Methods of Functional Analysis and Topology
Vol. 12 (2006), no. 1, pp. 74–81

TWO-WEIGHTED INEQUALITY FOR PARABOLIC SUBLINEAR OPERATORS IN LEBESGUE SPACES

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Abstract. In this paper, the author establishes the boundedness in weighted L_p spaces on $\mathbb{R}^{n+1}$ with a parabolic metric for a large class of sublinear operators generated by parabolic Calderon-Zygmund kernels. The conditions of these theorems are satisfied by many important operators in analysis. Sufficient conditions on weighted functions ω and ω_1 are given so that certain parabolic sublinear operator is bounded from the weighted Lebesgue spaces $L_{p,\omega}(\mathbb{R}^{n+1})$ into $L_{p,\omega_1}(\mathbb{R}^{n+1})$.

In this paper we shall prove the boundedness in weighted L_p spaces on $\mathbb{R}^{n+1}$ with a parabolic metric of some sublinear operators generated by parabolic Calderon-Zygmund kernels. We point out that the condition (2) (see below) was first introduced by Soria and Weiss in [11]. The condition (2) is satisfied by many interesting operators in harmonic analysis, such as the parabolic Calderon–Zygmund operators, parabolic maximal operators, parabolic Hardy–Littlewood maximal operators, and so on. See [11] for details.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x' = (x_1, \dots, x_n)$, $|x'|^2 = \sum_{i=1}^n x_i^2$ and denote by $x = (x', t) = (x_1, \dots, x_n, t)$ a point in $\mathbb{R}^{n+1}$. An almost everywhere positive and locally integrable function $\omega : \mathbb{R}^{n+1} \to R$ will be called a weight. We shall denote by $L_{p,\omega}(\mathbb{R}^{n+1})$ the set of all measurable function f on $\mathbb{R}^{n+1}$ such that the norm

$$\|f\|_{L_{p,\omega}(\mathbb{R}^{n+1})} \equiv \|f\|_{p,\omega;\mathbb{R}^{n+1}} = \left(\int_{\mathbb{R}^{n+1}} |f(x)|^p \omega(x)\, dx\right)^{1/p}, \quad 1 \le p < \infty,$$

is finite.

Let us now endow $\mathbb{R}^{n+1}$ with the following parabolic metric introduced by Fabes and Riviére in [4]:

$$d(x,y) = \rho(x-y), \quad \text{where} \quad \rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + 4t^2}}{2}}. \tag{1}$$

A ball with respect to the metric d centered at zero and of radius r is the ellipsoid

$$\mathcal{E}_r(0) = \left\{ x \in R^{n+1} : \ \frac{|x'|^2}{r^2} + \frac{t^2}{r^4} < 1 \right\}.$$

Obviously, the unit sphere with respect to this metric coincides with the unit sphere in $\mathbb{R}^{n+1}$, i.e.,

$$\partial\mathcal{E}_1(0) \equiv \Sigma_{n+1} = \left\{ x \in R^{n+1} : \ |x| = \Big(\sum_{i=1}^n x_i^2 + t^2\Big)^{1/2} = 1 \right\}.$$

Let

$$\widetilde{d}(x,y) = \widetilde{\rho}(x-y), \quad \widetilde{\rho}(x) = \max(|x'|, |t|^{1/2}),$$

2000 *Mathematics Subject Classification.* Primary 42B25.
Key words and phrases. Two-weighted inequality, weighted Lebesgue space, singular integral, parabolic Calderon-Zygmund kernel.

I be a *parabolic cylinder* centered at some point x of radius r, that is, $I \equiv I_r(x) = \{y = (y', \tau) \in \mathbb{R}^{n+1} \colon |x' - y'| < r, |t - \tau| < r^2\}$. It is easy to see that for any ellipsoid $\mathcal{E}_r$ there exist cylinders $\underline{I}$ and $\overline{I}$ with measures comparable with r^{n+2} and such that $\underline{I} \subset \mathcal{E}_r \subset \overline{I}$. Obviously, this implies an equivalence of both metrics and the topologies induced by them. Later we shall use this equivalence without making reference to, except if required.

It is worth noting that $\rho(x)$ has been employed in the study of singular integral operators with Calderón-Zygmund kernels of mixed homogeneity (see [4]).

Definition 1. *A function K defined on $\mathbb{R}^{n+1} \setminus \{0\}$ is said to be a parabolic Calderon-Zygmund (PCZ) kernel in the space $\mathbb{R}^{n+1}$ if*

i) $K \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$;

ii) $K(rx', r^2 t) = r^{-(n+2)} K(x', t)$ for each $r > 0$, $x = (x', t) \in \mathbb{R}^{n+1} \setminus \{0\}$;

iii) $\int_{\Sigma_{n+1}} K(x) d\sigma = 0$, where $d\sigma$ is the element of area of the sphere Σ_{n+1}.

First, we establish the boundedness in weighted L_p spaces for a large class of sublinear operators.

Theorem 2. *Let $p \in (1, \infty)$ and let T be a sublinear operator bounded from $L_p(\mathbb{R}^{n+1})$ to $L_p(\mathbb{R}^{n+1})$ such that, for any $f \in L_1(\mathbb{R}^{n+1})$ with compact support and $x \notin \operatorname{supp} f$,*

$$|Tf(x)| \le c \int_{\mathbb{R}^{n+1}} \frac{|f(y)|}{\rho^{n+2}(x-y)}\, dy, \tag{2}$$

where c is independent of f and x.

Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^{n+1}$ and the following three conditions be satisfied:

(a) there exist $b > 0$ such that

$$\sup_{\rho(x)/4 < \rho(y) \le 4\rho(x)} \omega_1(y) \le b\, \omega(x) \quad for \quad a.e. \quad x \in \mathbb{R}^{n+1},$$

$$(b) \qquad \mathcal{A} \equiv \sup_{r>0} \left(\int_{\rho(x) > 2r} \omega_1(x) \rho(x)^{-(n+2)p} dx \right) \left(\int_{\rho(x) < r} \omega^{1-p'}(x)\, dx \right)^{p-1} < \infty,$$

$$(c) \qquad \mathcal{B} \equiv \sup_{r>0} \left(\int_{\rho(x) < r} \omega_1(x)\, dx \right) \left(\int_{\rho(x) > 2r} \omega^{1-p'}(x) \rho(x)^{-(n+2)p'} dx \right)^{p-1} < \infty.$$

Then there exists a constant c_1 such that for all $f \in L_{p,\omega}(\mathbb{R}^{n+1})$

$$\int_{\mathbb{R}^{n+1}} |Tf(x)|^p \omega_1(x)\, dx \le c_1 \int_{\mathbb{R}^{n+1}} |f(x)|^p \omega(x)\, dx. \tag{3}$$

Moreover, condition (a) can be replaced by the condition

(a_1) there exist $b > 0$ such that

$$\omega_1(x) \left(\sup_{\rho(x)/4 \le \rho(y) \le 4\rho(x)} \frac{1}{\omega(y)} \right) \le b \quad for \quad a.e. \quad x \in \mathbb{R}^{n+1}.$$

Proof. For $k \in Z$ we define $E_k = \{x \in \mathbb{R}^{n+1} : 2^k < \rho(x) \le 2^{k+1}\}$, $E_{k,1} = \{x \in \mathbb{R}^{n+1} : \rho(x) \le 2^{k-1}\}$, $E_{k,2} = \{x \in \mathbb{R}^{n+1} : 2^{k-1} < \rho(x) \le 2^{k+2}\}$, $E_{k,3} = \{x \in \mathbb{R}^{n+1} : \rho(x) > 2^{k+2}\}$. Then $E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1}$ and the multiplicity of the covering $\{E_{k,2}\}_{k \in Z}$ is equal to 3.

Given $f \in L_{p,\omega}(\mathbb{R}^{n+1})$, we write

$$
\begin{aligned}
|Tf(x)| &= \sum_{k\in Z} |Tf(x)|\chi_{E_k}(x) \le \sum_{k\in Z} |Tf_{k,1}(x)|\,\chi_{E_k}(x) \\
&\quad + \sum_{k\in Z} |Tf_{k,2}(x)|\,\chi_{E_k}(x) + \sum_{k\in Z} |Tf_{k,3}(x)|\,\chi_{E_k}(x) \\
&\equiv T_1 f(x) + T_2 f(x) + T_3 f(x),
\end{aligned} \tag{4}
$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i} = f\chi_{E_{k,i}}$, $i = 1, 2, 3$.

First we estimate $\|T_1 f\|_{L_{p,\omega_1}}$. Note that for $x \in E_k$, $y \in E_{k,1}$ we have $\rho(y) \le 2^{k-1} \le \rho(x)/2$. Moreover, $E_k \cap \operatorname{supp} f_{k,1} = \emptyset$ and $\rho(x-y) \ge \rho(x)/2$. Hence by (2)

$$
\begin{aligned}
T_1 f(x) &\le c \sum_{k\in Z} \left(\int_{\mathbb{R}^n} \frac{|f_{k,1}(y)|}{\rho^{n+2}(x-y)}\, dy \right) \chi_{E_k}(x) \\
&\le c \int_{\rho(y)\le\rho(x)/2} \rho(x-y)^{-n-2} |f(y)|\, dy \le 2^{n+2} c \rho(x)^{-n-2} \int_{\rho(y)\le\rho(x)/2} |f(y)|\, dy
\end{aligned}
$$

for any $x \in E_k$. Hence we have

$$
\int_{\mathbb{R}^{n+1}} |T_1 f(x)|^p \omega_1(x)\, dx \le 2^{n+2} c \int_{\mathbb{R}^{n+1}} \left(\int_{\rho(y)<\rho(x)/2} |f(y)|\, dy \right)^p \rho(x)^{-(n+2)p} \omega_1(x)\, dx.
$$

Since $\mathcal{A} < \infty$, the Hardy inequality

$$
\int_{\mathbb{R}^{n+1}} \omega_1(x) \rho(x)^{-(n+2)p} \left(\int_{\rho(y)<\rho(x)/2} |f(y)|\, dy \right)^p dx \le C \int_{\mathbb{R}^{n+1}} |f(x)|^p \omega(x)\, dx
$$

holds and $C \le c'\mathcal{A}$ where c' depends on n and p. In fact the condition $\mathcal{A} < \infty$ is necessary and sufficient for the validity of this inequality (see [1], [7]). Hence, we obtain

$$
\int_{\mathbb{R}^{n+1}} |T_1 f(x)|^p \omega_1(x)\, dx \le c_2 \int_{\mathbb{R}^{n+1}} |f(x)|^p \omega(x)\, dx, \tag{5}
$$

where $c_2 > 0$ is independent of f.

Next we estimate $\|T_3 f\|_{L_{p,\omega_1}}$. As is easy to verify, for $x \in E_k$, $y \in E_{k,3}$ we have $\rho(y) > 2\rho(x)$ and $\rho(x-y) \ge \rho(y)/2$. Since $E_k \cap \operatorname{supp} f_{k,3} = \emptyset$, for $x \in E_k$ by (2) we obtain

$$
T_3 f(x) \le c \int_{\rho(y)>2\rho(x)} \frac{|f(y)|}{\rho(x-y)^{n+2}}\, dy \le 2^{n+2} c \int_{\rho(y)>2\rho(x)} |f(y)| \rho(y)^{-n-2} dy.
$$

Hence we have

$$
\int_{\mathbb{R}^{n+1}} |T_3 f(x)|^p \omega_1(x)\, dx \le 2^{n+2} c \int_{\mathbb{R}^{n+1}} \left(\int_{\rho(y)>2\rho(x)} |f(y)| \rho(y)^{-n-2} dy \right)^p \omega_1(x)\, dx.
$$

Since $\mathcal{B} < \infty$, the Hardy inequality

$$
\int_{\mathbb{R}^{n+1}} \omega_1(x) \left(\int_{\rho(y)<\rho(x)/2} |f(y)| \rho(y)^{-n-2} dy \right)^p dx \le C \int_{\mathbb{R}^{n+1}} |f(x)|^p \omega(x)\, dx
$$

holds and $C \le c'\mathcal{B}$, where c' depends on n and p. In fact the condition $\mathcal{B} < \infty$ is necessary and sufficient for the validity of this inequality (see [1], [7]). Hence, we obtain

$$
\int_{\mathbb{R}^{n+1}} |T_3 f(x)|^p \omega_1(x)\, dx \le c_3 \int_{\mathbb{R}^{n+1}} |f(x)|^p \omega(x)\, dx, \tag{6}
$$

where $c_3 > 0$ is independent of f.

Finally, we estimate $\|T_2 f\|_{L_{p,\omega_1}}$. By the $L_p(\mathbb{R}^{n+1})$ boundedness of T and condition (a) we have

$$\int_{\mathbb{R}^{n+1}} |T_2 f(x)|^p \omega_1(x)\,dx = \int_{\mathbb{R}^{n+1}} \left(\sum_{k\in Z} |Tf_{k,2}(x)|\,\chi_{E_k}(x)\right)^p \omega_1(x)\,dx$$
$$= \int_{\mathbb{R}^{n+1}} \left(\sum_{k\in Z} |Tf_{k,2}(x)|^p\,\chi_{E_k}(x)\right) \omega_1(x)\,dx = \sum_{k\in Z} \int_{E_k} |Tf_{k,2}(x)|^p\,\omega_1(x)\,dx$$
$$\le \sum_{k\in Z} \sup_{x\in E_k} \omega_1(x) \int_{\mathbb{R}^{n+1}} |Tf_{k,2}(x)|^p\,dx \le \|T\|^p \sum_{k\in Z} \sup_{x\in E_k} \omega_1(x) \int_{\mathbb{R}^{n+1}} |f_{k,2}(x)|^p\,dx$$
$$= \|T\|^p \sum_{k\in Z} \sup_{y\in E_k} \omega_1(y) \int_{E_{k,2}} |f(x)|^p dx,$$

where $\|T\| \equiv \|T\|_{L_p(\mathbb{R}^{n+1})\to L_p(\mathbb{R}^{n+1})}$. Since, for $x \in E_{k,2}$, $2^{k-1} < \rho(x) \le 2^{k+2}$, we have by condition (a)

$$\sup_{y\in E_k} \omega_1(y) = \sup_{2^{k-1}<\rho(y)\le 2^{k+2}} \omega_1(y) \le \sup_{\rho(x)/4<\rho(y)\le 4\rho(x)} \omega_1(y) \le b\omega(x)$$

for almost all $x \in E_{k,2}$. Therefore

$$\int_{\mathbb{R}^{n+1}} |T_2 f(x)|^p \omega_1(x)\,dx \le \|T\|^p b \sum_{k\in Z} \int_{E_{k,2}} |f(x)|^p \omega(x)\,dx \le c_4 \int_{\mathbb{R}^{n+1}} |f(x)|^p \omega(x)\,dx, \tag{7}$$

where $c_4 = 3\|T\|^p b$, since the multiplicity of covering $\{E_{k,2}\}_{k\in Z}$ is equal to 3.

Inequalities (4), (5), (6), (7) imply (3) which completes the proof. □

Let K be a parabolic Calderon–Zygmund kernel and T the corresponding integral operator

$$Tf(x) = p.v. \int_{\mathbb{R}^{n+1}} K(x-y) f(y)\,dy.$$

Then T satisfies the condition (2). See [3] for details. Thus, we have

Corollary 3. *Let $p \in (1,\infty)$, K be a parabolic Calderon–Zygmund kernel and T be the corresponding integral operator. Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^{n+1}$ and conditions (a), (b), (c) be satisfied. Then inequality (3) is valid.*

Note that Corollary 3 for singular integral operators with Calderon-Zygmund kernels was proved in [8] and for singular integral operators, defined on homogeneous groups, in [10], [6] (see also [5]).

Theorem 4. *Let $p \in (1,\infty)$, T be a sublinear operator satisfying (2). Moreover, let $\omega(t)$, $\omega_1(t)$ be weight functions on $\mathbb{R}$ and the following three conditions be satisfied:*

(a') *there exist $b > 0$ such that*

$$\sup_{|t|/4<|\tau|\le 4|t|} \omega_1(\tau) \le b\,\omega(t) \quad for \quad a.e. \quad t \in \mathbb{R},$$

(b')

$$\mathcal{A}' \equiv \sup_{\tau} \left(\int_{|t|>2|\tau|} \omega_1(t)|t|^{-p} d\tau\right) \left(\int_{|t|<|\tau|} \omega^{1-p'}(t)\,dt\right)^{p-1} < \infty,$$

(c')

$$\mathcal{B}' \equiv \sup_{\tau} \left(\int_{|t|<|\tau|} \omega_1(t)\,dt \right) \left(\int_{|t|>2|\tau|} \omega^{1-p'}(t)|t|^{-p'}\,dt \right)^{p-1} < \infty.$$

Then there exists a constant c_1, independent of f, such that for all $f \in L_{p,\omega}(\mathbb{R}^{n+1})$

$$\int_{\mathbb{R}^{n+1}} |Tf(x)|^p \omega_1(t)\,dx'dt \le c_1 \int_{\mathbb{R}^{n+1}} |f(x)|^p \omega(t)\,dx'dt. \tag{8}$$

Moreover, condition (a') can be replaced by the condition
(a'_1) *there exist $b > 0$ such that*

$$\omega_1(t) \left(\sup_{|t|/4 \le |\tau| \le 4|t|} \frac{1}{\omega(t)} \right) \le b \quad for \quad a.e. \quad t \in \mathbb{R}.$$

Proof. For $k \in Z$ we define $F_k = \{x = (x',t) \in R^{n+1} : 2^k < |t| \le 2^{k+1}\}$, $F_{k,1} = \{x = (x',t) \in R^{n+1} : |t| \le 2^{k-1}\}$, $F_{k,2} = \{x = (x',t) \in R^{n+1} : 2^{k-1} < |t| \le 2^{k+2}\}$, $F_{k,3} = \{x = (x',t) \in R^{n+1} : |t| > 2^{k+2}\}$. Then $F_{k,2} = F_{k-1} \cup F_k \cup F_{k+1}$ and the multiplicity of the covering $\{F_{k,2}\}_{k\in Z}$ is equal to 3.

Given $f \in L_{p,\omega}(\mathbb{R}^{n+1})$, we write

$$\begin{aligned} |Tf(x)| &= \sum_{k\in Z} |Tf(x)|\chi_{F_k}(x) \le \sum_{k\in Z} |Tf_{k,1}(x)|\,\chi_{F_k}(x) \\ &+ \sum_{k\in Z} |Tf_{k,2}(x)|\,\chi_{F_k}(x) + \sum_{k\in Z} |Tf_{k,3}(x)|\,\chi_{F_k}(x) \\ &\equiv T_1f(x) + T_2f(x) + T_3f(x), \end{aligned} \tag{9}$$

where χ_{F_k} is the characteristic function of the set F_k, $f_{k,i} = f\chi_{F_{k,i}}$, $i = 1,2,3$. We shall estimate $\|T_1f\|_{L_{p,\omega_1}}$. Note that for $x = (x',t) \in F_k$, $y = (y',\tau) \in F_{k,1}$ we have $|\tau| \le 2^{k-1} \le |t|/2$. Moreover, $F_k \cap \operatorname{supp} f_{k,1} = \emptyset$ and $|t-\tau| \ge |t|/2$. Hence by (2)

$$\begin{aligned} T_1f(x) &\le c \sum_{k\in Z} \left(\int_{\mathbb{R}^{n+1}} \frac{|f_{k,1}(y)|}{\rho(x-y)^{n+2}}\,dy \right) \chi_{F_k}(t) \\ &\le c \int_{\mathbb{R}^n} \int_{|\tau|<|t|/2} \frac{|f(y)|}{\rho(x-y)^{n+2}}\,dy \le c_5 \int_{\mathbb{R}^n} \int_{|\tau|<|t|/2} \frac{|f(y)|}{\left(|x'-y'| + |t|^{1/2}\right)^{n+2}}\,dy'd\tau \end{aligned}$$

for any $x \in F_k$. Using this last inequality we have

$$\begin{aligned} &\int_{\mathbb{R}^{n+1}} |T_1f(x)|^p \omega_1(t)\,dx'dt \\ &\quad \le c_5 \left\{ \int_{\mathbb{R}^{n+1}} \left(\int_{\mathbb{R}^n} \int_{|\tau|<|t|/2} \frac{|f(y)|}{\left(|x'-y'| + |t|^{1/2}\right)^{n+2}}\,dy'd\tau \right)^p \omega_1(t)\,dx \right\}^{1/p}. \end{aligned}$$

For $x = (x',t) \in \mathbb{R}^{n+1}$ let

$$\begin{aligned} I(t) &= \int_{\mathbb{R}^n} \left(\int_{|\tau|<|t|/2} \int_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'| + |t|^{1/2})^{n+2}}\,dy \right)^p dx' \\ &= \int_{\mathbb{R}^n} \left(\int_{|\tau|<|t|/2} \left(\int_{\mathbb{R}^n} \frac{|f(y',\tau)|}{(|x'-y'| + |t|^{1/2})^{n+2}}\,dy' \right) d\tau \right)^p dx'. \end{aligned}$$

Using the Minkowski and Young inequalities we obtain

$$
\begin{aligned}
I(t) &\le \left[\int_{|\tau|<|t|/2}\left(\int_{\mathbb{R}^n}|f(y',\tau)|^p dy'\right)^{1/p}\left(\int_{\mathbb{R}^n}\frac{dy'}{(|y'|+|t|^{1/2})^{n+2}}\right)d\tau\right]^p \\
&= \left(\int_{|\tau|<|t|/2}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}d\tau\right)^p\left(\int_{\mathbb{R}^n}\frac{dy'}{(|y'|+|t|^{1/2})^{n+2}}\right)^p \\
&= \frac{c_6}{|t|^p}\left(\int_{|\tau|<|t|/2}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}d\tau\right)^p\left(\int_{\mathbb{R}^n}\frac{dy'}{(|y'|+1)^{n+2}}\right)^p \\
&= \frac{c_7}{|t|^p}\left(\int_{|\tau|<|t|/2}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}d\tau\right)^p.
\end{aligned}
$$

Integrating over $\mathbb{R}$ we get

$$
\int_{\mathbb{R}^{n+1}}|T_1f(x)|^p\omega_1(t)\,dx'dt \le c_8\int_{\mathbb{R}}\omega_1(t)|t|^{-p}\left(\int_{|\tau|<|t|/2}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}d\tau\right)^p dt.
$$

Since $\mathcal{A}'<\infty$, the Hardy inequality

$$
\int_{\mathbb{R}}\omega_1(t)|t|^{-p}\left(\int_{|\tau|<|t|/2}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}d\tau\right)^p \le C\int_{\mathbb{R}}\|f(\cdot,\tau)\|^p_{p,\mathbb{R}^n}\omega(\tau)\,d\tau
$$

holds and $C\le c'\mathcal{A}'$ where c' depends only on p. In fact the condition $\mathcal{A}'<\infty$ is necessary and sufficient for the validity of this inequality, (see [2], [9]). Hence, we obtain

$$
\int_{\mathbb{R}^{n+1}}|T_1f(x)|^p\omega_1(t)\,dx'dt \le c_9\int_{\mathbb{R}}\|f(\cdot,\tau)\|^p_{p,\mathbb{R}^n}\omega(\tau)\,d\tau = c_9\|f\|^p_{L_{p,\omega}(\mathbb{R}^{n+1})}. \tag{10}
$$

Let us estimate $\|T_3f\|_{L_{p,\omega_1}}$. As is easy to verify, for $x\in F_k$, $y\in F_{k,3}$ we have $|\tau|>2|t|$ and $|t-\tau|\ge|\tau|/2$. For $x\in F_k$ we obtain

$$
T_3f(x)\le c_5\int_{\mathbb{R}^n}\int_{|\tau|>2|t|}\frac{|f(y)|}{\left(|x'-y'|+|\tau|^{1/2}\right)^{n+2}}\,dy'd\tau.
$$

Using this last inequality we have

$$
\|T_3f\|_{L_{p,\omega_1}(\mathbb{R}^{n+1})} \le c_5\left\{\int_{\mathbb{R}^{n+1}}\left(\int_{\mathbb{R}^n}\int_{|\tau|>2|t|}\frac{|f(y)|}{\left(|x'-y'|+|\tau|^{1/2}\right)^{n+2}}\,dy'd\tau\right)^p\omega(t)\,dx\right\}^{1/p}.
$$

For $x=(x',t)\in\mathbb{R}^{n+1}$ let

$$
\begin{aligned}
I_1(t) &= \int_{\mathbb{R}^n}\left(\int_{|\tau|>2|t|}\int_{\mathbb{R}^n}\frac{|f(y',\tau)|}{(|x'-y'|+|\tau|^{1/2})^{n+2}}\,dy\right)^p dx' \\
&= \int_{\mathbb{R}^n}\left(\int_{|\tau|>2|t|}\left(\int_{\mathbb{R}^n}\frac{|f(y',\tau)|}{(|x'-y'|+|\tau|^{1/2})^{n+2}}\,dy'\right)d\tau\right)^p dx'.
\end{aligned}
$$

Using the Minkowski and Young inequalities we obtain

$$\begin{aligned} I_1(t) &\le \left[\int_{|\tau|>2|t|}\left(\int_{\mathbb{R}^n}|f(y',\tau)|^p dy'\right)^{1/p}\left(\int_{\mathbb{R}^n}\frac{dy'}{(|y'|+|\tau|^{1/2})^{n+2}}\right)d\tau\right]^p \\ &= \left(\int_{|\tau|>2|t|}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}d\tau\right)^p\left(\int_{\mathbb{R}^n}\frac{dy'}{(|y'|+|\tau|^{1/2})^{n+2}}\right)^p \\ &= c_6\left(\int_{|\tau|>2|t|}|\tau|^{-p}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}d\tau\right)^p\left(\int_{\mathbb{R}^n}\frac{dy'}{(|y'|+1)^{n+2}}\right)^p \\ &= c_7\left(\int_{|\tau|>2|t|}|\tau|^{-p}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}d\tau\right)^p. \end{aligned}$$

Integrating over $\mathbb{R}$ we get

$$\int_{\mathbb{R}^{n+1}}|T_3f(x)|^p\omega_1(t)\,dx'dt \le c_8\int_{\mathbb{R}}\omega_1(t)\left(\int_{|\tau|>2|t|}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}|\tau|^{-p}\,d\tau\right)^p dt.$$

Since $\mathcal{B}' < \infty$, the Hardy inequality

$$\int_{\mathbb{R}}\omega_1(t)\left(\int_{|\tau|>2|t|}\|f(\cdot,\tau)\|_{p,\mathbb{R}^n}|\tau|^{-p}\,d\tau\right)^p \le C\int_{\mathbb{R}}\|f(\cdot,\tau)\|^p_{p,\mathbb{R}^n}\omega(\tau)\,d\tau$$

holds and $C \le c'\mathcal{B}'$ where c' depends on n and p. In fact the condition $\mathcal{B}' < \infty$ is necessary and sufficient for the validity of this inequality (see [2], [9]). Hence, we obtain

$$\|T_3f\|_{L_{p,\omega_1}(\mathbb{R}^{n+1})} \le c_9\left(\int_{\mathbb{R}}\|f(\cdot,\tau)\|^p_{p,\mathbb{R}^n}\omega(\tau)\,d\tau\right)^{1/p} = c_9\|f\|^p_{L_{p,\omega}(\mathbb{R}^{n+1})}. \tag{11}$$

Finally, we estimate $\|T_2f\|_{L_{p,\omega_1}}$. By the $L_p(\mathbb{R}^{n+1})$ boundedness of T we have

$$\begin{aligned} \int_{\mathbb{R}^{n+1}}|T_2f(x)|^p\omega_1(t)\,dx &= \int_{\mathbb{R}^{n+1}}\left(\sum_{k\in Z}|Tf_{k,2}(x)|\,\chi_{F_k}(t)\right)^p\omega_1(t)\,dx \\ &= \int_{\mathbb{R}^{n+1}}\left(\sum_{k\in Z}|Tf_{k,2}(x)|^p\,\chi_{F_k}(t)\right)\omega_1(t)\,dx = \sum_{k\in Z}\int_{F_k}|Tf_{k,2}(x)|^p\,\omega_1(t)\,dx \\ &\le \sum_{k\in Z}\sup_{x\in F_k}\omega_1(t)\int_{\mathbb{R}^{n+1}}|Tf_{k,2}(x)|^p\,dx \le \|T\|^p\sum_{k\in Z}\sup_{x\in F_k}\omega_1(t)\int_{\mathbb{R}^{n+1}}|f_{k,2}(x)|^p\,dx \\ &= \|T\|^p\sum_{k\in Z}\sup_{y\in F_k}\omega_1(\tau)\int_{F_{k,2}}|f(x)|^p dx, \end{aligned}$$

where $\|T\| \equiv \|T\|_{L_p(\mathbb{R}^{n+1})\to L_p(\mathbb{R}^{n+1})}$. Since, for $x \in F_{k,2}$, $2^{k-1} < |t| \le 2^{k+2}$, we have by condition (a')

$$\sup_{y\in F_k}\omega_1(\tau) = \sup_{2^{k-1}<|\tau|\le 2^{k+2}}\omega_1(\tau) \le \sup_{|t|/4<|\tau|\le 4|t|}\omega_1(\tau) \le b\omega(t)$$

for almost all $x \in F_{k,2}$. Therefore

$$\begin{aligned} \int_{\mathbb{R}^{n+1}}|T_2f(x)|^p\omega_1(t)\,dx &\le \|T\|^p b\sum_{k\in Z}\int_{F_{k,2}}|f(x)|^p\omega(t)\,dx \\ &\le c_{10}\int_{\mathbb{R}^{n+1}}|f(x)|^p\omega(t)\,dx, \end{aligned} \tag{12}$$

where $c_{10} = 3\|T\|^p b$, since the multiplicity of covering $\{F_{k,2}\}_{k\in Z}$ is equal to 3.

Inequalities (9), (10), (11), (12) imply (8) which completes the proof. □

Corollary 5. *Let $p \in (1, \infty)$, K be a parabolic Calderon–Zygmund kernel and T be the corresponding integral operator. Moreover, let $\omega(t)$,$\omega_1(t)$ be weight functions on $\mathbb{R}^{n+1}$ and conditions (a'), (b'), (c') be satisfied. Then inequality* (8) *is valid.*

Note that, two-weighted inequalities (3) for singular integrals were obtained in [12], [13], [14], [5], [15] and etc.

Acknowledgments. The author would like to express his thanks to Prof. V. I. Burenkov for many helpful discussions about this subject.

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Received 26/10/2004