_+(\mathcal{H})$. Then by Remark 4.4, i) there is a unique representation $\tau_+(\lambda)=\{K_0(\lambda),K_1(\lambda);\mathcal{H})\}$ such that

$$K_1(\lambda)+i\,K_0(\lambda)=I_{\mathcal{H}},\quad \lambda\in\mathbb{C}_+. \tag{4.21}$$

Let $\mathfrak{H}_1$ be a Hilbert space and let $\widetilde{\theta}\in\widetilde{\mathcal{C}}(\mathcal{H}\oplus\mathfrak{H}_1)$ be a selfadjoint linear relation such that (4.17) holds. Assume that $\widetilde{\theta}=\{\widetilde{K}_0,\widetilde{K}_1;\mathcal{H}\oplus\mathfrak{H}_1\}$ where $\widetilde{K}_1-i\,\widetilde{K}_0=I$ and the operators $\widetilde{K}_0$, $\widetilde{K}_1$ has the block-matrix representations (4.14). Denote by $\hat{K}_0(\lambda)$ and $\hat{K}_1(\lambda)$ the operator functions given by (4.15) and (4.16) respectively. It follows from (4.18) that

$$\hat{K}_1(\lambda)+i\,\hat{K}_0(\lambda)=I_{\mathcal{H}},\quad \lambda\in\mathbb{C}_+. \tag{4.22}$$

Moreover it was shown in the first part of the proof that the function $\hat{\tau}_+(\lambda):=\{\hat{K}_0(\lambda),\hat{K}_1(\lambda);\mathcal{H}\}$ satisfies (4.17). Hence $\hat{\tau}_+(\lambda)=\tau_+(\lambda)$ and by (4.21), (4.22) $K_j(\lambda)=\hat{K}_j(\lambda)$, $\lambda\in\mathbb{C}_+$, $j\in\{0,1\}$. Thus the functions $K_0(\lambda)$ and $K_1(\lambda)$ satisfy (4.15), (4.16).

2) Now assume that $\mathcal{H}_1\subset\mathcal{H}_0$, $\widetilde{\mathcal{H}}_j=\mathcal{H}_j\oplus\mathfrak{H}_1$, $j\in\{0,1\}$, $\widetilde{\theta}\in\mathrm{Self}(\widetilde{\mathcal{H}}_0,\widetilde{\mathcal{H}}_1)$ and prove the first statement of the theorem. It follows from Remark 3.7, ii) that there is a representation

$$\widetilde{\theta}=\{\widetilde{K}_0,\widetilde{K}_1;\widetilde{\mathcal{H}}_1\},\quad \widetilde{K}_0=(\widetilde{K}_{01}\ \ \widetilde{K}_{02})^{\top}:\widetilde{\mathcal{H}}_1\to\widetilde{\mathcal{H}}_1\oplus\mathcal{H}_2,\quad \widetilde{K}_1\in[\widetilde{\mathcal{H}}_1] \tag{4.23}$$

such that

$$\widetilde{K}_1-i\,\widetilde{K}_{01}=I_{\widetilde{\mathcal{H}}_1}. \tag{4.24}$$

Let (4.14) be the block-matrix representation of the operators $\widetilde{K}_0$ and $\widetilde{K}_1$. Using the decomposition $\mathcal{H}_0=\mathcal{H}_2\oplus\mathcal{H}_1$ one can rewrite the first equality in (4.14) as

$$\widetilde{K}_0=\begin{pmatrix}K_{12} & K_{22}\\ K_{11} & K_{21}\\ K_3 & K_4\end{pmatrix}:\underbrace{\mathcal{H}_1\oplus\mathfrak{H}_1}_{\widetilde{\mathcal{H}}_1}\to\mathcal{H}_2\oplus\underbrace{\mathcal{H}_1\oplus\mathfrak{H}_1}_{\widetilde{\mathcal{H}}_1}. \tag{4.25}$$

Hence the operators $\widetilde{K}_{01}\in[\widetilde{\mathcal{H}}_1]$ and $\widetilde{K}_{02}\in[\widetilde{\mathcal{H}}_1,\mathcal{H}_2]$ take the form

$$\widetilde{K}_{01}=\begin{pmatrix}K_{11} & K_{21}\\ K_3 & K_4\end{pmatrix}:\mathcal{H}_1\oplus\mathfrak{H}_1\to\mathcal{H}_1\oplus\mathfrak{H}_1,\quad \widetilde{K}_{02}=\begin{pmatrix}K_{12} & K_{22}\end{pmatrix}:\mathcal{H}_1\oplus\mathfrak{H}_1\to\mathcal{H}_2. \tag{4.26}$$

Moreover (4.25) and the first equality in (4.14) imply

$$K_1=(K_{12}\ \ K_{11})^{\top}:\mathcal{H}_1\to\mathcal{H}_2\oplus\mathcal{H}_1,\quad K_2=(K_{22}\ \ K_{21})^{\top}:\mathfrak{H}_1\to\mathcal{H}_2\oplus\mathcal{H}_1. \tag{4.27}$$

Next consider the operators (4.12) and (4.13) corresponding to the linear relation (4.23). It follows from (4.26) and (4.14) that

$$\widetilde{K}_0' = \left(\begin{array}{cc|cc} -\frac{i}{2}(V_1-I) & -\frac{i}{2}V_2 & 0 & 0 \\ 0 & \frac{i}{2}I_{\mathcal{H}_2} & -\frac{i}{\sqrt{2}}K_{12} & -\frac{i}{\sqrt{2}}K_{22} \\ \hline 0 & 0 & K_{11} & K_{21} \\ 0 & 0 & K_3 & K_4 \end{array}\right) \in [\mathfrak{H}' \oplus \mathcal{H}_2 \oplus \underbrace{\mathcal{H}_1 \oplus \mathfrak{H}_1}_{\widetilde{\mathcal{H}}_1}], \tag{4.28}$$

$$\widetilde{K}_1' = \left(\begin{array}{cc|cc} \frac{1}{2}(V_1+I) & \frac{1}{2}V_2 & 0 & 0 \\ 0 & \frac{1}{2}I_{\mathcal{H}_2} & \frac{1}{\sqrt{2}}K_{12} & \frac{1}{\sqrt{2}}K_{22} \\ \hline 0 & 0 & N_1 & N_2 \\ 0 & 0 & N_3 & N_4 \end{array}\right) \in [\mathfrak{H}' \oplus \mathcal{H}_2 \oplus \underbrace{\mathcal{H}_1 \oplus \mathfrak{H}_1}_{\widetilde{\mathcal{H}}_1}]. \tag{4.29}$$

Letting $\mathcal{H}' := \mathfrak{H}' \oplus \mathcal{H}_2 \oplus \mathcal{H}_1$ one presents (4.28) and (4.29) as

$$\widetilde{K}_0' = \begin{pmatrix} K_1' & K_2' \\ K_3' & K_4 \end{pmatrix} \in [\mathcal{H}' \oplus \mathfrak{H}_1], \quad \widetilde{K}_1' = \begin{pmatrix} N_1' & N_2' \\ N_3' & N_4 \end{pmatrix} \in [\mathcal{H}' \oplus \mathfrak{H}_1] \tag{4.30}$$

where

(4.31)
$$K_1' = \begin{pmatrix} -\frac{i}{2}(V_1-I) & -\frac{i}{2}V_2 & 0 \\ 0 & \frac{i}{2}I_{\mathcal{H}_2} & -\frac{i}{\sqrt{2}}K_{12} \\ 0 & 0 & K_{11} \end{pmatrix}, \quad K_2' = \begin{pmatrix} 0 \\ -\frac{i}{\sqrt{2}}K_{22} \\ K_{21} \end{pmatrix}, \quad K_3' = (0 \;\; 0 \;\; K_3),$$

(4.32)
$$N_1' = \begin{pmatrix} \frac{1}{2}(V_1+I) & \frac{1}{2}V_2 & 0 \\ 0 & \frac{1}{2}I_{\mathcal{H}_2} & \frac{1}{\sqrt{2}}K_{12} \\ 0 & 0 & N_1 \end{pmatrix}, \qquad N_2' = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}K_{22} \\ N_2 \end{pmatrix}, \qquad N_3' = (0 \;\; 0 \;\; N_3).$$

Since $\widetilde{\theta} \in \mathrm{Self}(\widetilde{\mathcal{H}}_0, \widetilde{\mathcal{H}}_1)$, it follows from Lemma 4.5 that

$$\widetilde{\theta}' := \{\widetilde{K}_0', \widetilde{K}_1'; \mathcal{H}' \oplus \mathfrak{H}_1\} \tag{4.33}$$

is a selfadjoint linear relation in $\mathcal{H}' \oplus \mathfrak{H}_1$. Moreover in view of (4.24) $\widetilde{K}_1' - i\,\widetilde{K}_0' = I$. This and part 1) of the proof imply that $0 \in \rho(N_4 - \lambda K_4),\ \lambda \in \mathbb{C}_+$ and the equalities

(4.34)
$$K_0'(\lambda) = -K_1' + K_2'(N_4 - \lambda K_4)^{-1}(N_3' - \lambda K_3'), \quad K_1'(\lambda) = N_1' - N_2'(N_4 - \lambda K_4)^{-1}(N_3' - \lambda K_3')$$

define operator functions $K_j'(\cdot) : \mathbb{C}_+ \to [\mathcal{H}'],\ j \in \{0,1\}$ such that the function $\tau_+'(\lambda) := \{K_0'(\lambda), K_1'(\lambda); \mathcal{H}'\}$ belongs to the class $\widetilde{R}_+(\mathcal{H}')$.

Assume now that $K_0(\lambda)$ and $K_1(\lambda)$ are operator functions (4.15), (4.16), $K_0(\lambda) = (K_{01}(\lambda) \;\; K_{02}(\lambda))$ ($\in [\mathcal{H}_1, \mathcal{H}_1 \oplus \mathcal{H}_2]$) is the block-matrix representation of $K_0(\lambda)$ and $\tau_+(\lambda) := \{K_0(\lambda), K_1(\lambda); \mathcal{H}_1\},\ \lambda \in \mathbb{C}_+$. It follows from (4.27) that

$$K_{01}(\lambda) = P_1 K_0(\lambda) = -K_{11} + K_{21}(N_4 - \lambda K_4)^{-1}(N_3 - \lambda K_3), \tag{4.35}$$

$$K_{02}(\lambda) = P_2 K_0(\lambda) = -K_{12} + K_{22}(N_4 - \lambda K_4)^{-1}(N_3 - \lambda K_3), \quad \lambda \in \mathbb{C}_+. \tag{4.36}$$

Now we are ready to prove the inclusion $\tau_+(\cdot) \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$. The immediate calculation with taking into account of (4.31) and (4.32) gives

$$\begin{gathered} -K_0'(\lambda) = \begin{pmatrix} -\frac{i}{2}(V_1-I) & -\frac{i}{2}V_2 & 0 \\ 0 & \frac{i}{2}I_{\mathcal{H}_2} & \frac{i}{\sqrt{2}}K_{02}(\lambda) \\ 0 & 0 & -K_{01}(\lambda) \end{pmatrix}, \\ K_1'(\lambda) = \begin{pmatrix} \frac{1}{2}(V_1+I) & \frac{1}{2}V_2 & 0 \\ 0 & \frac{1}{2}I_{\mathcal{H}_2} & -\frac{1}{\sqrt{2}}K_{02}(\lambda) \\ 0 & 0 & K_1(\lambda) \end{pmatrix}. \end{gathered} \tag{4.37}$$

Since $-\tau'_+(\lambda)=\{-K'_0(\lambda),K'_1(\lambda);\mathcal{H}'\}$ is a maximal accretive linear relation in $\mathcal{H}'$, it follows from (4.37) and Lemma 4.5 that $-\tau_+(\lambda)=\{-K_0(\lambda),K_1(\lambda);\mathcal{H}_1\}\in \mathrm{Ac}(\mathcal{H}_0,\mathcal{H}_1),\ \lambda\in\mathbb{C}_+$. Hence $\tau_+(\cdot)\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$.

3) Let us prove the second statement of the theorem. Assume that $\tau_+(\cdot)\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ and

$$\tau_+(\lambda)=\{K_0(\lambda),K_1(\lambda);\mathcal{H}_1\},\quad K_0(\lambda)=(K_{01}(\lambda)\ \ K_{02}(\lambda))^\top(\in[\mathcal{H}_1,\mathcal{H}_1\oplus\mathcal{H}_2]) \tag{4.38}$$

is a representation of $\tau_+(\cdot)$ such that $K_1(\lambda)+i\,K_{01}(\lambda)=I_{\mathcal{H}_1},\ \lambda\in\mathbb{C}_+$. Let $\mathfrak{H}'$ be a Hilbert space with $\dim\mathfrak{H}'=\dim(\mathfrak{H}'\oplus\mathcal{H}_2)$ and let $V=(V_1\ \ V_2)\in[\mathfrak{H}'\oplus\mathcal{H}_2,\mathfrak{H}']$ be a unitary operator from $\mathfrak{H}'\oplus\mathcal{H}_2$ onto $\mathfrak{H}'$ (here $\mathcal{H}_2=\mathcal{H}_0\ominus\mathcal{H}_1$). Put $\mathcal{H}'=\mathfrak{H}'\oplus\mathcal{H}_2\oplus\mathcal{H}_1$ and consider the operator functions $K'_j(\cdot):\mathbb{C}_+\to[\mathcal{H}'],\ j\in\{0,1\}$ defined by (4.37). It follows from Lemma 4.5 that the function $\tau'_+(\lambda):=\{K'_0(\lambda),K'_1(\lambda);\mathcal{H}'\}$ belongs to the class $\widetilde{R}_+(\mathcal{H}')$. Moreover the equality

$$K'_1(\lambda)+i\,K'_0(\lambda)=I_{\mathcal{H}'},\quad \lambda\in\mathbb{C}_+ \tag{4.39}$$

is valid. Therefore according to part 1) of the proof there exist a Hilbert space $\mathfrak{H}_1$ and operators $\widetilde{K}'_j,\ j\in\{0,1\}$ with the block-matrix representations (4.30) such that $\widetilde{\theta}':=\{\widetilde{K}'_0,\widetilde{K}'_1;\mathcal{H}'\oplus\mathfrak{H}_1\}$ is a selfadjoint linear relation in $\mathcal{H}'\oplus\mathfrak{H}_1$ and the operator functions $K'_j(\lambda),\ j\in\{0,1\}$ satisfy (4.34). Moreover the following equality holds

$$\widetilde{K}'_1-i\,\widetilde{K}'_0=I. \tag{4.40}$$

It follows from (4.30) and (4.40) that $N'_3-i\,K'_3=0$. Therefore $K'_0(i)=-K'_1,\ K'_1(i)=N'_1$ and by (4.37) the operators (4.30) take the form (in the decomposition $\mathcal{H}'\oplus\mathfrak{H}_1=\mathfrak{H}'\oplus\mathcal{H}_2\oplus\mathcal{H}_1\oplus\mathfrak{H}_1$)

$$\widetilde{K}'_0=\begin{pmatrix}-\frac{i}{2}(V_1-I) & -\frac{i}{2}V_2 & 0 & X_1\\ 0 & \frac{i}{2}I_{\mathcal{H}_2} & * & *\\ 0 & 0 & * & *\\ X_2 & X_3 & * & *\end{pmatrix},\quad \widetilde{K}'_1=\begin{pmatrix}\frac{1}{2}(V_1+I) & \frac{1}{2}V_2 & 0 & Y_1\\ 0 & \frac{1}{2}I_{\mathcal{H}_2} & * & *\\ 0 & 0 & * & *\\ Y_2 & Y_3 & * & *\end{pmatrix}. \tag{4.41}$$

Let $U:=(\widetilde{K}'_1+i\,\widetilde{K}'_0)(\widetilde{K}'_1-i\,\widetilde{K}'_0)^{-1}$ be a Cayley transform of $\widetilde{\theta}'$. Then by (4.40) $U=\widetilde{K}'_1+i\,\widetilde{K}'_0$, so that

$$U=\begin{pmatrix}V_1 & V_2 & 0 & Y_1+i\,X_1\\ 0 & 0 & * & *\\ 0 & 0 & * & *\\ Y_2+i\,X_2 & Y_3+i\,X_3 & * & *\end{pmatrix}. \tag{4.42}$$

Since $V=(V_1\ \ V_2)$ and U are unitary operators, the equality (4.42) gives $Y_j+i\,X_j=0,\ j\in\{1,2,3\}$. Moreover combining (4.40) and (4.41) one obtains $Y_j-i\,X_j=0,\ j\in\{1,2,3\}$. Hence $X_j=Y_j=0,\ j\in\{1,2,3\}$ and in view of (4.40) the equalities (4.41) can be rewritten as (4.28), (4.29).

Let now $\widetilde{K}_0$ be the operator (4.25) with entries taken from (4.28) and let $\widetilde{K}_1$ be the second operator in (4.14) with entries taken from (4.29). Furthermore let K_1 and K_2 be operators (4.27), so that $\widetilde{K}_0$ has the block-matrix representation (4.14). Then by Lemma 4.5 a linear relation $\widetilde{\theta}:=\{\widetilde{K}_0,\widetilde{K}_1;\mathcal{H}_1\oplus\mathfrak{H}_1\}$ belongs to the class $\mathrm{Self}\{\mathcal{H}_0\oplus\mathfrak{H}_1,\mathcal{H}_1\oplus\mathfrak{H}_1\}$. Moreover the same calculations as in the part 2) of the proof leads to the equalities (4.35), (4.36) for $K_{0j}(\lambda),\ j\in\{1,2\}$ and (4.16) for $K_1(\lambda)$. Thus the given operator functions $K_0(\cdot)$ and $K_1(\cdot)$ satisfy (4.15) and (4.16). $\square$

One can easily verify that under the assumptions of Theorem 4.6 a linear relation $\widetilde{\theta}$ and a function $\tau_+(\cdot)$ are connected via

$$-(\tau_+(\lambda)+\lambda P_1)^{-1} = P_{\mathcal{H}_0}(\widetilde{\theta}-\lambda\widetilde{P}_1)^{-1}\upharpoonright \mathcal{H}_1, \quad \lambda\in\mathbb{C}_+ \tag{4.43}$$

where $\widetilde{P}_1 \in [\widetilde{\mathcal{H}}_0, \widetilde{\mathcal{H}}_1]$ is the orthoprojector in $\widetilde{\mathcal{H}}_0$ onto $\widetilde{\mathcal{H}}_1$ and $P_1\in[\mathcal{H}_0,\mathcal{H}_1]$ is the orthoprojector in $\mathcal{H}_0$ onto $\mathcal{H}_1$. Therefore formulas (4.15) and (4.16) define the same function $\tau_+(\lambda)$ for different representations $\widetilde{\theta}=\{\widetilde{K}_0,\widetilde{K}_1;\widetilde{\mathcal{H}}_1\}$ of the linear relation $\widetilde{\theta}$. This allows is to introduce the following definition.

Definition 4.7. Let $\mathcal{H}_0$, $\mathfrak{H}_1$ be Hilbert spaces and let $\mathcal{H}_1$ be a subspace in $\mathcal{H}_0$. A linear relation $\widetilde{\theta}\in \mathrm{Self}(\mathcal{H}_0\oplus\mathfrak{H}_1,\mathcal{H}_1\oplus\mathfrak{H}_1)$ will be called a dilation of a $\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$-valued function $\tau_+(\cdot)\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$, if there exist representations $\widetilde{\theta}=\{\widetilde{K}_0,\widetilde{K}_1;\mathcal{H}_1\oplus\mathfrak{H}_1\}$ and $\tau_+(\lambda)=\{K_0(\lambda),K_1(\lambda);\mathcal{H}_1\}$, $\lambda\in\mathbb{C}_+$ such that the block-matrix representations (4.14) of $\widetilde{K}_0$ and $\widetilde{K}_1$ satisfy the equalities (4.15) and (4.16).

A function $\tau_+(\cdot)\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ will be called a compression of a linear relation $\widetilde{\theta}\in \mathrm{Self}(\mathcal{H}_0\oplus\mathfrak{H}_1,\mathcal{H}_1\oplus\mathfrak{H}_1)$, if $\widetilde{\theta}$ is a dilation of $\tau_+(\cdot)$.

It is clear that $\widetilde{\theta}$ is a dilation of $\tau_+(\cdot)$ (or, equivalently, $\tau_+(\cdot)$ is a compression of $\widetilde{\theta}$) if and only if the equality (4.43) holds. Moreover in view of Theorem 4.6 for every function $\tau_+(\cdot)\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ there exists a dilation $\widetilde{\theta}\in\mathrm{Self}(\mathcal{H}_0\oplus\mathfrak{H}_1,\mathcal{H}_1\oplus\mathfrak{H}_1)$. Note that for Nevanlinna functions with values in $\widetilde{\mathcal{C}}(\mathcal{H})$ this result is well known [11, 6].

Definition 4.8. A linear relation $\widetilde{\theta}\in\mathrm{Self}\{\mathcal{H}_0\oplus\mathfrak{H}_1,\mathcal{H}_1\oplus\mathfrak{H}_1\}$ will be called minimal (with respect to $\mathcal{H}_0$) if there are not decompositions

$$\mathfrak{H}_1=\mathfrak{H}_1'\oplus\mathfrak{H}_1'', \quad \widetilde{\theta}=\hat{\theta}\oplus\theta'' \tag{4.44}$$

with $\mathfrak{H}_1''\neq\{0\}$ and linear relations $\hat{\theta}\in\mathrm{Self}(\mathcal{H}_0\oplus\mathfrak{H}_1',\mathcal{H}_1\oplus\mathfrak{H}_1')$, $\theta''=(\theta'')^*\in\widetilde{\mathcal{C}}(\mathfrak{H}_1'')$.

It is well known that in the case $\mathcal{H}_0=\mathcal{H}_1:=\mathcal{H}$ a linear relation $\widetilde{\theta}=\widetilde{\theta}^*\in\widetilde{\mathcal{C}}(\mathcal{H}\oplus\mathfrak{H}_1)$ admits the unique representation (4.44) where $\hat{\theta}=\hat{\theta}^*\in\widetilde{\mathcal{C}}(\mathcal{H}\oplus\mathfrak{H}_1')$ is a minimal relation with respect to $\mathcal{H}$ and $\theta''=(\theta'')^*\in\widetilde{\mathcal{C}}(\mathfrak{H}_1'')$. Moreover the subspace $\mathfrak{H}_1'$ is defined by

$$\mathfrak{H}_1'=\overline{\mathrm{span}}\{P_{\mathfrak{H}_1}(\theta-\lambda)^{-1}\upharpoonright\mathcal{H}:\lambda\in\mathbb{C}_+\cup\mathbb{C}_-\}.$$

In the next lemma we prove similar result for the general case $\mathcal{H}_1\subset\mathcal{H}_0$.

Lemma 4.9. *For every* $\widetilde{\theta}\in\mathrm{Self}\{\mathcal{H}_0\oplus\mathfrak{H}_1,\mathcal{H}_1\oplus\mathfrak{H}_1\}$ *there exists the unique representation* (4.44) *where* $\hat{\theta}\in\mathrm{Self}(\mathcal{H}_0\oplus\mathfrak{H}_1',\mathcal{H}_1\oplus\mathfrak{H}_1')$ *is a minimal relation with respect to* $\mathcal{H}_0$ *and* $\theta''=(\theta'')^*\in\widetilde{\mathcal{C}}(\mathfrak{H}_1'')$.

Proof. Let (4.23) be the representation of $\widetilde{\theta}$ such that (4.24) holds. As in the proof of Theorem 4.6 consider the operators (4.28), (4.29) and the selfadjoint linear relation $\widetilde{\theta}'$ defined by (4.33). It is easily seen that every decomposition

$$\mathfrak{H}_1=\mathfrak{H}_1'\oplus\mathfrak{H}_1'', \quad \widetilde{\theta}'=\hat{\theta}'\oplus\theta''$$

with $\hat{\theta}'=(\hat{\theta}')^*\in\widetilde{\mathcal{C}}(\mathcal{H}'\oplus\mathfrak{H}_1')$ and $\theta''=(\theta'')^*\in\widetilde{\mathcal{C}}(\mathfrak{H}_1'')$ generates the decomposition (4.44) of $\widetilde{\theta}$ with $\hat{\theta}\in\mathrm{Self}\{\mathcal{H}_0\oplus\mathfrak{H}_1',\mathcal{H}_1\oplus\mathfrak{H}_1'\}$. This and the validity of the lemma for $\widetilde{\theta}'$ yield the desired statement for $\widetilde{\theta}$. □

Let $\mathcal{H}_0$, $\mathfrak{H}_1'$ and $\mathfrak{H}_1''$ be Hilbert spaces, let $\mathcal{H}_1$ be a subspace in $\mathcal{H}_0$ and let $\widetilde{\mathcal{H}}_j':=\mathcal{H}_j\oplus\mathfrak{H}_1'$, $\widetilde{\mathcal{H}}_j'':=\mathcal{H}_j\oplus\mathfrak{H}_1''$, $j\in\{0,1\}$. Clearly, $\widetilde{\mathcal{H}}_1'\subset\widetilde{\mathcal{H}}_0'$ and $\widetilde{\mathcal{H}}_1''\subset\widetilde{\mathcal{H}}_0''$. With a unitary operator $V\in[\mathfrak{H}_1',\mathfrak{H}_1'']$ we associate unitary operators $U_j\in[\widetilde{\mathcal{H}}_j',\widetilde{\mathcal{H}}_j'']$, $j\in\{0,1\}$ and $\widetilde{U}_V\in[\widetilde{\mathcal{H}}_0'\oplus\widetilde{\mathcal{H}}_1',\widetilde{\mathcal{H}}_0''\oplus\widetilde{\mathcal{H}}_1'']$ defined by

$$U_0=I_{\mathcal{H}_0}\oplus V, \quad U_1=U_0\upharpoonright\widetilde{\mathcal{H}}_1'=I_{\mathcal{H}_1}\oplus V, \quad \widetilde{U}_V=U_0\oplus U_1.$$

Definition 4.10. The dilations $\widetilde{\theta} \in \mathrm{Self}(\mathcal{H}_0 \oplus \mathfrak{H}_1', \mathcal{H}_1 \oplus \mathfrak{H}_1')$ and $\widetilde{\eta} \in \mathrm{Self}(\mathcal{H}_0 \oplus \mathfrak{H}_1'', \mathcal{H}_1 \oplus \mathfrak{H}_1'')$ of a function $\tau_+(\cdot) \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ will be called unitary equivalent if there is a unitary operator $V \in [\mathfrak{H}_1', \mathfrak{H}_1'']$ such that $\widetilde{\eta} = \widetilde{U}_V \widetilde{\theta}$.

It is known that every Nevanlinna function with values in $\widetilde{\mathcal{C}}(\mathcal{H})$ has a minimal dilation and every two such dilations are unitary equivalent. In the next proposition we generalize this assertion to the class $\widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$.

Proposition 4.11. *1) For every function $\tau_+(\cdot) \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ there exists a minimal dilation $\widetilde{\theta} \in \mathrm{Self}(\mathcal{H}_0 \oplus \mathfrak{H}_1, \mathcal{H}_1 \oplus \mathfrak{H}_1)$.*

2) Every two minimal dilations $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ of a function $\tau_+(\cdot) \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ are unitary equivalent.

Proof. The statement 1) directly follows from Theorem 4.6 and Lemma 4.9. The statement 2) can be proved similarly Lemma 4.9 by the passage to the selfadjoint dilations $\widetilde{\theta}_1'$ and $\widetilde{\theta}_2'$ of the form (4.33). ☐

Let $\mathcal{H}_0 = \mathcal{H}_1 := \mathcal{H}$ and let $\widetilde{\theta} \in \widetilde{\mathcal{C}}(\mathcal{H} \oplus \mathfrak{H}_1)$ be a selfadjoint dilation of a function $\tau_+(\cdot) \in \widetilde{R}_+(\mathcal{H})$. It is well known that $\widetilde{\theta}(0) \subset \mathfrak{H}_1$ if and only if $s - \lim\limits_{y \to +\infty} i\,y(\tau_+(iy) + iy)^{-1} = I_{\mathcal{H}}$. Similar result for the case $\mathcal{H}_1 \subset \mathcal{H}_0$ can be found in the next proposition.

Proposition 4.12. *Assume that $\tau_+(\cdot) \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ and let (4.1) be a representation of the function $\tau_+(\lambda)$. Then:*

1) a function

$$\tau_{1+}(\lambda) = \{K_{01}(\lambda), K_1(\lambda); H_+\}, \quad \lambda \in \mathbb{C}_+ \tag{4.45}$$

belongs to the class $\widetilde{R}_+(\mathcal{H}_1)$, so that there exists the strong limit

$$F_\infty := s - \lim_{y \to +\infty} i\,yK_{01}(i\,y)\big(K_1(i\,y) + i\,yK_{01}(i\,y)\big)^{-1} \big(= s - \lim_{y \to +\infty} i\,y(\tau_{1+}(i\,y) + i\,y)^{-1}\big) \tag{4.46}$$

and $0 \le F_\infty \le I_{\mathcal{H}_1}$.

2) if $\widetilde{\theta} \in \mathrm{Self}(\mathcal{H}_0 \oplus \mathfrak{H}_1, \mathcal{H}_1 \oplus \mathfrak{H}_1)$ is a dilation of $\tau_+(\cdot)$, then the following equivalence holds

$$\widetilde{\theta}(0) \subset \mathfrak{H}_1 \iff F_\infty = I_{\mathcal{H}_1}. \tag{4.47}$$

If in addition $\widetilde{\theta}$ is a simple dilation, then the condition $F_\infty = I_{\mathcal{H}_1}$ is necessary and sufficient for $\widetilde{\theta}$ to be an operator.

Proof. 1) It follows from (4.5) that $2\mathrm{Im}(K_{01}^*(\lambda)K_1(\lambda)) \ge 0$ and $0 \in \rho(K_1(\lambda) + i\,K_{01}(\lambda)),\ \lambda \in \mathbb{C}_+$. Hence $\tau_{1+}(\cdot) \in \widetilde{R}_+(\mathcal{H}_1)$

2) Assume that (4.23) is the representation of $\widetilde{\theta}$, (4.14) is the block-matrix representation of the operators $\widetilde{K}_j$ and $K_j(\lambda),\ j \in \{0,1\}$ are the operator functions (4.15), (4.16). Next consider the operators (4.28), (4.29), the Hilbert space $\mathcal{H}' := \mathfrak{H}' \oplus \mathcal{H}_2 \oplus \mathcal{H}_1$ and the operator functions $K_j'(\lambda),\ j \in \{0,1\}$ given by (4.34). Put $\widetilde{\theta}' = \{\widetilde{K}_0', \widetilde{K}_1'; \mathcal{H}' \oplus \mathfrak{H}_1\},\ \tau_+'(\lambda) = \{K_0'(\lambda), K_1'(\lambda); \mathcal{H}'\}$ and let

$$F_\infty' := s - \lim_{y \to +\infty} i\,yK_0'(i\,y)\big(K_1'(i\,y) + i\,yK_0'(i\,y)\big)^{-1}.$$

Since $\widetilde{\theta}'$ is a selfadjoint dilation of $\tau_+'(\cdot) \in \widetilde{R}_+(\mathcal{H}')$, the equivalence $\widetilde{\theta}'(0) \subset \mathfrak{H}_1 \iff F_\infty' = I_{\mathcal{H}'}$ is valid. Moreover in view of Lemma 4.5, 2) $\widetilde{\theta}'(0) = \widetilde{\theta}(0)$. This yields the equivalence

$$\widetilde{\theta}(0) \subset \mathfrak{H}_1 \iff F_\infty' = I_{\mathcal{H}'}.$$

Thus to prove (4.47) it is sufficient to show that

$$F'_\infty = I_{\mathcal{H}'} \iff F_\infty = I_{\mathcal{H}_1}. \tag{4.48}$$

Put in (4.37) $S' = \frac{i}{2}(V_1 - I),\ T' = \frac{1}{2}(V_1 + I)$ and consider a linear relation $\eta = \{S', T'; \mathfrak{H}'\}$. Since $V_1 \in [\mathfrak{H}']$ is an isometry and $\operatorname{Ker}(V_1 - I) = \{0\}$, it follows that η is a maximal symmetric operator in $\mathfrak{H}'$ and $\mathbb{C}_- \subset \rho(\eta)$. Therefore $0 \in \rho(T' + i\,yS'),\ y > 0$ and

$$s - \lim_{y \to +\infty} i\,yS'(T' + i\,yS')^{-1} = s - \lim_{y \to +\infty} i\,y(\eta + i\,y)^{-1} = I_{\mathfrak{H}'}. \tag{4.49}$$

The immediate calculation with taking into account of (4.37) gives

$$i\,yK'_0(i\,y)\big(K'_1(i\,y) + i\,yK'_0(i\,y)\big)^{-1} = \begin{pmatrix} i\,yS'(T' + i\,yS')^{-1} & * & * \\ 0 & \frac{y}{y+1} I_{\mathcal{H}_2} & * \\ 0 & 0 & \Phi(i\,y) \end{pmatrix} \tag{4.50}$$

where $\Phi(i\,y) = i\,yK_{01}(i\,y)\big(K_1(i\,y) + i\,yK_{01}(i\,y)\big)^{-1}$. Since $(F'_\infty)^* = F'_\infty$, it follows from (4.50) and (4.49) that $F'_\infty = \operatorname{diag}(I_{\mathfrak{H}'}, I_{\mathcal{H}_2}, F_\infty)$. This leads to (4.48) and, consequently, to (4.47).

Finally the last statement is implied by (4.47) and Corollary 3.5. $\square$

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