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SYSTEMS OF n SUBSPACES AND REPRESENTATIONS OF $*$-ALGEBRAS GENERATED BY PROJECTIONS

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Abstract. In the present work a relationship between systems of n subspaces and representations of $*$-algebras generated by projections is investigated. It is proved that irreducible nonequivalent $*$-representations of $*$-algebras $\mathcal{P}_{4,com}$ generate all nonisomorphic transitive quadruples of subspaces of a finite dimensional space.

1. Introduction

There are many articles that deal with a description of systems $S = (H; H_1, H_2, \dots, H_n)$ of n subspaces H_i, $i = \overline{1,n}$, of a Hilbert space H, which can be infinite or finite dimensional, up to an isomorphism or the unitary equivalence.

In particular, transitive quadruples of subspaces (see Section 2) of a finite dimensional space were described in [1], indecomposable quadruples were found in [2, 3].

For a pair of subspaces H_1, H_2 of a Hilbert space H there is a structure theorem (see, for example, [4]) that describes pairs of orthogonal projections onto these subspaces, up to the unitary equivalence, in terms of sums or integrals of irreducible one- or two-dimensional pairs of orthogonal projections. For three subspaces, to get such a theorem is unrealistic, — the problem of getting a unitary description of n orthogonal projections for $n \geq 3$ is $*$-wild (see [6, 7]). So, if we need to get a description of collections of n orthogonal projections up to the unitary equivalence, it is necessary to introduce additional relations. Recent works of Ukrainian mathematicians (see [9, 11] and the bibliography therein) make a study of irreducible systems of orthogonal projections $P_1, P_2, \dots, P_n$ such that their sum is a multiple of the identity operator.

In [10], the authors suspect that there is a relationship between systems of n subspaces and representations of $*$-algebras generated by projections, — "There seems to be interesting relations with the study of $*$-algebras generated by idempotents by S. Kruglyak and Yu. Samoilenko [7] and the study on sums of projections by S. Kruglyak, V. Rabanovich and Yu. Samoilenko [8]. But we do not know the exact implication …" [10]. This paper is devoted to a study of this relationship.

For an irreducible collection of orthogonal projections, P_1, P_2, …, P_n, on a Hilbert space H such that $\sum_{i=1}^{n} P_i = \alpha I_H$, consider the system of n subspaces

$$S = (H; P_1H, P_2H, \dots, P_nH).$$

Let us formulate the following hypothesis: collections of orthogonal projections such that their sum is a multiple of the identity operator, that is, irreducible nonequivalent

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$*$-representations of the $*$-algebras $\mathcal{P}_{n,com}$ (see Section 3), generate nonisomorphic transitive systems. In Section 4, we prove this hypothesis for $n = 1$ and $n = 2$. There, irreducible nonequivalent $*$-representations of the $*$-algebras $\mathcal{P}_{1,com}$ and $\mathcal{P}_{2,com}$ generate all nonisomorphic transitive systems of one or two subspaces in an arbitrary Hilbert space. We also prove there that, for $n = 3$, irreducible nonequivalent $*$-representations of the $*$-algebra $\mathcal{P}_{3,com}$ generate all nonisomorphic transitive systems of three subspaces of a finite dimensional linear space. Let us remark that it is an unsolved problem to describe irreducible triples of subspaces of an infinite dimensional space or even to prove their existence for $n = 3$ (see [5]). If $n = 4$, we prove in Section 4 that ireducible nonequivalent $*$-representations of the $*$-algebras $\mathcal{P}_{n,com}$ generate all nonisomorphic transitive systems for a finite dimensional space. Since irreducible nonequivalent $*$-representations of the $*$-algebra $\mathcal{P}_{4,com}$ can only be finite dimensional, irreducible nonequivalent $*$-representations of the $*$-algebra $\mathcal{P}_{4,com}$ already do not generate all nonisomorphic transitive systems of four subspaces if $n = 4$, see, for example, [10] and the bibliography therein.

2. Systems of n subspaces

2.1. **Definitions and main properties.** All statements of this section are regarded as known (see, for example, [10, 11]) and given without proofs. Let H be a Hilbert space, H_1, H_2, …, H_n be n subspaces of the space H. Denote by $S = (H; H_1, H_2, \dots, H_n)$ the system of n subspaces of the space H. Let $S = (H; H_1, H_2, \dots, H_n)$ be a system of n subspaces of the Hilbert space H and $\tilde{S} = (\tilde{H}; \tilde{H}_1, \tilde{H}_2, \dots, \tilde{H}_n)$ a system of n subspaces of the Hilbert space $\tilde{H}$.

Definition 1. A linear mapping $R : H \to \tilde{H}$ of the space H into the space $\tilde{H}$ is called a homomorphism of the system S into the system $\tilde{S}$ and denoted by $R : S \to \tilde{S}$, if

$$R(H_i) \subset \tilde{H}_i, \quad i = \overline{1, n}.$$

Definition 2. A homomorphism $R : S \to \tilde{S}$ of a system S into a system $\tilde{S}$ is called an isomorphism, and denoted by $R : S \to \tilde{S}$, if the mapping $R : H \to \tilde{H}$ is a bijection and $R(H_i) = \tilde{H}_i, \forall i = \overline{1, n}$.

Systems S and $\tilde{S}$ will be called isomorphic and denoted by $S \cong \tilde{S}$, if there exists an isomorphism $R : S \to \tilde{S}$.

Definition 3. We say that systems S and $\tilde{S}$ are unitary equivalent, or simply equivalent, if $S \cong \tilde{S}$ and the isomorphism $R : S \to \tilde{S}$ can be chosen as to be a unitary operator.

For each system $S = (H; H_1, H_2, \dots, H_n)$ of n subspaces of a Hilbert space H there is a naturally connected system of orthogonal projections P_1, P_2, …, P_n, where P_i is the orthogonal projection operator onto the subspace H_i, $i = \overline{1, n}$. A system of projections P_1, P_2, …, P_n on a Hilbert space H such that $\operatorname{Im} P_i = H_i$ for $i = \overline{1, n}$ will be called a system of orthogonal projections corresponding to the system of subspaces $S = (H; H_1, H_2, \dots, H_n)$. And conversely, for each system of projections there is a naturally connected system of subspaces. The system $S = (H; P_1H, P_2H, \dots, P_nH)$ will be called a system generated by the system of the projections P_1, P_2, …, P_n.

Definition 4. A system of orthogonal projections P_1, P_2, …, P_n on a Hilbert space H is called unitary equivalent to a system $\tilde{P}_1$, $\tilde{P}_2$, …, $\tilde{P}_n$ on a Hilbert space $\tilde{H}$ if there exists a unitary operator $R : H \to \tilde{H}$ such that $RP_i = \tilde{P}_iR$, $i = \overline{1, n}$.

It is clear that systems S and $\tilde{S}$ are unitary equivalent if and only if the corresponding systems of orthogonal projections are unitary equivalent.

Property 1. *Let $S = (H; H_1, H_2, \dots, H_n)$, $\tilde{S} = (\tilde{H}; \tilde{H}_1, \tilde{H}_2, \dots, \tilde{H}_n)$ be systems of n subspaces of Hilbert spaces H and $\tilde{H}$. Let P_i and $\tilde{P}_i$ be orthogonal projection operators* *onto H_i and $\tilde{H}_i$, correspondingly, $i = \overline{1,n}$. The systems S and $\tilde{S}$ are isomorphic if and only if there exists an invertible operator $T : H \to \tilde{H}$ such that*

$$P_i = T^{-1}\tilde{P}_i T P_i \qquad \tilde{P}_i = T P_i T^{-1}\tilde{P}_i, \quad i = \overline{1,n}.$$

Remark 1. If systems S $\tilde{S}$ are unitary equivalent, then $S \cong \tilde{S}$. The converse is not true.

Denote by $\mathrm{Hom}(S,\tilde{S})$ the set of homomorphisms of the system S into the system $\tilde{S}$, and by $\mathrm{End}(S) := \mathrm{Hom}(S,S)$ the algebra of endomorphisms from S into S, that is,

$$\mathrm{End}(S) = \{R \in B(H) | R(H_i) \subset H_i, i = \overline{1,n}\}.$$

Definition 5. A system $S = (H; H_1, H_2, \ldots, H_n)$ of n subspaces of a space H will be called transitive if $\mathrm{End}(S) = \mathbb{C}I_H$.

Remark 2. Isomorphic systems are simultaneously either transitive or nontransitive.

Let us introduce the notion of an indecomposable system, which is equivalent to the definition used in [2, 10]. Denote

$$\mathrm{Idem}(S) = \{R \in B(H) | R(H_i) \subset H_i, i = \overline{1,n}, R^2 = R\}.$$

Definition 6. A system $S = (H; H_1, H_2, \ldots, H_n)$ of n subspaces of a space H will be called indecomposable if $\mathrm{Idem}(S) = \{0, I_H\}$.

Remark 3. Isomorphic systems are simultaneously decomposable or indecomposable.

Definition 7. A system of orthogonal projections P_1, P_2, …, P_n on a Hilbert space H, which possesses only trivial invariant subspaces, is called irreducible.

Remark 4. Systems of unitary equivalent systems of orthogonal projections are simultaneously reducible or irreducible.

The following proposition answers the question about a relation between the notions of a transitive system, an indecomposable system, irreducibility of the corresponding system of orthogonal projections.

Proposition 1. *If a system of subspaces is transitive, then it is indecomposable. If a system of subspaces is indecomposable, then the corresponding system of orthogonal projections is irreducible.*

Proof. The first statement follows from the obvious inclusion $\mathrm{Idem}(S) \subset \mathrm{End}(S)$ and the definitions of a transitive and an indecomposable systems. To prove the second statement, we use the Schur's lemma (see, for example, [11]). A system of orthogonal projections P_1, P_2, …, P_n on a Hilbert space H is irreducible if and only if $\{R \in B(H) | RP_i = P_i R, i = \overline{1,n}, R^2 = R, R^* = R\} = \{0, I_H\}$. The identity $\{R \in B(H) | RP_i = P_i R, i = \overline{1,n}, R^2 = R, R^* = R\} = \{R \in B(H) | R(\mathrm{Im}\ P_i) \subset \mathrm{Im}\ P_i, i = \overline{1,n}, R^2 = R, R^* = R\}$, on the one hand, and the inclusion $\{R \in B(H) | R(H_i) \subset H_i, i = \overline{1,n}, R^2 = R, R^* = R\} \subset \mathrm{Idem}(S)$, on the other hand, finish the proof. □

Example 1. Let $S = (\mathbb{C}^2; \mathbb{C}(1,0), \mathbb{C}(\cos\theta, \sin\theta))$, $\theta \in (0, \pi/2)$ and $\tilde{S} = (\mathbb{C}^2; \mathbb{C}(1,0), \mathbb{C}(0,1))$. The decomposable system S, which corresponds to the irreducible pair of orthogonal projections, is isomorphic but not unitary equivalent to the decomposable system $\tilde{S}$ that corresponds to the reducible pair of orthogonal projections.

Definition 8. Let $S = (H; H_1, H_2, \ldots, H_n)$ be a system of n subspaces of a Hilbert space H. By an orthogonal complement to the system S, we will call the system $S^{\perp} = (H; H_1^{\perp}, H_2^{\perp}, \ldots, H_n^{\perp})$.

Property 2. *Let $S = (H; H_1, H_2, \dots, H_n)$ be a system of n subspaces of a Hilbert space H. Then S is transitive (indecomposable) if and only if $S^\perp$ is transitive (indecomposable).*

Property 2 follows directly, since if $R : S \to \tilde{S}$ is a homomorphism of the system S into $\tilde{S}$, then $R^* : \tilde{S}^\perp \to S^\perp$ is a homomorphism of the system $\tilde{S}$ into S, because, if $R : H \to \tilde{H}$ is a linear operator such that $R(H_i) \subset \tilde{H}_i, \forall i = \overline{1, n}$, then $R^* : \tilde{H} \to H$ and $R^*(\tilde{H}_i^\perp) \subset H_i^\perp, \forall i = \overline{1, n}$.

Definition 9. Let $S = (H; H_1, H_2, \dots, H_n)$ and $\tilde{S} = (\tilde{H}; \tilde{H}_1, \tilde{H}_2, \dots, \tilde{H}_n)$ be two systems of n subspaces. We say that $S \cong \tilde{S}$ up to a rearrangement of subspaces if there is a permutation $\sigma \in S_n$ such that the systems $\sigma(S)$ and $\tilde{S}$ are isomorphic, where $\sigma(S) = (H; H_{\sigma(1)}, H_{\sigma(2)}, \dots, H_{\sigma(n)})$, that is, there exists and invertible operator $R : H \to \tilde{H}$ such that $R(H_{\sigma(i)}) = \tilde{H}_i$, $\forall i = \overline{1, n}$.

2.2. **Transitive systems of one, two, and three subspaces.** In this section we give a description of transitive systems of one, two, and three subspaces up to an isomorphism. A list of nonisomorphic transitive systems of n subspaces will be called complete if, for any transitive system $S = (H; H_1, H_2, \dots, H_n)$ of n subspaces of the space H, there is in the list a system isomorphic to the system S.

Proposition 2. *If a system $S = (H; H_1)$ of a single subspace H_1 of the space H is transitive, then it is isomorphic to one of the following systems:*

$$S_1 = (\mathbb{C}; 0), \quad S_2 = (\mathbb{C}; \mathbb{C}).$$

Proof. Let $\dim H > 1$ and H_1 be an arbitrary proper subspace of the space H. Then the algebra $\mathrm{End}(S)$ corresponding to the system $S = (H; H_1)$ contains a nontrivial idempotent, for example, the operator of orthogonal projection onto $H_1^\perp$, and, consequently, the algebra is trivial. In the case where $\dim H > 1$ and H_1 is a trivial subspace of the space H, the algebra $\mathrm{End}(S) = B(H)$, that is, it coincides with the set of linear bounded operators from H into H. □

To construct lists of transitive systems of two and three subspaces, we use the description of the algebra $\mathrm{End}(S)$ for the system $S = (U; K_1, K_2, K_3)$ of 3 subspaces K_1, K_2, K_3 of a finite dimensional linear space U [1]. Let L be an arbitrary subspace complementary to the subspace $K_1 + K_2 + K_3$ in the space U, that is,

$$(K_1 + K_2 + K_3) \dot{+} L = U,$$

where $\dot{+}$ is the direct sum of vector spaces.

Denote $P = K_1 \cap K_2 \cap K_3$. Let M_1,M_2,M_3 be arbitrary subspaces complementary to the subspaces $K_1 \cap (K_2 + K_3)$, $K_2 \cap (K_1 + K_3)$, $K_3 \cap (K_1 + K_2)$ in K_1, K_2, K_3, correspondingly, that is,

$$\begin{aligned} K_1 \cap (K_2 + K_3) \dot{+} M_1 &= K_1, \\ K_2 \cap (K_1 + K_3) \dot{+} M_2 &= K_2, \\ K_3 \cap (K_1 + K_2) \dot{+} M_3 &= K_3. \end{aligned}$$

Denote by N_1,N_2,N_3 arbitrary complementary subspaces to the subspace P in $K_2 \cap K_3$, $K_1 \cap K_3$, $K_1 \cap K_2$, correspondingly, that is,

$$\begin{aligned} P \dot{+} N_1 &= K_2 \cap K_3, \\ P \dot{+} N_2 &= K_1 \cap K_3, \\ P \dot{+} N_3 &= K_1 \cap K_2. \end{aligned}$$

Let now Q_3 be an arbitrary subspace complementary to the subspace $K_3 \cap K_1 + K_3 \cap K_1$ in the subspace $K_3 \cap (K_1 + K_2)$. An arbitrary element x_3 of the subspace Q_3 is uniquely decomposed into the sum $x_3 = x_1 + x_2$, where $x_1 \in K_1$ and $x_2 \in K_2$ are such that if x_3 runs over a basis of Q_3, x_1 runs over a system of linearly independent vectors the linear span of which makes a subspace complementary to the subspace $K_1 \cap K_2 + K_1 \cap K_3$ in the space $K_1 \cap (K_2 + K_3)$, and x_2 runs over a system of linearly independent vectors that span a subspace complementary to the subspace $K_2 \cap K_1 + K_2 \cap K_3$ in the subspace $K_2 \cap (K_1 + K_3)$. Denote these complementary subspaces by Q_1 and Q_2, correspondingly. Thus,

$$\begin{aligned}(K_1 \cap K_2 + K_1 \cap K_3) \dot{+} Q_1 &= K_1 \cap (K_2 + K_3),\\ (K_2 \cap K_1 + K_2 \cap K_3) \dot{+} Q_2 &= K_2 \cap (K_1 + K_3),\\ (K_3 \cap K_1 + K_3 \cap K_2) \dot{+} Q_3 &= K_3 \cap (K_1 + K_2),\end{aligned}$$

and $\dim Q_1 = \dim Q_2 = \dim Q_3$. For the space U and the subspaces K_1, K_2, K_3, we have

$$\begin{aligned}U &= L \dot{+} M_1 \dot{+} M_2 \dot{+} M_3 \dot{+} Q_1 \dot{+} Q_2 \dot{+} N_1 \dot{+} N_2 \dot{+} N_3 \dot{+} P,\\ K_1 &= M_1 \dot{+} N_2 \dot{+} N_3 \dot{+} Q_1 \dot{+} P,\\ K_2 &= M_2 \dot{+} N_1 \dot{+} N_3 \dot{+} Q_2 \dot{+} P,\\ K_3 &= M_3 \dot{+} N_1 \dot{+} N_2 \dot{+} Q_3 \dot{+} P.\end{aligned} \tag{1}$$

Let now ℓ, m_i, q, n_i, p, u be dimensions of L, M_i, Q_i, N_i, P, and U, correspondingly. Then the dimension of the algebra $\mathrm{End}(S)$ that corresponds to the system $S = (U; K_1, K_2, K_3)$, considered as a linear space, can be calculated by the formula

$$\begin{aligned}\dim \mathrm{End}(S) &= \ell u + q^2 + q \sum_{i=1}^{3} (m_i + n_i) + \sum_{i=1}^{3} (m_i^2 + n_i^2) +\\ &+ \sum_{\substack{i \neq j\\ i,j=1}}^{3} m_i n_j + p^2.\end{aligned} \tag{2}$$

Proposition 3. *If a system $S = (H; H_1, H_2)$ of two subspaces of a space H is transitive, then it is isomorphic to one of the following system:*

$$\begin{aligned}S_1 &= (\mathbb{C}; 0, 0), \quad S_3 = (\mathbb{C}; 0, \mathbb{C}),\\ S_2 &= (\mathbb{C}; \mathbb{C}, 0), \quad S_4 = (\mathbb{C}; \mathbb{C}, \mathbb{C}).\end{aligned}$$

Proof. To make an analysis of a system of two subspaces in the case of a finite dimensional linear space, set $U = H$, $K_1 = H_1$, $K_1 = H_1$, $K_3 = 0$ in identities (1). We get

$$\begin{aligned}H &= L \dot{+} M_1 \dot{+} M_2 \dot{+} N_3,\\ H_1 &= M_1 \dot{+} N_3,\\ H_2 &= M_2 \dot{+} N_3.\end{aligned}$$

The formula for the dimension of the algebra $\mathrm{End}(S)$, for $K_3 = 0$, becomes

$$\dim \mathrm{End}(S) = \ell u + m_1^2 + m_2^2 + n_3^2.$$

Since the system $S = (H; H_1, H_2)$ is transitive, it follows that $\dim \mathrm{End}(S) = 1$ and, correspondingly, $\ell u + m_1^2 + m_2^2 + n_3^2 = 1$. This identity can hold only in the following four cases:

1) $\ell u = 1$. Hence, $\dim L = 1$, $H = L$, $H_1 = 0$, $H_2 = 0$ and, consequently, $S \cong S_1$.
2) $m_1^2 = 1$. Hence, $\dim M_1 = 1$, $H = M_1$, $H_1 = M_1$, $H_2 = 0$ and, consequently, $S \cong S_2$.
3) $m_2^2 = 1$. Hence, $\dim M_2 = 1$, $H = M_2$, $H_1 = 0$, $H_2 = M_2$ and, consequently, $S \cong S_3$.
4) $n_3^2 = 1$. Hence, $\dim N_3 = 1$, $H = N_3$, $H_1 = N_3$, $H_2 = N_3$ and, consequently, $S \cong S_4$.

It follows from Proposition 1 and [11] that if a pair of orthogonal projections on an infinite dimensional Hilbert space is reducible, then there do not exist transitive systems of two subspaces in an infinite dimensional Hilbert space. We remark that this fact can also be obtained from decomposability of a system of two subspaces in an infinite dimensional Hilbert space [10]. □

Proposition 4. *If a system $S = (U; K_1, K_2, K_3)$ of three subspaces of a finite dimensional linear space U is transitive, then it is isomorphic to one of the following systems:*

$$\begin{array}{ll} S_1 = (\mathbb{C}; 0, 0, 0), & S_5 = (\mathbb{C}; 0, \mathbb{C}, \mathbb{C}), \\ S_2 = (\mathbb{C}; \mathbb{C}, 0, 0), & S_6 = (\mathbb{C}; \mathbb{C}, 0, \mathbb{C}), \\ S_3 = (\mathbb{C}; 0, \mathbb{C}, 0), & S_7 = (\mathbb{C}; \mathbb{C}, \mathbb{C}, 0), \\ S_4 = (\mathbb{C}; 0, 0, \mathbb{C}), & S_8 = (\mathbb{C}; \mathbb{C}, \mathbb{C}, \mathbb{C}), \\ \multicolumn{2}{c}{S_9 = (\mathbb{C}^2; \mathbb{C}(1,0), \mathbb{C}(0,1), \mathbb{C}(1,1)).} \end{array}$$

Proof. Since the system $S = (U; K_1, K_2, K_3)$ is transitive, it follows that $\dim \operatorname{End}(S) = 1$ and, correspondingly,

$$\ell u + q^2 + q \sum_{i=1}^{3} (m_i + n_i) + \sum_{i=1}^{3} (m_i^2 + n_i^2) + \sum_{\substack{i \neq j \\ i,j=1}}^{3} m_i n_j + p^2 = 1.$$

The last identity can hold only in one of the following nine cases:

1) $\ell u = 1$. Hence, $\dim L = 1$, $U = L$, $K_1 = 0$, $K_2 = 0$, $K_3 = 0$. Thus $S \cong S_1$.
2) $m_1^2 = 1$. Hence, $\dim M_1 = 1$, $U = M_1$, $K_1 = M_1$, $K_2 = 0$, $K_3 = 0$ and thus $S \cong S_2$.
3) $m_2^2 = 1$. Hence, $\dim M_2 = 1$, $U = M_2$, $K_1 = 0$, $K_2 = M_2$, $K_3 = 0$, and thus $S \cong S_3$.
4) $m_3^2 = 1$. Hence, $\dim M_3 = 1$, $U = M_3$, $K_1 = 0$, $K_2 = 0$, $K_3 = M_3$, and thus $S \cong S_4$.
5) $n_1^2 = 1$. Hence, $\dim N_1 = 1$, $U = N_1$, $K_1 = 0$, $K_2 = N_1$, $K_3 = N_1$, and thus $S \cong S_5$.
6) $n_2^2 = 1$. Hence, $\dim N_2 = 1$, $U = N_2$, $K_1 = N_2$, $K_2 = 0$, $K_3 = N_2$, and thus $S \cong S_6$.
7) $n_3^2 = 1$. Hence, $\dim N_3 = 1$, $U = N_3$, $K_1 = N_3$, $K_2 = N_3$, $K_3 = 0$, and thus $S \cong S_7$.
8) $p^2 = 1$. Hence, $\dim P = 1$, $U = P$, $K_1 = P$, $K_2 = P$, $K_3 = P$, and thus $S \cong S_8$.
9) $q^2 = 1$. Hence, $\dim Q_1 = \dim Q_2 = 1$, $U = Q_1 \dot{+} Q_2$, $K_1 = Q_1$, $K_2 = Q_2$, $K_3 = Q_3$, and thus $S \cong S_9$.

□

We recall that the problem of even proving existence of transitive triples of subspaces of an infinite dimensional space is an open problem (see [5]).

2.3. **Transitive systems of four subspaces.** Following [2] let us introduce the notion of a defect of a system $S = (U; K_1, K_2, K_3, K_4)$ of four subspaces of a finite dimensional linear space U.

Definition 10. Let $S = (U; K_1, K_2, K_3, K_4)$ be a system of four subspaces of a finite dimensional linear space U. By a defect of the system S, we will call the number defined by

$$\rho(S) = \sum_{i=1}^{4} \dim K_i - 2 \dim U.$$

S. Brenner in [1] gave a description of a complete list of four distinct proper subspaces up to a rearrangement of the subspaces, and systems that have a nonnegative defect were written down explicitly. An explicit form for systems of four proper subspaces, with a negative defect, is given in this section by passing to orthogonal systems and choosing suitable isomorphic systems. We adopt the following notations used in [1]:

1 is the $r \times r$ identity matrix;
0 is the $r \times r$ zero matrix;
J is the $r \times r$ Jordan cell with zero on the diagonal;

ξ is the column of r zeros;
η is the row of r zeros;
b is the column of the first $(r-1)$ zeros and 1 as the last element;
d is the row with the first element equal 1 and other $r-1$ zeros.

The subspace K_i in the list is given by a matrix $\mathcal{K}_i$. Here the subspace K_i is set to be the linear span of rows of the matrix $\mathcal{K}_i$. Introduce two more notations, — $B(u,\rho)$ denotes the system $B=(U;K_1,K_2,K_3,K_4)$ of four subspaces of the space U of dimension u with defect ρ, and $B(u,\rho;\lambda)$ denotes the system $B=(U;K_1,K_2,K_3,K_4)$ of four subspaces of the spaces U of dimension u, with defect ρ, which depend on a parameter λ.

The following is a complete list of distinct proper subspaces, up to a rearrangement:

(1) $B(2,0;\lambda)$, $\lambda\in\mathbb{C},\lambda\neq 0,1$,

$$\mathcal{K}_1=\begin{pmatrix}1 & 0\end{pmatrix},\quad \mathcal{K}_2=\begin{pmatrix}0 & 1\end{pmatrix},\quad \mathcal{K}_3=\begin{pmatrix}1 & 1\end{pmatrix},\quad \mathcal{K}_4=\begin{pmatrix}1 & \lambda\end{pmatrix}.$$

(2) $B(2r,1)$, $r=2,3,\dots$,

$$\mathcal{K}_1=\begin{pmatrix}\mathbf{1} & \mathbf{0}\end{pmatrix},\quad \mathcal{K}_2=\begin{pmatrix}\mathbf{0} & \mathbf{1}\end{pmatrix},\quad \mathcal{K}_3=\begin{pmatrix}\mathbf{1} & \mathbf{1}\end{pmatrix},\quad \mathcal{K}_4=\begin{pmatrix}\mathbf{1} & \mathbf{J}\\ \eta & d\end{pmatrix}.$$

(3) $B(2r+2,-1)$, $r=1,2,\dots$,

$$\mathcal{K}_1=\begin{pmatrix}\mathbf{1} & \mathbf{0} & \xi & \xi\\ \eta & d & 0 & 0\end{pmatrix},\quad \mathcal{K}_2=\begin{pmatrix}\mathbf{0} & \mathbf{J} & b & \xi\\ \eta & \eta & 0 & 1\end{pmatrix},$$

$$\mathcal{K}_3=\begin{pmatrix}\mathbf{1} & \mathbf{J} & b & \xi\\ \eta & d & 0 & 1\end{pmatrix},\quad \mathcal{K}_4=\begin{pmatrix}\mathbf{1} & \xi & \xi & \mathbf{1}\end{pmatrix}.$$

(4a) $B(3,1)$,

$$\mathcal{K}_1=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},\quad \mathcal{K}_2=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$\mathcal{K}_3=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad \mathcal{K}_4=\begin{pmatrix}1 & 1 & 1\end{pmatrix}.$$

(4b) $B(2r+3,1)$, $r=1,2,\dots$,

$$\mathcal{K}_1=\begin{pmatrix}\mathbf{1} & \mathbf{0} & \xi & \xi & \xi\\ \eta & \eta & 1 & 0 & 0\\ \eta & \eta & 0 & 1 & 0\end{pmatrix},\quad \mathcal{K}_2=\begin{pmatrix}\mathbf{0} & \mathbf{1} & \xi & \xi & \xi\\ \eta & \eta & 1 & 0 & 0\\ \eta & \eta & 0 & 0 & 1\end{pmatrix},$$

$$\mathcal{K}_3=\begin{pmatrix}\mathbf{1} & \mathbf{1} & \xi & \xi & \xi\\ \eta & \eta & 0 & 1 & 0\\ \eta & \eta & 0 & 0 & 1\end{pmatrix},\quad \mathcal{K}_4=\begin{pmatrix}\mathbf{1} & \mathbf{J} & b & \xi & b\\ \eta & d & 0 & 1 & 0\end{pmatrix}.$$

(5a) $B(3,-1)$,

$$\mathcal{K}_1=\begin{pmatrix}0 & 1 & 0\end{pmatrix},\quad \mathcal{K}_2=\begin{pmatrix}0 & 0 & 1\end{pmatrix},$$

$$\mathcal{K}_3=\begin{pmatrix}1 & 0 & 0\end{pmatrix},\quad \mathcal{K}_4=\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\end{pmatrix}.$$

(5b) $B(2r+3,-1)$, $r=1,2,\dots$,

$$\mathcal{K}_1=\begin{pmatrix}\mathbf{1} & \mathbf{0} & \xi & \xi & \xi\\ \eta & \eta & 0 & 1 & 0\end{pmatrix},\quad \mathcal{K}_2=\begin{pmatrix}\mathbf{0} & \mathbf{1} & \xi & \xi & \xi\\ \eta & \eta & 0 & 0 & 1\end{pmatrix},$$

$$\mathcal{K}_3=\begin{pmatrix}\mathbf{1} & \mathbf{1} & \xi & \xi & \xi\\ \eta & \eta & 1 & 0 & 0\end{pmatrix},\quad \mathcal{K}_4=\begin{pmatrix}\mathbf{1} & \mathbf{J} & b & \xi & \xi\\ \eta & d & 0 & 1 & 0\\ \eta & \eta & 1 & 0 & 1\end{pmatrix}.$$

(6a) $B(3,2)$,

$$\mathcal{K}_1=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},\quad \mathcal{K}_2=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$\mathcal{K}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{K}_4 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(6b) $B(5,2)$,

$$\mathcal{K}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{K}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{K}_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{K}_4 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

(6c) $B(2r+3,2)$, $r = 2,3,\ldots$,

$$\mathcal{K}_1 = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \xi & \xi & \xi \\ \eta & \eta & 1 & 0 & 0 \\ \eta & \eta & 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{K}_2 = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \xi & \xi & \xi \\ \eta & \eta & 1 & 0 & 0 \\ \eta & \eta & 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{K}_3 = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \xi & \xi & \xi \\ \eta & \eta & 0 & 1 & 0 \\ \eta & \eta & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{K}_4 = \begin{pmatrix} \mathbf{1} & \mathbf{J}^2 & \mathbf{J}b & \xi & (\mathbf{J}+\mathbf{1})b \\ \eta & d & 0 & 0 & 0 \\ \eta & d\mathbf{J} & 0 & 1 & 0 \end{pmatrix}.$$

(7a) $B(3,-2)$,

$$\mathcal{K}_1 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{K}_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{K}_3 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{K}_4 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

(7b) $B(5,-2)$,

$$\mathcal{K}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{K}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{K}_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{K}_4 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

(7c) $B(2r+5,-2)$, $r = 1,2,\ldots$,

$$\mathcal{K}_1 = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \xi & \xi & \xi & \xi & \xi \\ \eta & d & 0 & 0 & 0 & 0 & 0 \\ \eta & \eta & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{K}_2 = \begin{pmatrix} \mathbf{0} & \mathbf{J} & b & \xi & \xi & \xi & \xi \\ \eta & \eta & 0 & 1 & 0 & 0 & 0 \\ \eta & \eta 0 & 0 & 0 & 0 & 1 & \end{pmatrix},$$

$$\mathcal{K}_3 = \begin{pmatrix} \mathbf{1} & \mathbf{J} & b & \xi & \xi & \xi & \xi \\ \eta & d & 0 & 1 & 0 & 0 & 0 \\ \eta & \eta & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{K}_4 = \begin{pmatrix} \mathbf{1} & \mathbf{J}^3 & \mathbf{J}^2 b & \mathbf{J}b & b & \xi & \xi \\ b^T & d & 0 & 0 & 0 & 0 & 1 \\ \eta & d\mathbf{J}^2 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Theorem 1 (S. Brenner)**.** *If a system $S = (U; K_1, K_2, K_3, K_4)$ of four distinct proper subspaces of a finite dimensional linear space U is transitive, then it is isomorphic, up to a rearrangement of the subspaces, to one of the following system:*

$$\begin{aligned} &B(2,0;\lambda), && \lambda \in \mathbb{C}, \lambda \neq 0,1, \\ &B(u,\pm 1), && u = 3,4,5,\ldots, \\ &B(u,\pm 2), && u = 3,5,7,\ldots. \end{aligned}$$

3. The algebra $\mathcal{P}_{n,com}$ and its $*$-representations

3.1. **Irreducible $*$-representations of the algebra $\mathcal{P}_{n,com}$.** For $n \in \mathbb{N}$, denote by Σ_n the set of $\alpha \in \mathbb{R}_+$ such that there exists at least one $*$-representation of the $*$-algebra $\mathcal{P}_{n,\alpha} = \mathbb{C} < p_1, p_2, \ldots, p_n | p_k^2 = p_k^* = p_k, \sum_{k=1}^n p_k = \alpha e >$, that is, the set of all real parameters α for which there exist n orthogonal projections P_1, P_2, …, P_n on a Hilbert space H satisfying the relation $\sum_{k=1}^n P_k = \alpha I_H$. Introduce an algebra, $\mathcal{P}_{n,com} = \mathbb{C} < p_1, p_2, \ldots, p_n | p_k^2 = p_k^* = p_k, [\sum_{k=1}^n p_k, p_i] = 0, \forall i = \overline{1,n} >$. All irreducible $*$-representations of $\mathcal{P}_{n,com}$ is a union over all $\alpha \in \Sigma_n$ of irreducible $*$-representations of $\mathcal{P}_{n,\alpha}$.

A description of the set Σ_n for all $n \in \mathbb{N}$ was obtained by S. A. Kruglyak, V. I. Rabanovich, and Yu. S. Samoĭlenko in [8], and is given by

$$\Sigma_1 = \{0,1\}, \quad \Sigma_2 = \{0,1,2\}, \quad \Sigma_3 = \{0,1,\tfrac{3}{2},2,3\},$$

$$\Sigma_n = \{\Lambda_n^0, \Lambda_n^1, \left[\tfrac{n-\sqrt{n^2-4n}}{2}, \tfrac{n+\sqrt{n^2-4n}}{2}\right], n-\Lambda_n^1, n-\Lambda_n^0\}, \quad n \geq 4,$$

$$\Lambda_n^0 = \{0, 1+\tfrac{1}{n-1}, 1+\tfrac{1}{(n-2)-\frac{1}{n-1}}, \ldots, 1+\cfrac{1}{(n-2)-\cfrac{1}{(n-2)-\cfrac{1}{\ddots-\frac{1}{n-1}}}}, \ldots\},$$

$$\Lambda_n^1 = \{0, 1+\tfrac{1}{n-2}, 1+\tfrac{1}{(n-2)-\frac{1}{n-2}}, \ldots, 1+\cfrac{1}{(n-2)-\cfrac{1}{(n-2)-\cfrac{1}{\ddots-\frac{1}{n-2}}}}, \ldots\}.$$

3.2. **Irreducible $*$-representations of the algebras** $\mathcal{P}_{1,com}$, $\mathcal{P}_{2,com}$, $\mathcal{P}_{3,com}$. Let us give a list of irreducible $*$-representations of the algebra $\mathcal{P}_{1,com}$. By [8], we have $\Sigma_1 = \{0,1\}$.

For $\alpha = 0$, the only irreducible representation of the algebra $\mathcal{P}_{1,0}$, up to equivalence, is the representation $P_1 = 0$ on the space $H = \mathbb{C}$. For $\alpha = 1$, the unique up to equivalence irreducible representation of the algebra $\mathcal{P}_{1,1}$ is the representation $P_1 = \mathbb{C}$ on the space $H = \mathbb{C}$.

For the algebra $\mathcal{P}_{2,com}$, we have $\Sigma_2 = \{0,1,2\}$ [8].

If $\alpha = 0$, there is a unique up to equivalence irreducible representation of the algebra $\mathcal{P}_{2,0}$ given by $P_1 = 0$, $P_2 = 0$ on the space $H = \mathbb{C}$. If $\alpha = 1$, there are two irreducible representations of the algebra $\mathcal{P}_{2,1}$, up to equivalence. The first one is given by $P_1 = I$, $P_2 = 0$ on the space $H = \mathbb{C}$, and the second one by $P_1 = 0$, $P_2 = I$ on the space $H = \mathbb{C}$. In the case where $\alpha = 2$, the only representation of the algebra $\mathcal{P}_{2,2}$, up to equivalence, is the representation $P_1 = I$, $P_2 = I$ on the space $H = \mathbb{C}$.

Now we give irreducible $*$-representations of the algebra $\mathcal{P}_{3,com}$. We have $\Sigma_3 = \{0,1,\frac{3}{2},2,3\}$.

If $\alpha = 0$, there is a unique up to equivalence irreducible representation of the algebra $\mathcal{P}_{3,0}$. It is given by $P_1 = 0$, $P_2 = 0$, $P_3 = 0$ on the space $H = \mathbb{C}$. If $\alpha = 1$, there are three inequivalent irreducible representations of the algebra $\mathcal{P}_{3,1}$. The first one is $P_1 = I$, $P_2 = 0$, $P_3 = 0$ on the space $H = \mathbb{C}$. The second one is $P_1 = 0$, $P_2 = I$, $P_3 = 0$ on $H = \mathbb{C}$. The third one is given by $P_1 = 0$, $P_2 = 0$, $P_3 = I$ on the space $H = \mathbb{C}$. If $\alpha = 3/2$, there is a unique up to equivalence irreducible representation of the algebra $\mathcal{P}_{3,3/2}$,

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{pmatrix},$$

which act on the space $H = \mathbb{C}^2$. If $\alpha = 2$, there are three inequivalent irreducible representations of the algebra $\mathcal{P}_{3,2}$. The first one is $P_1 = 0$, $P_2 = I$, $P_3 = I$ on $H = \mathbb{C}$, the second one is $P_1 = I$, $P_2 = 0$, $P_3 = I$ on $H = \mathbb{C}$, and the third one is $P_1 = I$, $P_2 = I$, $P_3 = 0$ on $H = \mathbb{C}$. For $\alpha = 3$, the unique up to equivalence irreducible representation of the algebra $\mathcal{P}_{3,3}$ is $P_1 = I$, $P_2 = I$, $P_3 = I$ on $H = \mathbb{C}$.

3.3. **Irreducible $*$-representations of the algebra $\mathcal{P}_{4,com}$.** We use the following notations:

$$A_{\ell,m} = \frac{1}{m}\begin{pmatrix} m-\ell & -\sqrt{\ell(m-l)} \\ -\sqrt{\ell(m-l)} & \ell \end{pmatrix},$$

$$B_{\ell,m} = \frac{1}{m}\begin{pmatrix} m-\ell & \sqrt{\ell(m-l)} \\ \sqrt{\ell(m-l)} & \ell \end{pmatrix},$$

$$C_{\ell,m} = I - A_{\ell,m} = \frac{1}{m}\begin{pmatrix} \ell & \sqrt{\ell(m-l)} \\ \sqrt{\ell(m-l)} & m-\ell \end{pmatrix},$$

$$D_{\ell,m} = I - B_{\ell,m} = \frac{1}{m}\begin{pmatrix} \ell & -\sqrt{\ell(m-l)} \\ -\sqrt{\ell(m-l)} & m-\ell \end{pmatrix}.$$

Let us consider a part of the unit sphere $\Omega \subset \mathbb{R}^3$, given by $\Omega = \{(a,b,c) \in \mathbb{R} | a^2+b^2+c^2 = 1, a > 0, b > 0, c \in (-1,1)\ a = 0, b^2+c^2 = 1, b > 0, c > 0\ b = 0, a^2+c^2 = 1, b > 0, c > 0\}$.

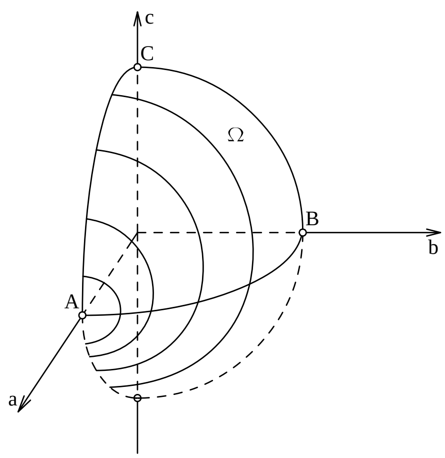

Figure 1

Since all irreducible $*$-representations of the algebra $\mathcal{P}_{4,com}$ are finite dimensional, denote the space of representations by U. Also denote by $S(u,\rho)$ the system $S = (U; \operatorname{Im} P_1, \operatorname{Im} P_2, \operatorname{Im} P_3, \operatorname{Im} P_4)$ of four subspaces of the space U of dimension u with defect ρ, which is generated by the representation P_1, P_2, P_3, P_4 on the space U, and by $S(u,\rho; a,b,c)$ the systems $S = (U; \operatorname{Im} P_1, \operatorname{Im} P_2, \operatorname{Im} P_3, \operatorname{Im} P_4)$ of four subspaces of the space U of dimension u with defect ρ, which are generated by the representation P_1, P_2,P_3, P_4 on U and depend on the parameters a,b,c. Using the results of [8, 11], we write a list of systems of four distinct proper subspaces, given up to a rearrangement of the subspaces, which are generated by irreducible inequivalent representations of the algebra $\mathcal{P}_{4,\alpha}$:

(1) $S(2,0;a,d,c)$, $(a,b,c) \in \Omega$,

$$P_1 = \frac{1}{2}\begin{pmatrix} 1+a & -b-ic \\ -b+ic & 1-a \end{pmatrix}, P_3 = \frac{1}{2}\begin{pmatrix} 1-a & -b+ic \\ -b-ic & 1+a \end{pmatrix},$$

$$P_2 = \frac{1}{2}\begin{pmatrix} 1-a & b-ic \\ b+ic & 1+a \end{pmatrix}, \qquad P_4 = \frac{1}{2}\begin{pmatrix} 1+a & b+ic \\ b-ic & 1-a \end{pmatrix}.$$

(2) $S(2r,1)$, $r=2,3,\ldots$,

$$\begin{aligned}P_1 &= A_{2r-1,4r}\oplus A_{2r-3,4r}\oplus\ldots\oplus A_{1,4r},\\ P_2 &= B_{2r-1,4r}\oplus B_{2r-3,4r}\oplus\ldots\oplus B_{1,4r},\\ U &= \underbrace{\mathbb{C}^2\oplus\ldots\oplus\mathbb{C}^2}_{r};\end{aligned}$$

$$\begin{aligned}P_3 &= 0\oplus B_{2r-2,4r}\oplus B_{2r-4,4r}\ldots\oplus B_{2,4r}\oplus 1,\\ P_4 &= 1\oplus A_{2r-2,4r}\oplus A_{2r-4,4r}\ldots\oplus A_{2,4r}\oplus 1,\\ U &= \mathbb{C}\oplus\underbrace{\mathbb{C}^2\oplus\ldots\oplus\mathbb{C}^2}_{r-1}\oplus\mathbb{C};\end{aligned}$$

(3) $S(2r,-1)$, $r=2,3,\ldots$,

$$\begin{aligned}P_1 &= C_{2r-1,4r}\oplus C_{2r-3,4r}\oplus\ldots\oplus C_{1,4r},\\ P_2 &= D_{2r-1,4r}\oplus D_{2r-3,4r}\oplus\ldots\oplus D_{1,4r},\\ U &= \underbrace{\mathbb{C}^2\oplus\ldots\oplus\mathbb{C}^2}_{r};\end{aligned}$$

$$\begin{aligned}P_3 &= 1\oplus D_{2r-2,4r}\oplus D_{2r-4,4r}\ldots\oplus D_{2,4r}\oplus 0,\\ P_4 &= 0\oplus C_{2r-2,4r}\oplus C_{2r-4,4r}\ldots\oplus C_{2,4r}\oplus 0,\\ U &= \mathbb{C}\oplus\underbrace{\mathbb{C}^2\oplus\ldots\oplus\mathbb{C}^2}_{r-1}\oplus\mathbb{C};\end{aligned}$$

(4) $S(2r+1,1)$, $r=1,2,\ldots$,

$$\begin{aligned}P_1 &= A_{2r,4r+2}\oplus A_{2r-2,4r+2}\oplus\ldots\oplus A_{2,4r+2}\oplus 1,\\ P_2 &= B_{2r,4r+2}\oplus B_{2r-2,4r+2}\oplus\ldots\oplus B_{2,4r+2}\oplus 1,\\ U &= \underbrace{\mathbb{C}^2\oplus\ldots\oplus\mathbb{C}^2}_{r}\oplus\mathbb{C};\end{aligned}$$

$$\begin{aligned}P_3 &= 1\oplus B_{2r-1,4r+2}\oplus B_{2r-3,4r+2}\ldots\oplus B_{1,4r+2},\\ P_4 &= 0\oplus A_{2r-1,4r+2}\oplus A_{2r-3,4r+2}\ldots\oplus A_{1,4r+2},\\ U &= \mathbb{C}\oplus\underbrace{\mathbb{C}^2\oplus\ldots\oplus\mathbb{C}^2}_{r}.\end{aligned}$$

(5) $S(2r+1,-1)$, $r=1,2,\ldots$,

$$\begin{aligned}P_1 &= C_{2r,4r+2}\oplus C_{2r-2,4r+2}\oplus\ldots\oplus C_{2,4r+2}\oplus 0,\\ P_2 &= D_{2r,4r+2}\oplus D_{2r-2,4r+2}\oplus\ldots\oplus D_{2,4r+2}\oplus 0,\\ U &= \underbrace{\mathbb{C}^2\oplus\ldots\oplus\mathbb{C}^2}_{r}\oplus\mathbb{C};\end{aligned}$$

$$\begin{aligned}P_3 &= 0\oplus D_{2r-1,4r+2}\oplus D_{2r-3,4r+2}\ldots\oplus D_{1,4r+2},\\ P_4 &= 1\oplus C_{2r-1,4r+2}\oplus C_{2r-3,4r+2}\ldots\oplus C_{1,4r+2},\\ U &= \mathbb{C}\oplus\underbrace{\mathbb{C}^2\oplus\ldots\oplus\mathbb{C}^2}_{r}.\end{aligned}$$

(6) $S(2r+1,2)$, $r=1,2,\ldots$,

$$\begin{aligned}P_1 &= 1\oplus A_{2r-1,2r+1}\oplus A_{2r-3,2r+1}\oplus\ldots\oplus A_{1,2r+1},\\ P_2 &= 1\oplus B_{2r-1,2r+1}\oplus B_{2r-3,2r+1}\oplus\ldots\oplus B_{1,2r+1},\\ U &= \mathbb{C}\oplus\underbrace{\mathbb{C}^2\oplus\ldots\oplus\mathbb{C}^2}_{r};\end{aligned}$$

$$\begin{aligned}P_3 &= B_{2r,2r+1}\oplus B_{2r-2,2r+1}\ldots\oplus B_{2,2r+1}\oplus 1,\\ P_4 &= A_{2r,2r+1}\oplus A_{2r-2,2r+1}\ldots\oplus A_{2,2r+1}\oplus 1,\\ U &= \underbrace{\mathbb{C}^2\oplus\ldots\oplus\mathbb{C}^2}_{r}\oplus\mathbb{C}.\end{aligned}$$

(7) $S(2r+1,-2)$, $r=1,2,\ldots$,

$$\begin{aligned} P_1 &= 0 \oplus C_{2r-1,2r+1} \oplus C_{2r-3,2r+1} \oplus \ldots \oplus C_{1,2r+1},\\ P_2 &= 0 \oplus D_{2r-1,2r+1} \oplus D_{2r-3,2r+1} \oplus \ldots \oplus D_{1,2r+1},\\ U &= \mathbb{C} \oplus \underbrace{\mathbb{C}^2 \oplus \ldots \oplus \mathbb{C}^2}_{r}; \end{aligned}$$

$$\begin{aligned} P_3 &= D_{2r,2r+1} \oplus D_{2r-2,2r+1} \ldots \oplus D_{2,2r+1} \oplus 0,\\ P_4 &= C_{2r,2r+1} \oplus C_{2r-2,2r+1} \ldots \oplus C_{2,2r+1} \oplus 0,\\ U &= \underbrace{\mathbb{C}^2 \oplus \ldots \oplus \mathbb{C}^2}_{r} \oplus \mathbb{C}. \end{aligned}$$

Hence, irreducible inequivalent representations, Rep $\mathcal{P}_{4,\alpha}$, give rise to the following list of systems of four distinct proper subspaces:

$$\begin{aligned} &S(2,0;a,b,c),\ (a,b,c) \in \Omega,\\ &S(u,\pm 1), \quad u = 3,4,5,\ldots,\\ &S(u,\pm 2), \quad u = 3,5,7,\ldots. \end{aligned} \tag{3}$$

4. Systems of subspaces generated by Rep $\mathcal{P}_{n,com}$, and transitive systems of n subspaces

4.1. **Transitive systems of subspaces generated by** Rep $\mathcal{P}_{1,com}$**,** Rep $\mathcal{P}_{2,com}$**,** Rep $\mathcal{P}_{3,com}$**.** In this section we show that irreducible nonequivalent $*$-representations of the $*$-algebras $\mathcal{P}_{1,com}$ and $\mathcal{P}_{2,com}$ generate all nonisomorphic transitive systems of one and two subspaces of an arbitrary Hilbert space. If $n=3$, irreducible nonequivalent $*$-representations of the $*$-algebra $\mathcal{P}_{3,com}$ give rise to all nonisomorphic transitive systems of three subspaces of a finite dimensional linear space.

Proposition 5. *Irreducible nonequivalent $*$-representations of $\mathcal{P}_{1,com}$ generate all transitive systems of one subspace of a Hilbert space.*

Proof. Using Proposition 2 we get a complete list of transitive systems of one subspaces as follows:

$$S_1 = (\mathbb{C}; 0), \quad S_2 = (\mathbb{C}; \mathbb{C}).$$

By the results of Section 3, we have $\Sigma_1 = \{0,1\}$.

If $\alpha = 0$, a unique up to equivalence irreducible representation of the algebra $\mathcal{P}_{1,0}$ is the representation $P_1 = 0$ on the space $H = \mathbb{C}$ and, consequently, a system of one subspace, induced by this representation, is isomorphic to S_1.

If $\alpha = 1$, there is only one, up to equivalence, irreducible representation of $\mathcal{P}_{1,1}$, $P_1 = \mathbb{C}$, on the space $H = \mathbb{C}$, and so a system of one subspace, corresponding to this representation, is isomorphic to S_2. ☐

Proposition 6. *Irreducible nonequivalent $*$-representations of $\mathcal{P}_{2,com}$ generate all transitive systems of two subspaces of a Hilbert space.*

Proof. By Proposition 3, a complete list of transitive systems of two subspaces has the form

$$\begin{aligned} S_1 &= (\mathbb{C}; 0, 0), \quad S_3 = (\mathbb{C}; 0, \mathbb{C}),\\ S_2 &= (\mathbb{C}; \mathbb{C}, 0), \quad S_4 = (\mathbb{C}; \mathbb{C}, \mathbb{C}). \end{aligned}$$

By Section 3, $\Sigma_2 = \{0,1,2\}$.

For $\alpha = 0$, the algebra $\mathcal{P}_{2,0}$ has, up to equivalence, a unique irreducible representation $P_1 = 0$, $P_2 = 0$ on the space $H = \mathbb{C}$ and, consequently, the system of subspaces generated by this representation is isomorphic to S_1.

If $\alpha = 1$, there are two inequivalent representations of $\mathcal{P}_{2,1}$. The first one is $P_1 = I$, $P_2 = 0$ on the space $H = \mathbb{C}$. A system of two subspaces that corresponds to this representation is isomorphic to S_2. The second representation is given by $P_1 = 0$, $P_2 = I$ on the space $H = \mathbb{C}$. The corresponding system of two subspaces is isomorphic to S_3.

If $\alpha = 2$, the only irreducible representation of the algebra $\mathcal{P}_{2,2}$ is $P_1 = I$, $P_2 = I$ on $H = \mathbb{C}$ and, consequently, the corresponding system of two subspaces is isomorphic to S_4. □

Proposition 7. *Irreducible nonequivalent $*$-representations of $\mathcal{P}_{3,com}$ generate all transitive systems of three subspaces of a finite dimensional linear space.*

Proof. By Proposition 4, a complete list of transitive systems of three subspaces has the following form:

$$\begin{aligned} S_1 &= (\mathbb{C}; 0, 0, 0), \quad S_5 = (\mathbb{C}; 0, \mathbb{C}, \mathbb{C}), \\ S_2 &= (\mathbb{C}; \mathbb{C}, 0, 0), \quad S_6 = (\mathbb{C}; \mathbb{C}, 0, \mathbb{C}), \\ S_3 &= (\mathbb{C}; 0, \mathbb{C}, 0), \quad S_7 = (\mathbb{C}; \mathbb{C}, \mathbb{C}, 0), \\ S_4 &= (\mathbb{C}; 0, 0, \mathbb{C}), \quad S_8 = (\mathbb{C}; \mathbb{C}, \mathbb{C}, \mathbb{C}), \\ S_9 &= (\mathbb{C}^2; \mathbb{C}(1, 0), \mathbb{C}(0, 1), \mathbb{C}(1, 1)). \end{aligned}$$

By the result of Section 3, $\Sigma_3 = \{0, 1, \frac{3}{2}, 2, 3\}$.

If $\alpha = 0$, the only representation of the algebra $\mathcal{P}_{3,0}$, up to equivalence, is $P_1 = 0$, $P_2 = 0$, $P_3 = 0$ on $U = \mathbb{C}$ and, consequently, the system of there subspaces generated by this representation is isomorphic to S_1.

If $\alpha = 1$ there are three inequivalent irreducible representations of the algebra $\mathcal{P}_{3,1}$. The first representation is $P_1 = I$, $P_2 = 0$, $P_3 = 0$ on the space $U = \mathbb{C}$. The system of three subspaces corresponding to this representation is isomorphic to S_2. The second representation is given by $P_1 = 0$, $P_2 = I$, $P_3 = 0$ on the space $U = \mathbb{C}$. The corresponding system of three subspaces is isomorphic to S_3. The third representation is $P_1 = 0$, $P_2 = 0$, $P_3 = I$ on $U = \mathbb{C}$. The corresponding system of three subspaces is isomorphic to S_4.

If $\alpha = 3/2$, there is a unique irreducible representation of the algebra $\mathcal{P}_{3,3/2}$. It is given by

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{pmatrix}$$

on $U = \mathbb{C}^2$. The system of three subspaces, corresponding to this representation, is transitive and is isomorphic to S_9, as follows from the complete list in Proposition 4 for a finite dimensional space.

If $\alpha = 2$, there are three inequivalent irreducible representations of $\mathcal{P}_{3,2}$. For the first representation, $P_1 = 0$, $P_2 = I$, $P_3 = I$ on the space $U = \mathbb{C}$, the system of subspaces is isomorphic to S_5. For the second representation, $P_1 = I$, $P_2 = 0$, $P_3 = I$ on $U = \mathbb{C}$, the corresponding system is isomorphic to S_6. The third representation is given by $P_1 = I$, $P_2 = I$, $P_3 = 0$ on the space $U = \mathbb{C}$. The system of three subspaces, generated by this representation, is isomorphic to S_7.

For $\alpha = 3$, the unique irreducible representation of $\mathcal{P}_{3,3}$, up to equivalence, is $P_1 = I$, $P_2 = I$, $P_3 = I$ on the space $U = \mathbb{C}$ and, hence, the corresponding system of three subspaces is isomorphic to S_8. □

4.2. **Transitive systems of subspaces, generated by** Rep $\mathcal{P}_{4,com}$**.** An important tool used for describing the set Σ_n for $n \geq 4$ and constructing the representations, Rep $\mathcal{P}_{4,\alpha}$, that generate systems of the subspaces $S(u, \pm 1)$, $u = 3, 4, 5, \ldots$, and $S(u, \pm 2)$, $u = 3, 5, 7, \ldots$, in the list (3) are the Coxeter functors, which were constructed in [8], between the categories of $*$-representations of $\mathcal{P}_{n,\alpha}$ for different values of the parameters.

Let us define a functor $\mathcal{T}$: Rep $\mathcal{P}_{n,\alpha} \to$ Rep $\mathcal{P}_{n,n-\alpha}$, which is the first functor constructed in [8]. Let the orthogonal projections P_1, P_2, $\ldots$, P_n be a representation in

Rep $\mathcal{P}_{n,\alpha}$ with the representation space H. Then the orthogonal projections $I-P_1$, $I-P_2$, …, $I-P_n$ constitute a representation in $\mathcal{T}(\text{Rep } \mathcal{P}_{n,\alpha})$ with the same representation space. The second functor in [8], $\mathcal{S} : \text{Rep } \mathcal{P}_{n,\alpha} \to \text{Rep } \mathcal{P}_{n,\frac{\alpha}{\alpha-1}}$, is defined as follows. Again denote by P_1, P_2, …, P_n the orthogonal projections in Rep $\mathcal{P}_{n,\alpha}$ with the representation space H. Let $\Gamma_k : \text{Im } P_k \to H$, $k=\overline{1,n}$, be the natural isometries and $\Gamma = [\Gamma_1, \Gamma_2, \dots, \Gamma_n] : \mathcal{H} = \text{Im } P_1 \oplus \text{Im } P_2 \oplus \dots \text{Im } P_n \to H$. Then the natural isometry $\sqrt{\frac{\alpha-1}{\alpha}}\Delta^*$ from the orthogonal complement in $\hat{H}$ to the subspace Im Γ^* in $\mathcal{H}$ gives the isometries $\Delta_k = \Delta|_{\text{Im } P_k} : \text{Im } P_k \to \hat{H}$, $k=\overline{1,n}$. The orthogonal projections $Q_k = \Delta_k\Delta_k^*$, $k=\overline{1,n}$, on the space $\hat{H}$ give the corresponding representation in $\mathcal{S}(\text{Rep } \mathcal{P}_{n,\alpha})$.

Lemma 1. *The functors $\mathcal{T}$ and $\mathcal{S}$ take representations that define transitive systems into representations that generate transitive systems.*

Proof. Property 2 immediately proves the statement for the functor $\mathcal{T}$.

Consider now the functor $\mathcal{S}$. Let a collection of orthogonal projections P_1, P_2, …, P_n on a Hilbert space H satisfy the condition $\sum_{i=1}^n P_i = \alpha I_H$ for some α, and the corresponding system of subspaces be transitive. Consider the representation $Q_1, Q_2, \dots,$ Q_n, $\sum_{k=1}^n Q_k = \frac{\alpha}{\alpha-1} I_{\hat{H}}$, with the representation space $\hat{H}$, into which the functor $\mathcal{S}$ maps the representation P_1, P_2, …, P_n. Let us prove that the system of subspaces generated by the representation $Q_1, Q_2, \dots, Q_n$, that is, the system $\hat{S} = (\hat{H}; Q_1\hat{H}, Q_2\hat{H}, \dots, Q_n\hat{H})$ is transitive. Let $R \in \text{End}(\hat{S})$. Then

$$Q_kRQ_k = RQ_k, \quad \forall k=\overline{1,n}. \tag{4}$$

Denote by $\hat{C}$ the operator such that $\hat{C} : \hat{H} \to \hat{H}$ and $\hat{C}^* = R$. It follows from (4) that

$$Q_k\hat{C}Q_k = Q_k\hat{C}, \quad \forall k=\overline{1,n}. \tag{5}$$

Consider the operators $C_k : \text{Im } P_k \to \text{Im } P_k$, $(k=\overline{1,n})$, given by

$$C_k = \Delta_k^*\hat{C}\Delta_k, \quad k=\overline{1,n}, \tag{6}$$

and show that the operator $\hat{C}$ can be represented as

$$\hat{C} = \frac{\alpha-1}{\alpha}\sum_{k=1}^n \Delta_kC_k\Delta_k^*. \tag{7}$$

Indeed, using (6) and the definition of Q_k we get

$$\frac{\alpha-1}{\alpha}\sum_{k=1}^n \Delta_kC_k\Delta_k^* = \frac{\alpha-1}{\alpha}\sum_{k=1}^n \Delta_k\Delta_k^*\hat{C}\Delta_k\Delta_k^* = \frac{\alpha-1}{\alpha}\sum_{k=1}^n Q_k\hat{C}Q_k =$$
$$= \frac{\alpha-1}{\alpha}\sum_{k=1}^n Q_k\hat{C} = \frac{\alpha-1}{\alpha}(\sum_{k=1}^n Q_k)\hat{C} = \hat{C}.$$

Now, (5) and (6) yield

$$\Delta_k^*\hat{C} = C_k\Delta_k^*, \quad \forall k=\overline{1,n}, \tag{8}$$

and

$$C_k\Delta_k^* = (\Delta_k^*\hat{C}\Delta_k)\Delta_k^* = \Delta_k^*\hat{C}(\Delta_k\Delta_k^*) = \Delta_k^*\hat{C}Q_k = I_{\text{Im } P_k}\Delta_k^*\hat{C}Q_k =$$
$$(\Delta_k^*\Delta_k)\Delta_k^*\hat{C}Q_k = \Delta_k^*(\Delta_k\Delta_k^*)\hat{C}Q_k = \Delta_k^*Q_k\hat{C}Q_k = \Delta_k^*Q_k\hat{C} =$$
$$= \Delta_k^*(\Delta_k\Delta_k^*)\hat{C} = (\Delta_k^*\Delta_k)\Delta_k^*\hat{C} = \Delta_k^*\hat{C}.$$

Consider the operator

$$C = \frac{1}{\alpha}\sum_{i=1}^{n} \Gamma_i C_i \Gamma_i^*. \tag{9}$$

Using properties of the operators $\{\Gamma_i\}_{i=1}^n$, $\{\Gamma_i^*\}_{i=1}^n$, $\{\Delta_i\}_{i=1}^n$, $\{\Delta_i^*\}_{i=1}^n$,

$$\sum_{i=1}^{n} \Gamma_i \Delta_i^* = 0, \tag{10}$$

$$\Gamma_i^* \Gamma_j = -(\alpha - 1)\Delta_i^* \Delta_j, \quad i \neq j, \tag{11}$$

it follows from [8] that

$$C\Gamma_k = \Gamma_k C_k \quad \forall k = \overline{1, n}, \tag{12}$$

$$C_k = \Gamma_k^* C \Gamma_k \quad \forall k = \overline{1, n}, \tag{13}$$

Indeed,

$$C\Gamma_k = \frac{1}{\alpha}\sum_{i=1}^{n} \Gamma_i C_i \Gamma_i^* \Gamma_k = \frac{1}{\alpha}\Gamma_k C_k + \frac{1}{\alpha}\sum_{\substack{i=1\\ i\neq j}}^{n} \Gamma_i C_i (\Gamma_i^* \Gamma_k) = \frac{1}{\alpha}\Gamma_k C_k -$$
$$\frac{\alpha-1}{\alpha}\sum_{\substack{i=1\\ i\neq j}}^{n} \Gamma_i (C_i \Delta_i^*) \Delta_k = \frac{1}{\alpha}\Gamma_k C_k - \frac{\alpha-1}{\alpha}\sum_{\substack{i=1\\ i\neq j}}^{n} \Gamma_i (\Delta_i^* \hat{C}) \Delta_k = \frac{1}{\alpha}\Gamma_k C_k +$$
$$+ \frac{\alpha-1}{\alpha}\Gamma_k \Delta_k^* \hat{C} \Delta_k = \Gamma_k C_k$$

and

$$\Gamma_k^* C \Gamma_k = \frac{1}{\alpha}\Gamma_k^* (\sum_{i=1}^{n} \Gamma_i C_i \Gamma_i^*) \Gamma_k = \frac{1}{\alpha} C_k + \frac{1}{\alpha}\sum_{\substack{i=1\\ i\neq j}}^{n} \Gamma_k^* \Gamma_i C_i \Gamma_i^* \Gamma_k = \frac{1}{\alpha} C_k +$$
$$+ \frac{(\alpha-1)^2}{\alpha}\sum_{\substack{i=1\\ i\neq j}}^{n} \Delta_k^* \Delta_i C_i \Delta_i^* \Delta_k = \frac{1}{\alpha} C_k + (\alpha-1)\Delta_k^* \hat{C} \Delta_k - \frac{(\alpha-1)^2}{\alpha} C_k = C_k.$$

It follows from (12), (13) that $CP_k = C\Gamma_k \Gamma_k^* = \Gamma_k C_k \Gamma_k^* = \Gamma_k \Gamma_k^* C_k \Gamma_k \Gamma_k^* = P_k C P_k$, which means that $C \in \mathrm{End}(S)$, where $S = (H; P_1 H, P_2 H, \ldots, P_n H)$. Because, by the assumption, the system S is transitive, we have $\mathrm{End}(S) = \mathbb{C} I_H$ and, consequently, C is a scalar operator. By (13), $C_k = \lambda I_{\mathrm{Im}\ P_k}$ $(k = \overline{1, n})$. Now, according to (7), $\hat{C} = \lambda I_{\hat{H}}$ and, correspondingly, R is a scalar operator. This ends the proof. □

Lemma 2. *The mapping*

$$\lambda = \frac{b^2 - a^2 c^2}{(1-a^2)^2} + i\frac{2abc}{(1-a^2)^2} \tag{14}$$

realizes a one-to-one correspondence between the region Ω *and the complex plain with the deleted points* 0 *and* 1*.*

Proof. Consider the points $A(1,0,0)$, $B(0,1,0)$, and $C(0,0,1)$ as in Fig. 1. The point C of the unit sphere, which does not belong to the region Ω, is mapped by (14) into the deleted point 0 of the complex plain (λ), see Fig. 2. The point B of the unite sphere does not belong to the region Ω and is mapped by (4) into the removed point 1. The points of the arc CB, which belong to the region Ω, that is, all the points of the arc except for the points C and B, are mapped by (4) in a one-to-one manner, into points of the interval $(0, 1)$ of the real axis.

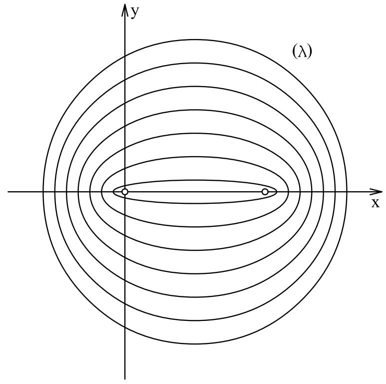

FIGURE 2

Let us fix $0 < a < 1$. Then $\Gamma_a = \{(a,b,c) \in \mathbb{R}^3 | b = \sqrt{1-a^2}\cos\varphi, c = \sqrt{1-a^2}\sin\varphi, \varphi \in (-\pi/2, \pi/2]\} \subset \Omega$. Denote $\tilde{a} = \frac{1}{2}\frac{1+a^2}{1-a^2}$ and $\tilde{b} = \frac{a}{1-a^2}$. For $x = \Re\lambda$ and $y = \Im\lambda$, we get

$$\frac{(x-1/2)^2}{\tilde{a}^2} + \frac{y^2}{\tilde{b}^2} = 1,$$

so that the mapping (4) takes points of the arc Γ_a, in a one-to-one manner, into an ellipse with center in the point $(1/2, 0)$, major semiaxis $\tilde{a}$ and minor semiaxis $\tilde{b}$.

As $a \in (0,1)$ ranges from zero to one, the major semiaxis is a strictly increasing function with values in the interval $(1/2, \infty)$. The minor semiaxis is also a strictly increasing function on the interval $(0,1)$ with values $\tilde{b}$ ranging over the interval $(0,\infty)$. □

Theorem 2. *Irreducible nonequivalent $*$-representations of $\mathcal{P}_{4,com}$ generate all transitive systems of four subspaces of a finite dimensional linear space.*

Proof. By Theorem 1, a complete list of nonisomorphic transitive systems of four distinct proper subspaces of a finite dimensional linear space is the following:

$$\begin{aligned} B(2,0;\lambda), &\quad \lambda \in \mathbb{C}, \lambda \neq 0, 1, \\ B(u, \pm 1), &\quad u = 3, 4, 5, \ldots, \\ B(u, \pm 2), &\quad u = 3, 5, 7, \ldots. \end{aligned}$$

Let us show that the systems $S(2,0;a,b,c)$ are isomorphic to the systems $B(2,0;\lambda)$ for $\lambda = \frac{b^2 - a^2c^2}{(1-a^2)^2} + i\frac{2abc}{(1-a^2)^2}$, up to a rearrangement of the subspaces. Denote $A = 1 + a$ and $B = b - ic$. Then

$$S(2,0;a,b,c) = (\mathbb{C}^2; \operatorname{Im} P_1, \operatorname{Im} P_2, \operatorname{Im} P_3, \operatorname{Im} P_4),$$

where

$$\begin{aligned} \operatorname{Im} P_1 = \mathbb{C}(A, -B), &\quad \operatorname{Im} P_3 = \mathbb{C}(B, A), \\ \operatorname{Im} P_2 = \mathbb{C}(B, -A), &\quad \operatorname{Im} P_4 = \mathbb{C}(A, B). \end{aligned}$$

Denote by $R \in M_2(\mathbb{C})$ a linear transformation from $\mathbb{C}^2$ to $\mathbb{C}^2$, such that $R(\operatorname{Im} P_1) \subset K_1$, $R(\operatorname{Im} P_2) \subset K_2$, $R(\operatorname{Im} P_4) \subset K_3$, $R(\operatorname{Im} P_3) \subset K_4$. The first three conditions give

$$R = \begin{pmatrix} 1 & \frac{B}{A} \\ \frac{A^2+B^2}{2A^2} & \frac{A^2+B^2}{2AB} \end{pmatrix}.$$

The matrix R satisfies the condition $R(\operatorname{Im} P_3) \subset K_4$ for $\lambda = \frac{b^2-a^2c^2}{(1-a^2)^2} + i\frac{2abc}{(1-a^2)^2}$. In virtue of Lemma 2, this gives an isomorphism, up to a rearrangement of the subspaces, between the systems $S(2,0;a,b,c)$, where $(a,b,c) \in \Omega$, and the systems $B(2,0;\lambda)$, where

$\lambda \in \mathbb{C}$, $\lambda \neq 0, 1$, for $\lambda = \frac{b^2 - a^2 c^2}{(1-a^2)^2} + i\frac{2abc}{(1-a^2)^2}$. This shows that systems that correspond to nonequivalent irreducible two-dimensional representations in Rep $\mathcal{P}_{4,2}$ are nonisomorphic and transitive.

By Lemma 1, we obtain transitivity, since the dimensions of the nonisomorphic systems

$$\begin{aligned} &S(u, \pm 1), \quad u = 3, 4, 5, \ldots, \\ &S(u, \pm 2), \quad u = 3, 5, 7, \ldots, \end{aligned}$$

are different. Since the list of transitive systems, given in Section 2, is complete, we have

$$\begin{aligned} &S(u, \pm 1) \cong B(u, \pm 1), \quad u = 3, 4, 5, \ldots, \\ &S(u, \pm 2) \cong B(u, \pm 2), \quad u = 3, 5, 7, \ldots, \end{aligned}$$

up to a rearrangement of the subspaces. $\square$

In confirmation of the hypothesis formulated in Introduction, Lemma 1 allows to conclude that the system of subspaces, generated by irreducible $*$-representations of $\mathcal{P}_{n,com}$ for $n \geq 5$ and $\alpha \in \{\Lambda_n^0, \Lambda_n^1, n - \Lambda_n^1, n - \Lambda_n^0\}$, is transitive.

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