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GENERALIZED ZEROS AND POLES OF $\mathcal{N}_\kappa$-FUNCTIONS: ON THE UNDERLYING SPECTRAL STRUCTURE

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To our friend and colleague Henk de Snoo on the occasion of his sixtieth birthday.

Abstract. Let q be a scalar generalized Nevanlinna function, $q \in \mathcal{N}_\kappa$. Its generalized zeros and poles (including their orders) are defined in terms of the function's operator representation. In this paper analytic properties associated with the underlying root subspaces and their geometric structures are investigated in terms of the local behaviour of the function. The main results and various characterizations are expressed by means of (local) moments, asymptotic expansions, and via the basic factorization of q. Also an inverse problem for recovering the geometric structure of the root subspace from an appropriate asymptotic expansion is solved.

1. Introduction

Let q belong to the class $\mathcal{N}_\kappa$ of scalar generalized Nevanlinna functions. Such a function is always meromorphic in $\mathbb{C} \setminus \mathbb{R}$ and admits a (minimal) representation by means of a self-adjoint relation A in a Pontryagin space $\mathcal{K}$ (with negative index κ). The *generalized zeros and poles* of q are defined in terms of such representations, as the eigenvalues of the corresponding self-adjoint relations, cf. [12]. The *order* of a generalized zero and pole is defined as the algebraic multiplicity of the corresponding eigenvalue. If the generalized zero (pole) is a point of holomorphy (meromorphy, respectively) of the function q then it is just an ordinary zero (pole) of q and the order coincides with the usual multiplicity of the zero (pole) of q. However, such points can also be embedded in the real spectrum.

In the literature a special attention has been paid to those generalized zeros and poles that are not of positive type, that is, the corresponding eigenvector is a non-positive element in $\mathcal{K}$. These poles and zeros make an essential difference when compared to the situation that appears in the Hilbert space case. At such a point the algebraic multiplicity of the corresponding eigenvalue can be strictly bigger than its geometric multiplicity and the root subspace can contain nontrivial non-positive vectors. The *degree of non-positivity* π_β of a generalized zero $\beta \in \mathbb{C}\cup\{\infty\}$ (ν_α of a generalized pole $\alpha \in \mathbb{C}\cup\{\infty\}$) of q describes the maximal dimension of a non-positive (invariant) subspace of the root subspace. Here the term "degree of non-positivity" will be used instead of the original term "multiplicity of a generalized zero of non-positive type" to make a clear distinction between this notion and the algebraic multiplicity of an eigenvalue (i.e. the order of a generalized zero). Both of these notions have an important role in the present paper. According to [12] the sum of the degrees π_β (ν_α) of all generalized zeros (poles, respectively), including the point $z = \infty$, is equal to the index κ. An analytic characterization for the degree of non-positivity of generalized zeros and poles of non-positive type was obtained in [13]. A

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comprehensive discussion of poles not of positive type in the case $\kappa = 1$ can be found in [4]. The subclass $\mathcal{N}_\kappa^\infty$, which contains functions with only one generalized pole at ∞, has been investigated in [7]. In particular, Theorem 3.1 includes as a special case the results [7, Theorem 4.2, Theorem 5.1] in a slightly different but equivalent form. An approach which uses asymptotic expansions and certain, not necessarily canonical (basic), factorization models for the functions $q \in \mathcal{N}_\kappa$ is developed in [5] and contains a detailed analysis of the underlying root subspace $\mathsf{R}_\infty(A)$ in a model space.

In this paper the most general situation is taken up by studying all generalized poles and zeros of an arbitrary scalar generalized Nevanlinna function $q \in \mathcal{N}_\kappa$. In particular, the spectral structure of a representing self-adjoint relation in the Pontryagin space $\mathcal{K}$ at the generalized zeros and poles is investigated. Characterizations associated with the underlying algebraic root subspaces and their geometric structure are given in analytic as well as spectral theoretical terms. For instance, it is shown that $\beta \in \mathbb{R}$ is a generalized zero of $q \in \mathcal{N}_\kappa$ geometrically means that q admits a minimal representation in a Pontryagin space $\mathcal{K}$ of the form

$$q(z) = (z-\beta)\big[\big(I + (z-\beta)(A-z)^{-1}\big)w, w\big],$$

where A is a self-adjoint relation in $\mathcal{K}$ with $\ker(A-\beta) = \{0\}$ and $w \in \mathcal{K}$. This leads to various descriptions concerning the structure of the underlying root subspace by means of the local behaviour of the function q at β. In particular, the following result holds:

Theorem. *Let a function q, $q(z) \not\equiv 0$, belong to the class $\mathcal{N}_\kappa$. If $\beta \in \mathbb{R}$ is a generalized zero of q, then q admits an asymptotic expansion of the form*

$$q(z) = s_{n-1}(z-\beta)^n + \cdots + s_{m-1}(z-\beta)^m + o((z-\beta)^m) \quad \text{as} \quad z \widehat{\to} \beta. \tag{1.1}$$

Conversely, if a function $q \in \mathcal{N}_\kappa$ admits an asymptotic expansion of the form (1.1) *and this expansion satisfies appropriate maximality conditions (see Section 5), then β is a generalized zero of q of order $d_\beta = \min\{n, [\frac{m+1}{2}]\}$ and with degree of non-positivity*

$$\pi_\beta = \begin{cases} \frac{n-1}{2}, & \text{if } n \text{ odd and } s_{n-1}(\beta) > 0, \\ \frac{n+1}{2}, & \text{if } n \text{ odd and } s_{n-1}(\beta) < 0, \\ \frac{n}{2}, & \text{if } n \text{ even}, \\ [\frac{n}{2}](= d_\beta), & \text{if } n = m+1. \end{cases}$$

Moreover, the inner product structure on the underlying root subspace is determined by the Hankel matrix $G_\beta = [s_{i+j-2}(\beta)]_{i,j=1}^{d_\beta}$, where $d_\beta = \min\{n, [\frac{m+1}{2}]\}$.

Here the real constants $s_i(\beta)$ can be interpreted as the local moments of q at β. Another way of characterizing the geometric structure of the underlying root subspace at β is using the function's basic factorization, where q is written as a product of a rational function and a usual Nevanlinna function q_0 (see Section 6). Then, for instance, the results in the above theorem can be stated in terms of the degree of non-positivity of β and the spectral properties of q_0.

The paper is organized as follows. In Section 2, necessary definitions and properties of generalized Nevanlinna functions and their operator representations are given. One of the main results in this paper is Theorem 3.1 in Section 3, where the order of a generalized zero β is characterized in terms of the (local) moments of q, which by definition describe the local behaviour of the function q at β. In Section 4, some further characterizations for the generalized zeros and poles, including their order, are given. Also the geometric structure (the inner product) of the underlying root subspace is described in a straightforward manner by using the (local) moments of q, which leads to a characterization for the degree of non-positivity of generalized zeros of q in terms of (local) moments. In Section 5, appropriate asymptotic expansions are introduced and, in particular, the results in the above theorem are established. Moreover, the announced inverse problem is solved in this section. Finally, in Section 6 the main results in the previous sections are translated into the basic factorization of q, leading to spectral theoretical characterizations involving the spectral measure of the factor $q_0 \in \mathcal{N}_0$.

2. Preliminaries

By definition a function $q : \mathcal{D} \subset \mathbb{C} \to \mathbb{C}$ belongs to the class of *generalized Nevanlinna functions* $\mathcal{N}_\kappa$ if it is meromorphic in $\mathbb{C} \setminus \mathbb{R}$ (with domain of holomorphy $\mathcal{D}$), if it is symmetric with respect to the real line, i. e., $q(\overline{z}) = \overline{q(z)}$ for $z \in \mathcal{D}$, and if the so-called Nevanlinna kernel

$$N_q(z,\zeta) := \frac{q(z) - q(\overline{\zeta})}{z - \overline{\zeta}} \quad \text{for} \quad z, \zeta \in \mathcal{D}$$

has κ negative squares. It is well known (see e.g. [11]) that these functions can also be characterized by their operator representations. That is, a function q belongs to the class of generalized Nevanlinna functions if and only if there exists a Pontryagin space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]\big)$ and a self-adjoint relation A in $\mathcal{K}$ such that q admits the representation

$$q(z) = \overline{q(z_0)} + (z - \overline{z_0})\big[\big(I + (z - z_0)(A - z)^{-1}\big)v, v\big], \tag{2.1}$$

where the point $z_0 \in \mathcal{D}$ is fixed and $v \in \mathcal{K}$. If, moreover, the representation (2.1) is minimal, i. e.,

$$\mathcal{K} = \text{c. l. s.}\,\big\{ \big(I + (z - z_0)(A - z)^{-1}\big)v : z \in \varrho(A) \big\},$$

then $q \in \mathcal{N}_\kappa$, where κ is the negative index of the space $\mathcal{K}$. Here c. l. s. stands for the closed linear span. In this case the representation of $q \in \mathcal{N}_\kappa$ is unique up to unitary equivalence.

Recall from [11] that q satisfies the condition

$$\lim_{y \uparrow \infty} \frac{q(iy)}{y} = 0 \quad \text{or equivalently} \quad \lim_{z \,\widehat{\to}\, \infty} \frac{q(z)}{\operatorname{Im} z} = 0 \tag{2.2}$$

if and only if in a minimal representation (2.1) of q the multi-valued part of the relation A is trivial, i. e., A is an operator. Here and in what follows $\lim_{z \,\widehat{\to}\, \beta}$ stands for the non-tangential limit if $\beta \in \mathbb{R} \cup \{\infty\}$ and the usual limit otherwise. If, in addition, q satisfies the growth condition

$$\lim_{y \uparrow \infty} y\,|\operatorname{Im} q(iy)| < \infty, \tag{2.3}$$

then $v \in \operatorname{dom} A$ and with $u := (A - \overline{z_0})v$ the operator representation (2.1) simplifies to

$$q(z) = s + \big[(A - z)^{-1}u, u\big], \tag{2.4}$$

where the real number s is given by $s := \overline{q(z_0)} - [(A - \overline{z_0})^{-1}u, u] = \lim_{z \,\widehat{\to}\, \infty} q(z)$. Conversely, if q is of the form (2.4) with a self-adjoint operator A in $\mathcal{K}$ then q satisfies the conditions (2.2) and (2.3), too. In particular, this gives

$$q(z) = \big[(A - z)^{-1}u, u\big] \quad \text{if and only if} \quad \lim_{y \uparrow \infty} y\,|q(iy)| < \infty. \tag{2.5}$$

Recall that an ordinary Nevanlinna function $q_0 \in \mathcal{N}_0$ admits also an integral representation

$$q_0(z) = a + bz + \int_{-\infty}^{\infty} \Big(\frac{1}{t - z} - \frac{t}{1 + t^2}\Big)\, d\sigma(t), \tag{2.6}$$

where $a \in \mathbb{R}$, $b \geq 0$, and σ is a real measure with $\int_{-\infty}^{\infty} \frac{1}{1+t^2}\, d\sigma(t) < \infty$. For a connection to the above minimal representation (2.1), see for instance [14]. The finite eigenvalues of A are then exactly those real points where σ has a point mass. Moreover, it follows from (2.6) that

$$b = \lim_{y\uparrow\infty} \frac{q_0(iy)}{y}, \tag{2.7}$$

which shows that A has a trivial multi-valued part exactly if $b = 0$. An analogue of (2.5) can be restated in terms of the integral representation (2.6) as follows:

$$q_0(z) = \int_{-\infty}^{\infty} \frac{d\sigma(t)}{t-z} \quad \text{if and only if} \quad \lim_{y\uparrow\infty} y\,|q_0(iy)| < \infty. \tag{2.8}$$

In this case,

$$-\lim_{y\uparrow\infty} y\,|q_0(iy)| = \int_{-\infty}^{\infty} d\sigma(t)\,(<\infty) \quad \text{and otherwise} \quad \lim_{y\uparrow\infty} y\,|q_0(iy)| = \infty. \tag{2.9}$$

It is useful to combine these basic observations with a recent factorization result concerning generalized Nevanlinna functions (see [6, 2] and also Section 6 of the present paper): every $q \in \mathcal{N}_\kappa$ admits a factorization of the form

$$q(z) = r(z)r^{\#}(z)q_0(z), \quad r(z) = \frac{a(z)}{b(z)},$$

where $a(z)$ and $b(z)$ are monic polynomials, $r^{\#}(z) = \overline{r(\overline{z})}$, and $q_0 \in \mathcal{N}_0$. One immediate consequence is the following result.

Corollary 2.1. *For every function $q \in \mathcal{N}_\kappa$, the following two non-tangential limits*

$$\lim_{z \widehat{\to} \infty} \frac{q(z)}{z}, \quad \lim_{z \widehat{\to} \infty} z\,q(z) \tag{2.10}$$

exist either as a real number or as ∞.

Proof. Since $2(\deg a - \deg b)$ is even the existence of the limits (2.10) follows from the fact that for every $k \in \mathbb{Z}$ the limit $\lim_{z \widehat{\to} \infty} z^{2k+1}\, q_0(z)$ either exists as a real number, or it is equal to ∞ (see (2.7), (2.9)). □

Definition 2.2. A point $\alpha \in \mathbb{C} \cup \{\infty\}$ is called a *generalized pole* of $q \in \mathcal{N}_\kappa$ if it is an eigenvalue of the relation A in a minimal representation (2.1). The dimension of the algebraic eigenspace at α is called its *order*.

Note that the definition does not depend on a particular representation. Clearly, ∞ is an eigenvalue of A if and only if $\alpha \in \mathbb{C}$ is an eigenvalue of the relation $A^{-1} + \alpha$. Moreover, the equivalence

$$\{u_1, u_2\} \in A \quad \Leftrightarrow \quad \{u_2, u_1 + \alpha u_2\} \in A^{-1} + \alpha \tag{2.11}$$

implies that a (maximal) Jordan chain of A at ∞ gives rise to a (maximal) Jordan chain of $A^{-1} + \alpha$ at α, and conversely.

Remark 2.3. For a scalar generalized Nevanlinna function q the minimality of the representation guarantees that the geometric eigenspace of A at α is one-dimensional. Hence the order of the generalized pole α is equal to the length of a maximal Jordan chain of A at α.

Every isolated pole $\alpha \in \mathbb{C} \cup \{\infty\}$ of A is just an ordinary pole of q and the order of α as a generalized pole of q coincides with the usual multiplicity of α as a pole of q, cf. [12]. This holds, in particular, for all non-real eigenvalues of A. The generalized poles of q which are not ordinary poles of q correspond to so-called embedded eigenvalues of A and they belong to $\mathbb{R} \cup \{\infty\}$.

If $q \in \mathcal{N}_\kappa$ then the function $\widetilde{q}(z) := -q(\frac{1}{z-\alpha})$ with $\alpha \in \mathbb{R}$ also belongs to the class $\mathcal{N}_\kappa$. The next lemma associates with q and $\widetilde{q}$ certain minimal representations involving the self-adjoint relations A and $A^{-1} + \alpha$, respectively.

Lemma 2.4. *Let $q \in \mathcal{N}_\kappa$ have a (minimal) representation of the form* (2.1). *Then the function $\widetilde{q}(\lambda) := -q(\frac{1}{\lambda-\alpha})$, $\alpha \in \mathbb{R}$, admits the (minimal) representation*

$$\widetilde{q}(\lambda) = \overline{\widetilde{q}(\lambda_0)} + (z - \overline{\lambda_0})\big[\big(I + (\lambda - \lambda_0)((A^{-1} + \alpha) - \lambda)^{-1}\big)\widetilde{v}, \widetilde{v}\big], \tag{2.12}$$

where $\widetilde{v} = \frac{v}{|\lambda_0 - \alpha|}$ and $\lambda_0 = \frac{1}{z_0} + \alpha$.

Proof. Observe that

$$\Big(A - \frac{1}{\lambda - \alpha}\Big)^{-1} = -(\lambda - \alpha)\Big(I + (\lambda - \alpha)\big((A^{-1} + \alpha) - \lambda\big)^{-1}\Big). \tag{2.13}$$

With $z = \frac{1}{\lambda-\alpha}$ and $z_0 = \frac{1}{\lambda_0-\alpha}$ one obtains from (2.13) the equality

$$I + (z - z_0)(A - z)^{-1} = \frac{\lambda - \alpha}{\lambda_0 - \alpha}\left(I + (\lambda - \lambda_0)((A^{-1} + \alpha) - \lambda)^{-1}\right). \tag{2.14}$$

With $\alpha \in \mathbb{R}$ this leads to

$$\widetilde{q}(\lambda) - \overline{\widetilde{q}(\lambda_0)} = \frac{1}{|\lambda_0 - \alpha|^2}(\lambda - \overline{\lambda_0})\big[\big(I + (\lambda - \lambda_0)((A^{-1} + \alpha) - \lambda)^{-1}\big)v, v\big],$$

which gives the representation (2.12) for $\widetilde{q}$. The minimality is clear from the identity (2.14). □

In particular, by Definition 2.2, the representation (2.12) in Lemma 2.4 implies that ∞ is a generalized pole of the function $q(z)$ if and only if $\alpha \in \mathbb{R}$ is a generalized pole of the function $\widetilde{q}(z) = -q(\frac{1}{z-\alpha})$. Furthermore, the orders of these poles coincide, cf. (2.11).

If $q \in \mathcal{N}_\kappa$ with $q(z) \not\equiv 0$, then the reciprocal function

$$\widehat{q}(z) := -\frac{1}{q(z)}$$

belongs to $\mathcal{N}_\kappa$ with the same index κ, which gives rise to the following notations, cf. [13], [1].

Definition 2.5. A point $\beta \in \mathbb{C} \cup \{\infty\}$ is called a *generalized zero* of $q \in \mathcal{N}_\kappa$ of *order* d if it is a generalized pole of $\widehat{q}$ of order d. Furthermore, q is said to *assume a generalized value* s at $\beta \in \mathbb{R} \cup \{\infty\}$ if the limit $\lim_{z \widehat{\to} \beta} q(z) =: s \in \mathbb{R}$ exists and the function $q(z) - s$ has a generalized zero at β.

Finally, recall the following technical fact, which will be used in the proof of the main theorem.

Remark 2.6. Let A be a self-adjoint operator in a Pontryagin space $\mathcal{K}$. Then A can be decomposed into an orthogonal sum,

$$A = A' \oplus A'', \tag{2.15}$$

where the operator A' is bounded and self-adjoint in $\mathcal{K}'$, and A'' is self-adjoint in $\mathcal{K}''$, with $\mathcal{K}'$ a Pontryagin space, $\mathcal{K}''$ a Hilbert space, and $\mathcal{K} = \mathcal{K}' \oplus \mathcal{K}''$, see [10].

3. The main characterization

For a formulation of the main theorem, the following notations are needed. Let a generalized Nevanlinna function $q \in \mathcal{N}_\kappa$ be given. Then with $\beta \in \mathbb{C}$ define

$$s_0(\beta) := \lim_{z \widehat{\to} \beta} \frac{q(z)}{z-\beta} \tag{3.1}$$

and then recursively

$$s_n(\beta) := \lim_{z \widehat{\to} \beta} \frac{1}{(z-\beta)^{n+1}} \left[q(z) - (z-\beta)s_0(\beta) - \cdots - (z-\beta)^n s_{n-1}(\beta) \right]. \tag{3.2}$$

Similarly, with $\beta = \infty$ define

$$s_0(\infty) := \lim_{z \widehat{\to} \infty} -z\, q(z)$$

and

$$s_n(\infty) := \lim_{z \widehat{\to} \infty} -z^{n+1} \left[q(z) + \frac{s_0(\infty)}{z} + \cdots + \frac{s_{n-1}(\infty)}{z^n} \right], \tag{3.3}$$

whenever these limits exist and are real if $\beta \in \mathbb{R} \cup \{\infty\}$. These numbers will be referred to as the *moments* of q at β, and if $\beta \in \mathbb{R}$ also as *local moments* of q. A justification for this notation is that for an ordinary Nevanlinna function $q_0 \in \mathcal{N}_0$ they coincide with the moments of the measure σ in the integral representation (2.6) with respect to $\beta \in \mathbb{R} \cup \{\infty\}$, whenever these moments exist as absolutely convergent integrals, cf. Lemma 6.1. The main theorem now reads as follows. Its proof will be presented at the end of this section after some preparations.

Theorem 3.1. *Let $q \in \mathcal{N}_\kappa$ with $q(z) \not\equiv 0$ be given. Then $\beta \in \mathbb{C} \cup \{\infty\}$ is a generalized zero of q of order $d_\beta \geq 1$ if and only if d_β is the maximal integer with the following properties:*

$$\exists\ s_0(\beta), s_1(\beta), \ldots, s_{2d_\beta - 2}(\beta) \quad \text{and} \quad s_0(\beta) = s_1(\beta) = \cdots = s_{d_\beta - 2}(\beta) = 0. \tag{3.4}$$

As a first step the following characterization of a generalized zero without taking into account its order will be established.

Lemma 3.2. *Let $q \in \mathcal{N}_\kappa$ with $q(z) \not\equiv 0$. Then the point $\beta \in \mathbb{C} \cup \{\infty\}$ is a generalized zero of q if and only if the moment $s_0(\beta)$ exists.*

Proof. For $\beta \in \mathbb{C} \setminus \mathbb{R}$ the statement is clear because of the meromorphy of q. Next consider the case $\beta = \infty$. By definition β is a generalized zero of q if and only if it is a generalized pole of the function $\widehat{q} = -\frac{1}{q}$. Combining the criterion (2.2) with Corollary 2.1 one concludes that the multi-valued part of the representing relation for $\widehat{q}$ is non-trivial exactly if the moment $s_0(\infty)$ exists. Finally consider the case $\beta \in \mathbb{R}$. By Lemma 2.4 q has a generalized zero (or pole) at ∞ if and only if the function $\widetilde{q}(z)$ has a generalized zero (pole) at $\beta \in \mathbb{R}$, since clearly $\widetilde{-\frac{1}{q}} = -\frac{1}{\widetilde{q}}$. It remains to observe that

$$\lim_{z \widehat{\to} \infty} -zq(z) = \lim_{z \widehat{\to} \beta} \frac{\widetilde{q}(z)}{z-\beta},$$

which means that q has the moment $s_0(\infty)$ if and only if $\widetilde{q}$ has the moment $s_0(\beta)$ and that these moments actually coincide. This completes the proof. □

The criterion (2.5) can now be reformulated as follows.

Corollary 3.3. *The point ∞ is a generalized zero of the function $q \in \mathcal{N}_\kappa$ if and only if q admits a minimal operator representation of the form* (2.4)

$$q(z) = [(A-z)^{-1}u, u].$$

In the next proposition, which extends a result that can be found in [15], the representing relation of the reciprocal function $\widehat{q}$ is described.

Proposition 3.4. *Let the function $q \in \mathcal{N}_\kappa$ be given with the (minimal) representation* (2.4) *with $u \in \mathcal{K} \setminus \{0\}$:*

$$q(z) = [(A - z)^{-1}u, u].$$

Then the reciprocal function $\widehat{q}(z) := -\dfrac{1}{q(z)}$ admits a (minimal) representation of the form (2.1) *with some $z_0 \in \rho(A)$, for which $q(z_0) \neq 0$:*

$$\widehat{q}(z) = \overline{\widehat{q}(z_0)} + (z - \overline{z_0})\big[\big(I + (z - z_0)(\widehat{A} - z)^{-1}\big)\widehat{v}, \widehat{v}\big]$$

with $\widehat{v} := \frac{1}{q(\overline{z_0})}(A - \overline{z_0})^{-1}u$ and the self-adjoint relation $\widehat{A}$ has the operator part $A\restriction_{\langle u\rangle^\perp}$ and its multi-valued part is spanned by the element u, that is,

$$\widehat{A} := A\restriction_{\langle u\rangle^\perp} \dot{+} \big(\{0\} \times \langle u\rangle\big). \tag{3.5}$$

Proof. Choose a non-real point z_0 from the domain of holomorphy of q such that $q(z_0) \neq 0$. Then q can be written in the form (2.1) with $v := (A - \overline{z_0})^{-1}u$ and the same relation A. According to [15, Proposition 2.1] the function $\widehat{q}$ admits the (minimal) representation

$$\widehat{q}(z) = \widehat{q}(\overline{z_0}) + (z - \overline{z_0})\big[\big(I + (z - z_0)(\widehat{A} - z)^{-1}\big)\widehat{v}, \widehat{v}\big], \tag{3.6}$$

with the element $\widehat{v} = \frac{1}{q(\overline{z_0})}v$ and where the self-adjoint relation $\widehat{A}$ is given by

$$(\widehat{A} - z_0)^{-1} := (A - z_0)^{-1} - \frac{1}{q(z_0)}[\,\cdot\,, v](I + (z_0 - \overline{z_0})(A - z_0)^{-1})v. \tag{3.7}$$

In order to show that the relation $\widehat{A}$ given by (3.7) has the form (3.5) calculate the resolvent $R(z_0)$ of $A\restriction_{\langle u\rangle^\perp} \dot{+} \big(\{0\} \times \langle u\rangle\big)$. Set $g := R(z_0)f$ for some element $f \in \mathcal{K}$, that is

$$\{g; f + z_0 g\} \in A\restriction_{\langle u\rangle^\perp} \dot{+} \big(\{0\} \times \langle u\rangle\big),$$

which means that the elements $g \in \operatorname{dom} A$ and u are orthogonal and there exists some $c \in \mathbb{R}$ such that $f + z_0 g - Ag = c\,u$. Since $z_0 \in \varrho(A)$ this can also be written as $g = (A - z_0)^{-1}f - c(A - z_0)^{-1}u$. From $[g, u] = 0$ it follows that $c = \frac{1}{q(z_0)}[(A - z_0)^{-1}f, u]$ and hence by the resolvent identity

$$\begin{aligned} g = R(z_0)f &= (A - z_0)^{-1}f - \frac{1}{q(z_0)}[f, (A - \overline{z_0})^{-1}u](A - z_0)^{-1}u \\ &= (A - z_0)^{-1}f - \frac{1}{q(z_0)}[f, v\,](I + (z_0 - \overline{z_0})(A - z_0)^{-1})v, \end{aligned}$$

which completes the proof. □

Remark 3.5. Note that if $q(z) = s + [(A - z)^{-1}u, u]$ with $s \neq 0$, then it holds

$$\widehat{q}(z) = -\frac{1}{s} + [(\widehat{A} - z)^{-1}\widehat{u}, \widehat{u}],$$

where the operator $\widehat{A}$ is given by $\widehat{A} = A + \frac{1}{s}[\,\cdot\,, u]u$ with $\widehat{u} := \frac{1}{s}u$.

Proof of Theorem 3.1. For every generalized zero that belongs to the domain of holomorphy of q the order (as in Definition 2.2) coincides with the usual multiplicity as a zero, and therefore (3.4) holds. What remains is to describe the order of a real generalized zero and of ∞. First the case $\beta = \infty$ is considered and then by using Möbius transformations the obtained results are carried over to an arbitrary real point.

If $\beta = \infty$ is a generalized zero of the function q, then according to Corollary 3.3 the function q admits a minimal representation of the form $q(z) = [(A - z)^{-1}u, u]$. To describe the order of the generalized zero ∞ one must characterize the maximal length of a Jordan chain at ∞ of the relation $\widehat{A}$ given in Proposition 3.4.

Recall that the nonzero vectors $x_0, x_1, \ldots, x_{l-1}$ form a (not necessarily maximal) Jordan chain of $\widehat{A}$ at ∞ if

$$\{0; x_0\} \in \widehat{A} \tag{3.8}$$

and

$$\{x_{j-1}; x_j\} \in \widehat{A} \quad \text{for} \quad j = 1, \ldots, l-1. \tag{3.9}$$

The relation (3.8) means that $x_0 = \lambda_0 u$ with some $\lambda_0 \neq 0$. For $j = 1$ (3.9) becomes $x_1 = \lambda_0 Au + \lambda_1 u$, where $u \in \mathrm{dom} A$ and $[u, u] = 0$. By induction one concludes that the existence of a Jordan chain $x_0, x_1, \ldots, x_{l-1}$ of $\widehat{A}$ at ∞ implies that

$$u \in \mathrm{dom} A^{l-1} \quad \text{and} \quad [A^j u, u] = 0 \quad \text{for} \quad j = 0, \ldots, l-2. \tag{3.10}$$

In this case the vectors are of the form $x_j = \sum_{i=0}^{j} \lambda_i A^{j-i} u$ with $\lambda_i \in \mathbb{C}$ and $\lambda_0 \neq 0$. Conversely, the conditions (3.10) imply that the vectors $u, Au, \ldots, A^{l-1}u$ form a Jordan chain of the relation $\widehat{A}$ at ∞. Note in particular, that $A^k u \neq 0$ for $k \leq l-1$. Indeed, for $A^k u = 0$ the minimality implies that $\mathcal{K}$ is spanned by the elements $u, Au, \ldots, A^{k-1}u$, but then (3.10) would give $u \in \mathcal{K} \cap \mathcal{K}^{[\perp]}$.

Therefore, there exists a Jordan chain of (not necessarily maximal) length l for $\widehat{A}$ at ∞ if and only if the conditions (3.10) are satisfied. Next it is shown that this is further equivalent to

$$\exists\, s_0(\infty), s_1(\infty), \ldots, s_{2l-2}(\infty) \quad \text{and} \quad s_0(\infty) = \cdots = s_{l-2}(\infty) = 0. \tag{3.11}$$

In fact, this is a consequence of [11, Satz 1.10]. For the convenience of the reader a full proof for the equivalence of (3.10) and (3.11) is presented here; many of the given formulas are also needed later on in the present paper. Now, first observe that by applying the formal expression

$$(A-z)^{-1} A^{2n} = z^{2n} \left((A-z)^{-1} + \sum_{i=0}^{2n-1} \frac{A^i}{z^{i+1}} \right) \tag{3.12}$$

one obtains with $u \in \mathrm{dom}\, A^n$ the formula

$$s_{2n}(\infty) = -\lim_{z \widehat{\to} \infty} z^{2n+1} \left(q(z) + \frac{[u,u]}{z} + \cdots + \frac{[A^n u, A^{n-1} u]}{z^{2n}} \right) = [A^n u, A^n u].$$

Hence, assuming (3.10) it follows that the moments $s_i(\infty)$ exist for all $i \leq 2l-2$ and that they are given by

$$s_i(\infty) = [A^i u, u], \quad s_{l-1+i}(\infty) = [A^{l-1} u, A^i u], \quad i = 0, \ldots, l-1, \tag{3.13}$$

and in particular $s_i(\infty) = 0$, for all $i = 0, \ldots, l-2$.

Conversely, assume that the moments $s_j(\infty)$ exists for all $j \leq 2l-2$. Then the definition of $s_{2l-2}(\infty)$ together with Lemma 3.2 shows that the function

$$q_R(z) := z^{2l-2} \left(q(z) + \sum_{i=0}^{2l-3} \frac{s_i(\infty)}{z^{i+1}} \right)$$

is a generalized Nevanlinna function of the form $q_R(z) = [(B-z)^{-1} w, w]$, where B is a self-adjoint operator in a Pontryagin space $\mathcal{M}$ and $w \in \mathcal{M}$. In view of (2.15) the generalized Stieltjes inversion formula (see [9]) applied to q_R shows that

$$\int_{|t|>K} t^{2l-2}\, d[E''(t) P'' u, P'' u] < \infty, \tag{3.14}$$

where $E''(t)$ denotes the spectral function of A'' in (2.15), P'' the orthogonal projection onto $\mathcal{K}''$, and $K > \|A'\|$. Thus, $u \in \operatorname{dom} A^{l-1}$ and the moments are given as in (3.13), and therefore (3.11) implies (3.10).

One concludes that the order of the generalized zero ∞ is the maximal number l such that (3.10), or equivalently (3.11), is satisfied. Finally, assume that $\beta \in \mathbb{R}$. As in the proof of Lemma 3.2 the characterization (3.4) is now obtained by considering the function $\widetilde{q}(z) = -q(\frac{1}{z-\beta})$ and observing that the moments of q at ∞ coincide with the corresponding moments of $\widetilde{q}$ at $\beta \in \mathbb{R}$. □

4. Some other characterizations

4.1. **Moments and representations by resolvents.** The characterization of generalized zeros and their orders in Theorem 3.1 can be reformulated equivalently by using minimal representations via resolvents of self-adjoint relations in a Pontryagin space as in (2.1). The following result is immediate from Theorem 3.1 by taking $\beta = \infty$.

Proposition 4.1. *Let $q \in \mathcal{N}_\kappa$ be a generalized Nevanlinna function. Then $\beta = \infty$ is a generalized zero of q of order $d_\infty \geq 1$ if and only if q admits a minimal operator representation*

$$q(z) = [(A-z)^{-1}u, u], \quad z \in \rho(A),$$

with a vector $u \in \operatorname{dom} S^{d_\infty - 1} \setminus \operatorname{dom} S^{d_\infty}$, where $S = A{\restriction} < u >^\perp$.

The result in Proposition 4.1 can be seen as an extension of [4, Theorem 3.5] from the case $\kappa = 1$ to the general case of $\mathcal{N}_\kappa$-functions.

In the proof of Theorem 3.1 the operator representation of q was investigated explicitly only in the case $\beta = \infty$. Some of the analogous results for $\beta \in \mathbb{R}$ are now given.

Let the function q be given with the minimal representation (2.1). Then the condition

$$\lim_{z \widehat{\to} \beta} (z-\beta) q(z) = 0$$

is equivalent to $\ker(A-\beta) = \{0\}$, that is $(A-\beta)^{-1}$ is an operator. If, in addition,

$$\lim_{z \widehat{\to} \beta} \frac{q(z) - q(\overline{z})}{z - \overline{z}} < \infty,$$

then $v \in \operatorname{ran}(A-\beta)$ and q assumes a generalized value at the point $\beta \in \mathbb{R}$. In this case the representation (2.1) simplifies to

$$q(z) = q(\beta) + (z-\beta)[\big(I + (z-\beta)(A-z)^{-1}\big)w, w],$$

where

$$w := \big(I + (\beta - z_0)(A-\beta)^{-1}\big)\, v \in \mathcal{K}$$

and

$$q(\beta) := q(\overline{z_0}) + (\beta - \overline{z_0})\left[\big(I + (\beta - z_0)(A-\beta)^{-1}\big)\, v, v\right] = \lim_{z \widehat{\to} \beta} q(z) \in \mathbb{R}.$$

The converse is also clear and thus one obtains the following characterization for q to assume a generalized value at $\beta \in \mathbb{R}$; for $\beta = \infty$ compare (2.4) and Corollary 3.3.

Corollary 4.2. *Let $q \in \mathcal{N}_\kappa$ be a generalized Nevanlinna function. Then q assumes the generalized value $s \in \mathbb{R}$ at $\beta \in \mathbb{R}$ if and only if q admits a minimal representation*

$$q(z) = s + (z-\beta)\left[\big(I + (z-\beta)(A-z)^{-1}\big)\, w, w\right], \tag{4.1}$$

where A is a self-adjoint relation in $\mathcal{K}$ with $\ker(A-\beta) = \{0\}$ and $w \in \mathcal{K}$.

Next an expression for the moments of q at $\beta \in \mathbb{R}$ and an analogue for Proposition 4.1 is established. For this purpose assume that in the representation (4.1) $w \in \operatorname{ran}(A-\beta)^n$, $n \geq 0$, and take $q(\beta) = 0$. From the formula (3.12) one obtains with a straightforward calculation the identity

$$\begin{aligned} q(z) - (z-\beta)[w,w] - \cdots - (z-\beta)^{2n}[(A-\beta)^{-n}w, (A-\beta)^{-n+1}w] \\ = (z-\beta)^{2n+1}\left[\left(I+(z-\beta)(A-z)^{-1}\right)(A-\beta)^{-n}v, (A-\beta)^{-n}w\right]. \end{aligned}$$

It follows that the moments of q at $\beta \in \mathbb{R}$ for $i = 0, \ldots, n$ are given by

$$s_i(\beta) = [(A-\beta)^{-i}w, w], \quad s_{n+i}(\beta) = [(A-\beta)^{-n}w, (A-\beta)^{-i}w]. \tag{4.2}$$

Furthermore, the order of a generalized zero $\beta \in \mathbb{R}$ can be characterized by means of a resolvent representation of the form (4.1).

Proposition 4.3. *Let $q \in \mathcal{N}_\kappa$ be a generalized Nevanlinna function. Then $\beta \in \mathbb{R}$ is a generalized zero of q of order $d_\beta \geq 1$ if and only if q admits a minimal representation*

$$q(z) = (z-\beta)\left[\left(I+(z-\beta)(A-z)^{-1}\right)w, w\right],$$

where A is a self-adjoint relation in $\mathcal{K}$ with $\ker(A-\beta) = \{0\}$ and $w \in \operatorname{dom} T^{d_\beta - 1} \setminus \operatorname{dom} T^{d_\beta}$, where $T = (A-\beta)^{-1} \restriction < w >^{\perp}$.

Proof. The condition (3.4) in Theorem 3.1 can be rewritten as

$$w \in \operatorname{ran}(A-\beta)^{d_\beta - 1}, \quad s_0(\beta) = \cdots = s_{d_\beta - 2}(\beta) = 0.$$

Now in view of (4.2) and the definition of the operator T one obtains for $i = 0, \ldots, d_\beta - 2$

$$0 = s_i(\beta) = [(A-\beta)^{-i}w, w] = [T^i w, w],$$

which means that $w \in \operatorname{dom} T^{i+1}$, $i = 0, \ldots, d_\beta - 2$. Moreover, the maximality of d_β is equivalent to $w \notin \operatorname{dom} T^{d_\beta}$. This completes the proof. □

4.2. **Moments and degree of non-positivity.** As a consequence of Theorem 3.1 the inner product on the root subspace $\mathsf{R}_\beta(\widehat{A})$ can be described in terms of moments. By definition β is a generalized zero of q of order d_β if and only if β is an eigenvalue of the relation $\widehat{A}$ in a minimal representation of $\widehat{q} = -1/q$ in (3.6) with algebraic multiplicity d_β. The inner product on the root subspace $\mathsf{R}_\beta(\widehat{A})$ can be described using a maximal Jordan chain $x_0, \ldots, x_{d_\beta - 1}$, which spans $\mathsf{R}_\beta(\widehat{A})$, by means of the corresponding Gram matrix $G_\beta := \left([x_i, x_j]\right)_{i,j=0}^{d_\beta - 1}$. If $\beta = \infty$ then, as the proof of Theorem 3.1 shows, the vectors

$$u, Au, \ldots, A^{d_\infty - 1}u \tag{4.3}$$

form a maximal Jordan chain of the relation $\widehat{A}$ at ∞. Similarly, if $\beta \in \mathbb{R}$ then the vectors

$$w, (A-\beta)^{-1}w, \ldots, (A-\beta)^{-d_\beta + 1}w \tag{4.4}$$

form a maximal Jordan chain for $\widehat{A}$ at β. Therefore, one obtains the following explicit form for the inner product structure on $\mathsf{R}_\beta(\widehat{A})$.

Corollary 4.4. *Let $\beta \in \mathbb{R} \cup \{\infty\}$ be a generalized zero of $q \in \mathcal{N}_\kappa$ of order d_β. The Gram matrix generated by the maximal Jordan chains (4.3) and (4.4) on $\mathsf{R}_\beta(\widehat{A})$ is given by*

$$G_\beta = \begin{pmatrix} 0 & \cdots & 0 & s_{d_\beta - 1}(\beta) \\ \vdots & & \iddots & s_{d_\beta}(\beta) \\ 0 & \iddots & \iddots & \vdots \\ s_{d_\beta - 1}(\beta) & s_{d_\beta}(\beta) & \cdots & s_{2d_\beta - 2}(\beta) \end{pmatrix}. \tag{4.5}$$

Proof. The statement is immediate from Theorem 3.1 in view of (3.13) and (4.2). □

Corollary 4.4 leads to a simple characterization for the degree of non-positivity of a generalized zero, which is similar to the characterization (3.4) given for the order of a generalized zero in Theorem 3.1. In fact, the result below can be interpreted also as a reformulation of the analytic characterization of the degree of non-positivity, originally proved in another way by H. Langer in [13]. First, recall the following definition.

Definition 4.5. Let $\beta \in \mathbb{R} \cup \{\infty\}$ be a generalized zero of $q \in \mathcal{N}_\kappa$. Then the dimension of a maximal non-positive invariant subspace of the root subspace $\mathsf{R}_\beta(\widehat{A})$ of $\widehat{A}$ at β is called its *degree of non-positivity*.

If β is an (ordinary) zero of Q then in general its multiplicity does not coincide with the degree of non-positivity. This is the reason for us not to use the notation multiplicity, that is used e.g. in [13].

Proposition 4.6. *Let $q \in \mathcal{N}_\kappa$ be given. Then the point $\beta \in \mathbb{R} \cup \{\infty\}$ is a generalized zero of q with degree of non-positivity $\pi_\beta > 0$ if and only if π_β is the maximal integer such that the moments $s_0(\beta), \dots, s_{2\pi_\beta-2}(\beta)$ exist and*

$$s_0(\beta) = \cdots = s_{2\pi_\beta-3}(\beta) = 0 \quad \text{and} \quad s_{2\pi_\beta-2}(\beta) \le 0. \tag{4.6}$$

In particular, if $\beta \in \mathbb{R} \cup \{\infty\}$ is a generalized zero of q of order d_β then the root subspace $\mathsf{R}_\beta(\widehat{A})$ is degenerate if and only if $s_{d_\beta-1}(\beta) = 0$.

Proof. If $\mathsf{R}_\beta(\widehat{A})$ is spanned by the maximal Jordan chain $x_0, \dots, x_{d_\beta-1}$, then the dimension π_β of a maximal non-positive invariant subspace of $\mathsf{R}_\beta(\widehat{A})$ is equal to the maximal integer for which the Gram matrix associated to the chain $x_0, \dots, x_{\pi_\beta-1}$ is non-positive. Therefore, the statements are clear from (4.5). □

Finally, observe that for a generalized zero $\beta \in \mathbb{C} \setminus \mathbb{R}$ the corresponding root subspace $\mathsf{R}_\beta(\widehat{A})$ is neutral, so that $d_\beta = \pi_\beta$ and in (3.1), (3.2) one automatically has $s_0(\beta) = \cdots = s_{2d_\beta-2}(\beta) = 0$.

5. Asymptotic expansions and an inverse problem

From the definition of the moments of $q \in \mathcal{N}_\kappa$ at a point $\beta \in \mathbb{R} \cup \{\infty\}$ it immediately follows that q admits a certain asymptotic expansion at β. By Theorem 3.1 this means that the generalized zeros of q, and thus also the generalized poles of q, can be characterized by means of such asymptotic expansions. These expansions in the case that $\beta = \infty$ were introduced in [11] and further studied for instance in [4], [5], and [7]. The main purpose in this section is to solve the following

Inverse Problem. *Reconstruct the geometric structure of the root subspaces of the representing self-adjoint relation A in* (2.1) *by means of the asymptotic behaviour of the function q.*

This problem concerns the generalized poles of q and the self-adjoint relation A in a minimal representation of q in (2.1). It can be reformulated also by means of generalized zeros of the function q and the root subspaces of the self-adjoint relation $\widehat{A}$ in a minimal representation of the function $\widehat{q}$ in (3.6). The problem is first solved in this second case.

It follows from Theorem 3.1 that a function $q \in \mathcal{N}_\kappa$ has a generalized zero at $\beta = \infty$ if and only if q admits a (non-tangential) *asymptotic expansion* at $\beta = \infty$ of the form

$$q(z) = -\frac{s_{n-1}(\infty)}{z^n} - \frac{s_n(\infty)}{z^{n+1}} - \cdots - \frac{s_{m-1}(\infty)}{z^m} + o\Big(\frac{1}{z^m}\Big) \quad \text{as} \quad z \widehat{\to} \infty \tag{5.1}$$

and a generalized zero at $\beta \in \mathbb{R}$, if and only if

$$q(z) = s_{n-1}(\beta)(z-\beta)^n + s_n(\beta)(z-\beta)^{n+1} + \cdots + s_{m-1}(\beta)(z-\beta)^m + o((z-\beta)^m) \tag{5.2}$$

as $z \hat{\to} \beta$, with some real numbers $s_j(\beta)$ for $j = n-1, \dots, m-1$. Clearly, these real numbers are exactly the moments as defined in Section 3. In order to reconstruct the geometric structure of the underlying root subspace suitable maximality conditions for these expansions are introduced.

Definition 5.1. The expansions (5.1) and (5.2) are called *maximal* if the following two conditions hold:

(i) Maximality of n: n is the largest index such that $s_i(\beta) = 0$ for all $i < n-1$.

(ii) Maximality of m: either there does not exist $s_m(\beta) \in \mathbb{R}$, i.e. q cannot be expanded any further, or for every $m \geq n$ there exists an expansion of the form (5.1) or (5.2), in which case every $m \geq 2n-1$ is called maximal.

Note that for $q \in \mathcal{N}_\kappa$ the maximality of these expansions is well defined due to Theorem 3.1, and in this case $n \leq m+1$.

The solution to the inverse problem in the case of generalized zeros of q and the self-adjoint relation $\widehat{A}$ can be based on Corollary 4.4. The question that needs to be solved first is to characterize the order d_β of the generalized zero β of q by means of appropriate asymptotic expansions. The answers are given in the next theorem, which expresses the order as well as the degree of non-positivity of a generalized zero from maximal expansions and describes the geometric structure of the root subspaces by means of the inner product.

Theorem 5.2. *Let $q \in \mathcal{N}_\kappa$ with $q(z) \not\equiv 0$ and $\beta \in \mathbb{R} \cup \{\infty\}$ be given and let $\widehat{A}$ be the self-adjoint relation in a minimal representation* (3.6). *Assume that the function q admits a maximal expansion of the form*

$$q(z) = -\frac{s_{n-1}(\infty)}{z^n} - \cdots - \frac{s_{m-1}(\infty)}{z^m} + o\left(\frac{1}{z^m}\right) \quad \text{as} \quad z \hat{\to} \infty,$$

if $\beta = \infty$, and a maximal expansion of the form

$$q(z) = s_{n-1}(\beta)(z-\beta)^n + \cdots + s_{m-1}(\beta)(z-\beta)^m + o((z-\beta)^m) \quad \text{as} \quad z \hat{\to} \beta,$$

if $\beta \in \mathbb{R}$, for some $n, m \geq 1$. Then:

(i) *β is a generalized zero of q of order $d_\beta = \min\left\{n, [\frac{m+1}{2}]\right\}$;*

(ii) *the inner product structure of the root subspace $\mathsf{R}_\beta(\widehat{A})$ is determined by the Hankel matrix $G_\beta = [s_{i+j-2}(\beta)]_{i,j=1}^{d_\beta}$ with d_β as given in part (i);*

(iii) *the degree of non-positivity of the generalized zero β of q is given by*

$$\pi_\beta = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ odd and } s_{n-1}(\beta) > 0 \\ \frac{n+1}{2} & \text{if } n \text{ odd and } s_{n-1}(\beta) < 0 \\ \frac{n}{2} & \text{if } n \text{ even} \\ [\frac{n}{2}](= d_\beta) & \text{if } n = m+1 \end{cases}.$$

In particular, the root subspace $\mathsf{R}_\beta(\widehat{A})$ is degenerate exactly if $m < 2n-1$.

Proof. (i) & (iii) First consider the case $\beta = \infty$. Clearly, for every $l \in \mathbb{N}$ the following equivalence holds:

$$\exists s_0(\beta), \dots, s_l(\beta) \quad \Leftrightarrow \quad q(z) = -\frac{s_0}{z} - \cdots - \frac{s_l}{z^{l+1}} + o\left(\frac{1}{z^{l+1}}\right) \quad \text{as} \quad z \hat{\to} \infty.$$

Now from Theorem 3.1 one concludes that $d_\beta = \min\left\{n, [\frac{m+1}{2}]\right\}$, while the formulas for π_β are obtained from Proposition 4.6.

In the case that $\beta \in \mathbb{R}$ one uses expansions of the form (5.2) instead of the expansions in (5.1). Otherwise the previous argument and the results remain the same in this case.

(ii) Since the numbers $s_i(\beta)$ in (5.1) and (5.2) coincide with the (local) moments of q at $\beta \in \mathbb{R} \cup \{\infty\}$, the statement follows immediately from part (i) and Corollary 4.4.

Finally, by Proposition 4.6 the corresponding root subspace is degenerate if and only if $s_{d_\beta-1}(\beta) = 0$. In the case that $n \le m$ the condition $s_{d_\beta-1}(\beta) = 0$ means that $d_\beta < n$, i.e., $[\frac{m+1}{2}] < n$, which is equivalent to $m+1 < 2n$. In the case that $n = m+1$ one has $m < 2n-1$ and now clearly $s_{d_\beta-1}(\beta) = 0$. This completes the proof. □

Theorem 5.2 shows that it suffices to consider maximal expansions where m satisfies the condition $m \le 2n-1$, in which case $d_\beta = \min\{n, [\frac{m+1}{2}]\} = [\frac{m+1}{2}]$.

In Theorem 5.2 the indices d_β and π_β are expressed with the numbers n and m in a maximal expansion. Observe that, conversely, for given d_β and π_β the numbers n and m of a maximal expansion cannot be recovered uniquely as the following example shows.

Example 5.3. If $q \in \mathcal{N}_\kappa$ admits a maximal expansion of the following type

$$q(z) = o\left(\frac{1}{z}\right), \quad q(z) = \frac{1}{z^2} + o\left(\frac{1}{z^2}\right) \quad \text{or} \quad q(z) = o\left(\frac{1}{z^2}\right) \quad \text{as} \quad z \widehat{\rightarrow} \infty,$$

then ∞ is a generalized zero of q with $d_\infty = \pi_\infty = 1$, and the corresponding root subspace is degenerate. If $q \in \mathcal{N}_\kappa$ satisfies

$$q(z) = \frac{1}{z} + o\left(\frac{1}{z}\right) \quad \text{as} \quad z \widehat{\rightarrow} \infty,$$

then this is a maximal expansion for q and therefore again $d_\infty = \pi_\infty = 1$, but now the corresponding root subspace is non-degenerate.

The solution to the inverse problem stated above is given in the next theorem.

Theorem 5.4. *Let $q \in \mathcal{N}_\kappa$ with $q(z) \not\equiv 0$ and $\alpha \in \mathbb{R} \cup \{\infty\}$ be given and let A be the self-adjoint relation in a minimal representation* (2.1) *of q. Assume that the function q admits a maximal expansion of the form*

$$q(z) = a_{j-1}(\infty)z^j + a_{j-2}(\infty)z^{j-1} + \cdots + a_{k-1}(\infty)z^k + o(z^k) \quad \text{as} \quad z \widehat{\rightarrow} \infty, \tag{5.3}$$

if $\alpha = \infty$, and a maximal expansion of the form

$$q(z) = -\frac{a_{j-1}(\alpha)}{(z-\alpha)^j} - \frac{a_{j-2}(\alpha)}{(z-\alpha)^{j-1}} - \cdots - \frac{a_{k-1}(\alpha)}{(z-\alpha)^k} + o\left(\frac{1}{(z-\alpha)^k}\right) \quad \text{as} \quad z \widehat{\rightarrow} \alpha, \tag{5.4}$$

if $\alpha \in \mathbb{R}$, for some $j, k \ge 1$. Then:

(i) *the point α is a generalized pole of q of order $p_\alpha = j - [\frac{k}{2}]$ if $j \ge k$ and $p_\alpha = [\frac{k}{2}]$ if $j = k-1$;*

(ii) *the inner product structure of the root subspace $\mathsf{R}_\alpha(A)$ is determined by the Hankel matrix $H_\alpha = [s_{i+j-2}(\alpha)]_{i,j=1}^{p_\alpha}$, where $s_0(\alpha) = \cdots = s_{j-2}(\alpha) = 0$ and the real numbers $s_{j-1}(\alpha), \ldots, s_{2p_\alpha-2}(\alpha)$ are determined by the convolution equations*

$$s_{j-1}(\alpha)a_{j-1}(\alpha) = 1, \quad \sum_{i=j}^{l} s_{i-1}(\alpha)a_{l+j-i-1}(\alpha) = 0, \quad l = j+1, \ldots, k, \tag{5.5}$$

when $a_{j-1}(\alpha) \ne 0$, and otherwise $s_0(\alpha) = \cdots = s_{2p_\alpha-2}(\alpha) = 0$;

(iii) *the degree of non-positivity of the generalized pole α of q is given by*

$$\nu_\alpha = \begin{cases} \frac{j-1}{2} & \text{if } j \text{ odd and } a_{j-1} > 0 \\ \frac{j+1}{2} & \text{if } j \text{ odd and } a_{j-1} < 0 \\ \frac{j}{2} & \text{if } j \text{ even} \\ [\frac{j+1}{2}] (= p_\alpha) & \text{if } j = k-1 \end{cases}.$$

In particular, the corresponding root subspace is degenerate if and only if $k > 1$.

Here the expansion is called maximal, if $j \geq 1$ is the smallest positive integer such that $a_i(\alpha) = 0$ for all $i \geq j$ and $k \geq 1$ is the smallest positive integer for which the indicated expansion exists.

Proof. (i) & (iii) Consider the function $\widehat{q} = -\frac{1}{q}$. Then the indices p_α and ν_α associated with the generalized pole $\alpha \in \mathbb{R} \cup \{\infty\}$ can be recovered from maximal expansions of the form (5.1) and (5.2) for the function $\widehat{q}$ according to Theorem 5.2. If $n \leq m$ then maximal expansions of the form (5.1) and (5.2) for $\widehat{q}$ with $m \leq 2n-1$ give by inversion maximal expansions of the form (5.3) and (5.4) for q with $j = n$ and $k = 2n-m$, respectively, so that $j \geq k \geq 1$. The converse is also true. Hence, in this case $p_\alpha = [\frac{m+1}{2}] = j - [\frac{k}{2}]$ and the formulas for ν_α coincide with those for π_α in Theorem 5.2.

Now consider the case $n = m+1$ with $\alpha = \infty$ (with $\alpha \in \mathbb{R}$ the argument is similar). Then the maximal expansion for $\widehat{q}$ is of the form $\widehat{q}(z) = o(\frac{1}{z^m})$. Since m is maximal and for every $l \in \mathbb{Z}$ the function $z^{2l}q(z)$ is a generalized Nevanlinna function, one concludes by Corollary 2.1 that q has the following maximal expansion

$$q(z) = o(z^{m+1}) \quad \text{or} \quad q(z) = o(z^{m+2}) \quad \text{as} \quad z\widehat{\rightarrow}\infty,$$

where the second expansion can occur only if m is an odd number. Hence, in this case $j = k-1$ and either $k = m+1$, or $k = m+2$ and this second case can occur only if m is an odd number. The converse is also true. Therefore, in this case $p_\alpha = [\frac{m+1}{2}] = [\frac{k}{2}] = \nu_\alpha$.

(ii) According to Corollary 4.4 the inner product on the root subspace $\mathsf{R}_\alpha(A)$ is determined by the (local) moments $s_i(\alpha)$ of the function $\widehat{q}$ at $\alpha \in \mathbb{R} \cup \{\infty\}$. The first part of the proof implies that these numbers are obtained by inverting the expansions of q in (5.3) and (5.4). The coefficients $a_i(\alpha)$ are all equal to zero exactly if the same is true for the (local) moments $s_i(\alpha)$ of $\widehat{q}$. If $a_{j-1}(\alpha) \neq 0$ so that $j \geq k$, then all the remaining coefficients in the expansion for $\widehat{q}$ are obtained by solving the convolution equations in (5.5), compare [8, Lemma 4.1].

As to the last statement, clearly the condition $m < 2n-1$ is equivalent to $k > 1$, when $j \geq k$. In the case that $j = k-1$ one has $k \geq 2$. This completes the proof. □

From Theorem 5.2, and also directly from (4.5), it can be seen that the degree of non-positivity π_β and the order d_β of a generalized zero β cannot be completely independent. The following inequalities are always satisfied:

$$\left[\frac{d_\beta}{2}\right] \leq \pi_\beta \leq d_\beta \quad \text{or equivalently} \quad \pi_\beta \leq d_\beta \leq 2\pi_\beta + 1.$$

As the following simple example shows, within these restrictions every combination is possible.

Example 5.5. Let d, $k \geq 1$ be integers satisfying $k \leq d \leq 2k-1$. Then the function

$$q(z) := \frac{1}{z^{2k-1}} + \frac{1}{z^{2d-1}}\frac{1}{\sqrt{z}} \tag{5.6}$$

is a generalized Nevanlinna function and has the generalized zero ∞, which is of order d and degree of non-positivity k. To see this first observe that the second summand in (5.6) belongs to the generalized Nevanlinna class $\mathcal{N}_d$ because of its basic factorization $\frac{1}{z^{2d-1}}\frac{1}{\sqrt{z}} = \frac{1}{z^{2d}}\sqrt{z}$, where $\sqrt{z} \in \mathcal{N}_0$. Hence, also q is a generalized Nevanlinna function as the sum of two such functions. Clearly, the maximal expansion for q is given by

$$q(z) = -\frac{-1}{z^{2k-1}} + o\Big(\frac{1}{z^{2d-1}}\Big) \quad \text{as} \quad z\widehat{\rightarrow}\infty.$$

Now the claim follows directly from Theorem 5.2.

As to the cases $d = 2k$ and $d = 2k+1$, consider the function $q(z) := -\frac{1}{z^d}$, which clearly has the desired properties.

It is mentioned that one obtains from parts (i) and (iii) of Theorem 5.4, for instance, the results in [3, Section 4] concerning non-degenerate root subspaces at $\alpha = \infty$. Moreover, if $\kappa = 1$ and $\alpha = \infty$ then part (i) of Theorem 5.4 reduces to [4, Corollary 6.3]. For a more specific classification of asymptotic expansions, see [5].

6. Characterizations via the basic factorization

In this section Theorem 3.1 will be expressed in terms of the function's basic factorization and some explicit formulas are derived from the corresponding integral representation. Recall that every $q \in \mathcal{N}_\kappa$ admits a factorization (see [6, 2]) of the form

$$q(z) = r(z)r^{\#}(z)q_0(z), \tag{6.1}$$

with $r(z) = \dfrac{\prod_{j=1}^{m}(z-\beta_j)^{\pi_{\beta_j}}}{\prod_{i=1}^{l}(z-\alpha_i)^{\nu_{\alpha_i}}}$, $r^{\#}(z) = \overline{r(\overline{z})}$, and $q_0 \in \mathcal{N}_0$. Here the points $\alpha_1, \dots, \alpha_l \in \mathbb{R} \cup \mathbb{C}^+$ are the finite generalized poles and $\beta_1, \dots, \beta_m \in \mathbb{R} \cup \mathbb{C}^+$ the finite generalized zeros of the function $q \in \mathcal{N}_\kappa$, which are not of positive type with degrees of non-positivity $\nu_{\alpha_1}, \dots, \nu_{\alpha_l}$ and $\pi_{\beta_1}, \dots, \pi_{\beta_m}$, respectively. Observe, that ∞ is a generalized zero not of positive type of q exactly if $\pi_\infty := \nu_{\alpha_1} + \cdots + \nu_{\alpha_l} - (\pi_{\beta_1} + \cdots + \pi_{\beta_m}) > 0$.

It is convenient to put $\pi_\beta = 0$ ($\nu_\alpha = 0$), whenever β is not a generalized zero (α not a generalized pole, respectively) of q not of positive type. In what follows factorization of the function q is augmented by incorporating in (6.1) the following standard integral representation of the factor $q_0 \in \mathcal{N}_0$:

$$q_0(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\sigma(t), \tag{6.2}$$

where $a \in \mathbb{R}$, $b \geq 0$, and $\int_{\mathbb{R}} \frac{d\sigma(t)}{t^2+1} < \infty$.

In order to distinguish the moments corresponding to the different functions the notation $s_n^{q_0}(\beta)$ is used to refer to a moment of the function q_0. In the next lemma the moments for q_0 are given in terms of its integral representation and via derivatives of q_0; these results can be considered to be well known at least for $\beta = \infty$, cf. e.g. [8].

Lemma 6.1. *Let the Nevanlinna function $q_0 \in \mathcal{N}_0$ be given. Then*

$$\exists\, s_0^{q_0}(\infty) \iff \int_{\mathbb{R}} d\sigma(t) < \infty, \quad b = 0, \quad \text{and} \quad \lim_{z \widehat{\to} \infty} q_0(z) = 0. \tag{6.3}$$

Furthermore, the moments of q_0 at $\beta = \infty$ are given by:

$$s_j^{q_0}(\infty) = \int_{\mathbb{R}} t^j \, d\sigma(t) \quad \text{when} \quad \int_{\mathbb{R}} |t|^j \, d\sigma(t) < \infty \quad \text{for} \quad j = 0, 1, \dots .$$

For $\beta \in \mathbb{R}$ one has:

$$\exists\, s_0^{q_0}(\beta) \iff \int_{\mathbb{R}} \frac{d\sigma(t)}{(t-\beta)^2} < \infty \quad \text{and} \quad \sigma(\{\beta\}) = 0 \quad \text{and} \quad \lim_{z \widehat{\to} \beta} q(z) = 0, \tag{6.4}$$

in which case

$$s_0^{q_0}(\beta) = b + \int_{\mathbb{R}} \frac{d\sigma(t)}{(t-\beta)^2},$$

and

$$s_j^{q_0}(\beta) = \int_{\mathbb{R}} \frac{d\sigma(t)}{(t-\beta)^{j+2}} \quad \text{when} \quad \int_{\mathbb{R}} \frac{d\sigma(t)}{|t-\beta|^{j+2}} < \infty \quad \text{for} \quad j = 1, 2, \dots$$

or - using derivatives -

$$s_j^{q_0}(\beta) = \lim_{z \widehat{\to} \beta} \frac{1}{(j+1)!} q_0^{(j+1)}(z) \quad \text{for} \quad j = 0, 1, \dots .$$

In fact, the results formulated in Lemma 6.1 can be derived directly from the integral representation (6.2). Alternatively, one can use the connection of the operator representation (2.1) to the integral representation (6.2). Then Corollary 3.3 (for $\beta = \infty$) and Corollary 4.2 (for $\beta \in \mathbb{R}$) give the characterizations (6.3) and (6.4) and the formulas for the higher moments correspond to (3.13) and (4.2), respectively.

The existence of the moments of q can now be characterized with integrability conditions on the measure σ in (6.2).

Lemma 6.2. *Let the function $q \in \mathcal{N}_\kappa$ be given and assume that $\beta \in \mathbb{R} \cup \{\infty\}$ is a generalized zero not of positive type of q; that is $\pi_\beta > 0$. Then:*

$$s_i(\beta) = 0 \quad for \quad i = 0, 1, \ldots, 2\pi_\beta - 3.$$

For $\beta = \infty$ and $l \geq \pi_\infty$ it holds

$$\exists\, s_0(\infty), s_1(\infty), \ldots, s_{2l-2}(\infty) \quad \Longleftrightarrow \quad \int\limits_{|t|>1} t^{2(l-1-\pi_\infty)}\, d\sigma(t) < \infty \tag{6.5}$$

and similarly for $\beta \in \mathbb{R}$ and $l \geq \pi_\beta$ one has

$$\exists\, s_0(\beta), s_1(\beta), \ldots, s_{2l-2}(\beta) \quad \Longleftrightarrow \quad \int\limits_{0<|t-\beta|<1} \frac{d\sigma(t)}{(t-\beta)^{2(l-\pi_\beta)}} < \infty.$$

Proof. The first statement is clear from Proposition 4.6. As to the second statement observe that, by the assumption $\pi_\infty > 0$, q admits an operator representation as in Corollary 3.3. Decompose $A = A' \oplus A''$ as in (2.15). Then an application of the generalized Stieltjes inversion formula yields

$$d[E''(t)P''u, P''u] = |r(t)|^2\, d\sigma(t), \quad |t| > K, \tag{6.6}$$

where $E''(t)$, P'', u, and K are as in (3.14). According to [11] (see also the proof of Theorem 3.1) the existence of the moments $s_0(\infty), \ldots, s_{2l-2}(\infty)$ is equivalent to $u \in \operatorname{dom} A^{l-1}$, which holds if and only if the integrability condition (3.14) is satisfied. Now (6.5) follows by comparing (3.14) with (6.6).

For $\beta \in \mathbb{R}$ the claim is obtained by applying the above proven result to the function $\widetilde{q}(z) := -q(\frac{1}{z} + \beta)$ and observing that the linear term in the integral representation of $\widetilde{q}_0$ is the point mass of σ at β in the integral representation of q_0 in (6.2). □

In order to simplify the formulation of the main result of this section the following notation is introduced. It is said that for $\pi_\infty \leq l \leq 2\pi_\infty + 1$ the function q has at ∞ the property $J(l)$

for $\pi_\infty \leq l \leq 2\pi_\infty - 1$: $\displaystyle\int\limits_{|t|>1} t^{2(l-\pi_\infty-1)} d\sigma(t) < \infty$

for $l = 2\pi_\infty$: $\begin{cases} \displaystyle\int\limits_{|t|>1} t^{2(\pi_\infty-1)} d\sigma(t) < \infty \\ \text{and } q_0 \text{ assumes a generalized value at } \infty \end{cases}$

for $l = 2\pi_\infty + 1$: $\begin{cases} \displaystyle\int\limits_{|t|>1} t^{2\pi_\infty} d\sigma(t) < \infty \\ \text{and } q_0 \text{ has a generalized zero at } \infty \end{cases}$

and for $\pi_\beta \le l \le 2\pi_\beta + 1$ it has at $\beta \in \mathbb{R}$ the property $J(l)$

$$\text{for } \pi_\beta \le l \le 2\pi_\beta - 1: \quad \int\limits_{0<|t-\beta|<1} \frac{d\sigma(t)}{(t-\beta)^{2(l-\pi_\beta)}} < \infty$$

$$\text{for } l = 2\pi_\beta: \quad \begin{cases} \int\limits_{0<|t-\beta|<1} \frac{d\sigma(t)}{(t-\beta)^{2\pi_\beta}} < \infty \\ \text{and } q_0 \text{ assumes a generalized value at } \beta \end{cases}$$

$$\text{for } l = 2\pi_\beta + 1: \quad \begin{cases} \int\limits_{0<|t-\beta|<1} \frac{d\sigma(t)}{(t-\beta)^{2(\pi_\beta+1)}} < \infty \\ \text{and } q_0 \text{ has a generalized zero at } \beta. \end{cases}$$

In particular, if $\pi_\beta = 0$ then q has the property $J(1)$ if and only if β is a generalized zero of q_0, which then has to be of positive type.

Remark 6.3. A Nevanlinna function q_0 has a generalized value at ∞ if and only if it is of the form $q_0(z) = s + \int_{\mathbb{R}} \frac{d\sigma(t)}{t-z}$ with a finite measure σ. Here $s = 0$ if and only if ∞ is a generalized zero of q_0.

Observe, that if q has at $\beta \in \mathbb{R} \cup \{\infty\}$ the property $J(l)$ then it has the property $J(l')$ also for every $l' < l$. The characterization of the order of a generalized zero can now be formulated as follows.

Theorem 6.4. *Let $q \in \mathcal{N}_\kappa$ with $q(z) \not\equiv 0$ be given. Then $\beta \in \mathbb{R} \cup \{\infty\}$ is a generalized zero of q of order $d_\beta \ge 1$ if and only if d_β is the maximal integer $\pi_\beta \le d_\beta \le 2\pi_\beta + 1$ for which q at β has the property $J(d_\beta)$.*

In particular, for $\pi_\beta = 0$ this gives: The function q has a generalized zero if and only if q_0 has a generalized zero, and in this case its order is 1.

Proof. Again first consider the case $\beta = \infty$. It is shown that for $l \ge \max\{1, \pi_\infty\}$ the function q has the property $J(l)$ at ∞ if and only if there exists a (not necessarily maximal) Jordan chain of length l of the representing relation $\widehat{A}$ (in Proposition 3.4) at ∞. That is we have to show that q has the property $J(l)$ if and only if (3.11) is satisfied. For $\pi_\infty \le l \le 2\pi_\infty - 1$ the claim follows directly from Lemma 6.2. For $l = 2\pi_\infty$ one has to show that $s_{l-2} = 0$ if and only if q_0 assumes a generalized value at ∞. Indeed,

$$0 = s_{2\pi_\infty - 2}(\infty) = \lim_{z \widehat{\to} \infty} z^{2\pi_\infty} r(z) r^{\#}(z) \frac{q_0(z)}{z} = b, \tag{6.7}$$

together with the integrability condition shows that q_0 assumes a generalized value at ∞, and conversely. For $l = 2\pi_\infty + 1$ the function q_0 assumes a generalized value and, in addition,

$$0 = s_{2\pi_\infty - 1}(\infty) = \lim_{z \widehat{\to} \infty} z^{2\pi_\infty} r(z) r^{\#}(z) q_0(z) = s. \tag{6.8}$$

This completes the proof in case $\beta = \infty$. Again, for $\beta \in \mathbb{R}$ the claim follows by applying this result to $\widetilde{q}(z) := -q(\frac{1}{z} + \beta)$. □

In what follows Theorem 6.4 is rewritten in a more explicit form by using the parameters in the integral representation (6.2). Moreover, a characterization for the root subspace to be degenerate is given. For $\beta = \infty$ one has the following result.

Corollary 6.5. *Let the function $q \in \mathcal{N}_\kappa$ with $q(z) \not\equiv 0$ be given. Then ∞ is a generalized zero of q of order d_∞ with*

$$\pi_\infty \le d_\infty \le 2\pi_\infty - 2 \iff \begin{cases} \displaystyle\int_{|t|>1} t^{2(d_\infty-\pi_\infty-1)}\, d\sigma(t) < \infty \quad \text{and} \\ \displaystyle\int_{|t|>1} t^{2(d_\infty-\pi_\infty)}\, d\sigma(t) = \infty \end{cases}$$

$$d_\infty = 2\pi_\infty - 1 \iff \begin{cases} \displaystyle\int_{|t|>1} t^{2(\pi_\infty-2)}\, d\sigma(t) < \infty \quad \text{and} \\ \Big(\displaystyle\int_{|t|>1} t^{2(\pi_\infty-1)}\, d\sigma(t) = \infty \quad \text{or} \quad b \ne 0 \Big) \end{cases}$$

$$d_\infty = 2\pi_\infty \iff \begin{cases} \displaystyle\int_{|t|>1} t^{2(\pi_\infty-1)}\, d\sigma(t) < \infty, \quad b = 0, \quad \text{and} \\ \Big(\displaystyle\int_{|t|>1} t^{2\pi_\infty}\, d\sigma(t) = \infty \quad \text{or} \quad s \ne 0 \Big) \end{cases}$$

$$d_\infty = 2\pi_\infty + 1 \iff \int_{|t|>1} t^{2\pi_\infty}\, d\sigma(t) < \infty \quad \text{and} \quad b = s = 0.$$

For $\pi_\infty \le d_\infty \le 2\pi_\infty - 2$ the root subspace of $\widehat{A}$ at ∞ is degenerate. It is non-degenerate for $d_\infty = 2\pi_\infty - 1$ and $d_\infty = 2\pi_\infty$ if and only if in the above characterizations the condition $b \ne 0$ and $s \ne 0$, respectively, holds true; for $d_\infty = 2\pi_\infty + 1$ it is always non-degenerate.

Proof. The first part of the statement is immediate from Theorem 6.4. According to Proposition 4.6 non-degeneracy of the underlying root subspace is equivalent to $s_{d_\beta - 1}(\beta) \ne 0$. Hence, for $d_\infty \le 2\pi_\infty$ the statements are obtained from (4.6), (6.7), (6.8), and for $d_\infty = 2\pi_\infty + 1$ from $s_{2\pi_\infty}(\infty) = s_0^{q_0}(\infty) > 0$, which follows from $b = s = 0$ and the assumption $q(z) \not\equiv 0$. □

From Corollary 6.5 one obtains, for instance, [3, Theorem 6.3] containing a characterization for the non-degeneracy of the root subspace $\mathsf{R}_\infty(\widehat{A})$ and for $\kappa = \pi_\infty = 1$ it reduces to [4, Corollary 4.2]. For some related results, involving a classification for generalized zeros and poles, see [5].

The corresponding result for $\beta \in \mathbb{R}$ is given in the following corollary, where the notation $q_0(\beta) := \lim_{z \hat{\to} \beta} q_0(z)$ is used.

Corollary 6.6. *Let the function $q \in \mathcal{N}_\kappa$ with $q(z) \not\equiv 0$ be given. Then $\beta \in \mathbb{R}$ is a generalized zero of q of order d_β with*

$$\pi_\beta \le d_\beta \le 2\pi_\beta - 2 \iff \begin{cases} \displaystyle\int_{0<|t-\beta|<1} \frac{d\sigma(t)}{(t-\beta)^{2(d_\beta-\pi_\beta)}} < \infty \quad \text{and} \\ \displaystyle\int_{0<|t-\beta|<1} \frac{d\sigma(t)}{(t-\beta)^{2(d_\beta-\pi_\beta+1)}} = \infty \end{cases}$$

$$d_\infty = 2\pi_\beta - 1 \iff \begin{cases} \displaystyle\int_{0<|t-\beta|<1} \frac{d\sigma(t)}{(t-\beta)^{2(\pi_\beta-1)}} < \infty \quad \text{and} \\ \Big(\displaystyle\int_{0<|t-\beta|<1} \frac{d\sigma(t)}{(t-\beta)^{2\pi_\beta}} = \infty \quad \text{or} \quad \sigma(\{\beta\}) \ne 0 \Big) \end{cases}$$

$$d_\beta = 2\pi_\beta \iff \begin{cases} \displaystyle\int_{0<|t-\beta|<1} \frac{d\sigma(t)}{(t-\beta)^{2\pi_\beta}} < \infty \quad \text{and} \quad \sigma(\{\beta\}) = 0 \quad \text{and} \\ \Big(\displaystyle\int \frac{d\sigma(t)}{(t-\beta)^{2(\pi_\beta+1)}} = \infty \quad \text{or} \quad q_0(\beta) \ne 0 \Big) \end{cases}$$

$$d_\infty = 2\pi_\beta + 1 \quad \Longleftrightarrow \quad \int\limits_{0<|t-\beta|<1} \frac{d\sigma(t)}{(t-\beta)^{2(\pi_\beta+1)}} < \infty \quad \textit{and} \quad \sigma(\{\beta\}) = q_0(\beta) = 0.$$

For $\pi_\beta \leq d_\beta \leq 2\pi_\beta - 2$ the root subspace of $\widehat{A}$ at β is degenerate. For $d_\beta = 2\pi_\beta - 1$ and $d_\beta = 2\pi_\beta$ it is non-degenerate if and only if in the above characterizations the condition $\sigma(\{\beta\}) \neq 0$ and $q_0(\beta) \neq 0$, respectively, holds true; for $d_\beta = 2\pi_\infty + 1$ it is always non-degenerate.

Finally, it is shown how to derive explicit formulas for the moments of the function q from its factorized integral representation. The result also completes the statements in Lemma 6.2 and, in fact, can be used to give a constructive proof for it.

Proposition 6.7. *Let $q \in \mathcal{N}_\kappa$ be factorized as in (6.1), (6.2), and let $\beta = \infty$ be a generalized zero of q of order $d_\infty \geq 1$. Then the following statements hold:*

(i) *if $d_\infty = \pi_\infty(q)$, then $\int_{\mathbb{R}} d\sigma(t) = \infty$ and q admits a maximal expansion of the form*

$$q(z) = \frac{b}{z^{2d_\infty - 1}} + o\left(\frac{1}{z^{2d_\infty-1}}\right), \quad z \widehat{\rightarrow} \infty; \tag{6.9}$$

(ii) *if $d_\infty > \pi_\infty(q)$, then $\int_{\mathbb{R}} d\sigma(t) < \infty$ and all the moments $s_n(\infty)$, $n \leq 2d_\infty - 2$, of q are determined by the asymptotic formula*

$$q(z) = r(z)r^{\#}(z)\left(s + bz - \sum_{i=0}^{2(d_\infty - \pi_\infty(q) - 1)} \frac{s_i(\sigma)}{z^{i+1}}\right) + o\left(\frac{1}{z^{2d_\infty - 1}}\right), \quad z \widehat{\rightarrow} \infty, \tag{6.10}$$

where $s_i(\sigma)$ are the moments of the measure $d\sigma(t)$ and $s = a - \int_{\mathbb{R}} \frac{t}{t^2+1}\, d\sigma(t)$.

Proof. If $\beta = \infty$ is a generalized zero of q of order $d_\infty \geq 1$ then by Theorem 3.1 the moments $s_i(\infty)$ of q at ∞ exist for all $i \leq 2d_\infty - 2$ and $s_i(\infty) = 0$ for $i = 0, \dots, d_\infty - 2$. Then by Lemma 6.2

$$\int_{|t|>1} t^{2(d_\infty - \pi_\infty(q) - 1)}\, d\sigma(t) < \infty. \tag{6.11}$$

In fact, this can also be seen by straightforward calculations using (6.1) and (6.2).

(i) Now assume that $d_\infty = \pi_\infty(q)$. Then $\int_{\mathbb{R}} d\sigma(t) = \infty$ (see Theorem 6.4) and it follows from (6.1) that

$$q(z) = r(z)r^{\#}(z)\, bz + o\left(\frac{1}{z^{2d_\infty - 1}}\right), \quad z \widehat{\rightarrow} \infty.$$

This leads to (6.9).

(ii) Next assume that $d_\infty > \pi_\infty(q)$. Then $\int_{\mathbb{R}} d\sigma(t) < \infty$ (cf. Theorem 6.4) and hence $s \in \mathbb{R}$ is well defined. Moreover, the condition (6.11) shows that the measure $d\sigma(t)$ in (6.1) has finite moments $s_i(\sigma)$ for $i \leq 2(d_\infty - \pi_\infty(q) - 1)$. This implies the following expansion

$$\int_{\mathbb{R}} \frac{1}{t-z}\, d\sigma(t) = -\sum_{i=0}^{2(d_\infty - \pi_\infty(q) - 1)} \frac{s_i(\sigma)}{z^{i+1}} + o\left(\frac{1}{z^{2(d_\infty - \pi_\infty(q)) - 1}}\right), \quad z \widehat{\rightarrow} \infty. \tag{6.12}$$

By incorporating (6.12) in the factorization (6.1), (6.2) and taking into account that

$$r(z)r^{\#}(z)\; o\left(\frac{1}{z^{2(d_\infty - \pi_\infty(q)) - 1}}\right) = o\left(\frac{1}{z^{2d_\infty - 1}}\right), \quad z \widehat{\rightarrow} \infty,$$

one arrives at the formula (6.10). □

Observe, that only in case the factor q_0 satisfies the condition (6.3) in Lemma 6.1 the corresponding moments $s_i^{q_0}(\infty)$ of q_0 are well defined, and in this case they coincide with the moments $s_i(\sigma)$ in Proposition 6.7.

The corresponding result holds also for the generalized zeros $\beta \in \mathbb{R}$ of q with obvious changes and, of course, the same is true for all the generalized poles of q, too. Proposition 6.7 shows, for instance, that the root subspace $\mathsf{R}_\infty(\widehat{A})$ is neutral if and only if in the factorization (6.1), (6.2) the conditions $b = 0$ and $\int_{\mathbb{R}} d\sigma(t) = \infty$ are satisfied. If $d_\infty > \pi_\infty(q)$ then all the moments $s_i(\infty)$, $i \leq 2d_\infty - 2$, of q can be calculated explicitly from the expansion in (6.10); details are left for the reader.

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