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ON EXISTENCE OF $*$-REPRESENTATIONS OF CERTAIN ALGEBRAS RELATED TO EXTENDED DYNKIN GRAPHS

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Abstract. For $*$-algebras associated with extended Dynkin graphs, we investigate a set of parameters for which there exist representations. We give structure properties of such sets and a complete description for the set related to the graph $\tilde{D}_4$.

0. Introduction

In [1, 2] (also see the bibliography therein) the following problems were studied. Let $M_i = \{0 = \alpha_0^{(i)} < \alpha_1^{(i)} < \dots < \alpha_{m_i}^{(i)}\}$, $i = 1, \dots, n$, be given finite subsets of $\mathbb{R}_+$ and $\gamma \in \mathbb{R}_+$. The problem is to determine whether there exist n-tuples of Hermitian operators $A_i = A_i^*$, $i = 1, \dots, n$, such that $\sigma(A_i) \subset M_i$ and

$$A_1 + A_2 + \dots + A_n = \gamma I,$$

and to describe all irreducible (up to a unitary equivalence) n-tuples of such operators. This problem could be reformulated in terms of $*$-algebras and their $*$-representations.

Consider the following $*$-algebra:

$$\mathcal{A}_{M_1,M_2,\dots,M_n;\gamma} = \mathbb{C}\langle a_1, \dots, a_n | a_i = a_i^*, (a_i - \alpha_0^{(i)}) \dots (a_i - \alpha_{m_i}^{(i)}) = 0, \\ a_1 + a_2 + \dots + a_n = \gamma e\rangle.$$

It is quite easy to show that such an algebra is isomorphic to the algebra generated by the projections

$$\mathcal{P}_{M_1,M_2,\dots,M_n;\gamma} = \mathbb{C}\langle p_1^{(1)}, \dots, p_{m_1}^{(1)}, \dots, p_1^{(n)}, \dots, p_{m_n}^{(n)} | p_i^{(k)} = p_i^{(k)2} = p_i^{(k)*}, \\ \sum_{i=1}^{n} \sum_{k=1}^{m_i} \alpha_k^{(i)} p_k^{(i)} = \gamma e, p_j^{(i)} p_k^{(i)} = 0\rangle.$$

To each algebra $\mathcal{P}_{M_1,M_2,\dots,M_n;\gamma}$, one can associate a connected non-oriented graph Γ that has n branches connected in a common vertex (the root), such that i-th branch has m_i vertices, $i = 1, \dots, n$. Starting with $\alpha_j^{(i)}$, we construct a function χ (we will call it a character of the algebra) on the set of vertices except for the root in the following way: $\chi_j^{(i)}$ (i-th branch, j-th vertex) equals to $\alpha_j^{(i)}$, the root of the tree corresponds to γ. The character χ could be written as the vector $\chi = (\alpha_1^{(1)}, \dots, \alpha_{m_1}^{(1)}; \dots; \alpha_1^{(n)}, \dots, \alpha_{m_n}^{(n)})$. The algebra $\mathcal{P}_{M_1,M_2,\dots,M_n;\gamma}$ is uniquely given by the graph Γ, the character χ, and γ, hence, we will denote it in the sequel by $\mathcal{P}_{\Gamma,\chi,\gamma}$.

The additive spectral problem is equivalent to the following:

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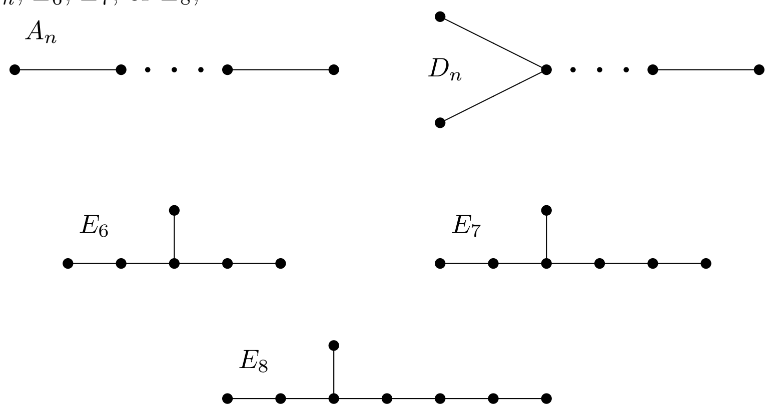
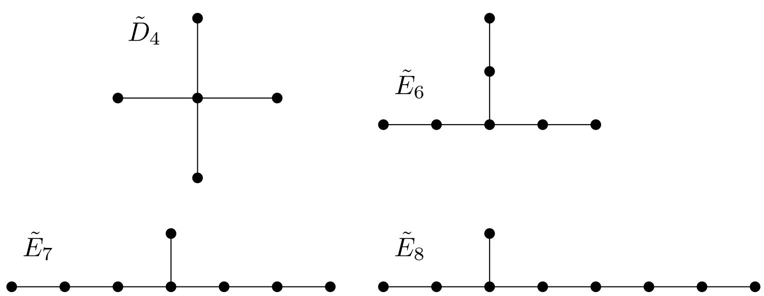
(1) a) to describe the set $\Sigma_\Gamma = \{(\chi;\gamma)|$ there exists a representation of the algebra $\mathcal{P}_{\Gamma,\chi,\gamma}\}$,
b) for each character χ, to describe the set $\Sigma_{\Gamma,\chi} = \{\gamma \in \mathbb{R}_+|$ there exists a representation of the algebra $\mathcal{P}_{\Gamma,\chi,\gamma}\}$;

(2) for every pair $(\chi;\gamma) \in \Sigma_\Gamma$ to describe all irreducible $*$-representation of $\mathcal{P}_{\Gamma,\chi,\gamma}$.

Depending on the properties of the graph Γ, the structure of representations of $\mathcal{P}_{\Gamma,\chi,\gamma}$ is quite different. The result of the recent paper [4] shows that if Γ is a Dynkin graph of the type A_n, D_n, E_6, E_7, or E_8,



then $\mathcal{P}_{\Gamma,\chi,\gamma}$ is finite dimensional, if Γ is an extended Dynkin graph of the type $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$,



then the algebra $\mathcal{P}_{\Gamma,\chi,\gamma}$ is infinite dimensional and of polynomial growth, and finally if Γ neither a Dynkin graph nor an extended Dynkin graph, then $\mathcal{P}_{\Gamma,\chi,\gamma}$ contains a free algebra with two generators (in this case, problem (2) could be too complicated).

In this paper, we study the sets $\Sigma_{\Gamma,\chi}$ in the case where Γ is an extended Dynkin graph, give a complete description of the set $\Sigma_{\tilde{D}_4,\chi}$, find conditions for the sets $\Sigma_{\Gamma,\chi}$ to be infinite and conditions for existence of $*$-representations of $\mathcal{P}_{\Gamma,\chi,\gamma}$ in a special case where $\gamma = \omega_\Gamma$ (see Section 2 for a definition of ω_Γ).

1. A description of the set $\Sigma_{\tilde{D}_4,\chi}$

As was shown in [11], the set $\Sigma_{\tilde{D}_4,\chi}$ can be reduced to the structure of set $\Sigma_{D_4,\chi}$. A complete description of the set $\Sigma_{D_4,\chi}$ was given in [7],

$$\Sigma_{D_4,(\alpha_1,\alpha_2,\alpha_3)} = \{0, (\alpha_1+\alpha_2+\alpha_3)/2\} \cup \left\{ \sum_{i\in J} \alpha_i, J \subset \{1,2,3\} \right\},$$

where we assume that $\alpha_3 < \alpha_1 + \alpha_2$, otherwise the set $\Sigma_{D_4,\chi}$ does not contain the point $(\alpha_1+\alpha_2+\alpha_3)/2$.

Let α_i, $i = 1, \dots, 4$, denote the i-th component of the character χ and $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. The set $\Sigma_{\tilde{D}_4,\chi}$ satisfies the following properties (see [11]):

(1) $\Sigma_{\tilde{D}_4,\chi} \subset [0, \alpha]$;
(2) $\Sigma_{\tilde{D}_4,\chi} \ni \sum_{i\in J} \alpha_i$, $J \subset \{0, 1, 2, 3, 4\}$;
(3) $\tau \in \Sigma_{\tilde{D}_4,\chi} \Longleftrightarrow \alpha - \tau \in \Sigma_{\tilde{D}_4,\chi}$.

The third property means that $\Sigma_{\tilde{D}_4,\chi}$ is symmetric with respect to $\frac{\alpha}{2}$, and therefore we will study the set $\Sigma_{\tilde{D}_4,\chi} \cap [0, \frac{\alpha}{2})$.

Notice that in the case where at least one of the components of the character $\chi_i \geqslant \frac{\alpha}{2}$, the corresponding projection in the representation equals 0 or I, hence the structure of set $\Sigma_{\tilde{D}_4,\chi}$ is the same as the structure of $\Sigma_{D_4,\chi}$. Therefore, it is interesting to study the case where all components of the character χ are less than $\frac{\alpha}{2}$. In this case, the set $\Sigma_{\tilde{D}_4,\chi}$ is quite different from $\Sigma_{D_4,\chi}$, furthermore, it is infinite. The following propositions give a complete description of the set $\Sigma_{\tilde{D}_4,\chi}$ (the exact technique and proofs can be found in [11]).

Lemma 1. *The set $\Sigma_{\tilde{D}_4,\chi}$ contains an infinite series Σ_∞ with the limit point $\frac{\alpha}{2}$ and the finite series Σ_0. These two series are described by the following:*

(1) *if $\alpha_2 + \alpha_3 > \alpha_1 + \alpha_4$, then*
$$\Sigma_\infty = \left\{ \frac{\alpha}{2} - \frac{\alpha_1}{2n} \mid n \in \mathbb{N} \right\},$$
$$\Sigma_0 = \left\{ \frac{\alpha}{2} - \frac{\alpha - 2\alpha_4}{2(2n-1)} \mid n < \frac{\alpha_1}{\alpha_2 + \alpha_3 - \alpha_1 - \alpha_4},\ n \in \mathbb{N} \right\};$$
(2) *if $\alpha_2 + \alpha_3 < \alpha_1 + \alpha_4$, then*
$$\Sigma_\infty = \left\{ \frac{\alpha}{2} - \frac{\alpha - 2\alpha_4}{2(2n-1)} \mid n \in \mathbb{N} \right\},$$
$$\Sigma_0 = \left\{ \frac{\alpha}{2} - \frac{\alpha_1}{2n} \mid n < \frac{\alpha_1}{\alpha_1 + \alpha_4 - \alpha_2 - \alpha_3},\ n \in \mathbb{N} \right\};$$
(3) *if $\alpha_2 + \alpha_3 = \alpha_1 + \alpha_4$, then*
$$\Sigma_\infty = \left\{ \frac{\alpha}{2} - \frac{\alpha_1}{n} \mid n \in \mathbb{N} \right\}, \quad \Sigma_0 = \varnothing.$$

Theorem 1.
$$\Sigma_{\tilde{D}_4,\chi} \cap [0; \alpha/2) = \Sigma_\infty \cup \Sigma_0 \cup \Sigma_1 \cup \Sigma_2^i \cup \Sigma_3 \cup \Sigma_4 \cup \Sigma_5^j, \quad i = 2, 3, 4, \quad j = 1, 2, 3,$$
where
$$\Sigma_1 = \left\{ \frac{\alpha}{2} - \frac{\alpha}{2(4n-1)} \mid n < \frac{\alpha_4}{4\alpha_4 - \alpha},\ n < \frac{\alpha - \alpha_1}{\alpha - 4\alpha_1},\ n \in \mathbb{N} \right\},$$
$$\Sigma_2^i = \left\{ \frac{\alpha}{2} - \frac{\alpha_i}{2n} \mid n < \frac{\alpha_i}{2\alpha_i + 2\alpha_4 - \alpha},\ n < \frac{\alpha_i}{\alpha_i - \alpha_1},\ n < \frac{\alpha_i}{4\alpha_i - \alpha},\ n \in \mathbb{N} \right\},\ i = 2, 3, 4,$$
$$\Sigma_3 = \left\{ \frac{\alpha}{2} - \frac{\alpha - 2\alpha_1}{2(2n+1)} \mid n < \frac{\alpha - \alpha_1}{\alpha - 4\alpha_1},\ n < \frac{\alpha_2 + \alpha_3}{2(\alpha_4 - \alpha_1)},\ n(4\alpha_i - \alpha) < \alpha_i,\ n \in \mathbb{N} \right\},$$
$$\Sigma_4 = \left\{ \frac{\alpha}{2} - \frac{\alpha}{2(4n+1)} \mid n < \frac{\alpha - \alpha_4}{4\alpha_4 - \alpha},\ n < \frac{\alpha_1}{\alpha - 4\alpha_1},\ n \in \mathbb{N} \right\},$$
$$\Sigma_5^i = \left\{ \frac{\alpha}{2} - \frac{\alpha - 2\alpha_i}{2(2n+1)} \mid n < \frac{\alpha_1}{\alpha - 2\alpha_i - 2\alpha_1}, n < \frac{\alpha_i}{\alpha - 4\alpha_i}, n < \frac{\alpha - \alpha_4 - \alpha_i}{2(\alpha_4 - \alpha_i)}, n \in \mathbb{N} \cup \{0\} \right\},$$
$$i = 1, 2, 3.$$

Corollary 1. *By using the structure of $\Sigma_{\tilde{D}_4,\chi}$ given by the latter theorem, one can show that the set $\Sigma_{\tilde{D}_4,\chi}$ contains the only limit point $\omega_{\tilde{D}_4}(\chi)$.*

2. Coxeter functors and evolution of characters

A powerful tool for investigating the algebras $\mathcal{P}_{\Gamma,\chi,\gamma}$ are the reflection (Coxeter) functors. Namely, there exist two functors [3] linear S and hyperbolic T which establish an equivalence between the categories of $*$-representations of the algebras $\mathcal{P}_{\Gamma,\chi,\gamma}$. These actions between the categories give rise to an action on the pairs $(\chi;\gamma)$ as follows:

$$S : (\chi;\gamma) \longmapsto (\chi';\gamma'),$$
$$\chi' = (\alpha^{(1)}_{m_1} - \alpha^{(1)}_{m_1-1}, \ldots, \alpha^{(1)}_{m_1}; \ldots; \alpha^{(n)}_{m_n} - \alpha^{(n)}_{m_n-1}, \ldots, \alpha^{(n)}_{m_n}),$$
$$\gamma' = \alpha^{(1)}_{m_1} + \cdots + \alpha^{(n)}_{m_n} - \gamma;$$

$$T : (\chi;\gamma) \longmapsto (\chi'';\gamma),$$
$$\chi'' = (\gamma - \alpha^{(1)}_{m_1}, \ldots, \gamma - \alpha^{(1)}_{1}; \ldots; \gamma - \alpha^{(n)}_{m_n}, \ldots, \gamma - \alpha^{(n)}_{1}).$$

The main idea is to take a pair $(\chi;\gamma)$ such that the structure of representations of $\mathcal{P}_{\Gamma,\chi,\gamma}$ is known and to apply the functors S and T to construct the whole series of algebras which have the same structure of representations as the algebra we started with.

To make a use of this technique, we first study the evolution of the pair $(\chi;\gamma)$ under the action of the Coxeter functors. In the general case, for an arbitrary graph Γ the formulas of the evolution could be complicated, but for the case where Γ is an extended Dynkin graph the results of [9] give an explicit formula for powers of the functor $(ST)^{k_\Gamma}$.

Let $\omega(\chi)$ be a positive functional on the set of characters. We say that $\omega(\chi)$ is an invariant functional if the following conditions holds:

$$S : (\chi;\omega(\chi)) \longmapsto (\chi';\omega(\chi')),$$
$$T : (\chi;\omega(\chi)) \longmapsto (\chi'';\omega(\chi'')).$$

The results [9] prove that if Γ is an extended Dynkin graph, then there is only one invariant functional,

$$\omega_{\tilde{D}_4}(\chi) = \frac{1}{2}(\alpha^{(1)}_1 + \alpha^{(2)}_1 + \alpha^{(3)}_1 + \alpha^{(4)}_1),$$
$$\omega_{\tilde{E}_6}(\chi) = \frac{1}{3}(\alpha^{(1)}_1 + \alpha^{(1)}_2 + \alpha^{(2)}_1 + \alpha^{(2)}_2 + \alpha^{(3)}_1 + \alpha^{(3)}_2),$$
$$\omega_{\tilde{E}_7}(\chi) = \frac{1}{4}(\alpha^{(1)}_1 + \alpha^{(1)}_2 + \alpha^{(1)}_3 + \alpha^{(2)}_1 + \alpha^{(2)}_2 + \alpha^{(2)}_3 + 2\alpha^{(3)}_1),$$
$$\omega_{\tilde{E}_8}(\chi) = \frac{1}{6}(\alpha^{(1)}_1 + \alpha^{(1)}_2 + \alpha^{(1)}_3 + \alpha^{(1)}_4 + \alpha^{(1)}_5 + 2\alpha^{(2)}_1 + 2\alpha^{(2)}_2 + 3\alpha^{(3)}_1).$$

Recall that in the case where $\gamma = \omega_\Gamma(\chi)$, the algebras $\mathcal{P}_{\Gamma,\chi,\omega_\Gamma(\chi)}$ are PI-algebra (see [5]) and their irreducible representations are of dimensions not greater than 2, 3, 4, and 6 for the graphs $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$, respectively.

Put $p_\Gamma = 2, 3, 4, 6$ for $\Gamma = \tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$, respectively. The evolution of the pair $(\chi;\gamma)$ under the action of powers of $(ST)^{p_\Gamma(p_\Gamma-1)}$ functor could be written as follows.

Theorem 2. (see [9]). *Let Γ be an extended Dynkin graph. Then the following formula holds:*

$$(ST)^{p_\Gamma(p_\Gamma-1)k}(\chi;\gamma) = (\chi - kp_\Gamma(\omega_\Gamma(\chi) - \gamma)\chi_\Gamma; \gamma - kp^2_\Gamma(\omega_\Gamma(\chi) - \gamma)), \tag{1}$$

where χ_Γ is a special character on Γ (see [10]).

Applying the functor (ST) to (1) we can obtain the evolution of the pair $(\chi;\gamma)$ under the action of an arbitrary power $k \in \mathbb{N}$ of $(ST)^k$ (in what follows we denote by $(\chi(k);\gamma(k))$ the image of the pair $(\chi;\gamma)$ under the action of the functor $(ST)^k$).

Proposition 1. *The action of the functor $(ST)^k$ on the pair $(\chi;\gamma)$ could be written in the following way:*

$$(ST)^k(\chi;\gamma) = (f_{1,k}(\chi) - (\omega_\Gamma(\chi)-\gamma)f_{2,k}(\chi_\Gamma); \psi_{1,k} - (\omega_\Gamma(\chi)-\gamma)\psi_{2,k}), \tag{2}$$

where the characters $f_{1,k}(\chi)$ and $f_{2,k}(\chi_\Gamma)$, and the numbers $\psi_{1,k}$ and $\psi_{2,k}$ satisfy the following properties:

(i) *if $k_1 \equiv k_2 \pmod{p_\Gamma(p_\Gamma - 1)}$, then $f_{1,k_1}(\chi) = f_{1,k_2}(\chi)$ and $\psi_{1,k_1} = \psi_{1,k_2}$;*

(ii) *the components of $f_{2,k}(\chi_\Gamma)$ and the numbers $\psi_{2,k}$ are defined in the following way:*

$$f_{2,k}(\chi_\Gamma)_i^{(j)} = \left[\frac{(\chi_\Gamma)_i^{(j)}}{p_\Gamma - 1}k\right], \quad \psi_{2,k} = \left[\frac{p_\Gamma}{p_\Gamma - 1}k\right];$$

(iii) *$f_{1,p_\Gamma(p_\Gamma-1)k} = \chi$, $f_{2,p_\Gamma(p_\Gamma-1)k} = kp_\Gamma\chi_\Gamma$, $\psi_{1,k} = \gamma$, and $\psi_{2,k} = kp_\Gamma^2$.*

Proof. Make a direct calculation using formula (1). □

Remark 1. Property (i) means that the orbits of $f_{1,0}(\chi)$ and $\psi_{1,0}$ under the action of the functor (ST) is finite and its length equals $p_\Gamma(p_\Gamma - 1)$. The formulas for the characters $f_{1,k}$ and the numbers $\psi_{1,k}$ are complicated (unlike for the characters $f_{2,k}(\chi_\Gamma)$ and the numbers $\psi_{2,k}$) and we do not give a list of their evolutions (this list contains 30 items in the case where $\Gamma = \tilde{E}_8$). Nevertheless, using (1) one can compute their values.

3. Structure properties of the set $\Sigma_{\Gamma,\chi}$

Definition 1. *Let $\pi : \mathcal{P}_{\Gamma,\chi,\gamma} \to L(\mathcal{H})$ be a finite dimensional ∗-representation on some Hilbert space $\mathcal{H}$. We call a vector d a generalized dimension of π, which is defined as follows:*

$$d_0 = \dim(\mathcal{H}), \quad d_i^{(j)} = \dim(\operatorname{Im}(\pi(p_i^{(j)}))), \quad j = 1,\dots,n, \quad i = 1,\dots,m_j.$$

One can extend the action of the functors S and T to the set of generalized dimensions (see for example [3]).

Lemma 2. *The sets $\Sigma_{\Gamma,\chi}$ satisfy the following properties:*

(i) $\Sigma_{\Gamma,\chi} \subset \left[0; \sum_{j=1}^{n} \chi_{m_j}^{(j)}\right]$,

(ii) $\Sigma_{\Gamma,\chi} \ni \sum_{j\in J} \chi_{m_j}^{(j)}, \quad J \subset \{1,\dots,n\}$.

Proof. Let us, for example, prove property (*ii*). It is clear that in this case, $P_{m_j}^{(j)} = I$, if $j \in J$, and $P_i^{(j)} = 0$, if $j \in J,\ i \neq m_j$ or $j \notin J,\ i \in 1,\dots,m_j$, form a representation of $\mathcal{P}_{\Gamma,\chi,\gamma}$. □

If knowing the set $\Sigma_{\Gamma,\chi} \cap [0, \omega_\Gamma(\chi)]$, one can restore the whole set $\Sigma_{\Gamma,\chi}$, since the functor S establishes a bijective correspondence between the set $\Sigma_{\Gamma,\chi} \setminus [0, \omega_\Gamma(\chi)]$ and the set $\Sigma_{\Gamma,\chi'} \cap [0, \omega_\Gamma(\chi'))$.

According to Lemma 1, the set $\Sigma_{\tilde{D}_4,\chi}$ is infinite if and only if all components of the character χ satisfy the condition $\chi_i < \frac{\alpha}{2}$ (in other words, this means that $\chi_i < \omega_{\tilde{D}_4}(\chi)$). Let us study a similar question for all extended Dynkin graphs.

Let χ_i be the i-th component of the character χ and χ_i' the corresponding component of the character χ' obtained by applying the functor S to the pair (χ, γ).

Theorem 3. *Let Γ be an extended Dynkin graph. The set $\Sigma_{\Gamma,\chi}$ is infinite if and only if all components of the character satisfy the two conditions $\chi_i < \omega_\Gamma(\chi)$ and $\chi_i' < \omega_\Gamma(\chi')$.*

Proof. We prove this theorem in several steps. At first we show that the conditions $\chi_i < \omega_\Gamma(\chi)$ and $\chi'_i < \omega_\Gamma(\chi')$ are both necessary for the set $\Sigma_{\Gamma,\chi}$ to be infinite.

Lemma 3. *If at least one component of the character χ or the character χ' satisfies the condition $\chi_i \geq \omega_\Gamma(\chi)$ or $\chi'_i \geq \omega_\Gamma(\chi')$, then the set $\Sigma_{\Gamma,\chi}$ is finite.*

Proof. It is not hard to check that the projection corresponding to a component of χ that satisfies the condition is equal to 0 or to I in a representation. Hence, the set $\Sigma_{\Gamma,\chi}$ has the same structure as the set $\Sigma_{\tilde{\Gamma},\chi}$, where $\tilde{\Gamma}$ is a proper subgraph of Γ. Since the set $\Sigma_{\Gamma,\chi}$ is always finite if Γ is a Dynkin graph (see [7]), the set $\Sigma_{\Gamma,\chi}$ is also finite. □

Remark 2. Conditions $\chi_i < \omega_\Gamma(\chi)$ and $\chi'_i < \omega_\Gamma(\chi')$ are equivalent if $\Gamma = \tilde{D}_4$, but are not, generally speaking, if $\Gamma \neq \tilde{D}_4$. For example, consider the character $\chi = (5,6;7,8;8,9)$ on the graph $\Gamma = \tilde{E}_6$. All components of χ satisfy $\chi_i < \omega_{\tilde{E}_6}(\chi)$ but, for the corresponding character $\chi' = (1,6;1,8;1,9)$, they do not.

To prove that the conditions $\chi_i < \omega_\Gamma(\chi)$ and $\chi'_i < \omega_\Gamma(\chi')$ are sufficient, we will consider a special procedure (e.g., for $\Gamma = \tilde{E}_6$) which allows to build infinite series in $\Sigma_{\Gamma,\chi}$ with the limit point $\omega_\Gamma(\chi)$.

Let $\Gamma = \tilde{E}_6$ and the latter inequalities hold. Consider the sets

$$A_{\chi_1^{(j)}} = \{(f_{1,k}(\chi))_1^{(j)} | k = 1, \dots, p_{\tilde{E}_6}(p_{\tilde{E}_6} - 1) = 6\},$$

which are the sets of orbits of the components $f_{1,0}(\chi)_1^{(j)}$ under the action of the functor (ST), and consider the set

$$A = \bigcup_{j=1}^{3} A_{\chi_1^{(j)}}.$$

Put $a = \min A$ and let l and m be such that $(f_{1,m}(\chi))_l = a$. Consider the sequence $\gamma_n = \omega_{\tilde{E}_6}(\chi) - \frac{a}{\varphi_n}$, where $\varphi_n = (f_{2,p_{\tilde{E}_6}(p_{\tilde{E}_6}-1)n+m}(\chi_{\tilde{E}_6}))_l) = \left[\frac{6n+m}{2}\right]$, and $n \in \mathbb{N}$.

Lemma 4. *The algebra $\mathcal{P}_{\tilde{E}_6,\chi,\gamma_n}$ has a representation for every natural n.*

Proof. Fix $n \in \mathbb{N}$ and apply the functor $(ST)^{6n+m-2}$ to the pair $(\chi; \omega_{\tilde{E}_6}(\chi) - \frac{a}{\varphi_n})$. To check that this action is correct, we have to show that, at each step $k \leqslant 6n+m-2$, we will get the pair $(\chi(k);\gamma(k))$, where $\gamma(k)$ and all components of the $\chi(k)$ are positive. Indeed, consider, for example, the component $\chi(k)_i^{(j)}$ at the step $k \leqslant 6n+m-2$. Using formula (2) we get

$$\chi(k)_i^{(j)} = (f_{1,k \bmod 6}(\chi))_i^{(j)} - (f_{2,k}(\chi_{\tilde{E}_6}))_i^{(j)} \frac{a}{\varphi_n} \geqslant a\left(1 - \left[\frac{k}{2}\right]\left[\frac{6n+m}{2}\right]^{-1}\right) \geqslant 0.$$

To complete the proof, it remains to note that $\chi(6n+m)_l = 0$, hence the corresponding component $\chi(6n+m-2)_l = \gamma(6n+m-2)$. According to Lemma 2, the algebra $\mathcal{P}_{\tilde{E}_6,\chi(6n+m-2),\gamma(6n+m-2)}$ has a representation. □

Notice that the same procedure was used to build infinite series in the case where $\Gamma = \tilde{D}_4$ (see Lemma 1). This procedure could be slightly modified for the cases of $\Gamma = \tilde{E}_7$ and $\Gamma = \tilde{E}_8$, hence the theorem holds. □

An interesting question is when there exists a $*$-representation of the algebras $\mathcal{P}_{\Gamma,\chi,\gamma}$ in the case where $\gamma = \omega_\Gamma(\chi)$. If $\Gamma = \tilde{D}_4$, this question was studied in [6] and the answer is as follows: a representation exists if all components of the character χ satisfy the condition $\chi_i < \omega_{\tilde{D}_4}(\chi)$. A similar answer appears to be correct for all extended Dynkin graphs.

Corollary 2. *(A representation in the case where $\gamma = \omega_\Gamma(\chi)$) Let Γ be an extended Dynkin graph, and χ be a character on Γ such that the conditions of Th. 3 are satisfied. Then there is a representation of the algebra $\mathcal{P}_{\Gamma,\chi,\omega_\Gamma(\chi)}$*

Proof. Since the conditions of Theorem 3 are satisfied, there exists a series in $\Sigma_{\Gamma,\chi}$ with the limit point $\omega_\Gamma(\chi)$. By Shulman's theorem (see [8]) the sets $\Sigma_{\Gamma,\chi}$ are closed, therefore, the set $\Sigma_{\Gamma,\chi}$ contains the point $\omega_\Gamma(\chi)$. □

Remark 3. Using the previous corollary one can determine if there is a ∗-representation of $\mathcal{P}_{\Gamma,\chi,\omega_\Gamma(\chi)}$ for a fixed character χ. Indeed, if the conditions of Theorem 3 are satisfied, then there exists a representation, if not we construct a proper graph by deleting vertices where the components of χ do not satisfy the desired conditions. Since we know an exact answer for all proper subgraphs of extended Dynkin graphs, we can determine whether there is a representation.

Theorem 4. *Let Γ be an extended Dynkin graph. If the set $\Sigma_{\Gamma,\chi}$ is infinite, then it contains the only limit point.*

Proof. Let us fix $\gamma < \omega_\Gamma(\chi)$ and apply the functor $(ST)^k$ to the pair $(\chi;\gamma)$. Using formulas (2) we see that there exists $k < \infty$ such that one of three following situations can occur (and all components of the character $\chi(\tilde{k})$ and $\gamma(\tilde{k})$, $\tilde{k} < k$ are positive):

(a) $\gamma(k) = 0$. In this case there exists a ∗-representation of the algebra $\mathcal{P}_{\Gamma,\chi(k),\gamma(k)}$ with the generalized dimension $d(k)$ defined by
$$d(k)_0 = 0, \quad d(k)_i^{(j)} = 0.$$

(b) $\chi(k)_1^{(j)} = \gamma(k)$, for $j \in J \subset \{1,\dots,n\}$. In this case, there exists a ∗-representation of the algebra $\mathcal{P}_{\Gamma,\chi(k),\gamma(k)}$ with the generalized dimension $d(k)$ defined by
$$\begin{aligned} d(k)_0 = 1,\ d(k)_1^{(j)} &= 1, \quad \text{if } \ j = m, \\ d(k)_i^{(j)} &= 0, \quad \text{if } \ j \in J \ \text{ and } \ j \neq m, \\ d(k)_i^{(j)} &= 0, \quad \text{if } \ i = 1,\ j \notin J \ \text{ or } \ i \neq 1, \end{aligned}$$
where $m = \min J$.

(c) $\chi(k)_i^{(j)} > \gamma(k)$ for $j \in J' \subset \{1,\dots,n\}$ and $i \in J'' \subset \{1,\dots,m_j\}$. In this case, we construct a new graph $\tilde{\Gamma}$ by deleting vertices from G that correspond to the sets J' and J'' (the branch from J' and vertex form J''). Since we know the exact structure of the set $\Sigma_{\tilde{\Gamma},\tilde{\chi}}$ (see [7]), where $\tilde{\chi}$ is the restriction of $\chi(k)$ to $\tilde{\Gamma}$, we can determine whether $\gamma(k)$ lies in $\Sigma_{\tilde{\Gamma},\tilde{\chi}}$, and if the latter is true, to build a corresponding vector of the generalized dimension $d(k)$.

Applying the above procedure rule to each pair $(\chi;\gamma) \in \Sigma_\Gamma,\ \gamma < \omega_\Gamma(\chi)$ we get a pair $(k; d(k))$, which means that we decompose the set
$$\Sigma_{\Gamma,\chi} \cap [0, \omega_\Gamma(\chi)] = \bigcup_{k,d} \Sigma_{k,d},$$
where the index d ranges over all possible generalized dimensions and $k = 0,\dots,p_\Gamma(p_\Gamma - 1) - 1$ (indeed according to (2) the functor (ST) acts cyclically, hence, in order to describe the sets $\Sigma_{\Gamma,\chi}$, we can take into consideration all possible vectors of the generalized dimensions along one cycle).

Lemma 5. *The index d ranges over a finite number of possible generalized dimensions.*

Proof. Each d_i corresponds to the case (a), (b) or (c). It is clear that in the cases (a), (b), the number of different generalized dimension is finite. In the case where $\tilde{\Gamma}$ is a proper subgraph of the extended Dynkin graph (see [7]), the number of different generalized dimensions is also finite, hence this proves the case where d_i corresponds to (c) and the lemma holds. □

Lemma 6. *The sets* $\Sigma_{k,d}$, $k \in \{0, \dots, p_\Gamma(p_\Gamma - 1) - 1\}$ *are either finite or have the only limit point* $\omega_\Gamma(\chi)$.

Proof. Direct calculation using (2). □

Now the theorem is an obvious corollary of Lemma 2 and Lemma 3. □

Remark 4. The previous theorem gives an exact algorithm for describing the set $\Sigma_{\Gamma,\chi}$, but it turns out to be combinatorially hard to get explicit formulas for the extended Dynkin graph $\Gamma \neq \tilde{D}_4$ (the case $\Gamma = \tilde{D}_4$ is simpler, and a complete description was done in [11] and is given in Section 1).

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