Methods of Functional Analysis and Topology
Vol. 12 (2006), no. 2, pp. 113–123

# SOME RESULTS ON THE SPACE OF HOLOMORPHIC FUNCTIONS TAKING THEIR VALUES IN B-SPACES

B. AQZZOUZ, M. T. BELGHITI, M. H. ELALJ, AND R. NOUIRA

Abstract. We define a space of holomorphic functions $O_1(U,E/F)$, where $U$ is an open pseudo-convex subset of $\mathbb{C}^n$, $E$ is a b-space and $F$ is a bornologically closed subspace of $E$, and we prove that the b-spaces $O_1(U,E/F)$ and $O(U,E)/O(U,F)$ are isomorphic.

## 1. Introduction and notations

The Bartle-Graves theorem [3] says that a surjective bounded linear mapping between Banach spaces has a right inverse which is continuous (but not necessary linear) and bounded on bounded subsets. In this paper we shall study a problem which arises from the Bartle-Graves theorem for holomorphic functions between b-spaces in the sense of L. Waelbroeck [12], which is a class more general than the class of bornological spaces (locally convex spaces), in the sense of N. Bourbaki [8]. More precisely, a b-space is a bornological inductive limit of Banach spaces.

Since the functor projective limit $\varprojlim$ is left exact and generally not exact on the category of b-spaces $\mathbf{b}$ [7], the bounded linear mapping $O(U,u) : O(U,E) \longrightarrow O(U,F)$, $f \longmapsto u \circ f$ is not necessary bornologically surjective whenever $u : E \longrightarrow F$ is a bornologically surjective bounded linear mapping between b-spaces. It follows that $O(U,E/F) \neq O(U,E)/O(U,F)$, where $U$ is an open pseudo-convex subset of $\mathbb{C}^n$, $E$ is a b-space, $F$ is a bornologically closed subspace of $E$ and $O(U,E) = \varprojlim_{V\in\mathcal{C}_U} (O(V)\,\varepsilon E)$.

In this paper, we shall construct a new b-space of holomorphic functions $O_1(U,E)$ as the kernel of the operator $\overline{\partial} : \mathcal{E}(U,E) \longrightarrow \mathcal{E}(U,E) \otimes \mathbb{C}^{n*}$, where $\mathbb{C}^{n*}$ is the space of antilinear forms on $\mathbb{C}^n$, and we will show that the two b-spaces $O(U,E)/O(U,F)$ and $O_1(U,E/F)$ are isomorphic where $\mathcal{E}(U,E)$ is the space of functions $f : U \longrightarrow E$ such that for all $x \in U$ there exist a coordinate neighbourhood $U_x$ of $x$ and a completant bounded subset $B_x$ of $E$ such that $f \in C^\infty(U_x, E_{B_x})$.

For this, we shall need to prove that if $U$ is an open subset of $\mathbb{R}^n$ and $u : E \longrightarrow F$ is a bornologically surjective bounded linear mapping between b-spaces, then the bounded linear mapping $\mathcal{E}(U,u) : \mathcal{E}(U,E) \longrightarrow \mathcal{E}(U,F), f \longmapsto u\circ f$ is bornologically surjective. As a consequence, if $E$ is a b-space and $F$ is a bornologically closed subspace of $E$, then $\mathcal{E}(U,E/F) = \mathcal{E}(U,E)/\mathcal{E}(U,F)$. For $r \in \mathbb{N}$, we define the space of functions of class $C^r$ taking their values in a b-space and we shall prove that if $U$ is a smooth manifold, $V$ is an open subset of $U$, which is relatively compact and $E$ is a b-space, then $C^r(\overline{V},E) \simeq C^r(\overline{V})\varepsilon E$ where $\varepsilon$ is the $\varepsilon$-product in the category of b-spaces. If $U$ is a smooth manifold, countable at infinity, we shall prove that the functors $C^\infty(U,.)$ and $\mathbf{b}(\mathcal{E}'(U)_i\,,.)$ are isomorphic on the category of b-spaces where $\mathcal{E}'(U)$ is the space of distribution with compact support on $U$, on which we put the equicontinuous boundedness. Next, if $U$ is a connected open subset of $\mathbb{C}^n$ and $E$ a b-space, we will define two b-spaces of holomorphic

2000 *Mathematics Subject Classification.* 46M05, 46M15, 46M40.
*Key words and phrases.* Holomorphic function, b-space, functor, category.

functions, the first one is $O(U,E) = \varprojlim_{V\in\mathcal{C}_U}(O(V)\,\varepsilon E)$, where $\mathcal{C}_U$ is the set of all open relatively compact subsets of $U$, and the second one is $O_1(U,E)$ defined as the kernel of the operator $\overline{\partial} : \mathcal{E}(U,E) \longrightarrow \mathcal{E}(U,E) \otimes \mathbb{C}^{n*}$. We will define a natural morphism $i : O(U,E)/O(U,F) \longrightarrow O_1(U,E/F)$ which is injective, and we shall prove that if $U$ is an open pseudo-convex subset of $\mathbb{C}^n$, $E$ is a b-space and $F$ a bornologically closed subspace of $E$, then $O_1(V,E/F)$ is naturally isomorphic to the projective limit of the projective system of b-spaces $(O(V,E)/O(V,F))_{V\in\mathcal{C}_U}$, where $V$ ranges over the set of open relatively compact subsets of $U$. Finally, we will deduce that the two b-spaces $O(U,E)/O(U,F)$ and $O_1(U,E/F)$ are isomorphic.

Let us fix some notations and recall some definitions that will be used in this paper. Let **E.V.** be the category of vector spaces and linear mappings over the scalar field $\mathbb{R}$ or $\mathbb{C}$, and **Ban** the category of Banach spaces and bounded linear mappings.

**1-** Let $E$ be a real or complex vector space, and let $B$ be an absolutely convex set of $E$. Let $E_B$ be the vector space generated by $B$ i.e. $E_B = \cup_{\lambda>0}\lambda B$. The Minkowski functional of $B$, $\|x\|_B = \inf\{\lambda > 0 : x \in \lambda B\}$ is a semi-norm on $E_B$. It is a norm if and only if $B$ does not contain any nonzero subspace of $E$. The set $B$ is completant if its Minkowski functional is a Banach norm.

A bounded structure $\beta$ on a vector space $E$ is defined by a set of "bounded" subsets of $E$ with the following properties:
**1)** Every finite subset of $E$ is bounded; **2)** every union of two bounded subsets is bounded; **3)** every subset of a bounded subset is bounded; **4)** a set homothetic to a bounded subset is bounded; **5)** each bounded subset is contained in a completant bounded subset.

A b-space $(E,\beta)$ is a vector space $E$ with a boundedness $\beta$. A subspace $F$ of a b-space $E$ is bornologically closed if the subspace $F \cap E_B$ is closed in $E_B$ for every completant bounded subset $B$ of $E$.

Given two b-spaces $(E,\beta_E)$ and $(F,\beta_F)$, a linear mapping $u : E \longrightarrow F$ is bounded, if it maps bounded subsets of $E$ into bounded subsets of $F$. The mapping $u : E \longrightarrow F$ is bornologically surjective if for every $B' \in \beta_F$, there exists $B \in \beta_E$ such that $u(B) = B'$. Let $(E,\beta_E)$ be a b-space. A b-subspace of $E$ is a subspace $F$ with a boundedness $\beta_F$ such that $(F,\beta_F)$ is a b-space and $\beta_F \subseteq \beta_E$. We denote by $\mathbf{b}(E_1,E_2)$ the space of all bounded linear mappings $E_1 \longrightarrow E_2$ and by $\mathbf{b}$ the category of b-spaces and bounded linear mappings.

Let $E$ be a topological vector space, a subset $B$ is bounded for the von Neumann boundedness of $E$ if it is absorbed by all neighbourhoods of the origin. The von Neumann boundedness is a vector boundedness, it is separated if and only if the topological vector space is separated. If $E$ is a locally convex space, its von Neumann boundedness is convex (i.e.is stable under the formation of convex hulls) but there exist topological vector spaces $E$ whose topologies are not locally convex but their von Neumann boundedness are convex (for example, take $I$ a set not countable and $\mathbb{C}^{(I)}$ with the strongest vector topology).

If $E$ is a locally convex space in which each bounded closed absolutely convex set is completant, then the space $E$ endowed with its von Neumann boundedness is a b-space. For more information about b-spaces we refer the reader to [6] and [12].

**2-** The $\varepsilon$-product of two Banach spaces $E$ and $F$ is the Banach space $E\varepsilon F$ of linear mappings $E' \longrightarrow F$ whose restrictions to the closed unit ball $B_{E'}$ of $E'$ are $\sigma(E',E)$-continuous, endowed with the norm of uniform convergence on $B_{E'}$ where $E'$ is the topological dual of $E$. It follows from Proposition 2 of [11] that the $\varepsilon$-product is symmetric i.e. the Banach spaces $E\varepsilon F$ and $F\varepsilon E$ are isometrically isomorphic. If $E_i$, $F_i$ are Banach spaces and $u_i : E_i \longrightarrow F_i$ are bounded linear mappings, $i = 1, 2$, the $\varepsilon$-product of $u_1$ and $u_2$ is the bounded linear mapping: $u_1\varepsilon u_2 : E_1\varepsilon E_2 \longrightarrow F_1\varepsilon F_2$, $f \longmapsto u_2 \circ f \circ u_1'$ , where

$u_1'$ is the dual mapping of $u_1$. It is clear that $u_1 \varepsilon u_2$ is injective whenever $u_i$ is injective for $i = 1, 2$.

If $G$ is a Banach space and $F$ is a closed subspace of a Banach space $E$, then $G\varepsilon F$ is a closed subspace of $G\varepsilon E$. For more information about the $\varepsilon$-product we refer the reader to [11] and [8].

3- Recall that a bounded linear mapping $u : E \longrightarrow F$ between two Banach spaces is nuclear if there exist bounded sequences $(x_n')_n \subset E'$ and $(y_n)_n \subset F$, and there exists $(\lambda_n)_n \subset l^1$ such that, for all $x \in E$ we have $u(x) = \sum_{n=1}^{+\infty} \lambda_n x_n'(x) y_n$. A b-space $G$ is nuclear if all bounded completant subset $B$ of $G$ is included in a bounded completant subset $A$ of $G$ such that the inclusion mapping $i_{AB} : G_B \longrightarrow G_A$ is nuclear. See [6] for more information about nuclear b-spaces.

## 2. Main results

Let $G$ be a b-space and $E$ a Banach space, if $A$ and $B$ are completant bounded subsets of $G$ such that $A \subset B$, the mapping $G_A\varepsilon E \longrightarrow G_B\varepsilon E$ is injective. So we define $G\varepsilon E = \cup_B (G_B\varepsilon E)$.

An element of $G\varepsilon E$ belongs to some Banach space $G_A\varepsilon E$, where $A$ is a completant bounded subset of $G$. So it is a bounded linear mapping $(G_A)^{'} \longrightarrow E$ whose restriction to the closed unit ball of $(G_A)^{'}$ is continuous for the weak topology $\sigma((G_A)^{'}, G_A)$ where $(G_A)^{'}$ is the topological dual of the Banach space $G_A$.

The b-space $G\varepsilon E$ is called the $\varepsilon$-product of $G$ and $E$. It is clear that if $F$ is a bornologically closed subspace of $G$, the space $F\varepsilon E$ is a bornologically closed subspace of $G\varepsilon E$.

Now if $G$ and $E$ are two b-spaces, the $\varepsilon$-product of $G$ and $E$ is the b-space $G\varepsilon E = \cup_{B,C} G_B\varepsilon E_C$, where $B$ and $C$ respectively ranges over the bounded completant subsets of $G$ and $E$ respectively.

Let $U$ be an open subset of $\mathbb{R}^n$ and let $\mathcal{C}_U$ be the set of all open relatively compact subsets of $U$. If $V \in \mathcal{C}_U$, the space $\mathcal{E}(V)$ with its von Neumann boundedness is a nuclear b-space, and then defines an exact functor $\mathcal{E}(V)\varepsilon. = \mathcal{E}(V, .)$ on the category **b**. If $E$ is a b-space and $F$ is a bornologically closed subspace of $E$, the b-space

$$E(V, E/F) = E(V)\varepsilon(E/F)$$

is defined as

$$(\mathcal{E}(V)\varepsilon E)/(\mathcal{E}(V)\varepsilon F) = \mathcal{E}(V, E)/\mathcal{E}(V, F).$$

If $W, V \in \mathcal{C}_U$ such that $W \subset V$, we have a bounded linear mapping

$$\Psi : \mathcal{E}(V) \longrightarrow \mathcal{E}(W), \quad f \longmapsto f_{|W}$$

where $f_{|W}$ is the restriction of $f$ to $W$. We can show that $(\mathcal{E}(V))_{V\in\mathcal{C}_U}$ is a projective system in the category **b**. If $E$ is a b-space the family $(\mathcal{E}(V)\varepsilon E)_{V\in\mathcal{C}_U}$ is also a projective system in **b**, and then has a projective limit in the category **b.**

We define

$$\mathcal{E}(U, E) = \varprojlim_{V\in\mathcal{C}_U} (\mathcal{E}(V)\varepsilon E,$$

where $\mathcal{C}_U$ is the set of all open relatively compact subsets of $U$.

**Proposition 2.1.** *Let $U$ be an open subset of $\mathbb{R}^n$ and $u : E \longrightarrow F$ a bornologically surjective bounded linear mapping between b-spaces. Then the bounded linear mapping $\mathcal{E}(U, u) :$ cal $\mathcal{E}(U, E) \longrightarrow \mathcal{E}(U, F)$, $f \longmapsto u \circ f$ is bornologically surjective.*

*Proof.* Let $(V_i)_{i\in I}$ be a locally finite open covering of $U$, such that, for all $i \in I$, the set $V_i$ is relatively compact. Let $(\varphi_i)_{i\in I}$ be a partition of unit subordinate to the covering $(V_i)_{i\in I}$ of $U$. Since each b-space $\mathcal{E}(V_i)$ is nuclear, the bounded linear mapping

$$\mathcal{E}(V_i, u) = Id_{\mathcal{E}(V_i)}\varepsilon u : E(V_i)\varepsilon E \longrightarrow E(V_i)\varepsilon F$$

is bornologically surjective. If we apply the projective limit functor $\varprojlim_{V_i}$, we obtain the following bounded linear mapping

$$E(U, u) = \varprojlim_{V_i}(Id_{\mathcal{E}(V_i)}\varepsilon u) : \varprojlim_{V_i}(\mathcal{E}(V_i)\varepsilon E) \longrightarrow \varprojlim_{V_i}(\mathcal{E}(V_i)\varepsilon F).$$

We shall prove that $\mathcal{E}(U, u)$ is bornologically surjective. Let $B$ be a bounded subset of $\varprojlim_{V_i}(\mathcal{E}(V_i)\varepsilon F)$. Since the set

$$B_i = \{g_i = g_{|_{V_i}} : g \in B\}$$

is bounded in $\mathcal{E}(V_i)\varepsilon F$, there exists a bounded subset $C_i$ of $\mathcal{E}(V_i)\varepsilon E$ such that $\mathcal{E}(V_i, u)(C_i) = B_i$.

Let

$$C = \Big\{ \sum_i \varphi_i f_i : \text{there exists } g \in B \text{ with } \mathcal{E}(V_i, u)(f_i) = g_i \text{ and for all } i \in I,\ f_i \in C_i \Big\}.$$

It is a bounded subset of $\mathcal{E}(U, E) = \varprojlim_{V_i}(\mathcal{E}(V_i)\varepsilon E)$ and $\mathcal{E}(U, u)(C) = B$. □

**Corollary 2.2.** *Let $U$ be an open subset of $\mathbb{R}^n$, $E$ a b-space and $F$ a bornologically closed subspace of $E$. Then $\mathcal{E}(U, E/F) = \mathcal{E}(U, E)/\mathcal{E}(U, F)$.*

*Proof.* In fact, in the category **b,** the b-space $E/F$ defines the following exact sequence:

$$(0, v, w, 0) : 0 \longrightarrow F \longrightarrow E \longrightarrow E/F \longrightarrow 0.$$

Its image by the functor $\mathcal{E}(U, .) : \mathbf{b} \longrightarrow \mathbf{b}$ is the following left exact sequence:

$$(0, \mathcal{E}(U, v), \mathcal{E}(U, w)) : 0 \longrightarrow \mathcal{E}(U, F) \longrightarrow \mathcal{E}(U, E) \longrightarrow \mathcal{E}(U, E/F).$$

We would like to prove the exactness of the sequence

$$(0, E(U, v), E(U, w), 0) : 0 \longrightarrow E(U, F) \longrightarrow E(U, E) \longrightarrow E(U, E/F) \longrightarrow 0$$

in the category **b**. It is clear that the mapping $\mathcal{E}(U, v)$ is injective, and by Proposition 2.1, the bounded linear mapping

$$\mathcal{E}(U, w) : \mathcal{E}(U, E) \longrightarrow \mathcal{E}(U, E/F)$$

is bornologically surjective.

It remains to show that the image of $\mathcal{E}(U, F)$ by the mapping $\mathcal{E}(U, v)$ coincides (vectorially and bornologically) with the kernel of $\mathcal{E}(U, w)$. This is clear, by what we have just proved in Proposition 2.1, the image of $\mathcal{E}(U, v)$ is $\mathcal{E}(U, v(E))$. But this space coincides with the kernel of $\mathcal{E}(U, w)$ (i.e. the b-space $\mathcal{E}(U, w^{-1}(0))$). Considered as a mapping from $F$ to $w^{-1}(0)$, $v$ is bornologically surjective, and by Proposition 2.1, the bounded linear mapping

$$E(U, v) : E(U, F) \longrightarrow E\left(U, w^{-1}(0)\right)$$

is bornologically surjective. But clearly

$$E\left(U, w^{-1}(0)\right) = (E(U, w))^{-1}(0),$$

and the sequence

$$0 \longrightarrow \mathcal{E}(U, F) \longrightarrow \mathcal{E}(U, E) \longrightarrow \mathcal{E}(U, E/F) \longrightarrow 0$$

is then exact in the category **b**. □

**Definition 2.3.** *Let $U$ be an open subset of $\mathbb{R}^n$, $r \in \mathbb{N}^*$ and $E$ a b-space. Then $f \in C^r(U,E)$ if for each $x \in U$, there exists a neighbourhood $U_x$ of $x$ and a bounded completant subset $B_x$ of $E$ such that $f_{|U_x} \in C_b^r(U_x, E_{B_x})$ where $C_b^r(U_x, E_{B_x})$ is the space of mappings $U_x \longrightarrow E_{B_x}$ of class $C^r$ such that the function and its derivatives (up to $r$) are bounded.*

*A subset $B$ of $C^r(U,E)$ is bounded if for each $x \in U$, there exists a neighbourhood $U_x$ of $x$ and a bounded completant subset $B_x$ of $E$ such that for any $k \in \mathbb{N}$ with $k \leq r$, the set $D^k B_{x_{|U_x}} = \{D^k f_{|U_x} : f \in B_x\}$ is bounded in $C_b^r(U_x, E_{B_x})$.*

**Proposition 2.4.** *Let $U$ be a smooth manifold, $V$ an open relatively compact subset of $U$ and $E$ a b-space. Then $C^r(\overline{V}, E) \simeq C^r(\overline{V})\varepsilon E$.*

*Proof.* For the Banach space $E_B$, it is well known that $C^r(\overline{V}, E_B) \simeq C^r(\overline{V})\varepsilon E_B$ where $B$ ranges over bounded completant subsets of $E$. If we apply the inductive limit $\varinjlim_B$, which is an exact functor on the category **b** [7], we obtain the following isomorphism

$$\varinjlim_B C^r(\overline{V}, E_B) \simeq \varinjlim_B (C^r(\overline{V})\varepsilon E_B).$$

By definition we have

$$C^r(\overline{V}, E) = \varinjlim_B C^r(\overline{V}, E_B)$$

and

$$C^r(\overline{V})\varepsilon E = \varinjlim_B (C^r(\overline{V})\varepsilon E_B).$$

This shows the Proposition. □

When the manifold $U$ is countable at infinity, the functor $C^r(U,.)$ is the projective limit of a countable family of functors $C^r(V_m,.)$, where $V_m$ is an open relatively compact subset of $U$.

More precisely, a manifold is countable at infinity if it is the union of an increasing sequence of open sets, each of them being relatively compact in the interior of the following one. Then the spaces $C^r(V_m, E)$ and the restriction mappings

$$C^r(V_m, E) \longrightarrow C^r(V_n, E) \quad (\text{for } n \leq m)$$

constitute a projective system, whose projective limit can be considered.

**Proposition 2.5.** *Let $U$ be a smooth manifold, countable at infinity. For all $r \in \mathbb{N}^*$, the functors $C^r(U,.)$, $\varprojlim_m C^r(V_m,.)$ and $\varprojlim_m C^r(\overline{V}_m,.)$ are naturally isomorphic on the category* **b.**

*Proof.* The isomorphism

$$\varprojlim_m C^r(V_m,.) \simeq \varprojlim_m C^r(\overline{V}_m,.)$$

is clear as $V_n$ is relatively compact in the interior of $V_{n+1}$, for any Banach space $E$, there exist restriction mappings

$$C^r(V_{n+1}, E) \longrightarrow C^r(V_n, E),$$
$$C^r(\overline{V}_{n+1}, E) \longrightarrow C^r(\overline{V}_n, E)$$

and

$$C^r(V_{n+1}, E) \longrightarrow C^r(\overline{V}_n, E)$$

which make commutative the following diagram:

$$\begin{array}{ccc} C^r(\overline{V}_{n+1}, E) & \longrightarrow & C^r(\overline{V}_n, E) \\ \downarrow & \nearrow & \downarrow \\ C^r(V_{n+1}, E) & \longrightarrow & C^r(V_n, E) \end{array}.$$

Now, if $E$ is a Banach space, it is clear that $C^r(U,E)$ is isomorphic to $\varprojlim_m C^r\left(\overline{V}_m, E\right)$. If $E$ is a b-space, let $f \in C^r(U,E)$. As $\overline{V}_m$ is compact for each $m$, there exists a bounded completant subset $B_m$ of $E$ such that $f_{|\overline{V}_m} \in C^r\left(\overline{V}_m, E_{B_m}\right)$.

Also, if $C$ is a bounded subset of $C^r(U,E)$, for each $m$, there exists a bounded completant subset $B_m$ of $E$ such that $C_{|\overline{V}_m} = \{f_{|\overline{V}_m}, f \in C\}$ is bounded in $C^r\left(\overline{V}_m, E_{B_m}\right)$. Thus a bounded linear mapping

$$C^r(U,E) \longrightarrow \varprojlim_m C^r\left(\overline{V}_n, E\right)$$

can be constructed. It is easy to see that this mapping is an isomorphism. □

Now we define a functor of functions of class $C^\infty$ taking their values in a b-space that we call $C^\infty(U,.)$. For each $r \in \mathbb{N}$, we have defined $C^r(U,.)$. If $r' \geq r$, we have a natural bounded linear mapping $C^{r'}(U) \longrightarrow C^r(U)$. The family $(C^r(U))_{r\in\mathbb{N}}$ is a projective system, and then has a projective limit in the category **b** [7].

**Definition 2.6.** *Let $U$ be a manifold. Then $C^\infty(U,.) \simeq \varprojlim_r C^r(U,.)$.*

For each $r \in \mathbb{N}$, the functor $C^r(U,.) : \mathbf{b} \longrightarrow \mathbf{b}$ is left exact. A projective limit of exact or of left exact functors is left exact. The functor $C^\infty(U,.) : \mathbf{b} \longrightarrow \mathbf{b}$ is therefore left exact which is not exact.

*Remark 2.7.*

**1-** By G. M. Khenkin [10] and W. Kaballo [9], if the dimension of the manifold $U$ is less than one, the functor $C^r(U,.)$ is not exact on the category **b.**

**2-** If $r \in \mathbb{R}^+\backslash\mathbb{N}$, we denote by $C^r(X)$ the space of functions of class $C^{[r]}$ on $X$ such that for all $k \in \mathbb{N}^n$, $|k| \leq [r]$, $D^k f$ is continuously o-Hölderian of exponent $r-[r]$. By J. Frampton and A. Tromba [4], and a paper of the first author [1], the functor $C^r(U,.) : \mathbf{b} \longrightarrow \mathbf{b}$ is exact, and it follows that if $E$ is a b-space and $F$ is a bornologically closed subspace of $E$, we have

$$C^r(X, E/F) = C^r(X,E)/C^r(X,F).$$

Let $U$ be a smooth manifold, countable at infinity, we denote by $\mathcal{E}_{Fré}(U)$ the space of infinitely differentiable mappings on $U$, with the topology of uniform convergence of the functions and their derivatives on compact subsets of $U$. It is clear that the topology of $\mathcal{E}_{Fré}(U)$ is defined by the family of semi-norms $(p_{n,r})_{(n,r)\in\mathbb{N}\times\mathbb{N}}$, where

$$p_{n,r}(f) = \sup\left\{D^k f(x) : |k| \leq r,\ x \in \overline{V}_n\right\}$$

and $(V_n)_n$ is a sequence of open relatively compact subsets of $U$ such that for all $n \in \mathbb{N}$, $V_n$ is contained in the interior of $V_{n+1}$.

It is also clear that $\mathcal{E}_{Fré}(U)$ is a Fréchet space which is a projective limit of the projective system $(C^r(\overline{V}_n))_{(n,r)\in\mathbb{N}\times\mathbb{N}}$ in the category of separated locally convex spaces **ELCS.**

Let $U$ be a smooth manifold, countable at infinity, we denote by $\mathcal{E}'(U)$, the space of distributions with compact support on $U$, on which we take the equicontinuous boundedness i.e. a subset $B$ of $\mathcal{E}'(U)$ is bounded if it is equicontinuous. It is a b-space which is the bornological dual of the Fréchet space $\mathcal{E}_{Fré}(U)$.

**Proposition 2.8.** *If $U$ is a smooth manifold, countable at infinity, the functors $C^\infty(U,.)$ and $\mathbf{b}(\mathcal{E}'(U),.)$ are isomorphic on the category* **b**.

*Proof.* The b-space $C^\infty(U,E)$ is the projective limit of the projective system

$$(C^r(\overline{V}_n, E))_{(n,r)\in\mathbb{N}\times\mathbb{N}}.$$

By Proposition 2.4, we have $C^r\left(\overline{V}_n, E\right) \simeq C^r\left(\overline{V}_n\right)\varepsilon E$, and hence we can write

$$\begin{aligned} C^\infty(U,E) &\simeq \varprojlim_{r\in\mathbb{N}}\varprojlim_{n\in\mathbb{N}} C^r\left(\overline{V}_n, E\right) \\ &\simeq \varprojlim_{r\in\mathbb{N}}\varprojlim_{n\in\mathbb{N}} \left(C^r\left(\overline{V}_n\right)\varepsilon E\right) \\ &\simeq \left(\varprojlim_{n\in\mathbb{N}}\varprojlim_{r\in\mathbb{N}} \left(C^r\left(\overline{V}_n\right)\right)\varepsilon E\right). \end{aligned}$$

On the other hand, the b-space $C^r\left(\overline{V}_n\right)\varepsilon E$ is isomorphic to the b-space of bounded linear mappings $\left(C^r\left(\overline{V}_n\right)\right)' \longrightarrow E$ which are weakly continuous on the unit ball $B_{\left(C^r\left(\overline{V}_n\right)\right)'}$, so it is included in $\mathbf{b}\left(C^r\left(\overline{V}_n\right)', E\right)$. Now, when $r' < r$, we have an inclusion mapping

$$v : C^r\left(\overline{V}_n\right) \longrightarrow C^{r'}\left(\overline{V}_n\right)$$

which is compact. So any bounded linear mapping

$$u : \left(C^r\left(\overline{V}_n\right)\right)' \longrightarrow E,$$

when composed with the adjoint mapping

$$v' : \left(C^{r'}\left(\overline{V}_n\right)\right)' \longrightarrow \left(C^r\left(\overline{V}_n\right)\right)'$$

of the inclusion mapping $v$ is weakly continuous and belongs to $C^{r'}\left(\overline{V}_n\right)\varepsilon E$. We have constructed the following mappings:

$$C^r\left(\overline{V}_n\right)\varepsilon E \longrightarrow \mathbf{b}\left(C^r\left(\overline{V}_n\right)', E\right) \longrightarrow C^{r'}\left(\overline{V}_n\right)\varepsilon E.$$

This shows that

$$\varprojlim_{r\in\mathbb{N}} C^r\left(\overline{V}_n\right)\varepsilon E \simeq \varprojlim_{r\in\mathbb{N}} \mathbf{b}\left(C^r\left(\overline{V}_n\right)', E\right)$$

and thus

$$C^\infty(U,E) \simeq \varprojlim_{n\in\mathbb{N}}\varprojlim_{r\in\mathbb{N}} \mathbf{b}\left(C^r\left(\overline{V}_n\right)', E\right).$$

Now, as the space $\mathcal{E}_{Fr\acute{e}}(U)$ is the projective limit of the system $\left(C^r\left(\overline{V}_n\right)\right)_{(n,r)\in\mathbb{N}\times\mathbb{N}}$, we obtain

$$\mathcal{E}'(U) \simeq \varinjlim_{n\in\mathbb{N}}\varinjlim_{r\in\mathbb{N}} C^r\left(\overline{V}_n\right)'$$

and hence

$$\mathbf{b}\left(\mathcal{E}'(U), E\right) \simeq \varprojlim_{n\in\mathbb{N}}\varprojlim_{r\in\mathbb{N}} \mathbf{b}\left(C^r\left(\overline{V}_n\right)', E\right).$$

This proves the proposition. □

*Remark 2.9.* If $E$ is a Banach space, we can show that $C^\infty(U,E) = \left(\mathcal{E}_{Fr\acute{e}}(U,E)\right)_b$, where $\left(\mathcal{E}_{Fr\acute{e}}(U,E)\right)_b$ is the space $\mathcal{E}_{Fr\acute{e}}(U,E)$ with its von Neumann boundedness.

Let $U$ be a connected open subset of $\mathbb{C}^n$ and let $\mathcal{C}_U$ be the set of all open relatively compact subsets of $U$. If $V \in \mathcal{C}_U$, the space $O(V)$ with its von Neumann boundedness is a nuclear b-space. If $E$ is a b-space and $F$ a bornologically closed subspace of $E$, $O(V)\varepsilon(E/F)$, which is isomorphic to $O(V, E/F)$, is defined as

$$\left(O(V)\varepsilon E\right)/\left(O(V)\varepsilon F\right) = O(V,E)/O(V,F).$$

As, we proved in [2], for nuclear b-spaces, the bounded linear mapping $\beta(Y,u)$ : $O(V,E) \longrightarrow O(V,F)$, $f \longmapsto u \circ f$ is bornologically surjective whenever $u : E \longrightarrow F$ is a bornologically surjective bounded linear mapping between two b-spaces. And hence, the functor $O(V,.) : \mathbf{b} \longrightarrow \mathbf{b}$ is exact. Since the sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow E/F \longrightarrow 0$$

is exact in the category $\mathbf{b}$, its image by the exact functor $O(V,.)$ is the following exact sequence:

$$0 \longrightarrow O(V,F) \longrightarrow O(V,E) \longrightarrow O(V,E/F) \longrightarrow 0,$$

and the result follows.

If $W, V \in \mathcal{C}_U$ are such that $W \subset V$, since $U$ is connected, we have an injective bounded linear mapping

$$\Psi : O\left(V\right) \longrightarrow O\left(W\right), \quad f \longmapsto f_{|W}.$$

We can show that $\left(O\left(V\right)\right)_{V \in \mathcal{C}_U}$ is a projective system in the category **b**. If $E$ is a b-space the family $\left(O\left(V\right) \varepsilon E\right)_{V \in \mathcal{C}_U}$ is also a projective system, and then has a projective limit in the category **b.**

**Definition 2.10.** *If $U$ is a connected open subset of $\mathbb{C}^n$ and $E$ is a b-space, we define the b-space $O\left(U, E\right) = \varprojlim_{V \in \mathcal{C}_U} \left(O\left(V\right) \varepsilon E\right)$.*

**Definition 2.11.** *We define another b-space $O_1\left(U, E\right)$ as the kernel of the following morphism $\overline{\partial} : \mathcal{E}\left(U, E\right) \longrightarrow \mathcal{E}\left(U, E\right) \otimes \mathbb{C}^{n*}$, where $\mathcal{E}\left(U, E\right) = \varprojlim_{V \in \mathcal{C}_U} \left(\mathcal{E}\left(V\right) \varepsilon E\right)$ and $\mathbb{C}^{n*}$ is the space of antilinear forms on $\mathbb{C}^n$.*

This defines the following left exact complex in the category **b** :

$$(0, i, \overline{\partial}) : 0 \longrightarrow O_1\left(U, E\right) \xrightarrow{i} \mathcal{E}\left(U, E\right) \xrightarrow{\overline{\overline{\partial}}} \mathcal{E}\left(U, E\right) \otimes \mathbb{C}^{n*}.$$

If $E$ is a b-space and $F$ a bornologically closed subspace of $E$, our objectif is to show that $O_1\left(U, E/F\right)$ is naturally isomorphic to $O\left(U, E\right)/O\left(U, F\right)$ where $U$ is a pseudo-convex and where $O\left(U, F\right)$ is a bornologically closed subspace of the b-space $O\left(U, E\right)$.

It follows from Corollary 2.2, that $\mathcal{E}\left(U, E/F\right) = \mathcal{E}\left(U, E\right)/\mathcal{E}\left(U, F\right)$. Then we can see that the b-space $O_1\left(U, E/F\right)$ is defined as the quotient of the b-space $(\overline{\partial})^{-1}(\mathcal{E}\left(U, E\right) \otimes \mathbb{C}^{n*})$ by the bornologically closed subspace $\mathcal{E}\left(U, F\right)$, where

$$(\overline{\partial})^{-1}(\mathcal{E}\left(U, E\right) \otimes \mathbb{C}^{n*}) = \{f \in \mathcal{E}\left(U, E\right) : \overline{\partial} f \in \mathcal{E}\left(U, F\right) \otimes \mathbb{C}^{n*}\}$$

which is a b-space for the following boundedness: a subset $B$ is bounded in $(\overline{\partial})^{-1}(\mathcal{E}\left(U, E\right) \otimes \mathbb{C}^{n*})$ if it is bounded in $\mathcal{E}\left(U, E\right)$ and $\overline{\partial} B = \{\overline{\partial} f : f \in B\}$ is bounded in $\mathcal{E}\left(U, F\right) \otimes \mathbb{C}^{n*}$.

**Proposition 2.12.** *A natural bounded linear mapping $i : O\left(U, E\right)/O\left(U, F\right) \longrightarrow O_1\left(U, E/F\right)$ exists which is injective.*

*Proof.* It follows from the following left exact complex:

$$(0, i, \overline{\partial}) : 0 \longrightarrow O_1\left(U, E/F\right) \longrightarrow \mathcal{E}\left(U, E/F\right) \longrightarrow \mathcal{E}\left(U, E/F\right) \otimes \mathbb{C}^{n*}$$

that inclusion mapping

$$O\left(U, E\right) \longrightarrow (\overline{\partial})^{-1}(\mathcal{E}\left(U, E\right) \otimes \mathbb{C}^{n*})$$

is bounded and its restriction $O\left(U, F\right) \longrightarrow \mathcal{E}\left(U, F\right)$ is also bounded. This induces a bounded linear mapping

$$O\left(U, E\right)/O\left(U, F\right) \longrightarrow (\overline{\partial})^{-1}(\mathcal{E}\left(U, E\right) \otimes \mathbb{C}^{n*})/\mathcal{E}\left(U, F\right).$$

Consequently, we obtain the following commutative diagram:

$$\begin{array}{ccccc} & & O\left(U, E\right)/O\left(U, F\right) & & \\ & \swarrow & & \searrow & \\ O_1\left(U, E/F\right) & & \longrightarrow & & \mathcal{E}\left(U, E/F\right) \end{array}$$

where the bounded linear mapping

$$O_1\left(U, E/F\right) \longrightarrow \mathcal{E}\left(U, E/F\right)$$

is injective. It follows that the bounded linear mapping

$$O\left(U, E\right)/O\left(U, F\right) \longrightarrow O_1\left(U, E/F\right)$$

is also injective. This ends the proof. □

**Proposition 2.13.** *Let $U$ be an open pseudo-convex subset of $\mathbb{C}^n$, $E$ a b-space and $F$ a bornologically closed subspace of $E$. Then the b-space $O_1(U, E/F)$ is naturally isomorphic to the projective limit of the family of b-spaces $(O(V,E)/O(V,F))_V$, where $V$ ranges over open relatively compact subsets of $U$.*

*Proof.* Consider $f \in O_1(U, E/F)$ and let $f_1$ be an element of the class of equivalence of $f$. Then $\overline{\partial} f_1 \in \mathcal{E}(U,F) \otimes \mathbb{C}^{*n}$. A function $g_V \in O(V,F)$ exists such that $\overline{\partial} {f_1}_{|_V} = \overline{\partial} g_V$ if $V$ is a relatively compact subset of $U$.

In other words, ${f_1}_{|_V} - g_V \in O(V,F)$, and this shows that $O_1(U, E/F)$ is the projective limit of the b-spaces $O(V,E)/O(V,F)$, where $V$ is an open relatively compact of $U$. □

**Proposition 2.14.** *Let $U$ be an open pseudo-convex subset of $\mathbb{C}^n$, $E$ a b-space and $F$ a bornologically closed subspace of $E$. Then the b-spaces $O_1(U, E/F)$ and $O(U,E)/O(U,F)$ are naturally isomorphic.*

*Proof.* To give an element of $O(U,E)/O(U,F)$ we must consider an open covering $(V_i)_{i\in I}$ of $U$, and for all $i \in I$, give an element $f_i$ of $O(V_i, E)$ in such a way that

$$ {f_i}_{|_{V_i \cap V_j}} - {f_j}_{|_{V_i \cap V_j}} \in O(V_i \cap V_j, F)\,. $$

Consider a partition of the unity, subordinate to the covering $(V_i)_{i\in I}$ of $U$. Let $f = \sum_i \varphi_i f_i$, this function belongs to $\mathcal{E}(U,E)$. Also from the fact that $\overline{\partial} f = 0$ and on $V_i \cap V_j$ we have $\overline{\partial} f = \sum_i f_i \overline{\partial} \varphi_i$ i.e.

$$ \overline{\partial} f = f_i \overline{\partial} \varphi_i + f_j \overline{\partial} \varphi_j = f_i \overline{\partial} \varphi_i + f_j \overline{\partial}(1-\varphi_i) = (f_i - f_j)\partial \varphi_i\,. $$

This shows that

$$ {f_i}_{|_{V_i \cap V_j}} - {f_j}_{|_{V_i \cap V_j}} \in O(V_i \cap V_j, F) $$

and then $\overline{\partial} f \in \mathcal{E}(U,F) \otimes \mathbb{C}^{*n}$ i.e. $f \in O_1(U, E/F)\,.$

In this way, we have a bounded linear mapping

$$ O(U,E)/O(U,F) \longrightarrow O_1(U, E/F)\,. $$

We want to find a bounded linear mapping

$$ O_1(U, E/F) \longrightarrow O(U,E)/O(U,F)\,. $$

Let $f \in O_1(U, E/F)$. Choose an open covering $(V_i)_i$ of $U$, such that each $V_i$ is a pseudo-convex and relatively compact subset in $U$. Let $f_1$ be an element of the class of equivalence of $f$. By Proposition 2.13, we find $f_i \in \mathcal{E}(V_i, E)$ such that $f_{|V_i} - f_i \in O(V_i, F)$. We see that

$$ {f_i}_{|_{V_i \cap V_j}} - {f_j}_{|_{V_i \cap V_j}} = ({f_i}_{|_{V_i \cap V_j}} - {f_1}_{|_{V_i \cap V_j}}) + ({f_1}_{|_{V_i \cap V_j}} - {f_j}_{|_{V_i \cap V_j}}) \in O(V_i \cap V_j, F)\,. $$

In this way, we have found a bounded linear mapping

$$ O_1(U, E/F) \longrightarrow O(U,E)/O(U,F) $$

which is the inverse of the above mapping. This shows the Proposition. □

*Remark 2.15.*

**1-** If $F$ is of finite dimension, then $E/F$ is isomorphic to a b-space $G$, and in this situation, we have $O_1(U, E/F) = O_1(U, G) \simeq O(U, G) = O(U,E)/O(U,F)$.

**2-** If $F$ is of finite codimension (for example $m$), then the b-space $E/F$ (which is of finite dimension) is isomorphic to the b-space $\mathbb{C}^m$, and hence

$$ \begin{aligned} O_1(U, \mathbb{C}^m) &= Ker\left(\overline{\partial} : \mathcal{E}(U, E/F) \longrightarrow \mathcal{E}(U, E/F) \otimes \mathbb{C}^{n*}\right) \\ &\simeq O(U, \mathbb{C}^m) = O(U, E/F) = O(U,E)/O(U,F). \end{aligned} $$

In fact, if $f \in O_1(U, \mathbb{C}^m)$, then $\overline{\partial} f = 0$, and hence $\sum_{i=1}^{m} \frac{\overline{\partial} f}{\overline{\partial} z_i} = 0$. Since the system $\{\overline{\partial} z_i : i = 1, ..., m\}$ is free, it follows that $\frac{\overline{\partial} f}{\overline{\partial} z_i} = 0$ for each $i = 1, ..., m$, and hence $f$ is holomorphic from $U$ into $\mathbb{C}^m$.

Finally, we give an example.

In $\mathbb{C}^2$, we take the sets

$$V_1 = \left\{ (z_1, z_2) : -1 < \operatorname{Re}(z_1) \leq 0, \ \ |\operatorname{Im}(z_1)| < 1, \ \frac{1}{2} < |z_2| < 1 \right\},$$

$$D = \{ z \in \mathbb{C} : -1 < \operatorname{Re}(z) < 1, \ \ -1 < \operatorname{Im}(z) < 1 \}$$

and

$$D_1 = \left\{ z \in \mathbb{C} : -\frac{1}{2} < \operatorname{Re}(z_1) < \frac{1}{2}, \ \ -\frac{1}{2} < \operatorname{Im}(z) < \frac{1}{2} \right\}.$$

Let $E/F$ be a b-space, there exists a bounded subset $B$ of $O(D, E)$ which is not bounded in $O(D, F)$ but its restriction to $D_1$ is bounded in $O(V_1, F)$.

We consider $\Psi \in \mathcal{E}(D)$ such that $\Psi(z) = 1$ when $z \in D \setminus D_1$ and $0 \notin \operatorname{supp}(\Psi)$. The mapping

$$g : (z_1, z_2) \longmapsto \Psi(z_2) \frac{f(z_1)}{z_2} = g(z_1, z_2)$$

belongs to $O_1(V_1, E/F)$. We have that $\Psi(z_2) = 1$ when $z_2 \in D$. There $f \in B \subset O(D, E)$. If $(z_1, z_2) \in V_1$ and $|z_2| < \frac{1}{2}$, we see that when $|z_1| < \frac{1}{2}$, we have $\overline{\partial} f \in \mathcal{E}(D, F) \otimes \mathbb{C}^{*n}$ and in this region, $\overline{\partial} g \in \mathcal{E}(D, F) \otimes \mathbb{C}^{*n}$.

If we consider a bounded subset $B$ of $O(D, E)$ whose restriction to $D \backslash D_1$ is bounded in $O(D \backslash D_1, F)$, we obtain a bounded subset $B_1$ of $O_1(V_1, E/F)$. This function should be an element of $O_1(V_1, E \mid F)$. Consider now the integral

$$\frac{1}{2\pi i} \int g\left(z_1, re^{it}\right) dt = -f_1(z_1) \ .$$

This function belongs to $O(D, E)$. For each $f \in B$, it is clear that $f - f_1 \in \mathcal{E}(D, F)$, the function $f_1$ vanishes on a not empty open set, i.e. $f_1 = 0$. We had assumed that $f \notin O(D, F)$. We have a contradiction.

## References


1. B. Aqzzouz, *Généralisations du Théorème de Bartle-Graves*, C. R. Acad. Sci. Paris **333** (2001), no. 10, 925–930.
2. B. Aqzzouz, *On an isomorphism in the category of b-spaces*, Siberian Math. J. **44** (2003), no. 5, 749–756.
3. R. G. Bartle and L. M. Graves, *Mappings between functions spaces*, Trans. Amer. Math. Soc. **72** (1952), 400–413.
4. J. Frampton and A. Tromba, *On the classification of spaces of Hölder continuous functions*, J. Funct. Anal. **10** (1972), 336–345.
5. R. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood. Cliffs, New Jersey, 1965.
6. H. Hogbe-Nlend, *Théorie des bornologies et applications*, Lecture Notes in Mathematics, vol. 213. Springer, Berlin, 1971.
7. C. Houzel, *Séminaire Banach*, Lecture Notes in Mathematics, vol. 227. Springer, Berlin, 1972.
8. H. Jarchow, *Locally convex spaces*, B.G. Teubner, Stuttgart, 1981.
9. W. Kaballo, *Lifting theorems for vector valued functions and the $\varepsilon$-product*, Proc. of the First Poderborn Conference on Functional Analysis, vol. 27, 1977, pp. 149–166.
10. G. M. Khenkin, *Impossibility of a uniform homeomorphism between spaces of smooth functions of one and of n variables* ($n \geq 2$), Mat. USSR-Sb. **3** (1967), 551–561.
11. L. Waelbroeck, *Duality and the injective tensor product*, Math. Ann. **163** (1966), 122–126.
12. L. Waelbroeck, *Topological vector spaces and algebras*, Lecture Notes in Mathematics, vol. 230. Springer, Berlin, 1971.

Université Ibn Tofail, Faculté des Sciences, Département de Mathématiques, Laboratoire d'Analyse Fonctionnelle, Harmonique et Complexe, B.P. 133, Kénitra, Morocco
*E-mail address*: baqzzouz@hotmail.com

Université Ibn Tofail, Faculté des Sciences, Département de Mathématiques, Laboratoire d'Analyse Fonctionnelle, Harmonique et Complexe, B.P. 133, Kénitra, Morocco

Université Ibn Tofail, Faculté des Sciences, Département de Mathématiques, Laboratoire d'Analyse Fonctionnelle, Harmonique et Complexe, B.P. 133, Kénitra, Morocco

Université Ibn Tofail, Faculté des Sciences, Département de Mathématiques, Laboratoire d'Analyse Fonctionnelle, Harmonique et Complexe, B.P. 133, Kénitra, Morocco

Received 03/06/2005; Revised 07/02/2006