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CONTINUOUS FRAMES IN HILBERT SPACES

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Abstract. In this paper we introduce a mean of a continuous frame which is a generalization of discrete frames. Since a discrete frame is a special case of these frames, we expect that some of the results that occur in the frame theory will be generalized to these frames. For such a generalization, after giving some basic results and theorems about these frames, we discuss the following: dual to these frames, perturbation of continuous frames and robustness of these frames to an erasure of some elements.

1. Introduction

The concept of frames (discrete frames) in Hilbert spaces has been introduced by Duffin and Schaeffer [11] in 1952 to study some deep problems in nonharmonic Fourier series, after the fundamental paper [10] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames.

Traditionally, frames have been used in signal processing, image processing, data compression and sampling in sampling theory. A discreet frame is a countable family of elements in a separable Hilbert space which allows for a stable, not necessarily unique, decomposition of an arbitrary element into an expansion of the frame elements. The concept of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure was proposed by G. Kaiser [15] and independently by Ali, Antoine and Gazeau [2]. These frames are known as continuous frames. Gabardo and Han in [14] called these frames *frames associated with measurable spaces*, Askari-Hemmat, Dehghan and Radjabalipour in [3] called them *generalized frames* and in mathematical physics they are referred to as *coherent states* [2]. For more details, the reader can refer to [1, 2, 3, 9, 12, 14, 17]. If in the definition of a continuous frame, the measure space $\Omega := \mathbb{N}$ and μ is the counting measure, the continuous frame will be a discrete frame and so we expect that some of the results obtained in the frame theory hold in the continuous frame theory.

In this paper, we focus on a continuous frame with a positive measure and $\mathcal{H}$ a complex Hilbert space.

The paper is organized as follows. In Section 2, we introduce some definitions and basic facts about continuous frames such as a continuous frame operator, a pre-frame operator, etc. In Section 3, we discuss the dual of continuous frames and further we present a condition that shows when a continuous frame is robust to erasure of some elements. A type of Paley-Wiener theorem about perturbation of continuous frames that was discussed by Gabardo and Han in [14] will be generalized in Section 4.

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2. Preliminaries

Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ will be complex Hilbert spaces.

Definition 2.1. Let $\mathcal{H}$ be a complex Hilbert space and (Ω, μ) be a measure space with positive measure μ. A mapping $F : \Omega \to \mathcal{H}$ is called a continuous frame with respect to (Ω, μ), if

(i) F is weakly-measurable, i.e., for all $f \in \mathcal{H}$, $\omega \to \langle f, F(\omega)\rangle$ is a measurable function on Ω;

(ii) there exist constants $A, B > o$ such that

$$A\|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega) \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{2.1}$$

The constants A and B are called continuous frame bounds. F is called a tight continuous frame if $A = B$. The mapping F is called *Bessel* if the second inequality in (2.1) holds. In this case, B is called the *Bessel constant.* If μ is a counting measure and $\Omega = \mathbb{N}$, F is called a discrete frame.

The first inequality in (2.1) shows that F is complete, i.e.,

$$\overline{\operatorname{span}}\{F(\omega)\}_{\omega\in\Omega} = \mathcal{H}.$$

The following are well known examples in wavelet frames and Gabor frames that are continuous frames.

Example 2.2. If $\psi \in L^2(\mathbb{R})$ is admissible, i.e.,

$$C_\psi := \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\gamma)|^2}{|\gamma|} \, d\gamma < +\infty$$

and, for $a, b \in \mathbb{R}$, $a \neq 0$,

$$\psi^{a,b}(x) := (T_b D_a \psi)(x) = \frac{1}{|a|^{\frac{1}{2}}} \psi\Big(\frac{x-b}{a}\Big), \quad \forall x \in \mathbb{R},$$

then $\{\psi^{a,b}\}_{a\neq 0, b\in\mathbb{R}}$ is a continuous frame for $L^2(\mathbb{R})$ with respect to $\mathbb{R}\setminus\{0\}\times\mathbb{R}$ equipped with the measure $\frac{da db}{a^2}$ and, for all $f \in L^2(\mathbb{R})$,

$$f = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_\psi(f)(a,b)\psi^{a,b} \, \frac{da db}{a^2},$$

where W_ψ is the continuous wavelet transform defined by

$$W_\psi(f)(a,b) := \langle f, \psi^{a,b}\rangle = \int_{-\infty}^{+\infty} f(x) \frac{1}{|a|^{\frac{1}{2}}} \overline{\psi\Big(\frac{x-b}{a}\Big)} \, dx.$$

For details, see the Proposition 11.1.1 and Corollary 11.1.2 in [8].

Definition 2.3. Fix a function $g \in L^2(\mathbb{R})\setminus\{0\}$. The *short-time Fourier transform* of a function $f \in L^2(\mathbb{R})$ with respect to the window function g is given by

$$\Psi_g(f)(y,\gamma) = \int_{-\infty}^{+\infty} f(x)\overline{g(x-y)} e^{-2\pi i x\gamma} dx, \quad y, \gamma \in \mathbb{R}.$$

Note that in terms of modulation operators and translation operators, $\Psi_g(f)(y,\gamma) = \langle f, E_\gamma T_y g\rangle$.

Example 2.4. Let $g \in L^2(\mathbb{R})\setminus\{0\}$. Then $\{E_b T_a g\}_{a,b\in\mathbb{R}}$ is a continuous frame for $\in L^2(\mathbb{R})$ with respect to $X = \mathbb{R}^2$ equipped with the Lebesgue measure. Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$. Then

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Psi_{g_1}(f_1)(a,b)\overline{\Psi_{g_2}(f_2)(a,b)} \, db \, da = \langle f_1, f_2\rangle\langle g_2, g_1\rangle.$$

For details see the Proposition 8.1.2 in [8].

The following proposition shows that it is enough to check the continuous frame condition on a dense set. One can find a discrete version of this proposition in ([8], Lemma 5.1.7).

Proposition 2.5. *Suppose that (Ω, μ) is a measure space and μ is σ-finite. Let $F : \Omega \to \mathcal{H}$ be a weakly-measurable vector-valued function and assume that there exist constants $A, B > 0$ such that (2.1) holds for all f in a dense subset V of $\mathcal{H}$. Then F is a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ with the bounds A, B.*

Proof. Let $\{\Omega_n\}_{n=1}^{\infty}$ be a family of disjoint measurable subsets of Ω such that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ with $\mu(\Omega_n) < \infty$ for each $n \geq 1$. Let

$$\Delta_m = \{\, \omega \in \Omega \mid m \leq \|F(\omega)\| < m+1 \,\}$$

for all integers $m \geq 0$. It is clear that $\Omega = \bigcup_{m=0,n=1}^{\infty} \big(\Omega_n \cap \Delta_m\big)$ where $\{\Omega_n \cap \Delta_m\}_{n=1}^{\infty}{}_{m=0}^{\infty}$ is a family of disjoint and measurable subsets of Ω. We show that F is Bessel. Suppose that there exists $f \in \mathcal{H}$ such that

$$\int_{\Omega} |\langle f, F(\omega)\rangle|^2 d\mu(\omega) > B\|f\|^2.$$

Therefore,

$$\sum_{m,n} \int_{\Delta_m \cap \Omega_n} \big|\langle F(\omega), f\rangle\big|^2 > B\|f\|^2$$

and thus there exist finite sets I, J such that

$$\sum_{m\in I} \sum_{n\in J} \int_{\Delta_m \cap \Omega_n} \big|\langle F(\omega), f\rangle\big|^2 > B\|f\|^2. \tag{2.2}$$

Let $\{f_k\}$ be a sequence in V such that $f_k \to f$ as $n \to \infty$. The assumption implies that

$$\sum_{m\in I} \sum_{n\in J} \int_{\Delta_m \cap \Omega_n} \big|\langle F(\omega), f_k\rangle\big|^2 \leq B\|f_k\|^2$$

which is a contradiction to (2.2) (by Lebesgue's Dominated Convergence Theorem). For the rest of the proof, we show that

$$\int_{\Omega} \big|\langle F(\omega), f_k\rangle\big|^2 \, d\mu(\omega) \;\to \int_{\Omega} \big|\langle F(\omega), f\rangle\big|^2 \, d\mu(\omega)$$

as $n \to \infty$. For this we have

$$\begin{aligned} &\Big| \int_{\Omega} \Big(\big|\langle F(\omega), f_k\rangle\big|^2 - \big|\langle F(\omega), f\rangle\big|^2 \Big) \, d\mu(\omega) \Big| \\ &\quad \leq \int_{\Omega} \big|\langle F(\omega), f_k - f\rangle\big|^2 \, d\mu(\omega) \; + 2 \int_{\Omega} \big|\langle F(\omega), f\rangle \langle F(\omega), f_k - f\rangle\big| \, d\mu(\omega) \\ &\quad \leq B\|f_k - f\|^2 + 2B\|f_k - f\|\|f\|, \end{aligned}$$

the last inequality follows from Cauchy-Schwarz' inequality. Since $f_k \to f$, the result is proved. □

Let F be a continuous frame with respect to (Ω, μ). Then the mapping

$$\Psi : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

defined by

$$\Psi(f, g) = \int_{\Omega} \langle f, F(\omega)\rangle \langle F(\omega), g\rangle \, d\mu(\omega)$$

is well defined, a sesquilinear form (i.e., linear in the first and conjugate-linear in the second variable) and is bounded. By Cauchy-Schwarz' inequality we get

$$\begin{aligned}|\Psi(f,g)| &\leq \int_\Omega |\langle f, F(\omega)\rangle\langle F(\omega), g\rangle|\, d\mu(\omega)\\ &\leq \left(\int_\Omega |\langle f, F(\omega)\rangle|^2\, d\mu(\omega)\right)^{\frac{1}{2}} \left(\int_\Omega |\langle F(\omega), g\rangle|^2\, d\mu(\omega)\right)^{\frac{1}{2}} \leq B\|f\|\|g\|.\end{aligned}$$

Hence, $\|\Psi\| \leq B$. By Theorem 2.3.6 in [16] there exists a unique operator $S_F : \mathcal{H} \to \mathcal{H}$ such that

$$\Psi(f,g) = \langle S_F f, g\rangle, \quad \forall f, g \in \mathcal{H}$$

and, moreover, $\|\Psi\| = \|S\|$.

Since $\langle S_F f, f\rangle = \int_\Omega |\langle f, F(\omega)|^2\, d\mu(\omega)$, we see that S_F is positive and $AI \leq S_F \leq BI$. Hence, S_F is invertible. We call S_F a continuous frame operator of F and use the notation $S_F f = \int_\Omega \langle f, F(\omega)\rangle F(\omega)\, d\mu(\omega)$. Thus, every $f \in \mathcal{H}$ has the representations

$$\begin{aligned}f &= S_F^{-1} S_F f = \int_\Omega \langle f, F(\omega)\rangle S_F^{-1} F(\omega)\, d\mu(\omega),\\ f &= S_F S_F^{-1} f = \int_\Omega \langle f, S_F^{-1} F(\omega)\rangle F(\omega)\, d\mu(\omega).\end{aligned}$$

The next theorem is analogous to Theorem 3.2.3 in [8].

Theorem 2.6. *Let (Ω, μ) be a measure space and let F be a Bessel mapping from Ω to $\mathcal{H}$. Then the operator $T_F : L^2(\Omega, \mu) \to \mathcal{H}$ weakly defined by*

$$\langle T_F \varphi, h\rangle = \int_\Omega \varphi(\omega)\langle F(\omega), h\rangle\, d\mu(\omega), \quad h \in \mathcal{H}$$

is well defined, linear, bounded, and its adjoint is given by

$$T_F^* : \mathcal{H} \to L^2(\Omega, \mu), \quad (T_F^* h)(\omega) = \langle h, F(\omega)\rangle, \quad \omega \in \Omega.$$

The operator T_F is called a pre-frame operator or synthesis operator and T_F^* is called an analysis operator of F.

Proof. The proof is straightforward. □

The converse of Theorem 2.6 holds when μ is a σ-finite measure.

Proposition 2.7. *Let (Ω, μ) be a measure space, where μ is a σ-finite measure and let $F : \Omega \to \mathcal{H}$ be a measurable function. If the mapping $T_F : L^2(\Omega, \mu) \mapsto \mathcal{H}$ defined by*

$$\langle T_F \varphi, h\rangle = \int_\Omega \varphi(\omega)\langle F(\omega), h\rangle\, d\mu(\omega), \quad \varphi \in L^2(\Omega, \mu), \quad h \in \mathcal{H}$$

is a bounded operator, then F is Bessel.

Proof. By Theorem 2.6, we have

$$(T^* h)(\omega) = \langle h, F(\omega)\rangle, \quad \omega \in \Omega.$$

Hence for each $h \in \mathcal{H}$

$$\int_\Omega |\langle h, F(\omega)\rangle|^2\, d\mu(\omega) = \|T^* h\|^2 \leq \|T\|^2\|h\|^2.$$

□

We now give a characterization of continuous frames in terms of the pre-frame operators. For the next theorem we will need the following lemma that is proved in [8].

Lemma 2.8. *Let $T : \mathcal{K} \to \mathcal{H}$ be a bounded operator with a closed range $\mathcal{R}_T$. Then there exists a bounded operator $T^\dagger : \mathcal{H} \to \mathcal{K}$ for which*

$$TT^\dagger f = f, \quad \forall f \in \mathcal{R}_T.$$

The next theorem gives an equivalent characterization of a continuous frame. For a discrete case of this theorem, see [5].

Theorem 2.9. *Let (Ω, μ) be a measure space where μ is a σ-finite measure. The mapping $F : \Omega \to \mathcal{H}$ is a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ if and only if the operator T_F as defined in Theorem (2.6) is a bounded and onto operator.*

Proof. Let F be a continuous frame. Then, by Theorem 2.6, T_F is bounded and

$$T_F^* : \mathcal{H} \to L^2(\Omega, \mu), \quad (T_F^* h)(\omega) = \langle h, F(\omega)\rangle, \quad h \in \mathcal{H}, \quad \omega \in \Omega.$$

Hence, for each $f \in \mathcal{H}$,

$$\|T_F^* f\|^2 = \int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega).$$

Therefore, T_F^* is one to one and so T_F is onto.

Conversely, let T_F be a bounded and onto operator. Then, by Lemma 2.8 there exists a bounded operator $T_F^\dagger : \mathcal{H} \to L^2(\Omega, \mu)$ such that $T_F T_F^\dagger f = f$ for all $f \in \mathcal{H}$. Since T_F is bounded, by Proposition 2.7, F is Bessel and

$$\|T_F^* f\|^2 = \int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega), \quad \forall f \in \mathcal{H}.$$

Let $f \in \mathcal{H}$, then $\|f\|^2 \leq \|T_F^\dagger\|^2 \|T_F^* f\|^2$. Hence,

$$\|T_F^\dagger\|^{-2}.\|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega), \quad \forall f \in \mathcal{H}.$$

□

As in the discrete case, we have the following lemma.

Lemma 2.10. *Let $F : \Omega \to \mathcal{H}$ be a Bessel function with respect to (Ω, μ). By the above notations, $S_F = T_F T_F^*$.*

Proof. For all $f, g \in \mathcal{H}$,

$$\langle T_F T_F^* f, g\rangle = \langle T_F^* f, T_F^* g\rangle = \int_\Omega \langle f, F(\omega)\rangle \langle F(\omega), g\rangle \, d\mu(\omega) = \langle Sf, g\rangle.$$

So $S_F = T_F T_F^*$. □

Proposition 2.11. *Let F be a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ with a frame operator S_F and let $V : \mathcal{H} \to \mathcal{K}$ be a bounded and invertible operator. Then VF is a continuous frame for $\mathcal{K}$ with the frame operator VS_FV^*.*

Proof. For each $f \in \mathcal{H}$, $\omega \to \langle V^* f, F(\omega)\rangle = \langle f, VF(\omega)\rangle$ is measurable. Let A and B be frame bounds for F. Therefore, for every $f \in \mathcal{H}$,

$$A\|V^* f\|^2 \leq \int_\Omega |\langle V^* f, F(\omega)\rangle|^2 \, d\mu(\omega) = \int_\Omega |\langle f, VF(\omega)\rangle|^2 \, d\mu(\omega) \leq B\|V^* f\|^2.$$

Hence,

$$\|f\| \leq \|V^{-1}\| \|V^* f\| \quad \text{and} \quad \|V^* f\| \leq \|V^*\| \|f\|, \quad \forall f \in \mathcal{H}.$$

Therefore, VF is a continuous frame with frame bounds $A\|V^{-1}\|^{-2}$ and $B\|V\|^2$. Let S_{VF} be the frame operator for VF. Then for each $f, g \in \mathcal{H}$,

$$\begin{aligned}\langle S_{VF} f, g\rangle &= \int_\Omega \langle f, VF(\omega)\rangle\langle VF(\omega), g\rangle\, d\mu(\omega) \\ &= \int_\Omega \langle V^* f, F(\omega)\rangle\langle F(\omega), V^* g\rangle\, d\mu(\omega) = \langle S_F V^* f, V^* g\rangle = \langle V S_F V^* f, g\rangle.\end{aligned}$$

So $S_{VF} = V S_F V^*$. □

Corollary 2.12. *Let F be a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ with a frame operator S_F. Then for all $\alpha \in \mathbb{R}$, $S_F^\alpha F$ is a continuous frame for $\mathcal{H}$ with the frame operator $S' = S_F^{2\alpha+1}$. In particular, $S_F^{-1} F$ is a continuous frame and we call it a standard dual frame for F. Also, $S_F^{-1/2} F$ is a normalized tight frame with the frame operator I.*

For later reference we state a general version of Proposition 2, where we only assume that U is a bounded operator with a closed range $\mathcal{R}_U$. The next proposition is analogous to Proposition 5.3.1 in [8].

Proposition 2.13. *Let F be a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ with bounds A, B and let $U : \mathcal{H} \to \mathcal{K}$ be a bounded operator with a closed range $\mathcal{R}_U$. Then UF is a continuous frame for $\mathcal{R}_U$ with the bounds $A\|U^\dagger\|^{-2}, B\|U\|^2$.*

Proof. It is clear that $\omega \to \langle f, VF(\omega)\rangle$ is measurable for all $f \in \mathcal{H}$. We may assume that U is onto. If $f \in \mathcal{K}$, then

$$\int_\Omega |\langle f, UF(\omega)\rangle|^2\, d\mu(\omega) = \int_\Omega |\langle U^* f, F(\omega)\rangle|^2\, d\mu(\omega) \le B\|U\|^2\|f\|^2,$$

which proves that UF is Bessel. For the lower frame condition, let $f \in \mathcal{K}$. Then $\|f\| \le \|U^\dagger\|\|U^* f\|$ and

$$\begin{aligned}\int_\Omega |\langle f, UF(\omega)\rangle|^2\, d\mu(\omega) &= \int_\Omega |\langle U^* f, F(\omega)\rangle|^2\, d\mu(\omega) \\ &\ge A\|U^* f\|^2 \ge A\|U^\dagger\|^{-2}\|f\|^2,\end{aligned}$$

which gives the result. □

Corollary 2.14. *If F is a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ with bounds A, B and $U : \mathcal{H} \to \mathcal{K}$ is a bounded surjective operator, then UF is a continuous frame with respect to (Ω, μ) for $\mathcal{K}$ with the bounds $A\|U^\dagger\|^{-2}, B\|U\|^2$.*

The following proposition is a criterion for a continuous frame for a closed subspace of $\mathcal{H}$ to be a continuous frame for $\mathcal{H}$. For the discrete, case see ([8], Lemma 5.2.1).

Proposition 2.15. *Suppose that F is a continuous frame with respect to (Ω, μ) for a closed subspace K of $\mathcal{H}$, where μ is a σ-finite measure. Let $T : L^2(\Omega, \mu) \to \mathcal{H}$ be the mapping defined by*

$$\langle T\varphi, h\rangle = \int_\Omega \varphi(\omega)\langle F(\omega), h\rangle\, d\mu(\omega), \quad \varphi \in L^2(\Omega, \mu), \quad h \in \mathcal{H}.$$

Then F is a continuous frame for $\mathcal{H}$ if and only if T^* is injective.

Proof. It is clear that T is well defined and bounded. Let F be a continuous frame for $\mathcal{H}$. Then, by Theorem 2.6, T^* is injective. Conversely, suppose that T^* is injective. Then T is onto and the result follows from Theorem 2.9. □

The next proposition is analogous to Proposition 5.3.5 in [8] and gives a relationship between continuous frames and orthogonal projections.

Proposition 2.16. *Let K be a closed subspace of $\mathcal{H}$ and let $P : \mathcal{H} \to K$ be an orthogonal projection. The the following holds:*

(i) *If F is a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ with bounds A and B, then PF is a continuous frame with respect to (Ω, μ) for K with the bounds A, B.*
(ii) *If F is a continuous frame with respect to (Ω, μ) for K with a frame operator S_F, then for each $f, g \in \mathcal{H}$,*

$$\langle Pf, g\rangle = \int_\Omega \langle f, S_F^{-1}F(\omega)\rangle\langle F(\omega), g\rangle \, d\mu(\omega).$$

Proof. The proof is easy. □

The proof of the following proposition is similar to the discrete case ([8], Proposition 5.3.6) and we omit it.

Proposition 2.17. *Let F be a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ with a synthesis operator T_F. The orthogonal projection Q from $L^2(\Omega, \mu)$ onto $\mathcal{R}_{T_F^*}$ is given by*

$$Q(\varphi)(\nu) = \int_\Omega \varphi(\omega)\langle S_F^{-1}F(\omega), F(\nu)\rangle \, d\mu(\omega), \quad \nu \in \Omega,$$

where S_F is the frame operator of F.

Let $F : \Omega \longrightarrow \mathcal{H}$ be a vector-valued function and $\varphi \in L^2(\Omega, \mu)$. It is natural to ask whether we can find $f \in \mathcal{H}$ such that

$$\langle f, F(\omega)\rangle = \varphi(\omega), \quad \forall \omega \in \Omega.$$

A problem of this type is called a *moment problem*. It is clear that there are cases where no solution exists: if, for example, $F(\omega) = F(\nu)$ for some $\omega \neq \nu$, a solution can only exist if $\varphi(\omega) = \varphi(\nu)$. Since the moment problem has no solution in general, there is a natural question of whether we can find an element in $\mathcal{H}$ which minimizes the function

$$f \to \int_\Omega |\varphi(\omega) - \langle f, F(\omega)\rangle|^2 \, d\mu(\omega).$$

The answer is given by the following theorem and is called a *best approximation solution*. For the discrete case, see ([8], Theorem 6.5.2).

Theorem 2.18. *Let F be a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ and $\varphi \in L^2(\Omega, \mu)$. Then there exists a unique vector in $\mathcal{H}$ which minimizes the map $f \to \int_\Omega |\varphi(\omega) - \langle f, F(\omega)\rangle|^2 \, d\mu(\omega)$; this vector is $f = S_F^{-1}T_F\varphi$. S_F and T_F are the frame operator and synthesis operators for F, respectively.*

Proof. Let Q be the orthogonal projection of $L^2(\Omega, \mu)$ onto $\mathcal{R}_{T^*}$. Then

$$\min_{\psi \in \mathcal{R}_{T^*}} \|\varphi - \psi\| = \|\varphi - Q(\varphi)\|.$$

Therefore,

$$\min_{g \in \mathcal{H}} \int_\Omega \left|\varphi(\omega) - \langle g, F(\omega)\rangle\right|^2 d\mu(\omega) = \|\varphi - Q(\varphi)\|^2.$$

Let $Q(\varphi) = T_F^*(f)$ for some $f \in \mathcal{H}$. By Proposition 2.17,

$$\int_\Omega \varphi(\omega)\langle S_F^{-1}F(\omega), F(\nu)\rangle \, d\mu(\omega) = \langle f, F(\nu)\rangle, \quad \forall \nu \in \Omega.$$

Hence $f = S_F^{-1}T_F\varphi$. Since T_F^* is injective, the minimization is unique. □

3. Dual of continuous frames

In this section, we mention some important properties of continuous frames and their dual. Gabardo and Han in [14] defined a dual frame for a continuous frame as follows.

Definition 3.1. Let F, G be a continuous frames with respect to (Ω, μ) for $\mathcal{H}$. We call G a *dual frame* if the following holds true:

$$\langle f, g\rangle = \int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), g\rangle \, d\mu(\omega), \quad \forall f, g \in \mathcal{H}. \tag{3.1}$$

In this case (F, G) is called a *dual pair.* If T_F and T_G denote the synthesis operators of F and G, respectively, then (3.1) is equivalent to $T_G T_F^* = I$.

It is certainly possible for a continuous frame F to have only one dual. In this case we call F a *Riesz-type frame.* The proof of the following proposition can be found in [14].

Proposition 3.2. *Let F be a continuous frame with respect to (Ω, μ) for $\mathcal{H}$. Then F is a Riesz-type frame if and only if $\mathcal{R}(T_F^*) = L^2(\Omega, \mu)$.*

Corollary 3.3. *Let $F : \Omega \to \mathcal{H}$ be a Riesz-type frame with respect to (Ω, μ) for $\mathcal{H}$, where μ is σ-finite. Then $F(\omega) \neq 0$ for all $\omega \in \Omega$.*

The next proposition shows that every continuous frame has a dual frame, as in the discrete case. For the discrete case, see ([11], Lemma VIII).

Proposition 3.4. *Let F be a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ with a frame operator S. Then $(S^\alpha F, S^{-1-\alpha} F)$ is a dual pair for each $\alpha \in \mathbb{R}$. In particular, $(F, S^{-1}F)$ is a dual pair and $S^{-1}F$ is called a standard dual frame of F.*

Proof. For any $f, g \in \mathcal{H}$ we have

$$\langle f, g\rangle = \langle S^\alpha S S^{-1-\alpha} f, g\rangle = \langle S S^{-1-\alpha} f, S^\alpha g\rangle = \int_\Omega \langle S^{-1-\alpha} f, F(\omega)\rangle\langle F(\omega), S^\alpha g\rangle \, d\mu(\omega).$$

□

The following proposition states an important property of a standard dual continuous frame of a given continuous frame. Its proof is similar to that in the discrete case ([11], Lemma VIII).

Proposition 3.5. *Let F be a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ with a frame operator S and let $f \in \mathcal{H}$. If there exists $\varphi \in L^2(\Omega, \mu)$ such that*

$$\langle f, g\rangle = \int_\Omega \varphi(\omega)\langle F(\omega), g\rangle \, d\mu(\omega), \quad \forall g \in \mathcal{H},$$

then

$$\int_\Omega |\varphi(\omega)|^2 \, d\mu(\omega) = \int_\Omega |\langle f, S^{-1}F(\omega)\rangle|^2 \, d\mu(\omega) + \int_\Omega |\varphi(\omega) - \langle f, S^{-1}F(\omega)\rangle|^2 \, d\mu(\omega).$$

The following theorem is analogous to Lemma IX in [11] and shows when we can remove some elements from a continuous frame so that the set still remains a continuous frame.

Theorem 3.6. *Let F be a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ with a frame operator S and let $\omega_0 \in \Omega$ such that*

$$\mu(\{\omega_0\}) \neq \frac{1}{\langle F(\omega_0), S^{-1}F(\omega_0)\rangle}.$$

Then F is robust to erasure of $F(\omega_0)$, i.e., $F : \Omega \setminus \{\omega_0\} \to \mathcal{H}$ is a continuous frame for $\mathcal{H}$.

Proof. It is clear that the upper frame condition holds. For the lower frame bound, we have

$$\langle F(\omega_0), f\rangle = \int_\Omega \langle F(\omega_0), S^{-1}F(\omega)\rangle\langle F(\omega), f\rangle\, d\mu(\omega), \quad \forall f \in \mathcal{H},$$

therefore,

$$\begin{aligned}\langle F(\omega_0), f\rangle = &\int_{\Omega\setminus\{\omega_0\}} \langle F(\omega_0), S^{-1}F(\omega)\rangle\langle F(\omega), f\rangle\, d\mu(\omega)\\ &+ \langle F(\omega_0), S^{-1}F(\omega_0)\rangle\langle F(\omega_0), f\rangle\mu(\{\omega_0\}).\end{aligned}$$

Hence, by the assumption of theorem,

$$\langle F(\omega_0), f\rangle = \frac{1}{1-\mu(\{\omega_0\})\langle S^{-1}F(\omega_0), F(\omega_0)\rangle}\int_{\Omega\setminus\{\omega_0\}} \langle F(\omega_0), S^{-1}F(\omega)\rangle\langle F(\omega), f\rangle\, d\mu(\omega).$$

Let A be the lower frame bound for F. For any $f \in \mathcal{H}$, the Cauchy-Schwartz inequality gives

$$|\langle f, F(\omega_0)\rangle|^2 \le K\int_{\Omega\setminus\{\omega_0\}} |\langle F(\omega), f\rangle|^2\, d\mu(\omega)$$

where

$$K := \int_{\Omega\setminus\{\omega_0\}} \frac{|\langle F(\omega_0), S^{-1}F(\omega)\rangle|^2}{|1-\mu(\{\omega_0\})\langle F(\omega_0), S^{-1}F(\omega_0)\rangle|^2}\, d\mu(\omega).$$

Therefore, for any $f \in \mathcal{H}$,

$$A\|f\|^2 \le \int_{\Omega\setminus\{\omega_0\}} |\langle F(\omega), f\rangle|^2\, d\mu(\omega) + |\langle F(\omega_0), f\rangle|^2\mu(\{\omega_0\})$$

and so

$$A\|f\|^2 \le (1+K\mu(\{\omega_0\}))\int_{\Omega\setminus\{\omega_0\}} |\langle F(\omega), f\rangle|^2\, d\mu(\omega).$$

Therefore, $F : \Omega\setminus\{\omega_0\} \to \mathcal{H}$ is a continuous frame for $\mathcal{H}$ with the lower frame bound $\frac{A}{1+K\mu(\{\omega_0\})}$. □

Corollary 3.7. *Let F be a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ with a frame operator S and let $\omega_0 \in \Omega$ such that $\mu(\{\omega_0\}) \ne \frac{1}{\langle F(\omega_0), S^{-1}F(\omega_0)\rangle}$. Then F is not a Riesz-type frame.*

Proof. By Theorem 3.6, $F : \Omega\setminus\{\omega_0\} \to \mathcal{H}$ is a continuous frame for $\mathcal{H}$. Let $G : \Omega\setminus\{\omega_0\} \to \mathcal{H}$ be a standard dual frame for $F_{|\Omega\setminus\{\omega_0\}}$ and let $G(\omega_0) = 0$. Then $S^{-1}F \ne G$ and for all $f, h \in \mathcal{H}$,

$$\langle f, h\rangle = \int_\Omega \langle f, G(\omega)\rangle\langle F(\omega), h\rangle.$$

Therefore, $G : \Omega \to \mathcal{H}$ is a dual frame for F which is different from $S^{-1}F$. □

4. Perturbation of continuous frames

A perturbation of discrete frames has been discussed in [4]. In this section we introduce and extent Christensen' works on a discrete frame, however, there is a similar theorem about perturbation of frames for measurable spaces in [14]. The following theorem is another version of a perturbation of continuous frames and its proof is based on the following lemma that is proved in [4].

Lemma 4.1. *Let U be a linear operator on a Banach space X and assume that there exist $\lambda_1, \lambda_2 \in [0, 1)$ such that*

$$\|x - Ux\| \le \lambda_1\|x\| + \lambda_2\|Ux\|$$

for all $x \in X$. Then U is bounded and invertible. Moreover,

$$\frac{1-\lambda_1}{1+\lambda_2}\|x\| \le \|Ux\| \le \frac{1+\lambda_1}{1-\lambda_2}\|x\|$$

and

$$\frac{1-\lambda_2}{1+\lambda_1}\|x\| \le \|U^{-1}x\| \le \frac{1+\lambda_2}{1-\lambda_1}\|x\|$$

for all $x \in X$.

Theorem 4.2. *Let F be a continuous frame with respect to (Ω, μ) for $\mathcal{H}$, where μ is σ-finite. Let $G : \Omega \to \mathcal{H}$ be a weakly-measurable vector-valued function and assume that there exist constants $\lambda_1, \lambda_2, \gamma \ge 0$ such that $\max\big(\lambda_1 + \gamma/\sqrt{A}, \lambda_2\big) < 1$ and*

$$\begin{aligned} &\left| \int_\Omega \varphi(\omega)\langle F(\omega) - G(\omega), f\rangle \, d\mu(\omega) \right| \\ &\quad \le \lambda_1 \left| \int_\Omega \varphi(\omega)\langle F(\omega), f\rangle \, d\mu(\omega) \right| + \lambda_2 \left| \int_\Omega \varphi(\omega)\langle G(\omega), f\rangle \, d\mu(\omega) \right| + \gamma\|\varphi\|_2 \end{aligned} \tag{4.1}$$

for all $\varphi \in L^2(\Omega, \mu)$ and for all f in the unit sphere in $\mathcal{H}$. Then $G : \Omega \to \mathcal{H}$ is a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ with the bounds

$$A\left[\frac{1 - \big(\lambda_1 + \gamma/\sqrt{A}\big)}{1+\lambda_2}\right]^2 \quad \text{and} \quad B\left[\frac{1+\lambda_1+\gamma/\sqrt{B}}{1-\lambda_2}\right]^2,$$

where A, B are frame bounds for F.

Proof. Let $\mathbb{S} = \{ f \in \mathcal{H} : \|f\| = 1\}$ be the unit sphere in $\mathcal{H}$. We first prove that G is Bessel. By the assumption, for any $f \in \mathbb{S}$ and $\varphi \in L^2(\Omega, \mu)$, we have

$$\begin{aligned} &\left| \int_\Omega \varphi(\omega)\langle G(\omega), f\rangle \, d\mu(\omega) \right| \\ &\le \left| \int_\Omega \varphi(\omega)\langle F(\omega) - G(\omega), f\rangle \, d\mu(\omega) \right| + \left| \int_\Omega \varphi(\omega)\langle F(\omega), f\rangle \, d\mu(\omega) \right| \\ &\le (1+\lambda_1) \left| \int_\Omega \varphi(\omega)\langle F(\omega), f\rangle \, d\mu(\omega) \right| + \lambda_2 \left| \int_\Omega \varphi(\omega)\langle G(\omega), f\rangle \, d\mu(\omega) \right| + \gamma\|\varphi\|_2, \end{aligned} \tag{4.2}$$

which implies that

$$\begin{aligned} \left| \int_\Omega \varphi(\omega)\langle G(\omega), f\rangle \, d\mu(\omega) \right| &\le \frac{1+\lambda_1}{1-\lambda_2} \left| \int_\Omega \varphi(\omega)\langle F(\omega), f\rangle \, d\mu(\omega) \right| + \frac{\gamma}{1-\lambda_2}\|\varphi\|_2 \\ &\le \left[\frac{1+\lambda_1}{1-\lambda_2}\sqrt{B} + \frac{\gamma}{1-\lambda_2}\right] \|\varphi\|_2. \end{aligned} \tag{4.3}$$

Let $U : L^2(\Omega, \mu) \to \mathcal{H}$ be defined by

$$\langle U\varphi, f\rangle = \int_\Omega \varphi(\omega)\langle G(\omega), f\rangle \, d\mu(\omega), \quad \forall f \in \mathcal{H}, \varphi \in L^2(\Omega, \mu).$$

Then

$$\begin{aligned} \|U\varphi\| = \sup_{\|f\|=1} |\langle U\varphi, f\rangle| &= \sup_{\|f\|=1} \left| \int_\Omega \varphi(\omega)\langle G(\omega), f\rangle \, d\mu(\omega) \right| \\ &\le \left[\frac{1+\lambda_1}{1-\lambda_2}\sqrt{B} + \frac{\gamma}{1-\lambda_2}\right] \|\varphi\|_2. \end{aligned}$$

Therefore U is bounded and so, by Proposition 2.7, G is Bessel with the required upper bound. Now we prove that G has the required lower frame bound. Let T_F and S_F be a synthesis operator and a frame operator for F, respectively. Let us define $V = UT_F^*S_F^{-1}$, then

$$\langle Vf, g\rangle = \int_\Omega \langle f, S_F^{-1}F(\omega)\rangle\langle G(\omega), g\rangle\, d\mu(\omega)$$

and

$$\langle f, g\rangle = \int_\Omega \langle f, S_F^{-1}F(\omega)\rangle\langle F(\omega), g\rangle\, d\mu(\omega)$$

for all $f, g \in \mathcal{H}$. For each $f \in \mathcal{H}$, let $\varphi_f : \Omega \to \mathbb{C}$ be a mapping defined by $\varphi_f(\omega) = \langle f, S^{-1}F(\omega)\rangle$. Since $S^{-1}F$ is a standard dual frame of F, we have $\varphi_f \in L^2(\Omega, \mu)$. Therefore, using the assumption, we deduce that

$$|\langle f - Vf, g\rangle| \leq \lambda_1|\langle f, g\rangle| + \lambda_2|\langle Vf, g\rangle| + \gamma\|\varphi_f\|_2$$

for all $f \in \mathcal{H}$ and $g \in \mathbb{S}$. Hence,

$$\begin{aligned}\|f - Vf\| = \sup_{\|g\|=1} |\langle f - Vf, g\rangle| &\leq \lambda_1\|f\| + \lambda_2\|Vf\| + \gamma\|\ \varphi_f\|_2 \\ &\leq \left(\lambda_1 + \frac{\gamma}{\sqrt{A}}\right)\|f\| + \lambda_2\|Vf\|\end{aligned}$$

for all $f \in \mathcal{H}$. By Lemma 4.1, V is invertible and

$$\|V\| \leq \frac{1 + \lambda_1 + \gamma/\sqrt{A}}{1 - \lambda_2}, \quad \|V^{-1}\| \leq \frac{1 + \lambda_2}{1 - \big(\lambda_1 + \gamma/\sqrt{A}\big)}.$$

Let $f \in \mathcal{H}$. Then

$$\langle f, f\rangle = \langle VV^{-1}f, f\rangle = \int_\Omega \langle V^{-1}f, S_F^{-1}F(\omega)\rangle\langle G(\omega), f\rangle\, d\mu(\omega),$$

and we obtain

$$\begin{aligned}\|f\|^4 = |\langle f, f\rangle|^2 &= \left|\int_\Omega \langle V^{-1}f, S_F^{-1}F(\omega)\rangle\langle G(\omega), f\rangle\, d\mu(\omega)\right|^2 \\ &\leq \int_\Omega \big|\langle V^{-1}f, S_F^{-1}F(\omega)\rangle\big|^2\, d\mu(\omega) \int_\Omega \big|\langle G(\omega), f\rangle\big|^2\, d\mu(\omega) \\ &\leq \frac{1}{A}\left[\frac{1 + \lambda_2}{1 - \big(\lambda_1 + \gamma/\sqrt{A}\big)}\right]^2 \|f\|^2. \int_\Omega \big|\langle G(\omega), f\rangle\big|^2\, d\mu(\omega).\end{aligned}$$

Therefore,

$$\int_\Omega \big|\langle G(\omega), f\rangle\big|^2\, d\mu(\omega) \geq A\left[\frac{1 - \big(\lambda_1 + \gamma/\sqrt{A}\big)}{1 + \lambda_2}\right]^2 \|f\|^2$$

for all $f \in \mathcal{H}$. □

Corollary 4.3. *Let F be a continuous frame with respect to (Ω, μ) for $\mathcal{H}$, where μ is σ-finite. Let $G : \Omega \to \mathcal{H}$ be a weakly-measurable vector-valued function and let A, B be frame bounds for F. If there exists $R < A$ such that*

$$\int_\Omega \big|\langle F(\omega) - G(\omega), f\rangle\big|^2\, d\mu(\omega) \leq R\|f\|^2, \quad \forall f \in \mathcal{H},$$

then G is a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ with the bounds

$$A\big(1 - \sqrt{R/A}\big)^2, \quad B\big(1 + \sqrt{R/B}\big)^2.$$

Corollary 4.4. *Let F and G be as in Theorem 4.2. Then G is similar to a dual of F.*

Proof. Let U and V be defined as in the proof of Theorem 4.2. Then $U = T_G$ and $VS_F = T_GT_F^*$. Therefore $T_GT_F^*$ is invertible. Let $D = (T_GT_F^*)^{-1}$. Then $T_{DG}T_F^* = DT_GT_F^* = I$. Therefore, DG is a dual of F. □

Corollary 4.5. *Let F and G be as in Theorem 4.2, and assume that F is a Riesz-type frame for $\mathcal{H}$. Then G is also a Riesz-type frame for $\mathcal{H}$.*

Proof. Since G is similar to $S_F^{-1}F$, it follows that G is similar to F and, thus, $\mathcal{R}_{T_F^*} = \mathcal{R}_{T_G^*}$. So the result follows from Proposition 3.2. □

Corollary 4.6. *Let F and G be as in Theorem 4.2 with $\gamma = 0$. Then G is similar to F.*

Proof. Since F and G are continuous frames for $\mathcal{H}$, we see that $\mathcal{R}_{T_F^*}$ and $\mathcal{R}_{T_g^*}$ are closed subspaces of $L^2(\Omega, \mu)$. Therefore, F and G are similar if and only if $\ker T_F = \ker T_G$. If $\varphi \in \ker T_F$, then (4.1) implies that

$$\|T_G\varphi\| \leq \lambda_2 \|T_G\varphi\|$$

and thus, $T_G\varphi = 0$, since $\lambda_2 < 1$. Therefore, $\ker T_F \subseteq \ker T_G$. Conversely, let $\varphi \in \ker T_G$. By (4.1) we have

$$\|T_F\varphi\| \leq \lambda_1 \|T_F\varphi\|.$$

Since $\lambda_1 < 1$, $T_F\varphi = 0$ and thus $\ker T_G \subseteq \ker T_F$. □

The proof of the following theorem is similar to discrete case and we refer to [8].

Theorem 4.7. *Let F be a continuous frame with respect to (Ω, μ) for $\mathcal{H}$, where μ is σ-finite. Let $G : \Omega \to \mathcal{H}$ be a weakly-measurable vector-valued function and assume that $K : L^2(\Omega, \mu) \to \mathcal{H}$ defined by*

$$\langle K\varphi, f\rangle = \int_\Omega \varphi(\omega)\langle F(\omega) - G(\omega), f\rangle \, d\mu(\omega), \quad f \in \mathcal{H}, \quad \varphi \in L^2(\Omega, \mu)$$

is a well defined compact operator. Then G is a continuous frame with respect to (Ω, μ) for $\overline{\operatorname{span}}_{\omega\in\Omega}\{G(\omega)\}$.

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