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LAGRANGIAN PAIRS IN HILBERT SPACES

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Abstract. Weakly Lagrangian pairs and Lagrangian pairs in a pair of Hilbert spaces $(\mathfrak{H}_1, \mathfrak{H}_2)$ are defined. The weakly Lagrangian pair and Lagrangian pair extensions in $(\mathfrak{H}_1, \mathfrak{H}_2)$ of a given weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$ are characterized and those extensions which are operators are identified. A description of all Lagrangian pair extensions in a larger pair of Hilbert spaces $(\widetilde{\mathfrak{H}}_1, \widetilde{\mathfrak{H}}_2)$ of a given weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$ is also given.

1. Introduction

Let A_1 and A_2 be two (not necessarily densely defined) operators in a Hilbert space $\mathfrak{H}$ and assume that U is a unitary operator in $\mathfrak{H}$ which commutes with A_1 and that A_2 is a restriction of the (linear) relation $U^*A_1^*$. Using the language of (linear) relations this assumption can be written as

$$UA_1 = A_1U, \quad A_2 \subset U^*A_1^*,$$

where the operator and its graph are identified. Let V_1 and V_2 be two operators from (the graph of) A_1 to (the graph of) A_2^* and from (the graph of) A_2 to (the graph of) A_1^*, which are defined as follows

$$V_1(x, A_1x) = (x, UA_1x), \quad V_2(y, A_2y) = (y, UA_2y), \quad x \in \operatorname{dom} A_1, \quad y \in \operatorname{dom} A_2,$$

so that V_1 and V_2 are isometric operators.

The above situation can be described in a general scheme using the concept of *weakly Lagrangian pair in a pair of Hilbert spaces.* This concept is the subject of the paper and is introduced as follows. Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be two Hilbert spaces and let A_1 and A_2 be two relations from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ and from $\mathfrak{H}_2$ to $\mathfrak{H}_1$, respectively. The pair (A_1, A_2) is said to be a *weakly dual pair* in the pair of Hilbert spaces $(\mathfrak{H}_1, \mathfrak{H}_2)$, if the following inclusions hold

$$\operatorname{dom} A_1 \subseteq \operatorname{dom} A_2^* \quad \text{and} \quad \operatorname{dom} A_2 \subseteq \operatorname{dom} A_1^*. \tag{1.1}$$

The pair (A_1, A_2) is said to be a *dual pair* if $A_1 \subseteq A_2^*$, cf. [13, 14]. Furthermore, a weakly dual pair is said to be a *weakly Lagrangian pair* if there are two isometries, $V_1 : A_1 \to A_2^*$ and $V_2 : A_2 \to A_1^*$ of the form

$$V_1(x_1, x_2) = (x_1, x_2'), \quad (x_1, x_2) \in A_1, \quad \|x_2\|_2 = \|x_2'\|_2,$$

and

$$V_2(y_2, y_1) = (y_2, y_1'), \quad (y_2, y_1) \in A_2, \quad \|y_1\|_1 = \|y_1'\|_1.$$

By definition, a *Lagrangian pair* is a weakly Lagrangian pair with

$$\operatorname{dom} A_1 = \operatorname{dom} A_2^* \quad \text{and} \quad \operatorname{dom} A_2 = \operatorname{dom} A_1^*, \tag{1.2}$$

and whose isometries V_1 and V_2 are from A_1 onto A_2^* and from A_2 onto A_1^*, so that

$$V_1A_1 = A_2^* \quad \text{and} \quad V_2A_2 = A_1^*.$$

When $\mathfrak{H}_1 = \mathfrak{H}_2$ and $A_1 = A_2$, a weakly Lagrangian pair becomes a *formally normal relation*, a notion which has been introduced by E. A. Coddington in [5]. Clearly, dual

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pairs and formally normal subspaces are particular cases of weakly Lagrangian pairs. Therefore, a densely defined formally normal operator N in a Hilbert space $\mathfrak{H}$ gives the weakly Lagrangian pair (N, N) in $(\mathfrak{H}, \mathfrak{H})$ with the isometries $V_1 = V_2 =: V$ given by

$$V(x, Nx) = (x, N^*x), \quad x \in \operatorname{dom} N,$$

cf. [5]. Furthermore, a normal operator $\widetilde{N}$ in a Hilbert space $\mathfrak{H}$ determines the Lagrangian pair $\left(\widetilde{N}, \widetilde{N}\right)$, cf. [3, 5].

Recently, the theory of dual pairs in Hilbert spaces has been developed by M. M. Malamud and V. I. Mogilevskii (see [13, 14]). Their treatment is mainly based on the concepts of boundary triplets and the Krein formula. In the present paper the duality of two relations in Hilbert spaces is reconsidered in order to study both dual pairs and formally normal subspaces. Some algebraic descriptions of weakly Lagrangian pairs and Lagrangian pairs extensions of a given weakly Lagrangian pair are proposed. The main results of this paper are parallel with the ones in [5] and complete the theory of dual pair of relations proposed by Malamud and Mogilevskii.

The so-called Dirac structure on a linear space has been introduced by T. J. Courant in [6], and both geometrical and functional analysis approaches on finite-dimensional differentiable manifolds or on Hilbert spaces have been developed, cf. [7, 16, 17, 18]. In fact a Dirac structure is a relation in a Hilbert space $\mathfrak{H}$ such that $A = -A^*$, so that it can be viewed as a particular Lagrangian pair. An approach to joint semi-normality based on the theory of Dirac and Laplace operators on a Dirac vector bundle has been developed in [15]. Using the concepts introduced in this paper, the theories proposed in the above mentioned references might be extended in order to develop new theories in geometry, physics and engineering.

The organization of the paper is as follows. In Section 2 some general facts concerning relations from a Hilbert space $\mathfrak{H}_1$ to a Hilbert space $\mathfrak{H}_2$ are presented. In Section 3 the notions of weakly Lagrangian pair and Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$ are analyzed. Section 4 gives a complete description of all weakly Lagrangian pairs and Lagrangian pairs extensions in $(\mathfrak{H}_1, \mathfrak{H}_2)$ of a weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$. Those extensions which are (graphs of) operators are explicitly characterized. This result contains the description of all Lagrangian pair extensions of a given densely defined weakly Lagrangian pair. Section 5 is devoted to a study of the weakly Lagrangian pair extensions in larger Hilbert spaces. In particular, Proposition 5.3 shows that a weakly Lagrangian pair need not have Lagrangian pair extensions in any larger pair of Hilbert spaces. The description of the possible Lagrangian pair extensions in a larger pair of Hilbert spaces of a weakly Lagrangian pair is also given. Finally, the last section contains an example of a weakly Lagrangian pair of differential operators.

2. Preliminaries

Let $\mathfrak{H}_i$, $i = 1, 2$, be two Hilbert spaces with the inner products denoted by $[\cdot, \cdot]_i$, $i = 1, 2$, and with the corresponding norms denoted by $\|\cdot\|_i$, $i = 1, 2$, respectively. A typical element of the Cartesian product $\mathfrak{H}_1 \times \mathfrak{H}_2$ is an ordered pair (f_1, f_2), $f_i \in \mathfrak{H}_i$, $i = 1, 2$. A relation A from $\mathfrak{H}_1$ to $\mathfrak{H}_2$, is by definition the linear subspace A of the Hilbert space $\mathfrak{H}_1 \times \mathfrak{H}_2$. The *domain* and the *kernel* of A are linear subspaces of $\mathfrak{H}_1$ which are denoted by $\operatorname{dom} A$ and $\ker A$, and are defined by

$$\operatorname{dom} A := \{f_1 \,:\, (f_1, f_2) \in A\}, \quad \ker A := \{f_1 \,:\, (f_1, 0) \in A\},$$

while the *range* and the *multivalued part* of A are linear subspaces of $\mathfrak{H}_2$ which are denoted by $\operatorname{ran} A$ and $\operatorname{mul} A$, and are defined by

$$\operatorname{ran} A := \{f_2 \,:\, (f_1, f_2) \in A\}, \quad \operatorname{mul} A := \{f_2 \,:\, (0, f_2) \in A\}.$$

A relation A is (the graph of) an operator precisely when $\operatorname{mul} A = \{0\}$. The inverse of A, is the relation A^{-1} from $\mathfrak{H}_2$ to $\mathfrak{H}_1$ defined by

$$A^{-1} = \{(f_2, f_1) : (f_1, f_2) \in A\}.$$

Clearly, $\operatorname{dom} A^{-1} = \operatorname{ran} A$ and $\ker A^{-1} = \operatorname{mul} A$.

A relation A is closed if it is closed as a subspace of $\mathfrak{H}_1 \times \mathfrak{H}_2$, in which case $\ker A$ and $\operatorname{mul} A$ are closed subspaces of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. The adjoint A^* of a relation A is the closed relation given by

$$A^* = \{(f_2, f_1) \in \mathfrak{H}_2 \times \mathfrak{H}_1 : \langle (f_2, f_1), (g_1, g_2) \rangle = 0 \text{ for all } (g_1, g_2) \in A\},$$

where

$$\langle (f_2, f_1), (g_1, g_2) \rangle = [f_1, g_1]_1 - [f_2, g_2]_2,$$

with (f_1, f_2), $(g_1, g_2) \in \mathfrak{H}_1 \times \mathfrak{H}_2$.

An important tool in the theory of relations in Hilbert spaces is the operator J_{12} defined on all of $\mathfrak{H}_1 \times \mathfrak{H}_2$ into $\mathfrak{H}_2 \times \mathfrak{H}_1$ as follows

$$J_{12}(f_1, f_2) = (f_2, -f_1), \quad (f_1, f_2) \in \mathfrak{H}_1 \times \mathfrak{H}_2.$$

Similarly, the operator J_{21} can be defined on all of $\mathfrak{H}_2 \times \mathfrak{H}_1$ into $\mathfrak{H}_1 \times \mathfrak{H}_2$, so that $J_{21} J_{12} = -I_{\mathfrak{H}_1 \times \mathfrak{H}_2}$ and $J_{12} J_{21} = -I_{\mathfrak{H}_2 \times \mathfrak{H}_1}$. Furthermore, if A is a relation from $\mathfrak{H}_1$ to $\mathfrak{H}_2$, then it is easily checked that

$$A^* = (\mathfrak{H}_2 \times \mathfrak{H}_1) \ominus (J_{12} A) = (J_{12} A)^{\perp} = J_{12}(A^{\perp}).$$

Finally, if $A \subset B$ are two closed relations from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ then

$$J_{12}\,(B \ominus A) = A^* \ominus B^*. \tag{2.1}$$

Let A be a closed relation from $\mathfrak{H}_1$ to $\mathfrak{H}_2$. Define the closed relation A_∞ to be the set of all elements of the form $(0, f_2)$ in A, and let $A_s := A \ominus A_\infty$. Then A_s is a closed operator from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ with $\operatorname{dom} A_s = \operatorname{dom} A$. The following result describes some simple facts about A_s and A_∞, which were noted for instance in [1].

Lemma 2.1. *Let $\mathfrak{H}_i$, $i = 1, 2$ be two Hilbert spaces. If A is a closed relation from $\mathfrak{H}_1$ to $\mathfrak{H}_2$, then*

(i) $\operatorname{mul} A = (\operatorname{dom} A^*)^{\perp}$*;*
(ii) $\operatorname{dom} A_s = \operatorname{dom} A$ *is dense in* $(\operatorname{mul} A^*)^{\perp}$*;*
(iii) $\operatorname{ran} A_s \subset (\operatorname{mul} A)^{\perp}$.

An object is said to be maximal if it is maximal with respect to the operation of inclusion of sets in the class of sets in which it is included. The formal identification of an operator with its graph is implicitly assumed, so that an operator is viewed as a relation and a maximal object will be maximal in the sense of relations. Throughout the paper, assume that the relations (operators) which are involved are closed.

3. Weakly Lagrangian pairs in Hilbert spaces

Let (A_1, A_2) be a weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$. Define the relations B_1 and B_2 as follows

$$B_1 := V_2 A_2, \quad B_2 := V_1 A_1.$$

Then $\operatorname{dom} B_1 = \operatorname{dom} A_2$, $\operatorname{dom} B_2 = \operatorname{dom} A_1$, and V_1, V_2 take A_1 onto B_2 and A_2 onto B_1, respectively, in a one to one way. Since $B_i \subseteq A_i^*$, $i = 1, 2$, it follows that $A_i \subseteq B_i^*$, $i = 1, 2$, which implies

$$\operatorname{dom} B_1 = \operatorname{dom} A_2 \subset \operatorname{dom} A_1^*, \quad \operatorname{dom} B_2 = \operatorname{dom} A_1 \subset \operatorname{dom} A_2^*. \tag{3.1}$$

Thus (B_1, B_2) can be viewed as a weakly Lagrangian pair with isometries V_2^{-1} and V_1^{-1}, respectively. Next, a simple non-trivial example is stated.

Example 3.1. Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two unitarily equivalent Hilbert spaces and let $\mathfrak{H} := \mathfrak{H}_1 \oplus \mathfrak{H}_2$ be their orthogonal sum. Assume that $A_{11} : \mathfrak{H}_1 \to \mathfrak{H}_1$ and $A_{21} : \mathfrak{H}_1 \to \mathfrak{H}_2$ are two bounded everywhere defined operators. Furthermore, consider U, V, W and T four unitary operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$. Define on $\mathfrak{H}$ the following two (non-densely defined) operators by

$$A_1 := \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}, \quad A_2 := \begin{pmatrix} A_{21}^* U \\ V A_{11}^* \end{pmatrix}, \tag{3.2}$$

with $\operatorname{dom} A_1 = \operatorname{dom} A_2 = \mathfrak{H}_1$. Furthermore, their adjoints are two (proper) linear relations in $\mathfrak{H}$ given by

$$A_1^* = \left\{ \left(\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} A_{11}^* h_1 + A_{21}^* h_2 \\ \beta \end{pmatrix} \right) : h_1 \in \mathfrak{H}_1,\, h_2,\, \beta \in \mathfrak{H}_2 \right\}, \tag{3.3}$$

and

$$A_2^* = \left\{ \left(\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} U^{-1} A_{21} h_1 + A_{11} V^{-1} h_2 \\ \beta \end{pmatrix} \right) : h_1 \in \mathfrak{H}_1,\, h_2,\, \beta \in \mathfrak{H}_2 \right\}. \tag{3.4}$$

Clearly, $\operatorname{dom} A_1^* = \operatorname{dom} A_2^* = \mathfrak{H}$ and $\operatorname{mul} A_1^* = \operatorname{mul} A_2^* = \mathfrak{H}_2$. Define the linear operator V_1 from A_1 to A_2^* by

$$V_1 \left(\begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \begin{pmatrix} A_{11} h_1 \\ A_{21} h_1 \end{pmatrix} \right) = \left(\begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \begin{pmatrix} U^{-1} A_{21} h_1 \\ T A_{11} h_1 \end{pmatrix} \right), \quad \text{for all } h_1 \in \mathfrak{H}_1,$$

and the linear operator V_2 from A_2 to A_1^* by

$$V_2 \left(\begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \begin{pmatrix} A_{21}^* U h_1 \\ V A_{11}^* h_1 \end{pmatrix} \right) = \left(\begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \begin{pmatrix} A_{11}^* h_1 \\ W A_{21}^* U h_1 \end{pmatrix} \right), \quad \text{for all } h_1 \in \mathfrak{H}_1.$$

It is easily seen that V_1 and V_2 are two isometries and that $\operatorname{dom} A_1 \subset \operatorname{dom} A_2^*$, and $\operatorname{dom} A_2 \subset \operatorname{dom} A_1^*$, so that the pair (A_1, A_2) is a weakly Lagrangian pair in $(\mathfrak{H}, \mathfrak{H})$ which, in general, is not reducible neither to a dual pair nor to a formally normal relation as the next result shows.

Lemma 3.2. *Let A_1 and A_2 be the linear operators given by* (3.2)*.*

(i) *The pair (A_1, A_2) is reducible to a dual pair if and only if $A_{21} = U A_{11}$;*

(ii) *The pair (A_1, A_2) is reducible to a formally normal relation if and only if $A_{21} = U A_{11}^* = V A_{11}^*$.*

Proof. (i) Clearly, (A_1, A_2) is reducible to a dual pair if and only if $A_1 \subset A_2^*$, equivalently $A_{21} = U A_{11}$.

(ii) Furthermore, (A_1, A_2) is reducible to a formally normal relation if and only if $A_1 = A_2$, equivalently $A_{21} = U A_{11}^* = V A_{11}^*$. □

A deep study of this example, involving the notion of boundary triplets adapted to the case of weakly Lagrangian pairs, will be done elsewhere.

In the paper i and j are mainly used in order to denote the indices of the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, and the indices of the relations A_1 and A_2. They run from 1 to 2, such that $i \neq j$, and, this fact is not mentioned whenever it is obvious.

The following result shows the behavior of a weakly Lagrangian pair with respect to the decomposition of a relation in its operator part and its multivalued part.

Lemma 3.3. *If $(A_1,\, A_2)$ is a weakly Lagrangian pair in $(\mathfrak{H}_1,\, \mathfrak{H}_2)$, then*

$$V_1 \left(A_1\right)_s = \left(V_1 A_1\right)_s = \left(B_2\right)_s, \quad V_2 \left(A_2\right)_s = \left(V_2 A_2\right)_s = \left(B_1\right)_s, \tag{3.5}$$

and

$$V_1 \left(A_1\right)_\infty = \left(V_1 A_1\right)_\infty = \left(B_2\right)_\infty, \quad V_2 \left(A_2\right)_\infty = \left(V_2 A_2\right)_\infty = \left(B_1\right)_\infty. \tag{3.6}$$

Proof. Since A_1 and A_2 are closed relations it follows that $A_i = \left(A_i\right)_s \oplus \left(A_i\right)_\infty$, $i = 1,\, 2$. As it can be easily verified, V_i, $i = 1,\, 2$, preserve the inner products, so that (3.5) and (3.6) follow. □

Theorem 3.4. *Let $(A_1,\, A_2)$ be a weakly Lagrangian pair in $(\mathfrak{H}_1,\, \mathfrak{H}_2)$ with the isometries V_1 and V_2. Then*

(i) *$(A_1,\, A_2)$ is a Lagrangian pair if and only if $A_i = B_i^*$, $i = 1, 2$.*

(ii) *If A_{0i}, $i = 1, 2$, are closed subspaces of A_i, $i = 1, 2$, then $(A_{01},\, A_{02})$ is a weakly Lagrangian pair with the isometries $V_{0i} := V_i \restriction A_i$, $i = 1, 2$.*

(iii) *A_1 and A_2 are both operators if and only if one of the following equivalent conditions are verified:*
  - (a) *B_2 and B_1 are both operators;*
  - (b) *$\operatorname{dom} A_2^*$ and $\operatorname{dom} A_1^*$ are dense in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively;*
  - (c) *$\operatorname{dom} B_1^*$ and $\operatorname{dom} B_2^*$ are dense in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively.*

(iv) *A_2^* and A_1^* are both operators if and only if one of the following equivalent conditions are verified:*
  - (a) *B_1^* and B_2^* are both operators;*
  - (b) *$\operatorname{dom} A_1$ and $\operatorname{dom} A_2$ are dense in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively.*

Proof. Let $i,\, j = 1,\, 2,\ i \neq j$. Clearly, $\{A_1, A_2\}$ is a Lagrangian pair if and only if $B_i = A_i^*$, $i = 1, 2$. Then, it is a Lagrangian pair if and only if $A_i = B_i^*$, $i = 1, 2$, which means that (i) holds. If $A_{0i} \subseteq A_i$, then $A_i^* \subseteq A_{0i}^*$, and

$$V_i A_{0i} = B_{0j} \subseteq V_i A_i = B_j \subseteq A_j^* \subset A_{0j}^*.$$

Thus $\operatorname{dom} A_{0i} = \operatorname{dom} B_{0j} \subseteq \operatorname{dom} A_j^*$, and V_{0i} are isometries of A_{0i} onto $V_{0i}A_{0i} = B_{0j} \subseteq A_{0j}^*$, so that (ii) follows. Furthermore, $\{0, y_j\} \in A_i$ if and only if $V_i\{0, y_j\} = \{0, y_j'\} \in B_j$ for some y_j' such that $\|y_j\|_j = \|y_j'\|_j$. Thus $(\operatorname{dom} A_i^*)^{\perp} = \operatorname{mul} A_i = \{0\}$ if and only if $\left(\operatorname{dom} B_j^*\right)^{\perp} = \operatorname{mul} B_j = \{0\}$, which leads to (iii). Finally,

$$\operatorname{mul} A_i^* = (\operatorname{dom} A_i)^{\perp} = \operatorname{mul} B_j^*.$$

Then A_2^* and A_1^* are both operators if and only if B_1^* and B_2^* are both operators, or, equivalently, $\operatorname{dom} A_1 = \operatorname{dom} B_2$ and $\operatorname{dom} A_2 = \operatorname{dom} B_1$ are dense in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. The proof is now complete. □

Corollary 3.5. *Let (A_1, A_2) be a weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$. Then both*

$$((A_1)_s, (A_2)_s) \quad \textit{and} \quad ((A_1)_\infty, (A_2)_\infty)$$

are weakly Lagrangian pairs in $(\mathfrak{H}_1, \mathfrak{H}_2)$.

Proof. A simple application of Lemma 3.3 and Theorem 3.4 leads to the statement of this result. □

Let $\mathfrak{K}_i := (\operatorname{mul} A_i)^{\perp}$, $i = 1, 2$. Since $\operatorname{dom}(A_i)_s$ is dense in $(\operatorname{mul} A_i^*)^{\perp}$ and $\operatorname{ran}(A_i)_s \subseteq (\operatorname{mul} A_i)^{\perp}$ it follows that

$$(A_i)_s \subseteq (\operatorname{mul} A_i^*)^{\perp} \times (\operatorname{mul} A_i)^{\perp}, \tag{3.7}$$

and similarly

$$(A_i^*)_s \subseteq (\operatorname{mul} A_i)^{\perp} \times (\operatorname{mul} A_i^*)^{\perp}. \tag{3.8}$$

The relation $\operatorname{dom} A_i \subseteq \operatorname{dom} A_j^*$ leads to

$$(\operatorname{mul} A_i^*)^{\perp} \subseteq (\operatorname{mul} A_j)^{\perp}. \tag{3.9}$$

It is easily seen from (3.7)-(3.9), that

$$(A_i)_s \subseteq \mathfrak{K}_j \times \mathfrak{K}_i, \quad (A_i^*)_s \subseteq \mathfrak{K}_i \times \mathfrak{K}_j.$$

Denote $(A_i)_\sigma := (A_i)_s \cap (\mathfrak{K}_j \times \mathfrak{K}_i)$. A natural question is when $((A_1)_\sigma,\, (A_2)_\sigma)$ is weakly Lagrangian in $(\mathfrak{K}_2,\, \mathfrak{K}_1)$. The key of the answer is given by $(A_i)_\sigma^{\otimes}$, which denotes the adjoint of $(A_i)_s$, viewed as a relation from $\mathfrak{K}_i$ to $\mathfrak{K}_j$. The next result gives an "estimation" of $(A_i)_\sigma^{\otimes}$.

Lemma 3.6. *If $(A_1,\, A_2)$ is a weakly Lagrangian pair in $(\mathfrak{H}_1,\, \mathfrak{H}_2)$ then*

$$(A_i^*)_s \subseteq (A_i)_\sigma^\otimes \subset A_i^*. \tag{3.10}$$

Proof. Let $(x,y) \in (A_i^*)_s \subseteq A_i^*$. Then, $(x,y) \in \mathfrak{K}_i \times \mathfrak{K}_j$ and

$$\langle (x,y),(a,b)\rangle = 0, \tag{3.11}$$

for all $(a,b) \in A_i$. In particular this is true for all $(a,b) \in (A_i)_\sigma$. Thus $(x,y) \in (A_i)_\sigma^\otimes$, and the first inclusion is established. Assume now that $(x,y) \in (A_i)_\sigma^\otimes$. Then $(x,y) \in \mathfrak{K}_i \times \mathfrak{K}_j$ and (3.11) holds for all $(a,b) \in (A_i)_\sigma$. Each $(a,b) \in A_i$ can be written as

$$(a,b) = (a,b_1) + (0,b_2), \tag{3.12}$$

with $(a,b_1) \in (A_i)_\sigma$, and $(0,b_2) \in (A_i)_\infty$. Then clearly

$$\begin{aligned} \langle (x,y),(a,b)\rangle &= [y,a]_{\mathfrak{K}_j} - [x,b_1]_{\mathfrak{K}_i} - [x,b_2]_{\mathfrak{K}_i} \\ &= [y,a]_{\mathfrak{K}_j} - [x,b_1]_{\mathfrak{K}_i} \\ &= \langle (x,y),(a,b_1)\rangle = 0, \end{aligned}$$

for all $(a,b) \in A_i$, which implies that $(x,y) \in A_i^*$, concluding the proof of the lemma. $\square$

Remark 3.7. A direct consequence of Lemma 3.6 is that $\operatorname{dom}(A_i)_\sigma \subseteq \operatorname{dom}(A_j)_\sigma^\otimes$ since

$$\operatorname{dom}(A_i)_\sigma = \operatorname{dom}(A_i)_s = \operatorname{dom} A_i \subseteq \operatorname{dom} A_j^* = \operatorname{dom}\left(A_j^*\right)_s \subseteq \operatorname{dom}(A_j)_\sigma^\otimes. \tag{3.13}$$

Moreover, there exists a natural isometry for $(A_i)_\sigma$: it is the isometry V_i restricted to $(A_i)_\sigma$, so that

$$V_i(A_i)_\sigma = V_i(A_i)_s = (B_j)_s.$$

Since $(B_j)_s \subseteq A_j^*$, it is known that if $(x,y) \in (B_j)_s$ then $\langle (x,y),(a,b)\rangle = 0$ for all $(a,b) \in (A_i)_\sigma$. However, the inclusion $V_i(A_i)_\sigma \subseteq (A_j)_\sigma^\otimes$ does not necessarily hold; although $x \in \operatorname{dom} A_i \in \mathfrak{K}_j$, it is not known that $y \in \mathfrak{K}_i$. No such problem arises in the case of a Lagrangian pair, as the following result shows.

Proposition 3.8. *Let (A_1, A_2) be a Lagrangian pair in $(\mathfrak{H}_1,\, \mathfrak{H}_2)$. Then*

(i) $(A_j)_\infty = (A_i^*)_\infty$ *;*

(ii) $((A_1)_\sigma,\, (A_2)_\sigma)$ *is a Lagrangian pair in* $(\mathfrak{K}_2,\, \mathfrak{K}_1)$, *whose components are densely defined operators. Moreover,*

$$(A_i)_\sigma^\otimes = (A_i^*)_s\,.$$

Proof. Clearly,

$$\operatorname{mul} A_j^* = (\operatorname{dom} A_j)^\perp = (\operatorname{dom} A_i^*)^\perp = \operatorname{mul} A_i,$$

which gives (i). Also, $\operatorname{dom}(A_i)_\sigma = \operatorname{dom} A_i = \operatorname{dom} A_j^*$ is dense in $\mathfrak{K}_i$, and (3.13) now implies that $\operatorname{dom}(A_i)_\sigma = \operatorname{dom}(A_j)_\sigma^\otimes$. From Lemma 3.6 it is known that

$$(A_i^*)_s \subseteq (A_i)_\sigma^\otimes. \tag{3.14}$$

Furthermore, the following two relations

$$(A_i)_\sigma^\otimes \subseteq ((A_i^*)_s \oplus (A_i^*)_\infty) \cap (\mathfrak{K}_i \times \mathfrak{K}_j)\,,$$

and

$$(A_i^*)_\infty = (A_j)_\infty,$$

imply that $(A_i^*)_\infty$ is orthogonal to $\mathfrak{K}_i \times \mathfrak{K}_j$, and

$$(A_i)_\sigma^\otimes \subseteq (A_i^*)_s \cap (\mathfrak{K}_i \times \mathfrak{K}_j) = (A_i^*)_s\,. \tag{3.15}$$

Now, the relations (3.14) and (3.15) lead to the identity $(A_i)_\sigma^\otimes = (A_i^*)_s$. The isometry $(V_i)_\sigma$ for $(A_i)_\sigma$, defined by

$$(V_i)_\sigma = V_i \restriction (A_i)_\sigma = V_i \restriction (A_i)_s,$$

is such that

$$V_i(A_i)_\sigma = V_i(A_i)_s = \left(A_j^*\right)_s = (A_j)_\sigma^\otimes.$$

Therefore $(V_i)_\sigma$ is an isometry of $(A_i)_\sigma$ onto $(A_j)_\sigma^\otimes$. The proof is now complete. $\square$

4. Lagrangian extensions of a weakly Lagrangian pair

Assume that $(A_1,\, A_2)$ is a weakly Lagrangian pair in $(\mathfrak{H}_1,\, \mathfrak{H}_2)$ and let $\left(\widetilde{A}_1,\, \widetilde{A}_2\right)$ be a weakly Lagrangian extension of $(A_1,\, A_2)$. It is the purpose of this section to give an algebraic characterization of all such extensions. If $\widetilde{V}_1$ and $\widetilde{V}_2$ are the isometries for $\left(\widetilde{A}_1,\, \widetilde{A}_2\right)$, then the isometries V_1 and V_2 for (A_1, A_2) are $V_i = \widetilde{V}_i \restriction A_i$, $i = 1, 2$. The following relation

$$B_j = V_i A_i = \widetilde{V}_i A_i \subseteq \widetilde{V}_i \widetilde{A}_i = \widetilde{B}_j, \quad i,\, j = 1,\, 2, \quad i \neq j,$$

holds, and thus

$$A_i \subseteq \widetilde{A}_i \subseteq \widetilde{B}_i^* \subseteq B_i^*, \quad i = 1,\, 2.$$

In particular,

$$A_i \subseteq \widetilde{A}_i \subseteq B_i^*, \quad B_i \subseteq \widetilde{A}_i^* \subseteq A_i^*, \quad i = 1,\, 2.$$

This implies that $\widetilde{A}_i = A_i \oplus C_i$, where $C_i := \widetilde{A}_i \ominus A_i$ is a subspace of B_i^*, which added orthogonally to A_i, give rise to weakly Lagrangian extension $\left(\widetilde{A}_1,\, \widetilde{A}_2\right)$ of $(A_1,\, A_2)$. The next result gives a purely algebraic characterization of all weakly Lagrangian pair extensions of a weakly Lagrangian pair.

Theorem 4.1. *Let (A_1, A_2) be a weakly Lagrangian pair in $(\mathfrak{H}_1,\, \mathfrak{H}_2)$ with the isometries V_1 and V_2, and let $X_i := B_i^* \ominus A_i$, $i = 1,\, 2$. Then $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ is a weakly Lagrangian pair extension of $(A_1,\, A_2)$ in $(\mathfrak{H}_1,\, \mathfrak{H}_2)$ with the isometries $\widetilde{V}_1$ and $\widetilde{V}_2$ if and only if the following items are valid:*

(i) $X_i = X_{1i} \oplus X_{2i}$, $i = 1,\, 2$;
(ii) $\widetilde{A}_i = A_i \oplus X_{1i}$, $i = 1,\, 2$;
(iii) $\operatorname{dom} X_{1i} \subseteq \operatorname{dom}(J_{ji} X_{2j})$, $i,\, j = 1,\, 2$, $i \neq j$;
(iv) $\widetilde{V}_i = V_i \oplus V_i'$, *where V_i' is an isometry of X_{1i} into $J_{ji} X_{2j}$ of the form*

$$V_i'(\varphi, \psi) = (\varphi, \psi'), \quad \|\psi\|_j = \|\psi'\|_j, \quad i,\, j = 1,\, 2, \quad i \neq j.$$

Proof. Assume that (A_1, A_2) is a weakly Lagrangian pair with the isometries V_1 and V_2 and $\left(\widetilde{A}_1,\, \widetilde{A}_2\right)$ is a weakly Lagrangian pair extension with the isometries $\widetilde{V}_1$ and $\widetilde{V}_2$, respectively. Thus $V_i = \widetilde{V}_i \restriction A_i$, $i = 1, 2$. Clearly, with $X_{1i} := \widetilde{A}_i \ominus A_i$ and $X_{2i} := B_i^* \ominus \widetilde{A}_i$, the items (i) and (ii) are valid. Moreover, for $i,\, j = 1,\, 2,\, i \neq j$,

$$\begin{aligned} \widetilde{V}_i X_{1i} &= \widetilde{V}_i(\widetilde{A}_i \ominus A_i) = \widetilde{V}_i \widetilde{A}_i \ominus \widetilde{V}_i A_i \\ &= \widetilde{V}_i \widetilde{A}_i \ominus V_i A_i = \widetilde{B}_j \ominus B_j \\ &\subseteq \widetilde{A}_j^* \ominus B_j = J_{ji}(B_j^* \ominus \widetilde{A}_j) = J_{ji} X_{2j}. \end{aligned}$$

Thus $\operatorname{dom} X_{1i} = \operatorname{dom} \widetilde{V}_i X_{1i} \subseteq \operatorname{dom} J_{ji} X_{2j}$, proving (iii). If $V_i' := \widetilde{V}_i \restriction X_{1i}$ then clearly V_i' satisfies the conditions in (iv).

For the converse implication, assume that X_i can be decomposed as an orthogonal sum as in (i), and $\left(\widetilde{A}_1,\, \widetilde{A}_2\right)$ is defined by (ii), with X_{1i} and X_{2i} satisfying (iii) and (iv). It will be shown that $\left(\widetilde{A}_1,\, \widetilde{A}_2\right)$ is a weakly Lagrangian pair extension of (A_1, A_2). Clearly, $A_i \subseteq \widetilde{A}_i \subseteq B_i^*$ and consequently $B_i \subseteq \widetilde{A}_i^* \subseteq A_i^*$. Moreover, $X_{2i} = B_i^* \ominus \widetilde{A}_i = X_i \ominus X_{1i}$ and using (2.1) it follows that $\widetilde{A}_i^* = B_i \oplus J_{ij} X_{2i}$. By (iii), the relations $\operatorname{dom} A_i = \operatorname{dom} B_j$ and $\operatorname{dom} X_{1i} \subseteq \operatorname{dom} J_{ji} X_{2j}$ lead to

$$\operatorname{dom} \widetilde{A}_i = \operatorname{dom} A_i + \operatorname{dom} X_{1i} \subseteq \operatorname{dom} B_j + \operatorname{dom} J_{ji} X_{2j} = \operatorname{dom} \widetilde{A}_j^*,$$

where the sums are algebraic ones. Consequently, $\operatorname{dom} \widetilde{A}_i \subseteq \operatorname{dom} \widetilde{A}_j^*$. Now $\widetilde{V}_i := V_i \oplus V_i'$ maps $\widetilde{A}_i$ isometrically into $\widetilde{A}_j^*$ in the prescribed manner, and thus $\left(\widetilde{A}_1,\, \widetilde{A}_2\right)$ is a weakly

Lagrangian pair with isometries $\widetilde{V}_i$, $i=1, 2$, and, since $V_i = \widetilde{V}_i \restriction A_i$ it follows that $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ is a weakly Lagrangian extension of (A_1, A_2). $\square$

The next result describes those weakly Lagrangian pair extensions $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ of a weakly Lagrangian pair (A_1, A_2) which are Lagrangian pairs.

Theorem 4.2. *Let (A_1, A_2) be a weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$ with the isometries V_1 and V_2 and let $X_i := B_i^* \ominus A_i$, $i=1, 2$. Then $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ is a Lagrangian pair extension of (A_1, A_2) in $(\mathfrak{H}_1, \mathfrak{H}_2)$ with the isometries $\widetilde{V}_1$ and $\widetilde{V}_2$ if and only if the following items are valid:*

(i) $X_i = X_{1i} \oplus X_{2i}$, $i=1, 2$;
(ii) $\widetilde{A}_i = A_i \oplus X_{1i}$, $i=1, 2$;
(iii) $\operatorname{dom} X_{1i} = \operatorname{dom}(J_{ji}X_{2j})$, $i, j = 1, 2$, $i \neq j$;
(iv) $\widetilde{V}_i = V_i \oplus V_i'$, $i=1, 2$, *where V_i' is an isometry of X_{1i} onto $J_{ji}X_{2j}$ of the form*

$$V_i'(\varphi, \psi) = \{\varphi, \psi'\}, \quad \|\psi\|_j = \|\psi'\|_j .$$

Proof. Assume $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ is a Lagrangian pair extension of (A_1, A_2). Then (i) and (ii) are valid due to Theorem 4.1. Since $\widetilde{V}_i\widetilde{A}_i = \widetilde{B}_j = \widetilde{A}_j^*$ it follows that $\widetilde{V}_iX_{1i} = \widetilde{A}_j^* \ominus B_j = J_{ji}X_{2j}$, and thus (iii) and (iv) hold true.

Conversely, assume that (A_1, A_2) is weakly Lagrangian with some X_i, $i=1, 2$ satisfying (i)-(iv). It is known from Theorem 4.1 that $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ is a weakly Lagrangian extension of (A_1, A_2) with

$$\widetilde{A}_i = A_i \oplus X_{1i}, \quad \widetilde{A}_i^* = B_i \oplus J_{ij}X_{2i}.$$

Then $\operatorname{dom}\widetilde{A}_i = \operatorname{dom}\widetilde{A}_j^*$, since

$$\operatorname{dom}\widetilde{A}_i = \operatorname{dom}A_i + \operatorname{dom}X_{1i} = \operatorname{dom}B_j + \operatorname{dom}J_{ji}X_{2j} = \operatorname{dom}\widetilde{A}_j^*.$$

Moreover, $\widetilde{V}_i$, $i=1, 2$, give the desired isometries of $\widetilde{A}_i$ onto $\widetilde{A}_j^*$ and thus $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ is a Lagrangian pair extension of (A_1, A_2). $\square$

Theorem 4.2 differs from Theorem 4.1 in that equality occurs in (iii) and V_i' is now onto $J_{ji}X_{2j}$. The condition (iv) implies that $\dim X_{1i} = \dim X_{2j}$. Moreover, it is possible to specify those extensions $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ of (A_1, A_2) in Theorems 4.1 and 4.2 which are operators. The following result is useful in this respect.

Lemma 4.3. *Let A and $\widetilde{A}$ be relations from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ and let B be a relation from $\mathfrak{H}_2$ to $\mathfrak{H}_1$, such that $A \subset \widetilde{A} \subset B^*$. Denote*

$$X_1 := \widetilde{A} \ominus A, \quad X_2 := B^* \ominus \widetilde{A}, \quad X := X_1 \oplus X_2,$$

and

$$X' := P_X(B^*)_\infty,$$

where P_X is the orthogonal projection of $\mathfrak{H}_1 \times \mathfrak{H}_2$ onto X. Then $\widetilde{A}$ is an operator if and only if the following two items are satisfied:

(i) *A is an operator ;*
(ii) $X_1 \cap X' = \{(0,0)\}$.

Proof. It is easily seen that

$$\operatorname{mul}\widetilde{A} = \operatorname{mul}B^* \cap (\operatorname{dom}J_{12}X_2)^\perp ,$$

so that the conclusion follows. $\square$

Proposition 4.4. *Let (A_1, A_2) be a weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$, and let $X_i = B_i^* \ominus A_i$, $i = 1, 2$. Assume that $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ is a weakly Lagrangian pair extension of (A_1, A_2) in $(\mathfrak{H}_1, \mathfrak{H}_2)$ with $\widetilde{A}_i = A_i \oplus X_{1i}$, $X_{1i} \subset X_i$. Then $\widetilde{A}_1$ and $\widetilde{A}_2$ are both operators if and only if the following two conditions are satisfied:*

(i) *A_1 and A_2 are both operators;*
(ii) *$X_{1i} \cap X_i' = \{(0, 0)\}$, where $X_i' := P_{X_i}(B_i^*)_\infty$, $i = 1, 2$.*

Proof. Apply Lemma 4.3 successively with $A := A_i$, $\widetilde{A} := \widetilde{A}_i$ and $B := B_i$. □

Corollary 4.5. *If (A_1, A_2) is a weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$ with $\operatorname{dom} A_i$ dense in $\mathfrak{H}_i$, $i = 1, 2$, then (A_1, A_2) and every weakly Lagrangian pair extension of (A_1, A_2) in $(\mathfrak{H}_1, \mathfrak{H}_2)$ are pairs of operators.*

Proof. Condition (ii) in Theorem 3.4 is trivially satisfied if B_i^*, $i = 1, 2$ are operators. From Theorem 3.4, this is the case if and only if $\operatorname{dom} A_i$, $i = 1, 2$ are dense in $\mathfrak{H}_i$, $i = 1, 2$. □

A weakly Lagrangian pair (A_1, A_2) in $(\mathfrak{H}_1, \mathfrak{H}_2)$ is said to be a *weakly Lagrangian densely defined pair* if $\operatorname{dom} A_i$ are dense in $\mathfrak{H}_i$, $i = 1, 2$. The next characterization of the Lagrangian pair extensions of a weakly Lagrangian densely defined pair (A_1, A_2) in $(\mathfrak{H}_1, \mathfrak{H}_2)$ follows from Proposition 4.2 and Corollary 4.5.

Proposition 4.6. *A densely defined weakly Lagrangian pair (A_1, A_2) in $(\mathfrak{H}_1, \mathfrak{H}_2)$ has a Lagrangian pair extension $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ in $(\mathfrak{H}_1, \mathfrak{H}_2)$ if and only if the following conditions are satisfied:*

(i) *$\mathcal{X}_i = \mathcal{X}_{1i} + \mathcal{X}_{2i}$, a direct sum, where $\mathcal{X}_i = \ker\,(I + A_i^* B_i^*)$, $i = 1, 2$;*
(ii) *$(B_i^* \upharpoonright \mathcal{X}_{1i}) \perp (B_i^* \upharpoonright \mathcal{X}_{2i})$, $i = 1, 2$;*
(iii) *$B_i^* \mathcal{X}_{2i} = \mathcal{X}_{1j}$, $i, j = 1, 2$, $i \neq j$;*
(iv) *$\|A_i^* \alpha\|_i = \|B_j^* \alpha\|_i$, $\alpha \in \mathcal{X}_{1j}$, $i, j = 1\,2$, $i \neq j$.*

Proof. From Corollary 4.5 it follows that all Lagrangian pairs extensions $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ of (A_1, A_2) are operators. Apply Theorem 4.2, and with $\mathcal{X}_i = \operatorname{dom} X_i$, $\mathcal{X}_{1i} = \operatorname{dom} X_{1i}$, $\mathcal{X}_{2i} = \operatorname{dom} X_{2i}$, it follows that

$$X_i = B_i^* \upharpoonright \mathcal{X}_i, \quad X_{ki} = B_i^* \upharpoonright \mathcal{X}_{ki}, \quad i, k = 1, 2.$$

Clearly, $\mathcal{X}_i = \ker\,(I + A_i^* B_i^*)$. Condition (i) in Theorem 4.2 gives (i) and (ii) of this result and (ii) in Theorem 4.2 implies the last statement describing $\left(\widetilde{A}_1, \widetilde{A}_2\right)$. Since

$$X_{1i} = \{(\alpha, B_i^* \alpha),\, \alpha \in \mathcal{X}_{1i}\}, \quad i = 1, 2,$$

$$X_{2i} = \{(\beta, B_i^* \beta),\, \beta \in \mathcal{X}_{2i}\}, \quad i = 1, 2,$$

$$J_{ij} X_{2i} = \{(B_i^* \beta, -\beta\},\, \beta \in \mathcal{X}_{2i}\}, \quad i = 1, 2, \quad i, j = 1, 2,$$

the condition $\operatorname{dom} X_{1j} = \operatorname{dom} J_{ij} X_{2i}$ assures that $\alpha \in \mathcal{X}_{1j}$ if and only if $\alpha = B_i^* \beta$ for some $\beta \in \mathcal{X}_{2i}$. Then $A_i^* \alpha = A_i^* B_i^* \beta = -\beta$, which shows that

$$J_{ij} X_{2i} = \{(\alpha, A_i^* \alpha),\, \alpha \in \mathcal{X}_{1i}\}.$$

Condition (iv) of Theorem 4.2 shows that the isometry V_i of X_{1i} onto $J_{ji} X_{2j}$ implies that $\|A_i^* \alpha\|_i = \|B_j^* \alpha\|_i$, $\alpha \in \mathcal{X}_{1j}$. □

5. Extension of weakly Lagrangian pairs in larger Hilbert spaces

Let (A_1, A_2) be a weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$. It will be shown how the results of Section 4 may be applied to investigate the Lagrangian pair extensions of (A_1, A_2) in a pair of larger Hilbert spaces $\left(\widetilde{\mathfrak{H}}_1, \widetilde{\mathfrak{H}}_2\right)$, where $\mathfrak{H}_i \subset \widetilde{\mathfrak{H}}_i$, $i = 1, 2$.

Assume the weakly Lagrangian pair $(A_1,\, A_2)$ with the isometries V_1 and V_2 has a Lagrangian pair extension $\left(\widetilde{A}_1,\, \widetilde{A}_2\right)$ in $\left(\widetilde{\mathfrak{H}}_1,\, \widetilde{\mathfrak{H}}_2\right)$, with the isometries $\widetilde{V}_1$ and $\widetilde{V}_2$, where $\widetilde{\mathfrak{H}}_i = \mathfrak{H}_i \oplus \mathfrak{H}'_i$, $i = 1, 2$, are the orthogonal sums of the Hilbert spaces $\mathfrak{H}_i$ and $\mathfrak{H}'_i$, respectively. The Hilbert space $\mathfrak{H}_1 \times \mathfrak{H}_2$ can be identified with $(\mathfrak{H}_1 \oplus \{0\}) \times (\mathfrak{H}_2 \oplus \{0\})$ and $\mathfrak{H}'_1 \times \mathfrak{H}'_2$ can be identified with $(\{0\} \oplus \mathfrak{H}'_1) \times (\{0\} \oplus \mathfrak{H}'_2)$ in $\widetilde{\mathfrak{H}}_1 \times \widetilde{\mathfrak{H}}_2$, and then (A_1, A_2) is identified with $(A_1 \oplus \{(0,0)\},\, A_2 \oplus \{(0,0)\})$. Then $A_i \subset \widetilde{A}_i$ as a subspace of $\widetilde{\mathfrak{H}}_i \times \widetilde{\mathfrak{H}}_j$ and $V_i = \widetilde{V}_i \restriction A_i$. Define the pair $(A'_1,\, A'_2)$ as follows

$$A'_i = \left\{(x'_i, y'_j) \in \widetilde{A}_i \cap \left(\mathfrak{H}'_i \times \mathfrak{H}'_j\right) \;:\; \widetilde{V}_i(x'_i, y'_j) \in \mathfrak{H}'_i \times \mathfrak{H}'_j\right\}, \quad i,\, j = 1,\, 2, \quad i \neq j.$$

Note that $(x'_i, y'_j) \in \mathfrak{H}'_i \times \mathfrak{H}'_j$ is identified with $\big((0, x'_i), (0, y'_j)\big) \in \widetilde{\mathfrak{H}}_i \times \widetilde{\mathfrak{H}}_j$, $i,\, j = 1,\, 2,\, i \neq j$. Let P_i and P'_i be the orthogonal projections from $\widetilde{\mathfrak{H}}_i \times \widetilde{\mathfrak{H}}_j$ onto $\mathfrak{H}_i \times \mathfrak{H}_j$ and $\mathfrak{H}'_i \times \mathfrak{H}'_j$, respectively, and, p_i and p'_i be the orthogonal projections from $\widetilde{\mathfrak{H}}_i$ onto $\mathfrak{H}_i$ and $\mathfrak{H}'_i$, $i = 1,\, 2$, respectively.

Theorem 5.1. *Let $(A_1,\, A_2)$ be a weakly Lagrangian pair in $(\mathfrak{H}_1,\, \mathfrak{H}_2)$ with the isometries V_1 and V_2, and let $\left(\widetilde{A}_1,\, \widetilde{A}_2\right)$ be a Lagrangian pair extension of $(A_1,\, A_2)$ in $\left(\widetilde{\mathfrak{H}}_1,\, \widetilde{\mathfrak{H}}_2\right)$ with the isometries $\widetilde{V}_1$ and $\widetilde{V}_2$. Then (A'_1, A'_2) is a weakly Lagrangian pair in $(\mathfrak{H}'_1,\, \mathfrak{H}'_2)$ with the isometries $V'_i := \widetilde{V}_i \restriction A'_i$. Moreover, the following relations hold:*

$$A_i \subseteq \widetilde{A}_i,\ A_i \subseteq P_i\widetilde{A}_i \subseteq B_i^*, \quad i = 1,\, 2, \tag{5.1}$$

$$B_i \subseteq \widetilde{A}_i^*,\ B_i \subseteq P_j\widetilde{A}_i^* \subseteq A_i^*, \quad i,\, j = 1,\, 2, \quad i \neq j, \tag{5.2}$$

and

$$A'_i \subseteq \widetilde{A}_i, \quad A'_i \subseteq P'_i\widetilde{A}_i \subseteq B_i'^*, \quad i = 1,\, 2, \tag{5.3}$$

$$B'_i \subseteq \widetilde{A}_i^*, \quad B'_i \subseteq P'_j\widetilde{A}_i^* \subseteq A_i^*, \quad i,\, j = 1, \quad 2,\, i \neq j. \tag{5.4}$$

Proof. Since $V_i = \widetilde{V}_i \restriction A_i$ and $B_i = V_jA_j \subset A_i^*$ it follows that $B_i = \widetilde{V}_jA_j \subset \widetilde{V}_j\widetilde{A}_j = \widetilde{A}_i^*$, and hence $B_i = P_jB_i \subseteq P_j\widetilde{A}_i^*$. Now let $(a, b) \in \widetilde{A}_i^*$ and consider $P_j(a, b) = (p_ja, p_ib)$. If $(x, y) \in A_i \subset \widetilde{A}_i$, then

$$\begin{aligned} \langle (p_ja, p_ib), (x, y)\rangle &= [p_ib, x]_i - [p_ja, y]_j \\ &= [b, x]_i - [a, y]_j \\ &= \langle (a, b), (x, y)\rangle = 0, \end{aligned}$$

which shows that $P_j\widetilde{A}_i^* \subseteq A_i^*$. Similarly, if $(c, d) \in \widetilde{A}_i$ and $(\alpha, \beta) \in B_i \subset \widetilde{A}_i^*$, then

$$\begin{aligned} \langle (p_ic, p_jd), (\alpha, \beta)\rangle &= [p_jd, \alpha]_j - [p_ic, \beta]_i \\ &= [d, \alpha]_j - [c, \beta]_i \\ &= \langle (c, d), (\alpha, \beta)\rangle = 0, \end{aligned}$$

which gives $P_i\widetilde{A}_i \subset B_i^*$. Thus (5.1) and (5.2) are verified. Moreover, A'_i, $i = 1, 2$ are closed relations. Indeed, if $(x'_n, y'_n) \in A'_i$ and $(x'_n, y'_n) \to (x', y') \in \mathfrak{H}'_i \times \mathfrak{H}'_j$, then $(x', y') \in \widetilde{A}_i \cap \left(\mathfrak{H}'_i \times \mathfrak{H}'_j\right)$, since $\widetilde{A}_i$ and $\mathfrak{H}'_i \times \mathfrak{H}'_j$ are closed. Now, the following inequality

$$\begin{aligned} \|P_i\widetilde{V}_i(x', y')\| &= \|P_i\widetilde{V}_i(x', y') - P_i\widetilde{V}_i(x'_n, y'_n)\| \\ &= \|P_i\widetilde{V}_i(x' - x'_n, y' - y'_n)\| \\ &\le \|(x' - x'_n, y' - y'_n)\|, \end{aligned} \tag{5.5}$$

shows that $P_i\widetilde{V}_i(x', y') = 0$, or that $\widetilde{V}_i(x', y') \in \mathfrak{H}'_i \times \mathfrak{H}'_j$, and hence $(x', y') \in A'_i$.

Let $(y'_j, x'_i) \in A'_j \subseteq \widetilde{A}_j$ and let $x''_i \in \operatorname{dom} A'_i \subseteq \operatorname{dom} \widetilde{A}_i = \operatorname{dom} \widetilde{A}^*_j$, with $(x''_i, y_j) \in \widetilde{A}^*_j$. Then

$$\begin{aligned} \left\langle (y'_j, x'_i), (x''_i, p'_j y_j) \right\rangle &= [x'_i, x''_i]_i - \left[y'_j, p'_j y_j\right]_j \\ &= [x'_i, x''_i]_j - \left[y'_j, y_j\right]_j \\ &= \left\langle (y'_j, x'_i), (x''_i, y_j) \right\rangle = 0, \end{aligned}$$

which implies $(x''_i, p'_j y_j) \in {A'_j}^*$, and, in particular, $\operatorname{dom} A'_i \subset \operatorname{dom} {A'_j}^*$. Clearly, the pair (A'_i, A'_2) with the isometries $V'_i = \widetilde{V}_i \restriction A'_i$, $i = 1, 2$, is a weakly Lagrangian pair in $(\mathfrak{H}'_i, \mathfrak{H}'_j)$, and $B'_j := V'_i A'_i$ is given by

$$B'_j = \left\{ (x, y) \in \widetilde{A'}^*_j \cap \left(\mathfrak{H}'_i \times \mathfrak{H}'_j\right) \,:\, \widetilde{V}_i^{-1}(x, y) \in \mathfrak{H}'_i \times \mathfrak{H}'_j \right\}.$$

The inclusions (5.3) and (5.4) follow using similar arguments as for the proof of (5.1) and (5.2), respectively. □

In general it can not be asserted that $A'_i = \widetilde{A}_i \cap \left(\mathfrak{H}'_i \times \mathfrak{H}'_j\right)$, that is, for $\{x'_i, y'_j\} \in \widetilde{A}_i \cap \left(\mathfrak{H}'_i \times \mathfrak{H}'_j\right)$ it is not possible to guarantee that $\widetilde{V}_i(x'_i, y'_j) \in \mathfrak{H}'_i \times \mathfrak{H}'_j$.

Clearly, $\widetilde{V}_i(x'_i, y'_j) = (x'_i, z'_j) \in \widetilde{A}^*_j$, and $\|y'_j\|_j^2 = \|z'_j\|_j^2 = \|p_j z'_j\|_j^2 + \|p'_j z'_j\|_j^2$. Since $P_i \widetilde{A}^*_j \subset A^*_j$ it follows for any $(x_j, y_i) \in A_j$ that $P_i(x'_i, z'_j) = (0, p_j z'_j) \in A^*_j$, and thus,

$$[y_i, 0]_i = [x_j, p_j z'_j]_j = 0,$$

which implies $p_j z'_j \in (\operatorname{dom} A_j)^{\perp}$. Consequently, $A'_i = \widetilde{A}_i \cap \left(\mathfrak{H}'_i \times \mathfrak{H}'_j\right)$ if $\operatorname{dom} A_i$ is dense in $\mathfrak{H}_i$, $i = 1, 2$.

Define the subspaces A^+_i by

$$A^+_i := \left\{ (x_i, y_j) \in \widetilde{A}_i \cap (\mathfrak{H}_i \times \mathfrak{H}_j) \,:\, \widetilde{V}_i(x_i, y_j) \in \mathfrak{H}_i \times \mathfrak{H}_j \right\}.$$

Then clearly $\left(A^+_1, A^+_2\right)$ is a weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$ with the isometries $\widetilde{V}_i \restriction A^+_i$, so that it is a weakly Lagrangian pair extension of (A_1, A_2). If (A_1, A_2) is a maximal weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$, then $A_i = A^+_i$ and $A_i = \widetilde{A}_i \cap (\mathfrak{H}_i \times \mathfrak{H}_j)$. Moreover, if $\operatorname{dom} A'_i$ are dense in $\mathfrak{H}'_i$, then $A^+_i = \widetilde{A}_i \cap (\mathfrak{H}_i \times \mathfrak{H}_j)$, $i = 1, 2$.

The next result gives a necessary condition for the existence of a Lagrangian pair extension $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ in $(\widetilde{\mathfrak{H}}_1, \widetilde{\mathfrak{H}}_2)$.

Proposition 5.2. *Let (A_1, A_2) be a weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$, with the isometries V_1 and V_2, and suppose that $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ is a Lagrangian pair extension of (A_1, A_2) in $(\widetilde{\mathfrak{H}}_1, \widetilde{\mathfrak{H}}_2)$, with the isometries $\widetilde{V}_1$ and $\widetilde{V}_2$, where $\widetilde{\mathfrak{H}}_i = \mathfrak{H}_i \oplus \mathfrak{H}'_i$. Then*

$$\widetilde{A}_i = A_i \oplus C_i, \quad \widetilde{A}^*_j = B_j \oplus D_j, \tag{5.6}$$

where $C_i = \widetilde{A}_i \ominus A_i$, $D_j = \widetilde{A}^*_j \ominus B_j$ and

$$\widetilde{V}_i C_i = D_j, \quad P_i C_i \subseteq E_i, \quad P_i D_j \subseteq F_j, \tag{5.7}$$

where $E_i = B^*_i \ominus A_i$, $F_j = A^*_j \ominus B_j$. In particular,

$$\operatorname{dom} C_i = \operatorname{dom} D_j,$$

and

$$p_i \operatorname{dom} C_i = p_i \operatorname{dom} D_j \subseteq \operatorname{dom} E_i \cap \operatorname{dom} E_j.$$

Proof. Clearly,

$$\begin{aligned} \widetilde{V}_i C_i &= \widetilde{V}_i \left(\widetilde{A}_i \ominus A_i\right) = \widetilde{V}_i \widetilde{A}_i \ominus \widetilde{V}_i A_i \\ &= \widetilde{B}_j \ominus B_j = \widetilde{A}^*_j \ominus B_j = D_j. \end{aligned}$$

If $(x_i, y_j) \in C_i$ then (5.1) leads to $P_i(x_i, y_j) \in P_i\widetilde{A}_i \subseteq B_i^*$. Let $(x_{1i}, y_{1j}) \in A_i \subseteq \mathfrak{H}_i \times \mathfrak{H}_j$. Then the identity

$$[P_i(x_i, y_j), (x_{1i}, y_{1j})] = [(x_i, y_j), (x_{1i}, y_{1j})] = 0,$$

shows that $P_iC_i \subset B_i^* \ominus A_i = E_i$. From (5.2) it follows that

$$P_iD_j = P_i\widetilde{V}_iC_i \subseteq P_i\widetilde{A}_j^* \subseteq A_j^*,$$

and using similar arguments as above it follows that $P_iD_j \subseteq F_j$, completing the proof. □

Proposition 5.3. *Let (A_1, A_2) be a weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$ such that* $\operatorname{dom} A_i$ *is dense in $\mathfrak{H}_i$, $i = 1, 2$. With the notations from Proposition 5.2, assume that*

$$\operatorname{dom} E_i \cap \operatorname{dom} F_j = \{0\}, \quad i, j = 1, 2, \quad i \neq j. \tag{5.8}$$

Then (A_1, A_2) is a maximal weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$. If (A_1, A_2) is not a Lagrangian pair, it has no Lagrangian pair extension in any $\left(\widetilde{\mathfrak{H}}_1, \widetilde{\mathfrak{H}}_2\right)$.

Proof. Assume (A_1, A_2) has a weakly Lagrangian pair extension $\left(A_1^+, A_2^+\right)$ in $(\mathfrak{H}_1, \mathfrak{H}_2)$. Then

$$A_i \subseteq A_i^+ \subseteq (B_i^+)^* \subseteq B_i^*, \quad B_i \subset B_i^+ \subseteq (A_i^+)^* \subseteq A_i^*, \quad i = 1, 2,$$

and thus A_i^+, B_i^+, B_i^* and $(B_i^+)^*$ are all operators. Using analogous notations as in Proposition 5.2 it follows that

$$A_i^+ = A_i \oplus C_i^+, \quad C_i^+ = A_i^+ \ominus A_i, \quad (A_j^+)^* = B_j \oplus D_j^+, \quad D_j^+ = (A_j^+)^* \ominus B_j,$$

where $\operatorname{dom} C_i^+ \subseteq \operatorname{dom} D_j^+$. Thus

$$\operatorname{dom} C_i^+ \subseteq \operatorname{dom} C_i^+ \cap \operatorname{dom} D_j^+ \subseteq \operatorname{dom} E_i \cap \operatorname{dom} F_j,$$

and (5.8) leads to $\operatorname{dom} C_i^+ = \{0\}$. Since C_i^+ is an operator, $C_i^+ = \{(0, 0)\}$, proving $A_i^+ = A_i$.

Assume now that (A_1, A_2) has a Lagrangian pair extension $(\widetilde{A}_1, \widetilde{A}_2)$ in $\left(\widetilde{\mathfrak{H}}_1, \widetilde{\mathfrak{H}}_2\right)$. Since B_i^* and A_j^* are operators, E_i and F_j are operators, and the relations (5.7) and (5.8) lead to

$$p_i \operatorname{dom} C_i = p_i \operatorname{dom} D_j = \{0\},$$

where C_i and D_j are as in Proposition 5.2. Then $P_iC_i \subseteq E_i$ and $P_iD_j \subseteq F_j$ imply that $P_iC_i = P_iD_j = \{(0, 0)\}$. Therefore $A_j^* \subseteq P_i\widetilde{A}_j^*$. Indeed, let $(x_i, y_j) \in A_j^*$, and $(a_j, b_i) \in \widetilde{A}_j$. Then the decomposition

$$(a_j, b_i) = (a_{1j}, b_{1i}) + (\alpha, \beta),$$

holds with $(a_{1j}, b_{1i}) \in A_j$ and $(\alpha, \beta) \in C_j$. Clearly,

$$P_j(\alpha, \beta) = (p_j\alpha, p_i\beta) = (0, 0),$$

so that

$$\langle (a_j, b_i), (x_i, y_j) \rangle = \langle (a_{1j}, b_{1i}), (x_i, y_j) \rangle = 0,$$

showing that $A_j^* \subseteq P_i\widetilde{A}_j^*$. Thus $\widetilde{A}_j^* \subseteq P_i\widetilde{A}_j^* = P_i(B_j \oplus D_j) = B_j$, for $P_iD_j = \{(0, 0)\}$. Now $B_j \subseteq A_j^*$ and therefore $B_j = A_j^*$, which means that (A_1, A_2) has to be a Lagrangian pair. If (A_1, A_2) is not a Lagrangian pair, this contradiction shows that (A_1, A_2) has no Lagrangian pair extension in any $\widetilde{\mathfrak{H}}_1 \times \widetilde{\mathfrak{H}}_2$. □

The purpose of the next part of this section is to describe the Lagrangian pair extensions in $\left(\widetilde{\mathfrak{H}}_1, \widetilde{\mathfrak{H}}_2\right)$ of a weakly Lagrangian pair (A_1, A_2) in $(\mathfrak{H}_1, \mathfrak{H}_2)$. For such $\left(\widetilde{A}_1, \widetilde{A}_2\right)$, with the isometries $\widetilde{V}_1$ and $\widetilde{V}_2$, consider (A_1', A_2') as in Theorem 5.1, and define $\mathcal{A}_i := A_i \oplus A_i'$ in $\widetilde{\mathfrak{H}}_i \times \widetilde{\mathfrak{H}}_j$, by

$$\mathcal{A}_i = \left\{ \left((x_i, x_i'), (y_j, y_j')\right) \in \widetilde{\mathfrak{H}}_i \times \widetilde{\mathfrak{H}}_j \,:\, (x_i, y_j) \in A_i, \, (x_i', y_j') \in A_i' \right\},$$

and define the isometries $\mathcal{V}_i := V_i \oplus V_i'$ on $\mathcal{A}_i$, where $V_i = \widetilde{V}_i \upharpoonright A_i$ and $V_i' = \widetilde{V}_i \upharpoonright A_i'$. Furthermore, define $\mathcal{B}_j := \mathcal{V}_i \mathcal{A}_i$.

Proposition 5.4. *The pair $(\mathcal{A}_1, \mathcal{A}_2)$ is a weakly Lagrangian pair in $\left(\widetilde{\mathfrak{H}}_1, \widetilde{\mathfrak{H}}_2\right)$ with the isometries $\mathcal{V}_1$ and $\mathcal{V}_2$. Moreover, the following relations hold true:*

$$\mathcal{A}_i^* = A_i^* \oplus (A_i')^*, \quad \mathcal{B}_i = B_i \oplus B_i', \quad \mathcal{B}_i^* = B_i^* \oplus (B_i')^*, \tag{5.9}$$

$$\mathcal{A}_i \subseteq \widetilde{A}_i \subseteq \mathcal{B}_i^*, \quad \mathcal{B}_i \subseteq \widetilde{A}_i^* \subseteq \mathcal{A}_i^*, \tag{5.10}$$

$$\mathcal{E}_i := \mathcal{B}_i^* \ominus \mathcal{A}_i = E_i \oplus E_i', \tag{5.11}$$

where $E_i = B_i^* \ominus A_i$ and $E_i' = (B_i')^* \ominus A_i'$,

$$(\mathcal{B}_i^*)_\infty = (B_i^*)_\infty \oplus ((B_i')^*)_\infty, \tag{5.12}$$

$$P_{\mathcal{E}_i}(\mathcal{B}_i^*)_\infty = P_{E_i}(B_i^*)_\infty \oplus P_{E_i'}((B_i')^*)_\infty, \tag{5.13}$$

where $P_{\mathcal{E}_i}$ is the orthogonal projection from $\widetilde{\mathfrak{H}}_1 \times \widetilde{\mathfrak{H}}_2$ onto $\mathcal{E}_i$, P_{E_i} is the orthogonal projection from $\mathfrak{H}_1 \times \mathfrak{H}_2$ onto E_i and $P_{E_i'}$ is the orthogonal projection from $\mathfrak{H}_1' \times \mathfrak{H}_2'$ onto E_i', respectively.

Proof. Clearly, $\mathcal{A}_i = A_i \oplus A_i'$ implies that $\mathcal{A}_i^* = A_i^* \oplus (A_i')^*$, and since (A_1, A_2) and (A_1', A_2') are weakly Lagrangian pairs in $(\mathfrak{H}_1, \mathfrak{H}_2)$ and $(\mathfrak{H}_1', \mathfrak{H}_2')$, respectively, it follows that $(\mathcal{A}_1, \mathcal{A}_2)$ is a weakly Lagrangian pair in $\left(\widetilde{\mathfrak{H}}_1, \widetilde{\mathfrak{H}}_2\right)$ with the isometries $\mathcal{V}_i = \widetilde{V}_i \upharpoonright \mathcal{A}_i = V_i \oplus V_i'$, $i = 1, 2$. Moreover,

$$\mathcal{B}_j = \mathcal{V}_i \mathcal{A}_i = V_i A_i \oplus V_i' A_i' = B_j \oplus B_j',$$

and therefore $\mathcal{B}_j^* = B_j^* \oplus (B_j')^*$, proving (5.9).

The inclusion $\mathcal{A}_i \subset \widetilde{A}_i$ implies $\widetilde{A}_i^* \subset \mathcal{A}_i^*$. Also $\mathcal{B}_i \subset \widetilde{A}_i^*$ since $B_i \subset \widetilde{A}_i^*$ and $B_i' \subset \widetilde{A}_i^*$ and thus $\mathcal{B}_i = B_i \oplus B_i' \subset \widetilde{A}_i^*$. Then $\widetilde{A}_i \subset \mathcal{B}_i^*$, and then (5.10) is verified. The relations (5.11) and (5.12) follow directly from the definition of $\mathcal{A}_i$ and the last equality in (5.9). Finally, (5.13) is easily verified from the definitions which are involved. □

Theorem 5.5. *Let (A_1, A_2) be a weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$ with the isometries V_1 and V_2. Assume that $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ is a Lagrangian pair extension of (A_1, A_2) in $\left(\widetilde{\mathfrak{H}}_1, \widetilde{\mathfrak{H}}_2\right)$ with the isometries $\widetilde{V}_1$ and $\widetilde{V}_2$. If $\mathcal{A}_i = A_i \oplus A_i'$, $\mathcal{E}_i = \mathcal{B}_i^* \ominus \mathcal{A}_i$, as in Proposition 5.4, then there exists a decomposition*

$$\mathcal{E}_i = \mathcal{E}_{1i} \oplus \mathcal{E}_{2i}, \tag{5.14}$$

such that

$$\operatorname{dom} \mathcal{E}_{1i} = \operatorname{dom}(J_{ji}\mathcal{E}_{2j}), \tag{5.15}$$

$$\widetilde{A}_i = \mathcal{A}_i \oplus \mathcal{E}_{1i}, \tag{5.16}$$

$$\widetilde{V}_i = \mathcal{V}_i \oplus \mathcal{V}_i', \tag{5.17}$$

where

$$\mathcal{V}_i = \widetilde{V}_i \upharpoonright \mathcal{A}_i, \tag{5.18}$$

and $\mathcal{V}_i'$ is an isometry from $\mathcal{E}_{1i}$ onto $J_{ji}\mathcal{E}_{2j}$ of the form

$$\mathcal{V}_i'\{\alpha_i, \beta_j\} = \{\alpha_i, \beta_j'\}, \quad \|\beta_j\|_j = \|\beta_j'\|_j. \tag{5.19}$$

Conversely, if (A_1', A_2') is a weakly Lagrangian pair in $(\mathfrak{H}_1', \mathfrak{H}_2')$ with the isometries V_1' and V_2', and $\mathcal{V}_i = V_i \oplus V_i'$, $i = 1, 2$, are such that there exists a decomposition of $\mathcal{E}_i$ as in (5.14)*, and the pair $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ with the isometries $\widetilde{V}_1$ and $\widetilde{V}_2$ defined by* (5.17)–(5.19)*, then $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ is a weakly Lagrangian pair extension of $(\mathcal{A}_1, \mathcal{A}_2)$ in $\left(\widetilde{\mathfrak{H}}_1, \widetilde{\mathfrak{H}}_2\right)$, which is a Lagrangian pair extension of (A_1, A_2).*

Furthermore, $\widetilde{A}_1$ and $\widetilde{A}_2$ are both operators if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ are both operators, and

$$\mathcal{E}_{1i} \cap P_{\mathcal{E}_i}(B_i^*)_\infty = \{(0,0)\}\,.$$

Proof. A direct application of Theorem 4.2 and Lemma 4.3 leads to the statement of this theorem. □

The next result deals with the particular case when $A_i' = \widetilde{A}_i \cap (\mathfrak{H}_i \times \mathfrak{H}_j)$.

Proposition 5.6. *Let (A_1, A_2) be a weakly Lagrangian pair in $(\mathfrak{H}_1,\, \mathfrak{H}_2)$, and let $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ be a Lagrangian pair extension of (A_1, A_2) in $\left(\widetilde{\mathfrak{H}}_1,\, \widetilde{\mathfrak{H}}_2\right)$, where $\widetilde{\mathfrak{H}}_i = \mathfrak{H}_i \oplus \mathfrak{H}_i'$, $i = 1, 2$. Then, $A_i' = \widetilde{A}_i \cap (\mathfrak{H}_i' \times \mathfrak{H}_j')$ if and only if the projection P_i is one-to-one from $\mathcal{E}_{1i}$ onto $P_i\mathcal{E}_{1i}$, $i = 1, 2$. If $A_i' = \widetilde{A}_i \cap (\mathfrak{H}_i' \times \mathfrak{H}_j')$, then*

$$\dim E_i' \leq \dim E_i.$$

Similarly, $A_i = \widetilde{A}_i \cap (\mathfrak{H}_i' \times \mathfrak{H}_j')$ if and only if P_i' is one-to-one from $\mathcal{E}_{1i}$ onto $P_{1i}'\mathcal{E}_{1i}$, and, in this case

$$\dim E_i \leq \dim E_i'.$$

Moreover, if $A_i = \widetilde{A}_i \cap (\mathfrak{H}_i \times \mathfrak{H}_j)$ and $A_i' = \widetilde{A}_i \cap (\mathfrak{H}_i' \times \mathfrak{H}_j')$, then

$$\dim E_i = \dim E_i' = \dim \mathcal{E}_{1i} = \dim \mathcal{E}_{2i}.$$

Proof. Assume $A_i' = \widetilde{A}_i \cap (\mathfrak{H}_i' \times \mathfrak{H}_j')$ and let $(\alpha, \beta) \in \mathcal{E}_{1i}$, $P_i(\alpha, \beta) = (0, 0)$. Then $(\alpha, \beta) = P_i'(\alpha, \beta) \in E_i'$. Since A_i' and E_i' are orthogonal it follows that $(\alpha, \beta) = (0, 0)$, showing that P_i is one-to-one on $\mathcal{E}_i$. Conversely, assume P_i is one-to-one on $\mathcal{E}_{1i}$. Let $(a, b) \in \widetilde{A}_i$, $P_i(a, b) = (0, 0)$, that is

$$(a, b) \in \widetilde{A}_i \cap \left(\mathfrak{H}_i' \times \mathfrak{H}_j'\right).$$

Then

$$(a, b) = (x, y) + (u, v),$$

where $(x, y) \in \mathcal{A}_i$, $(u, v) \in \mathcal{E}_{1i}$, and

$$(0, 0) = P_i(a, b) = P_i(x, y) + P_i(u, v).$$

But, $P_i(x, y) \in A_i$, $P_i(u, v) \in P_i\mathcal{E}_{1i} \subseteq E_i$, and A_i orthogonal to E_i imply that

$$P_i(x, y) = P_i(u, v) = (0, 0).$$

This implies $(u, v) = (0, 0)$, or

$$(x, y) = P_i'(x, y) \in A_i',$$

that is

$$\widetilde{A}_i \cap \left(\mathfrak{H}_i' \times \mathfrak{H}_j'\right) \subseteq A_i'.$$

Clearly, $A_i' \subseteq \widetilde{A}_i \cap \left(\mathfrak{H}_i' \times \mathfrak{H}_j'\right)$, and thus

$$A_i' = \widetilde{A}_i \cap \left(\mathfrak{H}_i' \times \mathfrak{H}_j'\right).$$

Assume now that $A_i' = \widetilde{A}_i \cap \left(\mathfrak{H}_i' \times \mathfrak{H}_j'\right)$. Then $P_i\mathcal{E}_{1i} \subseteq \mathcal{E}_{1i}$, and, since P_i is one-to-one on $\mathcal{E}_{1i}$, it follows that $\dim \mathcal{E}_{1i} = \dim P_i\mathcal{E}_{1i} \leq \dim E_i$. Then

$$\mathcal{E}_i = \mathcal{E}_{1i} \oplus \mathcal{E}_{2i} = E_i \oplus E_i',$$

and $\dim \mathcal{E}_{1i} = \dim \mathcal{E}_{2i}$ (which follows from (5.15)) lead to

$$\dim \mathcal{E}_i = \dim E_i + \dim E_i' = 2\dim \mathcal{E}_{1i} \leq 2\dim E_i,$$

or, $\dim E_i' \leq \dim E_i$, as stated. The others statements have a similar proof. □

Corollary 5.7. *If $\operatorname{dom} A_i$ is dense in $\mathfrak{H}_i$, then*

$$A_i' = \widetilde{A}_i \cap \left(\mathfrak{H}_i' \times \mathfrak{H}_j'\right),$$

P_i is one-to-one on $\mathcal{E}_{1i}$ and

$$\dim E_i' \leq \dim E_i.$$

The next result identifies those Lagrangian pairs $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ which are operators in the case when $\operatorname{dom} A_i$ is dense in $\mathfrak{H}_i$, $i = 1, 2$.

Proposition 5.8. *Let (A_1, A_2) be a weakly Lagrangian pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$, let $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ be a Lagrangian pair extension of (A_1, A_2) in $(\mathfrak{H}_1', \mathfrak{H}_2')$, and assume that $\operatorname{dom} A_i$ is dense in $\mathfrak{H}_i$, $i = 1, 2$. Then $\widetilde{A}_1$ and $\widetilde{A}_2$ are both operators if and only if A_1' and A_2' are both operators.*

Proof. If $\widetilde{A}_i$ is an operator then A_i' is an operator as well since $A_i' \subset \widetilde{A}_i$. Assume now that A_1' and A_2' are both operators. Since $\operatorname{dom} A_i$ is dense in $\mathfrak{H}_i$ it follows that B_i^* is an operator and thus A_i is an operator. Therefore $\mathcal{A}_i$ is an operator. In order to show that $\widetilde{A}_i$ is an operator, let $(\alpha, \beta) \in \mathcal{E}_{1i} \cap P_{\mathcal{E}_i}(B_i^*)_\infty$. Since

$$(0,0) = P_i(\alpha, \beta) \in P_{\mathcal{E}_{1i}}(B_i^*),$$

and P_i is one-to-one on $\mathcal{E}_{1i}$ it follows that $(\alpha, \beta) = (0,0)$, showing that

$$\mathcal{E}_{1i} \cap P_{\mathcal{E}_i}(B_i^*)_\infty = (0,0),$$

which completes the proof. □

The operator version of Theorem 5.5 is now stated. Its proof follows immediately from Theorem 5.5 – Proposition 5.8.

Theorem 5.9. *Let (A_1, A_2) be a weakly Lagrangian densely defined pair in $(\mathfrak{H}_1, \mathfrak{H}_2)$. Assume that $\left(\widetilde{A}_1, \widetilde{A}_1\right)$ is a Lagrangian pair extension of (A_1, A_2) in $(\widetilde{\mathfrak{H}}_1, \widetilde{\mathfrak{H}}_2)$, where $\widetilde{\mathfrak{H}}_i = \mathfrak{H}_1 \oplus \mathfrak{H}_i'$. If $A_i' = \widetilde{A}_i \cap \left(\mathfrak{H}_i' \times \mathfrak{H}_j'\right)$, then (A_1', A_2') is a weakly Lagrangian pair in $(\mathfrak{H}_1', \mathfrak{H}_2')$. Let $\mathcal{A}_i = A_i \oplus A_i'$, $i = 1, 2$. Then (A_1, A_2) is a weakly Lagrangian operator pair in $\left(\widetilde{\mathfrak{H}}_1, \widetilde{\mathfrak{H}}_2\right)$ such that*

$$\mathcal{A}_i \subseteq \widetilde{A}_i \subseteq \mathcal{B}_i^*, \tag{5.20}$$

$$\mathcal{E}_i = \mathcal{B}_i^* \ominus \mathcal{A}_i = E_i \oplus E_i', \tag{5.21}$$

where,

$$E_i = B_i^* \ominus A_i, \quad E_i' = {B_i'}^* \ominus A_i'. \tag{5.22}$$

Moreover,

- (i) *$\mathcal{E}_i = \mathcal{E}_{1i} \oplus \mathcal{E}_{2i}$;*
- (ii) *$\operatorname{dom} \mathcal{E}_{1i} = \operatorname{dom} J_{ji} \mathcal{E}_{2j}$;*
- (iii) *P_i is one-to-one from $\mathcal{E}_{1i}$ onto $P_i \mathcal{E}_{1i} \subseteq E_{1i}$;*
- (iv) *$\widetilde{A}_i = \mathcal{A}_i \oplus \mathcal{E}_{1i}$;*
- (v) *$\dim E_i' \leq \dim E_i$;*
- (vi) *There is an isometry $\mathcal{V}_{1i}$ of $\mathcal{E}_{1i}$ onto $J_{ji} \mathcal{E}_{2j}$ of the form*

$$\mathcal{V}_{1i}\{\alpha, \beta\} = \{\alpha, \beta'\}, \quad \|\beta\|_j = \|\beta'\|_j, \quad \alpha \in \operatorname{dom} \mathcal{E}_{1i}.$$

Conversely, assume for a given weakly Lagrangian pair (A_1, A_2) that there exists a weakly Lagrangian operator pair (A_1', A_2') in $(\mathfrak{H}_1', \mathfrak{H}_2')$ such that (5.20)-(5.22) are satisfied, and the pair $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ and the isometries $\mathcal{V}_{1i}$, $i = 1, 2$, exist, satisfying (i)–(vi). Then $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ is a Lagrangian pair in $\left(\widetilde{\mathfrak{H}}_1, \widetilde{\mathfrak{H}}_2\right)$, and

$$A_i' = \widetilde{A}_i \cap \left(\mathfrak{H}_i' \times \mathfrak{H}_j'\right).$$

Remark 5.10. Assume that (A_1, A_2) is a maximal weakly Lagrangian operator pair in Theorem 5.9. Then for any Lagrangian operator pair extension $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ it follows that

$$A_i = \widetilde{A}_i \cap (\mathfrak{H}_i \times \mathfrak{H}_j),$$

and Proposition 5.6 then implies that $\dim \mathcal{E}_i = \dim E_i' = \dim \mathcal{E}_{1i} = \dim \mathcal{E}_{2i}$.

6. A class of weakly Lagrangian pairs of differential operators

Let $\mathfrak{H} = L^2(0,1)$ and let $\mathfrak{D}_0$ be the space of all absolutely continuous functions f on the interval $0 \le x \le 1$ satisfying the condition $f(0) = f(1)$, and such that its derivative f' is also an element of $\mathfrak{H}$. Clearly, $\mathfrak{D}_0$ is a dense subspace of $\mathfrak{H}$. Define two (linear) operators A_{j0}, $1 \le j \le 2$ in $\mathfrak{H}$ by $\operatorname{dom} A_{10} = \operatorname{dom} A_{20} = \mathfrak{D}_0$ and

$$A_{j0} f = L_{j0} f := \Omega_j f' + a_j f, \quad f \in \mathfrak{D}_0,$$

where Ω_j and a_j, $1 \le j \le 2$ are some non-zero complex numbers which satisfy the following conditions

$$|\Omega_1| = |\Omega_2|, \quad |a_1| = |a_2|, \quad \Omega_j \overline{a_j} \in \mathbb{R}, \quad 1 \le j \le 2.$$

Let $\mathfrak{D}_0^+$ be the set of all absolutely continuous continuous functions on the interval $0 \le x \le 1$ such that $f' \in \mathfrak{H}$. It is now easily seen that the operators A_{10}^* and A_{20}^* have the domains given by $\operatorname{dom} A_{10}^* = \operatorname{dom} A_{20}^* = \mathfrak{D}_0^+$ and are defined by

$$A_j^* f = L_j^+ f := -\overline{\Omega_j} f' + \overline{a_j} f, \quad f \in \mathfrak{D}_0^+, \quad 1 \le j \le 2.$$

From now on the indices j and k will run from 1 to 2 and are different whenever they appear in the same sentence. Since $\|A_j f\| = \|A_k^* f\|$ for all $f \in \mathfrak{D}_0$ it follows that the operators $V_{10} : A_{10} \to A_{20}^*$ and $V_{20} : A_{20} \to A_{10}^*$, defined by

$$V_{10}(f, L_{10} f) = (f, L_{20}^+), \quad V_{20}(f, L_{20} f) = (f, L_{10}^+), \quad f \in \mathfrak{D}_0$$

are two isometries. Therefore the pair (A_{10}, A_{20}) is a densely defined weakly Lagrangian pair in $(\mathfrak{H}, \mathfrak{H})$. Furthermore, the corresponding operators B_{10} and B_{20} are defined on $\mathfrak{D}_0$ and are given by

$$B_{10} f = L_{20}^+ f, \quad B_{20} f = L_{10}^+ f, \quad f \in \mathfrak{D}_0,$$

while their adjoints B_{10}^* and B_{20}^* are defined on $\mathfrak{D}_0^+$ and are given by

$$B_{10}^* f = L_{20} f, \quad B_{20}^* f = L_{10} f, \quad f \in \mathfrak{D}_0^+.$$

Let $\mathfrak{H}_0$ be a finite-dimensional subspace of $\mathfrak{H}$ and define the operators A_1 and A_2 to be the restrictions of A_{10} and A_{20} to $\operatorname{dom} A_1 = \operatorname{dom} A_2 = \mathfrak{D} := \mathfrak{D}_0 \cap \mathfrak{H}_0^\perp$. Thus, $A_1 \subset A_{10}$, $A_2 \subset A_{20}$, so that

$$A_{10}^* \subset A_1^*, \quad A_{20}^* \subset A_2^*. \tag{6.1}$$

Furthermore,

$$\operatorname{mul} A_j^* = (\operatorname{dom} A_j)^\perp = \operatorname{clos}(\mathfrak{D}_0^\perp \,\widehat{+}\, \mathfrak{H}_0),$$

which implies that

$$\mathfrak{H}_0 \subset \operatorname{mul} A_j^*, \quad 1 \le j \le 2. \tag{6.2}$$

It follows from (6.1) and (6.2) that

$$A_{j0}^* \,\widehat{+}\, (\{0\} \times \mathfrak{H}_0) \subset A_j^*, \quad 1 \le j \le 2.$$

Using similar arguments as in [5] it can be shown that in fact there is equality in the above inclusions, namely,

$$A_j^* = A_{j0}^* \,\widehat{+}\, (A_j^*)_\infty, \quad (A_j^*)_\infty := \{0\} \times \mathfrak{H}_0. \tag{6.3}$$

Furthermore, the single-valued part of A_j^* is given by

$$(A_j^*)_s f = A_{0j}^* f - P_0 A_{0j}^* f, \quad f \in \mathfrak{D}_0^+,$$

where P_0 is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_0$. Indeed, let $(f, g) \in (A_j^*)_s$, so that $(f, g) = (f, A_{j0}^* f + \varphi)$, for some $\varphi \in \mathfrak{H}_0$, and (f, g) is orthogonal to $(0, \psi)$ for all $\psi \in \mathfrak{H}_0$, that is

$$[A_{j0}^* f + \varphi, \psi] = 0,$$

for all $\psi \in \mathfrak{H}_0$, or $\varphi = -P_0 A_{j0}^* f$.

Consider V_1 and V_2 the restrictions of V_{10} and V_{20} to A_1 and A_2, respectively. Therefore, the pair (A_1, A_2) is now a (non-densely defined) Lagrangian pair in $(\mathfrak{H}, \mathfrak{H})$. It is clear that the corresponding operators B_1 and B_2 are the restrictions of B_{10} and B_{20} to $\mathfrak{D}$. Moreover,

$$\operatorname{mul} B_k^* = (\operatorname{dom} B_k)^\perp = (\operatorname{dom} A_j)^\perp = \operatorname{mul} A_j^* = \mathfrak{H}_0,$$

and

$$B_k^* = B_{k0}^* \mathbin{\widehat{+}} (B_k^*)_\infty, \quad (B_k^*)_s f = B_{k0}^* f - P_0 B_{k0}^* f, \quad f \in \mathfrak{D}_0^+, \tag{6.4}$$

where $(B_k^*)_\infty = (A_j^*)_\infty = \{0\} \times \mathfrak{H}_0$.

Remark 6.1. The fact that $A_j \subset B_j^*$, and A_j and $(A_k^*)_s$ are operators does not necessarily imply that $A_j \subset (B_j^*)_s$. Indeed, let $\mathfrak{H}_0$ be the one-dimensional space spanned by the function $\varphi(x) = x$ for all $0 \le x \le 1$. Assume that $A_j \subset (B_j^*)_s$, so that $\operatorname{ran} A_j \subset \operatorname{ran}(B_j^*)_s$, or

$$\mathfrak{K}_0 := (\operatorname{ran} A_j)^\perp \supset (\operatorname{ran}(B_j^*)_s)^\perp \supset \mathfrak{H}_0.$$

Thus, $\varphi \in \mathfrak{K}_0 = \ker A_j^*$, or by (6.3)

$$A_{j0}^* \varphi = -\overline{\Omega_j} \varphi' + \overline{a_j} \varphi = -b_j \varphi,$$

for some complex constant b_j. Since

$$-\overline{\Omega_j} \varphi' + \overline{a_j} \varphi + b_j \varphi = -\overline{\Omega_j} + (\overline{a_j} + b_j) x$$

is not the zero function it follows that the assumption is false. Thus, $A_j \not\subset (B_j^*)_s$.

Define now the subspaces $X_j = B_j^* \ominus A_j$ and $Y_j = A_j^* \ominus B_j$. Since

$$X_j = B_j^* \cap A_j^\perp = B_j^* \cap J_{kj} A_j^*,$$

it follows that $(f_j, g_j) \in X_j$ if and only if $(f_j, g_j) \in B_j^*$ and $(g_j, -f_j) \in A_j^*$. Furthermore, (6.3) and (6.4) imply that $(f_j, g_j) \in X_j$ if and only if $f_j \in \operatorname{dom} B_j^* = \mathfrak{D}_0^+$, $g_j \in \operatorname{dom} A_j^* = \mathfrak{D}_0^+$, and

$$g_j = L_j f_j - \varphi_j, \quad -f_j = L_j^+ g_j - \psi_j, \tag{6.5}$$

for some φ_j, $\psi_j \in \mathfrak{H}_0$. This system of equations has a solution for every pair $(\varphi_j, \psi_j) \in \mathfrak{H}_0 \times \mathfrak{H}_0$. When $\varphi_j = \psi_j = 0$ the pair (f_j, g_j) satisfies the system

$$g_j = L_j f_j, \quad -f_j = L_j^+ g_j,$$

and hence f_j and g_j are solutions of the same second order differential equation

$$(L_j^+ L_j + I) u = -\overline{\Omega_j} \Omega_j u'' + (1 + |a_j|^2) = 0.$$

It follows that

$$\dim X_j = 2 + 2 \dim \mathfrak{H}_0 = \dim Y_j$$

and $Y_j = J_{jk} X_j$ is the set of all (f_j, g_j) such that f_j, $g_j \in \mathfrak{D}_0^+$ and

$$g_j = L_j^+ f_j - \varphi_j, \quad -f_j = L_j g_j - \psi_j,$$

for some φ_j, $\psi_j \in \mathfrak{H}_0$. Applying Theorem 4.2, the Lagrangian extensions $\left(\widetilde{A}_1, \widetilde{A}_2\right)$ of (A_1, A_2) in $(\mathfrak{H}, \mathfrak{H})$ will be next described. In fact $\widetilde{A}_j = A_j \oplus X_{1j}$, where $X_j = X_{1j} \oplus X_{2j}$ with $\dim X_{1j} = \dim X_{2j} = 1 + \dim \mathfrak{H}_0$, and $\operatorname{dom} X_{1j} = \operatorname{dom} J_{kj} X_{2k}$, and there are two isometries $\widetilde{V}_j = V_j \oplus V_j'$, $1 \le j \le 2$, where V_j' is an isometry from X_{1j} onto $J_{kj} X_{2k}$ of the form

$$V_j'(\alpha_j, \beta_j) = (\alpha_j, \beta_j'), \quad \|\beta_j\| = \|\beta_j'\|.$$

Here, V_j is the isometry of A_j onto B_k given by

$$V_j(f, A_j f) = (f, B_k f), f \in \mathfrak{D}.$$

Since $\widetilde{A}_j = B_j^* \ominus X_{2j}$ it follows that $\widetilde{A}_j$ can be described as the set of all $(f_j, g_j) \in B_j^*$ such that

$$[g_j, \alpha_j] - [f_j, \beta_j] = 0, \tag{6.6}$$

for all $(\alpha_j, \beta_j) \in J_{jk}X_{1j}$. Since $B_j^* = B_{j0}^* \,\widehat{+}\, (\{0\} \times \mathfrak{H}_0)$ it follows that the pairs (f_j, g_j) and (α_j, β_j) can be expressed by

$$(f_j, g_j) = (f_j, L_j f_j + \psi_j), \quad (\alpha_j, \beta_j) = (\alpha_j, L_j^+ \alpha_j + \chi_j),$$

with $f_j \in \mathfrak{D}_0^+$, $\alpha_j \in \operatorname{dom} J_{jk}X_{2j}$, and for some elements ψ_j, $\chi_j \in \mathfrak{H}_0$. The condition (6.6) may then be written as

$$\langle f_j \alpha_j \rangle_j + [\psi_j, \alpha_j] - [f_j, \chi_j] = 0, \tag{6.7}$$

where

$$\langle f_j \alpha_j \rangle_j = [L_j f_j, \alpha_j] - [f_j, L_j^+ \alpha_j] = \Omega_j[\overline{\alpha_j(1)} f_j(1) - \overline{\alpha_j(0)} f_j(0)].$$

In (6.7) are in fact $1 + \dim \mathfrak{H}_0$ independent boundary-integral conditions specifying the elements of $\widetilde{A}_j$.

Assume now that $\dim \mathfrak{H}_0 = 1$ and that $\mathfrak{H}_0$ is spanned by a function $\varphi \in \mathfrak{H}$ with $\|\varphi\| = 1$. Then in the above $\psi_j = c_j \varphi$, $\chi_j = d_j \varphi$ for some complex constants c_j and d_j. Let $(\alpha_{1j}, L_j^+ \alpha_{1j} + d_{1j}\varphi)$, $(\alpha_{2j}, L_j^+ \alpha_{2j} + d_{2j}\varphi)$ be a basis for the subspace $J_{jk}X_{2j}$. Then (6.7) is equivalent to the following equations

$$\langle f_j \alpha_{lj} \rangle_j + c_j[\varphi, \alpha_{lj}] - \overline{d_{lj}}[f_j, \varphi] = 0, \quad 1 \le l \le 2, \tag{6.8}$$

for $(f_j, L_j f_j + c_j \varphi) \in \widetilde{A}_j$. Since $(\widetilde{A}_j)_\infty \subset (B_j^*)_\infty = \{0\} \times \mathfrak{H}_0$ it follows that $\widetilde{A}_j$ is not an operator if and only if $(o, \varphi) \in \widetilde{A}_j$, and using (6.8) it follows that this is the case if and only if

$$[\varphi, \alpha_{1j}] = [\varphi, \alpha_{2j}] = 0, \tag{6.9}$$

that is $\varphi \in \operatorname{dom}(J_{jk}X_{2j})^\perp = \operatorname{dom}(X_{1k})^\perp$. Therefore, two cases have to be considered:

A. $\widetilde{A}_j$ is an operator and, for instance, $[\varphi, \alpha_{1j}] \neq 0$;

B. $\widetilde{A}_j$ is not an operator, and (6.9) holds true.

In the first case it follows from (6.8) that c_j can be uniquely obtained, and then $\widetilde{A}_j$ can be specified as the set of all pairs $(f_j, L_j f_j + c_j \varphi) \in B_{j0}^*$ such that the following relations hold:

$$\langle f_j \alpha_j \rangle_j + [f_j, d_j \varphi] = 0, \quad c_j = \frac{1}{[\varphi, \alpha_{1j}]} \left([f_j, d_j \varphi] - \langle f_j \alpha_{1j} \rangle_j \right),$$

where

$$\alpha_j = [\alpha_{1j}, \varphi]\alpha_{2j} - [\alpha_{2j}, \varphi]\alpha_{1j}, \quad d_j = [\alpha_{2j}, \varphi] d_{1j} - [\alpha_{1j}, \varphi] d_{2j}.$$

In the latter case, namely **B**, it follows that $[f_j, \varphi] = 0$ for all $f_j \in \operatorname{dom} \widetilde{A}_j$, since

$$(\operatorname{dom} \widetilde{A}_j)^\perp = \operatorname{mul} \widetilde{A}_j^* = \operatorname{mul} \widetilde{A}_j = \mathfrak{H}_0,$$

and this relation can be used in (6.8) to obtain $\widetilde{A}_j$ as the set of all pairs $(f_j, L_j f_j + c_j \varphi) \in B_{0j}^*$ satisfying the following relation

$$\langle f_j \alpha_{1j} \rangle_j = 0, \quad \langle f_j \alpha_{2j} \rangle_j = 0, \quad [f_j, \varphi] = 0. \tag{6.10}$$

It is easily seen that the second condition in (6.10) is superfluous, and thus $(f_j, L_j f_j + c_j \varphi)$ is an element of $\widetilde{A}_j$ if and only if

$$\langle f_j \alpha_{1j} \rangle_j = 0, \quad [f_j, \varphi] = 0, \quad f_j \in \mathfrak{D}_0^+, \tag{6.11}$$

and c_j is a complex number. The single-valued parts of $\widetilde{A}_1$ and $\widetilde{A}_2$, namely $(\widetilde{A}_1)_s$ and $(\widetilde{A}_2)_s$ determine a densely defined Lagrangian pair in $(\mathfrak{H}_0^\perp, \mathfrak{H}_0^\perp)$, and it is specified by the boundary-integral conditions (6.11) and

$$(\widetilde{A}_j)_s f_j = L_j f_j - (L_j f_j, \varphi)\varphi, \quad 1 \le j \le 2.$$

Notice that since $\alpha_{1j} \in \operatorname{dom} X_{1j} \subset \operatorname{dom} \widetilde{A}_j$ it follows that α_{1j} is a function satisfying

$$\langle \alpha_{1j} \alpha_{1j} \rangle_j = \Omega_j \left(|\alpha_{1j}(1)|^2 - |\alpha_{1j}(0)|^2 \right) = 0.$$

Hence, $\alpha_{1j}(0) = \overline{\theta_{1j}} \alpha_{1j}(1)$, where $|\theta_{11}| = |\theta_{12}| = 1$, and the conditions in (6.11) become

$$f_j(1) = \theta_{1j} f_j(0), \quad [f_j, \varphi] = 0 \quad \text{for all} \quad f_j \in \mathfrak{D}_0^+.$$

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