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A SPECTRAL ANALYSIS OF SOME INDEFINITE DIFFERENTIAL OPERATORS

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Abstract. We investigate the main spectral properties of quasi–Hermitian extensions of the minimal symmetric operator $L_{\min}$ generated by the differential expression $-\frac{\operatorname{sgn} x}{|x|^\alpha}\frac{d^2}{dx^2}$ $(\alpha > -1)$ in $L^2(\mathbb{R}, |x|^\alpha)$. We describe their spectra, calculate the resolvents, and obtain a similarity criterion to a normal operator in terms of boundary conditions at zero. As an application of these results we describe the main spectral properties of the operator $\frac{\operatorname{sgn} x}{|x|^\alpha}\left(-\frac{d^2}{dx^2}+c\delta\right)$, $\alpha > -1$.

1. Introduction

Let us recall that two closed operators T_1 and T_2 acting in a Hilbert space $\mathfrak{H}$ are called similar if there exists a bounded operator C with a bounded inverse C^{-1} such that $T_1 = C^{-1}T_2C$.

Denote by $L^2(\mathbb{R},\ |x|^\alpha)$, $\alpha > -1$, the Hilbert function space of equivalence classes of Lebesgue measurable functions $f(\cdot)$ such that $\int_{\mathbb{R}}|f(x)|^2|x|^\alpha dx < \infty$; the inner-product of $f, g \in L^2(\mathbb{R},\ |x|^\alpha)$ is defined by $(f, g) := \int_{\mathbb{R}} f(x)\overline{g(x)}|x|^\alpha dx$. Let us consider in $L^2(\mathbb{R},\ |x|^\alpha)$ the following symmetric operator:

$$(1)\qquad L_{\min} = -\frac{\operatorname{sgn} x}{|x|^\alpha}\frac{d^2}{dx^2}, \quad \operatorname{dom}(L_{\min}) = \{f \in \operatorname{dom}(L) : f(0) = f'(0) = 0\}.$$

Here $\operatorname{dom}(L)$ stands for a domain of the operator

$$(2)\quad L := -\frac{\operatorname{sgn} x}{|x|^\alpha}\frac{d^2}{dx^2},$$
$$\operatorname{dom}(L) := \{f \in L^2(\mathbb{R},\ |x|^\alpha) :\ f,\ f' \in W^1_{1,loc}(\mathbb{R}),\ Lf \in L^2(\mathbb{R},\ |x|^\alpha)\}.$$

The aim of the paper is to describe all quasi–Hermitian extensions $\tilde{L}$ of $L_{\min}$ (see [1]) similar to a self–adjoint operator.

Let A be an elliptic operator and let $r(\cdot)$ be an indefinite weight. The Riesz basis property of eigenfunctions of the weighted spectral problem

$$(Ay)(x) = \lambda r(x)y(x)$$

has been investigated in connection with some mechanical and physical problems (see [2, 24] and references therein). If the operator $\frac{1}{r}A$ has a nonempty continuous spectrum, then in place of the Riesz basis property it is naturally to consider a problem of similarity to a self–adjoint (normal) operator.

In particular, the model operator L of the form (2) has been studied by B. Ćurgus and B. Najman [4] ($\alpha = 0$) and by A. Fleige and B. Najman [9] ($\alpha > -1$). Using the Krein–Langer theory of definitizable operators in Krein spaces (see [19]), they proved the

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similarity of L to a self–adjoint operator. Different proofs and generalizations of these facts have been proposed in [7, 8, 13, 14, 17] (see also references therein).

In recent papers of M. M. Faddeev and R. G. Shterenberg [7] and I. M. Karabash and M. M. Malamud [17], the operator

$$L_q := \operatorname{sgn} x \left(-\frac{d^2}{dx^2} + q(x) \right) \tag{3}$$

with a nonconstant potential $q(\cdot)$ has been investigated. More precisely, necessary and sufficient conditions for operator (3) to be similar to a self–adjoint one are obtained in [7] (the case of a decaying potential) and in [17] (the cases of both decaying and finite zone potentials $q(\cdot)$).

In the paper [15] proper extensions of $L_{\min}$ which are similar to a self–adjoint or normal operator have been described for the case $\alpha = 0$.

Differential operators with an indefinite weight are of interest for one more reason. The characteristic function $W(\cdot)$ of the operator $\frac{1}{r}A$ as well as the corresponding J-form $J - W^*JW$ is unbounded in $\mathbb{C}_+$ (see Remark 3 in Section 6). Therefore, the known sufficient conditions of similarity to a self–adjoint operator (see [21] and the references therein) cannot be applied here.

The paper is organized as follows. Section 2 is preparatory. Here we present the Naboko–Malamud resolvent similarity criterion ([20, 23]) and necessary facts concerning boundary triplets and the corresponding Weyl functions ([5, 6]).

In Section 3 we investigate the Krein–Feller differential operator L_+ naturally connected with the operator $L_{\min}$. In Theorem 3 we find an explicit form of one of the Weyl functions corresponding to the operator L_+,

$$M(\lambda) = \frac{1}{(-\lambda)^{1/(2+\alpha)}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+.$$

It allows us to describe the main spectral properties of $L_{\min}$ and its quasi–Hermitian extensions. More precisely, in Section 4 we construct a boundary triplet for $L_{\min}$ and obtain the corresponding Weyl function (Theorem 4). Moreover, we describe the spectra of proper extensions and calculate their resolvents (Lemmas 1–2). Finally, in Section 5 we prove our main result (Theorem 5). This is a similarity criterion to a self–adjoint operator. In order to illustrate these results, in Section 6 (see Theorem 8) we obtain a simple similarity criterion for operators with local point interactions at zero,

$$L_c := \frac{\operatorname{sgn} x}{|x|^\alpha} \left(-\frac{d^2}{dx^2} + c\delta \right), \quad c \in \mathbb{C} \setminus \{0\}.$$

The results of the paper have been announced in [18].

Throughout the paper we use the following notation: $\mathfrak{H}, \mathcal{H}$ denote separable Hilbert spaces. The set of all bounded linear operators from $\mathfrak{H}$ to $\mathcal{H}$ is denoted by $[\mathfrak{H}, \mathcal{H}]$ or $[\mathfrak{H}]$ if $\mathfrak{H} = \mathcal{H}$. $\mathcal{C}(\mathfrak{H})$ stands for the set of closed densely defined operators in $\mathfrak{H}$. Let T be a linear operator in a Hilbert space $\mathfrak{H}$. In what follows $\operatorname{dom}(T)$, $\ker(T)$, $\operatorname{ran}(T)$ are the domain, kernel, range of T, respectively. We denote by $\sigma(T)$, $\sigma_r(T)$, $\sigma_c(T)$ the point, residual and continuous spectra of T. By $\sigma_p(T)$ the set of eigenvalues of T is indicated. We denote the resolvent set by $\rho(T)$; $R_T(\lambda) := (T - \lambda I)^{-1}$, $\lambda \in \rho(T)$, is the resolvent of T. Recall that $\sigma_r(T) = \{\lambda \in \sigma(T) \setminus \sigma_p(T) : \overline{\operatorname{ran}(T - \lambda I)} \neq \mathfrak{H}\}$, $\sigma_c(T) = \sigma(T) \setminus (\sigma_p(T) \bigcup \sigma_r(T))$.

If T is a symmetric operator, we denote by $\mathfrak{N}_\lambda := \ker(T^* - \lambda)$ the deficiency subspaces of T and by $n_\pm(A) := \dim \mathfrak{N}_{\pm i}$ its deficiency indices.

We set $\mathbb{C}_\pm := \{\lambda \in \mathbb{C} \ : \ \pm \operatorname{Im} \lambda > 0\}$, $\mathbb{R}_+ := [0, +\infty)$, $\mathbb{R}_- := (-\infty, 0]$. By $\chi_{\mathcal{I}}(t)$ we denote the characteristic function of the interval $\mathcal{I}$, i.e., $\chi_{\mathcal{I}}(t) = 1$ for $t \in \mathcal{I}$, $\chi_{\mathcal{I}}(t) = 0$ for $t \notin \mathcal{I}$. Finally, we set $\chi_\pm(t) := \chi_{\mathbb{R}_\pm}(t)$.

2. Preliminaries

2.1. **Similarity criterion.** Our approach is based on the concept of boundary triplets (see [10], [6]) and the resolvent similarity criterion obtained by S. N. Naboko [23] and M. M. Malamud [20] (in [3] this criterion was obtained under an additional assumption that the operator $T \in \mathcal{C}(\mathfrak{H})$ is a generator of C_0-group).

Theorem 1. ([20, 23]). *A closed operator T in a Hilbert space $\mathfrak{H}$ is similar to a self-adjoint one if and only if $\sigma(A) \subset \mathbb{R}$ and for all $f \in \mathfrak{H}$ the inequalities*

$$\begin{aligned} &\sup_{\varepsilon>0} \int_{-\infty}^{+\infty} \varepsilon \left\| R_T(\mu + i\varepsilon) f \right\|^2 d\mu \le C \left\| f \right\|^2, \\ &\sup_{\varepsilon>0} \int_{-\infty}^{+\infty} \varepsilon \left\| R_{T^*}(\mu + i\varepsilon) f \right\|^2 d\mu \le C_* \left\| f \right\|^2, \end{aligned} \tag{4}$$

are valid with constants C and C_* independent of f.

2.2. **Boundary triplets and Weyl functions.** Let $A \in \mathcal{C}(\mathfrak{H})$ be a closed symmetric operator with equal deficiency indices $n_+(A) = n_-(A)$. Without loss of generality we may assume that A is simple. This means that A has no self-adjoint parts.

We recall the definition of a boundary triplet which may be considered as an abstract version of the second Green formula.

Definition 1. ([10]). *A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of an auxiliary Hilbert space $\mathcal{H}$ and linear mappings*

$$\Gamma_j : \operatorname{dom}(A^*) \longrightarrow \mathcal{H}, \quad j \in \{0, 1\}, \tag{5}$$

is called a boundary triplet for the adjoint operator A^* of A if the following two conditions are satisfied:

(i) *The second Green formula*

$$(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \operatorname{dom}(A^*), \tag{6}$$

takes place and

(ii) *the mapping*

$$\Gamma : \operatorname{dom}(A^*) \longrightarrow \mathcal{H} \oplus \mathcal{H}, \quad \Gamma f := \{\Gamma_0 f, \Gamma_1 f\}, \tag{7}$$

is surjective.

Note that the boundary triplet for the adjoint A^* of the symmetric operator A is not unique. With each boundary triplet we associate two self–adjoint extensions $A_i := A^* | \ker(\Gamma_i),\ i \in \{0, 1\}$.

Definition 2. ([6]). *The proper extension $\tilde{A} \supset A$ is called an almost solvable if there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and an operator $B \in [\mathcal{H}]$ such that*

$$\operatorname{dom}(\tilde{A}) = \operatorname{dom}(A_B) := \ker(\Gamma_1 - B\Gamma_0). \tag{8}$$

The set of almost solvable extensions is denoted by $\mathcal{A}s_A$. Note that the class $\mathcal{A}s_A$ is sufficiently wide. According to [6] any proper extension having two regular points $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\operatorname{Im}\lambda_1 \cdot \operatorname{Im}\lambda_2 < 0$, belongs to $\mathcal{A}s_A$. All proper (in other terminology quasi-Hermitian, see [1]) extensions belong to the class $\mathcal{A}s_A$ if $n_\pm(A) < \infty$ (see [6]).

Weyl function is an important tool in the direct and inverse spectral theory of singular Sturm–Liouville operators. In [5, 6] a concept of Weyl function was introduced for an arbitrary symmetric operator A with infinite deficiency indices $n_+(A) = n_-(A)$.

Definition 3. ([5]). *Let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for the operator* A^**. The Weyl function of* A *corresponding to the boundary triplet* $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is a unique mapping*

$$M(\cdot) : \rho(A_0) \longrightarrow [\mathcal{H}] \tag{9}$$

satisfying

$$\Gamma_1 f_\lambda = M(\lambda)\Gamma_0 f_\lambda, \quad f_\lambda \in \mathfrak{N}_\lambda = \ker(A^* - \lambda I), \quad \lambda \in \rho(A_0). \tag{10}$$

It is well known (see [5, 6]) that the above implicit definition of the Weyl function is correct and $M(\cdot)$ is an R-function obeying $0 \in \rho(\mathrm{Im}(M(i)))$. The Weyl function immediately provides some information about the "spectral properties" of almost solvable extensions of the symmetric operator A.

Proposition 1. ([6]). *Suppose that* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is a boundary triplet for* A^**,* $M(\cdot)$ *is the corresponding Weyl function,* $\lambda \in \rho(A_0)$ *and* $B \in [\mathcal{H}]$*. Then:*
1) $\lambda \in \rho(A_B)$ *if and only if* $0 \in \rho(B - M(\lambda))$*;*
2) $\lambda \in \sigma_i(A_B)$ *if and only if* $0 \in \sigma_i(B - M(\lambda))$*,* $i \in \{p, r, c\}$*.*

We also need the following connection discovered in [5, 6] between the Krein formula for resolvents and boundary triplets.

Theorem 2. ([5, 6]). *Let* $\tilde{A}$ *be an almost solvable extension of* A *(*$\tilde{A} \in \mathcal{A}s_A$*), i.e.,* $\tilde{A} = A_B$ *with* $B \in [\mathcal{H}]$ *for some boundary triplet* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$*. Then*

$$(A_B - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)(B - M(\lambda))^{-1}\gamma^*(\bar{\lambda}), \quad \lambda \in \rho(A_B). \tag{11}$$

Here $\gamma(\lambda) := (\Gamma_0 | \mathfrak{N}_\lambda)^{-1}$ *is a so–called gamma–field of the operator* A *corresponding to the boundary triplet* Π*.*

3. Krein–Feller differential operator

We start with the operator $L_+ := P_+ \cdot L_{\min}$; here P_+ denote the orthogonal projection in $L^2(\mathbb{R}; |x|^\alpha)$ onto $L^2(\mathbb{R}_+; |x|^\alpha)$. Evidently, L_+ is a minimal Krein–Feller differential operator in $L^2(\mathbb{R}_+;\ |x|^\alpha)$ corresponding to the string S_α with the mass distribution $m_\alpha(x)$, i.e.,

$$\begin{gathered} L_+ := -\frac{d^2}{dm_\alpha(x)dx}, \quad \mathrm{dom}(L_+) = P_+(\mathrm{dom}(L_{\min})); \\ m_\alpha(x) = x^{1+\alpha}/(1+\alpha), \quad \alpha > -1. \end{gathered} \tag{12}$$

Notice that L_+ is a simple closed symmetric operator.

Following [12] we denote by $\varphi_\alpha(x, \lambda)$ and $\psi_\alpha(x, \lambda)$ the solutions of the equation

$$-\frac{d^2 u(x)}{dm_\alpha(x)\ dx} - \lambda\, u(x) = 0, \quad x > 0, \tag{13}$$

satisfying boundary conditions at zero $u(0) = 1,\ u'(0) = 0$, and $u(0) = 0,\ u'(0) = 1$, respectively. Then the following limit exists

$$\Gamma_\alpha(\lambda) := \lim_{x \to +\infty} \frac{\psi_\alpha(x, \lambda)}{\varphi_\alpha(x, \lambda)} = \lim_{x \to +\infty} \frac{\psi'_\alpha(x, \lambda)}{\varphi'_\alpha(x, \lambda)}, \quad \lambda \notin [0, +\infty), \tag{14}$$

and the function $\Gamma_\alpha(\cdot)$ belongs to the Krein–Stieltjes class S (see [11]), i.e., it admits the representation

$$\Gamma_\alpha(\lambda) = \int_0^{+\infty} \frac{d\tau_\alpha(s)}{s - \lambda}, \quad \lambda \notin [0, +\infty); \quad \int_0^{+\infty} \frac{d\tau_\alpha(s)}{1 + s} < +\infty. \tag{15}$$

Here $\tau_\alpha(\cdot)$ is a nondecreasing function defined on $\mathbb{R}_+$, obeying

$$\tau_\alpha(0) = 0, \quad \tau_\alpha(s) = \frac{1}{2}\left(\tau_\alpha(s+0) + \tau_\alpha(s-0)\right).$$

$\tau_\alpha(\cdot)$ is called a general spectral function of the string S_α.

We denote by $z^{1/(2+\alpha)}$, $z \in \mathbb{C}$, the branch of the complex root with a cut along the negative semi-axis $\mathbb{R}_-$ such that $(-1+i0)^{1/(2+\alpha)} = e^{i\pi/(2+\alpha)}$.

Proposition 2. ([18]). *Let the operator* L_+ *be defined by* (12). *Then*

$$\Gamma_\alpha(\lambda) = \frac{C_\alpha}{(-\lambda)^{1/(2+\alpha)}}, \quad C_\alpha := \Gamma_\alpha(-1) > 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+. \tag{16}$$

Moreover,

$$\tau_\alpha(x) = C_\alpha \cdot \frac{(2+\alpha)\cdot sin(\pi/(2+\alpha))}{\pi(1+\alpha)} \cdot x^{(1+\alpha)/(2+\alpha)}, \quad x \geq 0, \tag{17}$$

is the general spectral function of the string S_α.

Proof. It is obvious that

$$\frac{dm_\alpha(\rho x)}{dx} = (\rho x)^\alpha = \rho^\alpha \frac{dm_\alpha(x)}{dx}, \quad \rho > 0.$$

Hence

$$\varphi(\rho x,\ \lambda) = \varphi(x,\ \rho^{2+\alpha}\lambda)\ , \quad \psi(\rho x,\ \lambda) = \rho\psi(x,\ \rho^{2+\alpha}\lambda)\ . \tag{18}$$

By (14), we obtain

$$\Gamma(\lambda) = \rho\Gamma(\rho^{2+\alpha}\lambda), \quad \rho > 0. \tag{19}$$

Putting $\lambda = -1$, we get

$$\Gamma(-1) = \rho^{1/(2+\alpha)}\Gamma(-\rho), \quad \rho > 0. \tag{20}$$

Finally, (20) yields (16), since $\Gamma(\cdot)$ is analytic in $\mathbb{C} \setminus \mathbb{R}_+$.

Equation (17) follows from (15)–(16) and the Stieltjes inversion formula (see [11]). □

Theorem 3. *A triplet* $\Pi_+ = \{\mathbb{C}, \Gamma_0^+, \Gamma_1^+\}$, *where*

$$\Gamma_j^+ :\ \mathrm{dom}(L_+^*) \to \mathbb{C},\ j \in \{0,1\}, \quad \Gamma_1^+ f = f(0), \quad \Gamma_0^+ f = -f'(0), \quad f \in \mathrm{dom}(L_+^*), \tag{21}$$

forms a boundary triplet for L_+^*. *The corresponding Weyl function* $M_+(\cdot)$ *is*

$$M_+(\lambda) := \Gamma_\alpha(\lambda) = \frac{C_\alpha}{(-\lambda)^{1/(2+\alpha)}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+. \tag{22}$$

Proof. Since $\int_0^{+\infty} t^2 dm(t) = +\infty$, we have $n_\pm(L_+) = 1$ (see [12]). Furthermore, the Lagrange identity holds for the adjoint operator L_+^* (see [12]), i.e.,

$$\begin{aligned}(L_+^* u,\ v) - (u,\ L_+^* v) &= u'(0)\overline{v(0)} - u(0)\overline{v'(0)} \\ &= \Gamma_1^+(u)\cdot\overline{\Gamma_0^+(v)} - \Gamma_0^+(u)\cdot\overline{\Gamma_1^+(v)}, \quad u, v \in \mathrm{dom}(L_+^*).\end{aligned} \tag{23}$$

Then, by Definition 1, Π_+ is a boundary triplet for L_+^*.

Note also that for all $f \in \mathrm{dom}(L_+^*)$ the following limit exists: $\lim_{x\to+\infty} f(x) = 0$. Using (14), one gets

$$y_\alpha(\cdot,\lambda) := \psi(\cdot,\lambda) - \Gamma_\alpha(\lambda)\varphi(\cdot,\lambda) \in \mathfrak{N}_\lambda, \quad \mathfrak{N}_\lambda = \{c y_\alpha(\cdot,\lambda) : c \in \mathbb{C}\}. \tag{24}$$

Combining Definition 3 with (21), we obtain

$$M_+(\lambda) = \frac{\Gamma_1^+(y_\alpha(\cdot,\lambda))}{\Gamma_0^+(y_\alpha(\cdot,\lambda))} = \frac{-\Gamma_\alpha(\lambda)}{-1} = \Gamma_\alpha(\lambda). \tag{25}$$

□

Corollary 1. *Let the function* $y_\alpha(\cdot,\lambda)$ *be of the form* (24) *and* $\lambda=\mu+i\varepsilon\in\mathbb{C}\setminus\mathbb{R}_+$. *Then*

$$\|y_\alpha(\cdot,\mu+i\varepsilon)\|^2_{L^2(\mathbb{R}_+,x^\alpha)}=\frac{C_\alpha}{\varepsilon}\cdot\operatorname{Im}\frac{1}{(-\mu-i\varepsilon)^{1/(2+\alpha)}}. \tag{26}$$

Proof. Using the functional model of the operator L_+ in the Hilbert space $L^2(\mathbb{R}_+,d\tau_\alpha)$ (see [6, 22]), we obtain

$$\begin{aligned}\|y_\alpha(\cdot,\lambda)\|^2_{L^2(\mathbb{R}_+,x^\alpha)}&=\int_0^{+\infty}\frac{1}{|t-\lambda|^2}d\tau_\alpha(t)\\&=\frac{1}{\operatorname{Im}\lambda}\int_0^{+\infty}\operatorname{Im}\frac{1}{t-\lambda}d\tau_\alpha(t)=\frac{1}{\operatorname{Im}\lambda}\operatorname{Im}\Gamma_\alpha(\lambda).\end{aligned} \tag{27}$$

Equality (16) completes the proof. □

Corollary 2. *The spectral kernel of the operator* L_+ *is continuous and coincides with positive semi-axis* $\mathbb{R}_+$.

Proof. By (26), we see that $\mathbb{R}_+$ does not belong to the set of regular type points of the operator L_+. Hence $\mathbb{R}_+$ is a spectral kernel of L_+. Moreover, L_+ is a simple symmetric operator with deficiency indices $n_\pm(L_+)=1$, then (see [1]) the spectral kernel is continuous. □

4. Proper extensions of $L_{\min}$

The following result is a simple corollary of Theorem 3.

Theorem 4. (*i*) *The operator* $L_{\min}$ *of the form* (1) *is a simple closed symmetric operator in* $L^2(\mathbb{R},\ |x|^\alpha)$ *with deficiency indices* $n_\pm(L_{\min})=2$.

(*ii*) *Let mappings* $\Gamma_i:\operatorname{dom}(L^*_{\min})\to\mathbb{C}^2$, *be given by*

$$\Gamma_1 f=\begin{pmatrix} f'(+0)\\ -f(-0)\end{pmatrix},\quad \Gamma_0 f=\begin{pmatrix} f(+0)\\ f'(-0)\end{pmatrix}. \tag{28}$$

Then $\Pi=\{\mathbb{C}^2,\Gamma_0,\Gamma_1\}$ *is a boundary triplet for* $L^*_{\min}$.

(*iii*) *The corresponding Weyl function* $M(\cdot)$ *is*

$$M(\lambda):=\begin{pmatrix} -1/\Gamma_\alpha(\lambda) & 0\\ 0 & -\Gamma_\alpha(-\lambda)\end{pmatrix},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{29}$$

Here $\Gamma_\alpha(\cdot)$ *is defined by* (16).

Proof. (*i*) is obvious. Moreover, by (23), one gets for all $f,g\in\operatorname{dom}(L_{\min^*})$

(30)

$$\begin{aligned}(L^*_{\min}f,\ g)-(f,\ L^*_{\min}g)&=f'(+0)\overline{g(+0)}+f'(-0)\overline{g(-0)}-f(+0)\overline{g'(+0)}-f(-0)\overline{g'(-0)}\\&=(\Gamma_1 f,\Gamma_0 g)_{\mathbb{C}^2}-(\Gamma_0 f,\Gamma_1 g)_{\mathbb{C}^2}.\end{aligned}$$

This proves (*ii*).

Note that the defect subspace of the operator $L_{\min}$ has the form

$$\mathfrak{N}_\lambda(L_{\min})=\operatorname{span}\{y^+_\alpha(x,\lambda);\ \ y^-_\alpha(x,\lambda)\}, \tag{31}$$

where

$$y^+_\alpha(x,\lambda):=\begin{cases}\psi_\alpha(x,\lambda)-\Gamma_\alpha(\lambda)\varphi_\alpha(x,\lambda), & x>0,\\ 0, & x<0;\end{cases} \tag{32}$$

$$y_\alpha^-(x,\lambda) := \begin{cases} 0, & x>0, \\ \psi_\alpha(-x,-\lambda) - \Gamma_\alpha(-\lambda)\varphi_\alpha(-x,-\lambda), & x<0. \end{cases} \tag{33}$$

By Definition 3, after simple calculations one obtains (29). □

Corollary 3. *The gamma–field* $\gamma(\cdot)$ *corresponding to the triplet* Π *is*

$$\gamma(\lambda)\begin{pmatrix} c^+ \\ c^- \end{pmatrix} = -\frac{c^+}{\Gamma_\alpha(\lambda)} y_\alpha^+(x,\lambda) - c^- y_\alpha^-(x,\lambda). \tag{34}$$

Here $\Gamma_\alpha(\lambda)$ *and* $y_\alpha^\pm(x,\lambda)$ *are defined by* (16) *and* (32)–(33)*, respectively.*

Let us remark that all quasi–Hermitian extensions of the operator $L_{\min}$ are almost solvable, because $n_\pm(L_{\min}) = 2 < \infty$. In what follows we confine ourselves to the almost solvable extensions described by the boundary triplet Π of the form (28):

$$L_B = -\frac{\operatorname{sgn} x}{|x|^\alpha}\frac{d^2}{dx^2}, \quad \operatorname{dom}(L_B) = \{f \in \operatorname{dom}(L_{\min}^*) :\ \Gamma_1 f = B\Gamma_0 f\}. \tag{35}$$

Here $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ is a 2×2–matrix with complex coefficients. In this case boundary conditions at zero take the form

$$\begin{cases} f'(+0) = b_{11} f(+0) + b_{12} f'(-0), \\ -f(-0) = b_{21} f(+0) + b_{22} f'(-0). \end{cases} \tag{36}$$

Equality (29) allows us to describe the spectrum of the operator L_B. Let us determine the function $\Psi_B(\cdot) : \overline{\mathbb{C}_+} \to \overline{\mathbb{C}}$ by the following way

$$\Psi_B(\lambda) := \det(B - M(\lambda)), \quad \lambda \in \mathbb{C}_+; \tag{37}$$

$$\Psi_B(\mu) := \lim_{\varepsilon\downarrow +0} \det(B - M(\mu + i\varepsilon)), \quad \mu \in \mathbb{R}\cup\{\infty\}, \quad \varepsilon > 0. \tag{38}$$

Here $M(\cdot)$ is given by (29) and (16).

Note that $\Psi_B(\cdot)$ is analytic on $\mathbb{C}_+$ and continuous on $\mathbb{R}\setminus\{0\}$. Furthermore, it is obvious that

$$\Psi_B(\lambda) = \frac{e^{-i\pi/(2+\alpha)} b_{22}}{C_\alpha}\lambda^{1/(2+\alpha)} + b_{11}b_{22} - b_{12}b_{21} + e^{-i\pi/(2+\alpha)} + b_{11} C_\alpha \frac{1}{\lambda^{1/(2+\alpha)}}, \quad \lambda \in \overline{\mathbb{C}_+}\setminus\{0\}. \tag{39}$$

It follows from (39) that $\Psi_B(\cdot)$ has at most two zeroes (a zero of multiplicity k is counted as k zeroes).

Lemma 1. ([18]). *Let* $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2}$ *and* $|b_{12}| + |b_{21}| \neq 0$*; let* L_B *be an almost solvable extension of the form* (35)–(36)*. Then:*

(i) $\sigma_c(L_B) = \mathbb{R}$ *and* $\sigma_p(L_B) \cap \mathbb{R} = \emptyset$*;*
(ii) $\sigma_p(L_B) \cap \mathbb{C}_+ = \{\lambda \in \mathbb{C}_+ :\ \Psi_B(\lambda) = 0\}$ *and* $\sigma_p(L_B) \cap \mathbb{C}_- = \{\lambda \in \mathbb{C}_- :\ \Psi_{B^*}(\overline{\lambda}) = 0\}$.

Proof. (i) The spectrum of a quasi–Hermitian extension of a simple symmetric operator A with finite deficiency indices consists of the spectral kernel of A and the eigenvalues (see [1]). It is obvious that the spectral kernel of $L_{\min}$ is continuous and coincides with $\mathbb{R}$ (cf. Corollary 2). Hence $\sigma_c(L_B) = \mathbb{R}$. Moreover, the condition $|b_{12}| + |b_{21}| \neq 0$ implies $\sigma_p(L_B) \cap \mathbb{R} = \emptyset$.

(ii) trivially, follows from Proposition 1. □

Remark 1. Since $L_B^* = L_{B^*}$, the operator L_B is self–adjoint exactly when $B = B^*$. If B is a self–adjoint matrix and the condition $|b_{12}| + |b_{21}| \neq 0$ holds, then $\sigma(L_B) = \sigma_c(L_B) = \mathbb{R}$. In other words, the spectrum of the self–adjoint extension type (35)–(36) with nonseparate boundary conditions is continuous and coincide with $\mathbb{R}$.

Remark 2. Suppose that $|b_{12}| + |b_{21}| = 0$, i. e., the operator L_B has a separate boundary conditions. Such type operators are well studied and this case is not of interest.

Let us determine the following functions for all $f \in L^2(\mathbb{R}, |x|^\alpha)$

$$F_\alpha^+(f,\lambda) := \int_0^{+\infty} f(t) y_\alpha^+(t,\lambda)dt, \quad F_\alpha^-(f,\lambda) := \int_{-\infty}^{0} f(t) y_\alpha^-(t,\lambda)dt, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{40}$$

Lemma 2. *Let $B \in \mathbb{C}^{2\times 2}$ and $|b_{12}| + |b_{21}| \neq 0$; let L_B be of the form* (35)–(36) *and $\tilde{L}_0 = L_{\min}|\ker(\Gamma_0)$. Then*

$$\begin{aligned}(L_B - \lambda I)^{-1} f(x) &= (\tilde{L}_0 - \lambda I)^{-1} f(x) \\ &+ \frac{y_\alpha^+(x,\lambda)}{\Psi_B(\lambda)\cdot\Gamma_\alpha(\lambda)} \cdot \left(\frac{b_{22} + \Gamma_\alpha(-\lambda)}{\Gamma_\alpha(\lambda)} F_\alpha^+(f,\lambda) - b_{12} F_\alpha^-(f,\lambda)\right) \\ &- \frac{y_\alpha^-(x,\lambda)}{\Psi_B(\lambda)} \cdot \left(b_{21} \frac{F_\alpha^+(f,\lambda)}{\Gamma_\alpha(\lambda)} - (b_{11} + 1/\Gamma_\alpha(\lambda)) F_\alpha^-(f,\lambda)\right), \\ & f \in L^2(\mathbb{R}, |x|^\alpha), \quad \lambda \in \rho(L_B).\end{aligned} \tag{41}$$

Proof. Note that (see Corollary 1)

$$\gamma^*(\overline{\lambda}) : L^2(\mathbb{R}, |x|^\alpha) \to \mathbb{C}^2 \quad \text{and} \quad \gamma^*(\overline{\lambda}) f = \begin{pmatrix} -F_\alpha^+(f,\lambda)/\Gamma_\alpha(\lambda) \\ -F_\alpha^-(f,\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{42}$$

Combining (29), (34), (42) with Theorem 2, we obtain (41). □

5. Similarity to a self–adjoint operator

Here we present the main result of the paper, a criterion of similarity of L_B to a self–adjoint operator. To prove the main result we need two lemmas. The first of them is known and trivial.

Lemma 3. *If an operator $T \in \mathcal{C}(\mathfrak{H})$ is similar to a normal one, then the inequality*

$$\|(T - \lambda I)^{-1}\|_{\mathfrak{H}} \leq \frac{C}{\operatorname{dist}(\lambda,\ \sigma(T))} \tag{43}$$

holds with some constant $C > 0$.

Lemma 4. *Let the functions $\Gamma_\alpha(\lambda)$, $y_\alpha^\pm(x,\lambda)$ and $F_\alpha^\pm(\cdot,\lambda)$ be defined by* (16), (32)–(33) *and* (40), *respectively. Then the inequalities*

$$\sup_{\varepsilon>0} \int_{-\infty}^{+\infty} \varepsilon \left\| \frac{y_\alpha^\pm(x,\mu+i\varepsilon) F_\alpha^\pm(f,\mu+i\varepsilon)}{\Gamma_\alpha(\mu+i\varepsilon)} \right\|^2 d\mu \leq C\|f\|^2, \quad f \in L^2(\mathbb{R}, |x|^\alpha); \tag{44}$$

$$\sup_{\varepsilon>0} \int_{-\infty}^{+\infty} \varepsilon \left\| \frac{y_\alpha^\pm(x,\mu+i\varepsilon) F_\alpha^\mp(f,\mu+i\varepsilon)}{\Gamma_\alpha(\mu+i\varepsilon)} \right\|^2 d\mu \leq C\|f\|^2, \quad f \in L^2(\mathbb{R}, |x|^\alpha); \tag{45}$$

hold for all $f \in L^2(\mathbb{R}, |x|^\alpha)$ with some constant $C > 0$ independent of f.

Proof. Let us consider the self–adjoint operator L_{B_0} of the form (35)-(36) with $B_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is clear that

$$\frac{\Gamma_\alpha(-\lambda)}{\Gamma_\alpha(\lambda)} \equiv e^{-i\pi/(2+\alpha)}, \quad \Psi_{B_0}(\lambda) \equiv -1 + e^{-i\pi/(2+\alpha)}, \quad \lambda \in \mathbb{C}_+.$$

Substituting $f_\pm(x) = f(x)\cdot\chi_{\mathbb{R}_\pm}(x)$ for f in (41), we obtain

$$\left\|(L_{B_0}-\lambda)^{-1}f_\pm - (\tilde{L}_0-\lambda)^{-1}f_\pm\right\|^2 = \left\|\frac{y_\alpha^\pm(x,\lambda)F_\alpha^\pm(f,\lambda)}{\Psi_0\cdot\Gamma_\alpha(\lambda)}\right\|^2 + \left\|\frac{y_\alpha^\mp(x,\lambda)F_\alpha^\mp(f,\lambda)}{\Psi_0\cdot\Gamma_\alpha(\lambda)}\right\|^2, \quad \lambda\in\mathbb{C}_+. \tag{46}$$

Finally note that the operators L_{B_0} and $\tilde{L}_0$ are self–adjoint, hence, by Theorem 1, they satisfy inequalities (4). Combining this fact with (46), we get (44)–(45). □

Theorem 5. (Main Theorem). *Let*

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix} \in \mathbb{C}^{2\times 2} \quad \textit{and} \quad |b_{12}|+|b_{21}|\neq 0;$$

and let $\Psi_B(\cdot)$ be defined by (37)–(38). *The operator L_B of the form* (35)–(36) *is similar to a self–adjoint operator if and only if $\Psi_B(\cdot)$ and $\Psi_{B^*}(\cdot)$ have no zeroes in the closed upper half-plane $\overline{\mathbb{C}_+}$.*

Proof. (Necessity). Suppose that L_B is similar to a self-adjoint operator. Then L_B has a real spectrum, i. e., $\sigma(L_B)\subset\mathbb{R}$. Hence the functions $\Psi_B(\cdot)$ and $\Psi_{B^*}(\cdot)$ have no zeroes in $\mathbb{C}_+$ (cf. Lemma 1 (ii)).

Let us prove that $\Psi_B(\cdot)$ and $\Psi_{B^*}(\cdot)$ have no zeroes in $\mathbb{R}\cup\{\infty\}$. Assume the converse. Without loss of generality we suppose that $\Psi_B(\cdot)$ has a zero in $\mathbb{R}\cup\{\infty\}$ and $b_{12}\neq 0$. Let us consider three cases.

i) Suppose that $\mu_0 = \overline{\mu_0}\neq 0$ is a zero of the function $\Psi_B(\cdot)$. We put $f_\lambda^-(x) := y_\alpha^-(x,\overline{\lambda})/\|y_\alpha^-(x,\overline{\lambda})\|$. Substituting $f_\lambda^-(\cdot)$ for $f(\cdot)$ in (41), we obtain (in what follows $\|\cdot\|$ stands for the norm in the corresponding Hilbert space $L^2(\mathbb{R},\ |x|^\alpha)$)

$$\begin{aligned}\left\|(L_B-\lambda I)^{-1}f_\lambda^- - (\tilde{L}_0-\lambda I)^{-1}f_\lambda^-\right\|^2 &= \left\|\frac{y_\alpha^+(x,\lambda)}{\Psi_B(\lambda)\cdot\Gamma_\alpha(\lambda)}b_{12}F_\alpha^-(f_\lambda^-,\lambda)\right\|^2 \\ &+ \left\|-\frac{y_\alpha^-(x,\lambda)}{\Psi_B(\lambda)}\cdot(b_{11}+1/\Gamma_\alpha(\lambda))F_\alpha^-(f_\lambda^-,\lambda)\right\|^2 \\ &\geq |b_{12}|^2\cdot\frac{\|y_\alpha^+(x,\lambda)\|^2\cdot\|y_\alpha^-(x,\lambda)\|^2}{|\Psi_B(\lambda)\cdot\Gamma_\alpha(\lambda)|^2}, \quad \lambda\in\rho(L_B).\end{aligned} \tag{47}$$

Since the operator $\tilde{L}_0$ is self–adjoint, then

$$\|(\tilde{L}_0-(\mu_0+i\varepsilon)I)^{-1}\|\leq\frac{1}{\varepsilon}.$$

The function $\Psi_B(\cdot)$ admits an analytic continuation into $\mu_0\neq 0$ (see (39)), hence we get

$$|\Psi_B(\mu_0+i\varepsilon)| = O(|\varepsilon|), \quad \varepsilon\to 0. \tag{48}$$

Combining (47), (48) and (26), we obtain

$$\begin{aligned}&\left\|(L_B-(\mu_0+i\varepsilon)I)^{-1}\right\|^2 \geq \left\|(L_B-(\mu_0+i\varepsilon)I)^{-1}f_\lambda^-(x)\right\|^2 \\ &\quad\geq |b_{12}|^2\cdot\frac{\|y_\alpha^+(x,\mu_0+i\varepsilon)\|^2\cdot\|y_\alpha^-(x,\mu_0+i\varepsilon)\|^2}{|\Psi_B(\mu_0+i\varepsilon)\cdot\Gamma_\alpha(\mu_0+i\varepsilon)|^2} - \frac{1}{\varepsilon^2} \\ &\quad= \frac{|b_{12}|^2\cdot|(\mu_0+i\varepsilon)|^{2/(2+\alpha)}}{|\varepsilon|^2|\Psi_B(\mu_0+i\varepsilon)|^2}\operatorname{Im}\frac{1}{(-\mu_0-i\varepsilon)^{1/(2+\alpha)}}\operatorname{Im}\frac{1}{(\mu_0+i\varepsilon)^{1/(2+\alpha)}} - \frac{1}{\varepsilon^2} \\ &\quad= \frac{|b_{12}|^2\cdot\operatorname{Im}(-\mu_0+i\varepsilon)^{1/(2+\alpha)}\operatorname{Im}(\mu_0-i\varepsilon)^{1/(2+\alpha)}}{|\varepsilon|^2|(\mu_0+i\varepsilon)|^{2/(2+\alpha)}|\Psi_B(\mu_0+i\varepsilon)|^2} - \frac{1}{\varepsilon^2} = O\left(\frac{1}{|\varepsilon|^3}\right), \quad \varepsilon\to 0.\end{aligned} \tag{49}$$

By Lemma 3, the operator L_B is not similar to a normal operator. This is a contradiction.

ii) If $\mu_0 = 0$ is a zero of $\Psi_B(\cdot)$, then $\Psi_B(\lambda) = b_{22}/\Gamma_\alpha(\lambda)$ (cf. (39)). Arguing as above, we obtain

$$
\begin{aligned}
&\left\|(L_B - i\varepsilon)^{-1} f_\lambda^- - (\tilde{L}_0 - i\varepsilon)^{-1} f_\lambda^-\right\|^2 \geq \left|\frac{b_{12}}{b_{22}}\right|^2 \cdot \|y_\alpha^+(x, i\varepsilon)\|^2 \cdot \|y_\alpha^-(x, i\varepsilon)\|^2 \\
&\quad = \left|\frac{b_{12}}{b_{22}}\right|^2 \frac{1}{|\varepsilon|^2} \operatorname{Im} \frac{1}{(-i\varepsilon)^{1/(2+\alpha)}} \operatorname{Im} \frac{1}{(i\varepsilon)^{1/(2+\alpha)}} = O\left(\frac{1}{|\varepsilon|^{2+2/(2+\alpha)}}\right), \quad \varepsilon \to 0.
\end{aligned} \tag{50}
$$

This contradicts Lemma 3.

iii) Finally let $\Psi_B(\infty) = 0$. Then $\Psi_B(\lambda) = b_{11}\Gamma_\alpha(-\lambda)$. Using (41), one gets

$$
\begin{aligned}
&|\operatorname{Im}\lambda|^2 \left\|(L_B - \lambda)^{-1} f_\lambda^- - (\tilde{L}_0 - \lambda)^{-1} f_\lambda^-\right\|^2 \\
&\quad \geq |\operatorname{Im}\lambda|^2 \left|\frac{b_{12}}{b_{11}}\right|^2 \frac{\|y_\alpha^+(x,\lambda)\|^2 \cdot \|y_\alpha^-(x,\lambda)\|^2}{|\Gamma_\alpha(\lambda)\Gamma_\alpha(-\lambda)|^2} = \left|\frac{b_{12}}{b_{11}}\right|^2 \frac{|\operatorname{Im}\Gamma_\alpha(\lambda)\operatorname{Im}\Gamma_\alpha(-\lambda)|}{|\Gamma_\alpha(\lambda)\Gamma_\alpha(-\lambda)|^2} \\
&\quad = \left|\frac{b_{12}}{b_{11}c_\alpha}\right|^2 \left|\operatorname{Im}(\overline{-\lambda})^{1/(2+\alpha)} \operatorname{Im}(\overline{\lambda})^{1/(2+\alpha)}\right|, \quad \lambda \in \mathbb{C}_+.
\end{aligned} \tag{51}
$$

It is obvious that the right part of (51) is unbounded in $\mathbb{C}_+$.

This contradiction concludes the proof of necessity.

(Sufficiency).

Suppose that the functions $\Psi_B(\cdot)$ and $\Psi_{B^*}(\cdot)$ have no zeroes in the closed upper half-plane. Then, by Lemma 1, the operator L_B has a real spectrum and $\sigma(L_B) = \mathbb{R}$. Moreover, the inequalities

$$
\frac{1}{|\Psi_B(\lambda)|} \leq C_1, \quad \frac{1}{|\Psi_{B^*}(\lambda)|} \leq C_1, \quad \lambda \in \overline{\mathbb{C}_+}, \quad C_1 > 0, \tag{52}
$$

are valid. Using (52) and Lemma 4, we obtain that the resolvents of the operators L_B and L_{B^*} (see (41)) satisfy estimates (4). By Theorem 1, the operator L_B is similar to a self–adjoint one.

This concludes the proof of Theorem 5. □

At the end of this section we formulate the criterion of similarity of L_B to a normal operator.

Theorem 6. *Let*

$$
B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2} \quad \textit{and} \quad |b_{12}| + |b_{21}| \neq 0;
$$

and let $\Psi_B(\cdot)$ *be defined by* (37)–(38). *Then the operator* L_B *of the form* (35)–(36) *is similar to a normal operator if and only if the following conditions hold*

(1) $\Psi_B(\cdot)$ *and* $\Psi_{B^*}(\cdot)$ *have no zeroes in* $\mathbb{R} \cup \{\infty\}$*;*

(2) $\Psi_B(\cdot)$ *and* $\Psi_{B^*}(\cdot)$ *have no zeroes of the second order in* $\mathbb{C}_+$*.*

Let us make several comments. Necessity is obvious. Actually, if the condition (1) is failed, then the resolvent of the operator L_B or L_{B^*} has a nonlinear growth in some neighborhood of a real zero (see the proof of necessity of Theorem 5). If the function $\Psi_B(\cdot)$ or $\Psi_{B^*}(\cdot)$ has a zero of the second order in $\mathbb{C}_+$, then it is not hard to show that the resolvent of L_B has a pole of the second order. This contradicts similarity of L_B to a normal operator.

The proof of sufficiency is similar to that contained in [16].

6. Examples

6.1. **On similarity of L to a self–adjoint operator.** Let the matrix B be of the form $B = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix}$. Then, by (39), we get

$$\Psi_B(\lambda) \equiv b_{12} \cdot b_{21} + e^{-i\pi/(\alpha+2)}, \quad \Psi_{B^*}(\lambda) \equiv \overline{b_{12} \cdot b_{21}} + e^{-i\pi/(\alpha+2)}, \quad \lambda \in \overline{\mathbb{C}_+}.$$

Hence, by Theorems 5–6, we easily obtain

Theorem 7. *Suppose that*

$$B = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2} \quad \textit{and} \quad b_{12} \cdot b_{21} \neq e^{\pm i\pi/(\alpha+2)}.$$

Then
(i) $\sigma(L_B) = \sigma_c(L_B) = \mathbb{R}$;
(ii) the operator L_B *is similar to a self–adjoint one.*

Note that in the case $b_{12} \cdot b_{21} = e^{\pm i\pi/(\alpha+2)}$ the point spectrum of L_B coincides with the upper or lower half-plane. Hence the operator L_B is not similar to a normal one.

Remark 3. The result of A. Fleige and B. Najman [9] immediately follows from Theorem 7. Actually, the operator L of the form (2) is the operator L_B with $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Let us calculate the characteristic function $W_L(\cdot)$ of the operator L. According to [6], the characteristic function $W_L(\cdot)$ has the form

$$W_L(\lambda) = (B - M(\lambda))(B^* - M(\lambda))^{-1},$$

since $\det((B_0 - B_0^*)/2i) = 1 \neq 0$. Here $M(\cdot)$ is defined by (29) and (16). After simple calculations we get

$$W_L(\lambda) = \begin{pmatrix} 1 - 2/D_\alpha & 2/(D_\alpha \cdot \Gamma_\alpha(\lambda)) \\ -2 \cdot \Gamma_\alpha(-\lambda)/D_\alpha & 1 - 2/D_\alpha \end{pmatrix}, \quad D_\alpha := 1 + e^{-i\pi/(\alpha+2)}. \tag{53}$$

We see that the characteristic function $W_L(\cdot)$ as well as its J-form $J - W_L^*(\cdot)JW_L(\cdot)$, where $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, are unbounded in $\mathbb{C}_+$. Therefore known sufficient conditions of similarity to a self–adjoint operator in terms of the characteristic functions (see [21]) are not applicable.

Note that L is similar to an operator $R = R^*$ with absolutely continuous spectrum $R = R_{ac}$ since $W_L(\cdot)$ is unbounded only in neighborhoods of zero and infinity.

6.2. **On similarity of $\frac{\operatorname{sgn} x}{|x|^\alpha}\left(-\frac{d^2}{dx^2} + c\delta\right)$ to a self–adjoint operator.** Let δ be the Dirac delta. The differential expression

$$l_c := \frac{1}{|x|^\alpha}\left(-\frac{d^2}{dx^2} + c\delta\right), \quad c \in \mathbb{C} \setminus \{0\}, \tag{54}$$

generates in $L^2(\mathbb{R},\ |x|^\alpha)$ the following operator

$$\begin{aligned} &A_c := -\frac{1}{|x|^\alpha}\frac{d^2}{dx^2}, \\ &\operatorname{dom}(A_c) = \{f \in \operatorname{dom}(L_{\min}^*) : \ f(+0) = f(-0), \quad f'(+0) - f'(-0) = cf(0)\}. \end{aligned} \tag{55}$$

Let us consider the operator $L_c := JA_c$, where $J : y(x) \to \operatorname{sgn} x \cdot y(x)$.

Theorem 8. *Let* $L_c = \frac{\operatorname{sgn} x}{|x|^\alpha}\left(-\frac{d^2}{dx^2} + c\delta\right)$ *be the operator defined by* (54) *and* (55) *and* $c \in \mathbb{C} \setminus \{0\}$. *Then*

(*i*) L_c *is similar to a normal operator if and only if*

$$\operatorname{Re} c \neq -\frac{1+\cos(\pi/(\alpha+2))}{\sin(\pi/(\alpha+2))} \cdot |\operatorname{Im} c|; \tag{56}$$

(*ii*) L_c *is similar to a self–adjoint operator if and only if*

$$\operatorname{Re} c > -\frac{1+\cos(\pi/(\alpha+2))}{\sin(\pi/(\alpha+2))} \cdot |\operatorname{Im} c|. \tag{57}$$

Proof. It is obvious that L_c is an almost solvable extension L_B of the form (35)–(36) with $B = \begin{pmatrix} c & 1 \\ -1 & 0 \end{pmatrix}$. By (39), we get

(58)

$$\Psi_B(\lambda) = 1 + e^{-i\pi/(\alpha+2)} + \frac{c \cdot C_\alpha}{\lambda^{1/(\alpha+2)}}, \quad \Psi_{B^*}(\lambda) = 1 + e^{-i\pi/(\alpha+2)} + \frac{\overline{c} \cdot C_\alpha}{\lambda^{1/(\alpha+2)}}, \quad \lambda \in \overline{\mathbb{C}_+}.$$

The function $\Psi_B(\cdot)$ or $\Psi_{B^*}(\cdot)$ has a real zero if and only if (56) does not valid. $\Psi_B(\cdot)$ and $\Psi_{B^*}(\cdot)$ have no zeroes in the closed upper half-plane exactly when c satisfy (57). Theorems 5–6 complete the proof. □

Remark 4. Note that $c \in \mathbb{C}$ satisfy (57) if it belongs to the angle G_β with vertex zero and the half–angle

$$\beta := \operatorname{arcctg}\left(-\frac{1+\cos(\pi/(\alpha+2))}{\sin(\pi/(\alpha+2))}\right).$$

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