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HIGHER POWERS OF q-DEFORMED WHITE NOISE

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Dedicated to Professor Yuri M. Berezansky on the occasion of his 80-th birthday.

Abstract. We introduce the renormalized powers of q-deformed white noise, for any q in the open interval $(-1, 1)$, and we extend to them the no–go theorem recently proved by Accardi–Boukas–Franz in the Boson case. The surprising fact is that the lower bound (6.5), which defines the obstruction to the positivity of the sesquilinear form, uniquely determined by the renormalized commutation relations, is independent of q in the half-open interval $(-1, 1]$, thus including the Boson case. The exceptional value $q = -1$, corresponding to the Fermion case, is dealt with in the last section of the paper where we prove that the argument used to prove the no–go theorem for $q \neq 0$ does not extend to this case.

1. Introduction

The construction of the Fock representation for the renormalized square of white noise (RSWN), obtained in [2], led to an unexpected connection with the Meixner classes in [3]. These classes also appeared, in a different context, in a series of papers by Berezansky and his school (see [9], [8], [14], [15] and the references therein). After these results the problem to extend the above analysis to higher powers of white noise naturally arised. Up to now all the attempts in this direction have run into the obstruction of the no–go theorems. Let us briefly outline the essence of the problem.

Once proved that both the first and second order Boson white noise admit a Fock representation, it was natural to ask whether there exists a Fock representation for the combination of the two. This question was raised by P. Sniady in [17] who observed that it has a negative answer. This is due to the fact that, if such a representation exists, then the scalar product of the corresponding "Fock space" is uniquely determined by the commutation relations. But it turns out that the renormalization procedure perturbs the algebra in such a way that this "scalar product" cannot be positive semi-definite.

Now the first order white noise over $\mathbb{R}$ can be described as the current algebra of the 1-mode CCR algebra

$$[a, a^+] = 1$$

over the algebra

$$\mathcal{C} := \{\text{step functions on } \mathbb{R}^d\} \subseteq L^\infty(\mathbb{R}^d, dx)$$

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(cf. Section 2 for this notion). Similarly the second order white noise can be described as the current algebra of the sl$(2, \mathbb{R})$-Lie algebra

$$[a^2, a^{+2}] = c + 4a^+a,$$
$$[a^2, a^+a] = 2a^+a$$

based on the same algebra $\mathcal{C} \subseteq L^\infty(\mathbb{R}^d, dx)$ as above.

The combined 1-st and 2-d order Boson white noise can be defined as the current algebra over the Lie algebra, with generators:

$$a, a^+, 1, a^+a^+, a^2, a^{+2}$$

also called the Schrödinger algebra [12], based on the same algebra $\mathcal{C}$ as above. Clearly the 1-mode CCR gives a Fock representation for this algebra.

Therefore Sniady's theorem can be rephrased as follows: the Schrödinger algebra has a Fock representation but its current algebra over $\mathcal{C}$ has no Fock representation.

The problem of lifting a representation of a $*$-Lie algebra to an associated current algebra, is deeply related to an old open problem of quantum field theory: giving a meaning to the "higher powers of free fields". Since a Boson free field is nothing but a Boson white noise, this problem is equivalent to constructing a mathematical object which has the right to be called "a higher (≥ 2) power of white noise".

In the above notations, a naive way to state the problem of constructing higher powers of white noise would be: consider the 1-dimensional Schrödinger representation of the CCR:

$$[a, a^+] = 1$$

acting on $L^2(\mathbb{R})$. The linear span of the non-commutative monomials in a, a^+ is an infinite dimensional Lie algebra, called the full oscillator algebra, which by construction admits a Fock representation.

Does there exist a Fock representation of the current algebra of the oscillator algebra over on $\mathbb{R}^d$ with the Lebesgue measure?

Since the full oscillator algebra contains the Schrödinger algebra, we know that the answer to this question is negative.

However the analogy with the second order case suggests the possibility that a more restricted program might be realized, namely that representations of some special (but nontrivial) $*$-Lie sub-algebras of the oscillator algebra might exist.

The most natural thing to do was to start with the cubic powers a^3, a^{+3} and consider the $*$-Lie algebra generated by them; then take the smallest power not contained in this algebra and iterate the procedure.

Accardi, Boukas and Franz proved in [1] that even the first step of this restricted program, i.e. the case of a^3, a^{+3}, faces a no–go theorem. Recently Accardi and Boukas proved in [4] that the same negative result holds for a^n, a^{+n} for any $n \geq 3$.

More precisely they proved that, if a $*$-Lie sub-algebra of the oscillator algebra contains a^n for some $n \geq 3$, then the associated current algebra based on a algebra $\mathcal{C}$ of step functions over a measure space $(S, \mathcal{B}, \mu)$ cannot have a Fock representation unless there exits a fundamental volume $\kappa > 0$ with the property that, if $I \subseteq S$ is a measurable set satisfying $\chi_I \in \mathcal{C}$, then $\mu(I) > \kappa$. Moreover this fundamental volume is the inverse of the renormalization constant, introduced in [2], thus in particular, it is independent of n.

A natural next question is if these no–go theorems are specific to the Boson case, i.e. if they still hold under more general commutation relations.

In the present paper we introduce the algebra of renormalized powers of q-deformed white noise and we extend the above mentioned no–go theorem to the q-oscillator algebra, with $q \in (-1, 1)$.

A surprising result in this extension is that the fundamental volume κ is the same as in the Boson case. In particular it does not depend on q.

2. Current representations of Lie algebras

Current representations of Lie algebras were first introduced by Araki in [7] and [6] and then studied by several authors. The monographs of Parthasarathy and Schmidt [16] and of Guichardet [13] are good and complementary introductions to the subject. Here we recall the terminology introduced in [5].

Definition 1. *Let $\mathcal{G}$ be a complex $*$-Lie algebra. A canonical set of generators of $\mathcal{G}$ is a linear basis of $\mathcal{G}$*

$$l_\alpha^+, l_\alpha^-, l_\beta^0, \quad \alpha \in I, \quad \beta \in I_0$$

where I_0, I are sets, satisfying the following conditions:

$$(l_\beta^0)^* = l_\beta^0, \quad \forall\, \beta \in I_0,$$
$$(l_\alpha^+)^* = l_\alpha^-, \quad \forall\, \alpha \in I$$

and all the central elements among the generators are of l^0-type (i.e. self-adjoint).

We will denote $c_{\alpha\beta}^\gamma(\varepsilon, \varepsilon', \delta)$ the structure constants of $\mathcal{G}$ with respect to the generators (l_α^ε):

$$[l_\alpha^\varepsilon, l_\beta^{\varepsilon'}] = c_{\alpha\beta}^\gamma(\varepsilon, \varepsilon', \delta) l_\gamma^\delta := \sum_{\gamma \in I_0} c_{\alpha\beta}^\gamma(\varepsilon, \varepsilon', 0) l_\gamma^0 + \sum_{\gamma \in I} c_{\alpha\beta}^\gamma(\varepsilon, \varepsilon', +) l_\gamma^+ + \sum_{\gamma \in I} c_{\alpha\beta}^\gamma(\varepsilon, \varepsilon', -) e_\gamma^- .$$

In the following we will consider only *locally finite* Lie algebras, i.e. those such that, for any pair $\alpha, \beta \in I \cup I_0$ only a finite number of the structure constants $c_{\alpha\beta}^\gamma(\varepsilon, \varepsilon', \delta)$ is different from zero.

Definition 2. *Let there be given*
– a $*$-Lie algebra $\mathcal{G}$,
– a measurable space $(S, \mathcal{B})$,
– a $*$-sub-algebra $\mathcal{C} \subseteq L_{\mathbb{C}}^\infty(S, \mathcal{B})$ for the pointwise operations.
The current algebra of $\mathcal{G}$ over $\mathcal{C}$ is the $*$-Lie algebra

$$\mathcal{G}(\mathcal{C}) := \{\mathcal{C} \otimes \mathcal{G}, [\,\cdot\,,\,\cdot\,]\}$$

where $\mathcal{C} \otimes \mathcal{G}$ is the algebraic tensor product, the Lie brackets $[\,\cdot\,,\,\cdot\,]$ are given by

$$[f \otimes l, g \otimes l'] := fg \otimes [l, l'], \quad f, g \in \mathcal{C}, \quad l, l' \in \mathcal{G}$$

and the involution $*$ is given by

$$(f \otimes l)^* := \overline{f} \otimes l^*, \quad f \in \mathcal{C}, \quad l \in \mathcal{G}$$

where $\overline{f}$ denotes complex conjugate.

Remark. If (l_γ^ε) is a canonical set of generators of $\mathcal{G}$ then the set

$$\{f \otimes l_\alpha^+\ ,\ f^* \otimes l_\alpha^-\ ,\ \mathrm{Re}(f) \otimes l_\gamma^0\ :\ \alpha \in I\ ,\ \gamma \in I_0\ ,\ f \in \mathcal{C}\}$$

$(\mathrm{Re}(f) := (f + \overline{f})/2)$ is a canonical set of generators of $\mathcal{G}(\mathcal{C})$.

In the following we will use the notations:

$$l_\alpha^+(f) := f \otimes l_\alpha^+\ , \quad l_\alpha^-(f) := f^* \otimes l_\alpha^-\ , \quad l_\alpha^0(f) := f \otimes l_\gamma^0$$

and, when no confusion can arise, we will often speak of *the current algebra* $(l_\alpha^\varepsilon(f))$.

Definition 3. *A representation of a $*$-Lie algebra $\mathcal{G}$ is a triple*

$$\{\mathcal{H}, \mathcal{D}, \pi\}$$

where

– *$\mathcal{H}$ is an Hilbert space ,*
– *$\mathcal{D}$ is a total subset of $\mathcal{H}$,*
– *$\pi : \mathcal{D} \to \mathcal{H}$ is a map such that:*
(i) *for any $l \in \mathcal{G}$, $\pi(l)$ is a pre-closed operator on $\mathcal{D}$ with adjoint $\pi(l^*)$,*
(ii) *for any $l, l' \in \mathcal{G}$*

$$\pi([l, l']) = [\pi(l), \pi(l')]$$

where the commutator on the right hand side is meant weakly on $\mathcal{D}$.

Remark. At the algebraic level the existence of representations of current algebras over $\{\mathcal{G}, (l^\varepsilon_\alpha)\}$ is easily established for an arbitrary measure space $(S, \mathcal{B}, \mu)$ and sub-$*$-algebra

$$\mathcal{C} \subseteq L^\infty(S, \mathcal{B}, \mu).$$

In fact, if $\{\pi, \mathcal{K}\}$ is any representation of $\mathcal{G}$ one can define a structure of $*$-Lie algebra on

$$\mathcal{C} \otimes \pi(\mathcal{G}) \in \mathcal{L}(L^2(S, \mathcal{B}, \mu) \otimes \mathcal{K})$$

in terms of the brackets

$$[f \otimes \pi(e), g \otimes \pi(l')] := fg \otimes \pi([l, l']).$$

Therefore, defining

$$l^+_\alpha(f) := f \otimes \pi(l^+_\alpha),$$
$$l^\circ_\alpha(f) := f \otimes \pi(l^\circ_\alpha)$$

one has

$$\begin{aligned} l^+_\alpha(f)^* &= \overline{f} \otimes \pi(l^+_\alpha)^* = \overline{f} \otimes \pi((l^+_\alpha)^*) = \overline{f} \otimes \pi(l^-_\alpha), \\ (l^0_\beta(f))^* &= \overline{f} \otimes \pi(l^0_\beta)^* = \overline{f} \otimes \pi(l^0_\beta) = l^0_\beta(\overline{f}), \\ [l^\varepsilon_\alpha(f), l^{\varepsilon'}_\beta(g)] &= [f^\varepsilon \otimes \pi(l^\varepsilon_\alpha), g^{\varepsilon'} \otimes \pi(l^{\varepsilon'}_\beta)] = f^\varepsilon g^{\varepsilon'} \otimes [\pi(l^\varepsilon_\alpha), \pi(l^{\varepsilon'}_\beta)] \\ &= f^\varepsilon g^{\varepsilon'} \otimes \pi[l^\varepsilon_\alpha, l^{\varepsilon'}_\beta] = c^\gamma_{\alpha\beta}(\varepsilon, \varepsilon', \delta) f^\varepsilon g^{\varepsilon'} \otimes l^\delta_\gamma. \end{aligned}$$

Thus the current algebra relations are verified. Thus any representation of $\mathcal{G}$ can be lifted to a representation of the current algebra $\mathcal{G} \otimes \mathcal{G}$. We will see the following that, in some cases, there exists no lifting which preserves some special property of the representation of $\mathcal{G}$, e.g. the property of being a Fock representation with respect to a given canonical set of generators.

Definition 4. *Let $\mathcal{G}$ be a $*$-Lie algebra with a canonical set of generators (l^ε_α). A representation $\{\mathcal{K}, \mathcal{D}, \pi\}$ of $\mathcal{G}$ on a Hilbert space $\mathcal{K}$ is called a Fock representation if*

(i) there exists a unit vector $\Phi \in \mathcal{K}$ such that $\forall\, \alpha \in I$ and $\forall\, \beta \in I_0$, with the exception of those $\beta \in I_0$ which correspond to central elements, one has

$$\pi(l^-_\alpha)\Phi = \pi(l^0_\beta)\Phi = 0,$$

(ii) the set

$$\{\pi(l^+_\alpha)^n \Phi \ : \ \alpha \in I \ , \ n \in \mathbb{N}\}$$

is total in $\mathcal{K}$.

Problem. Let $\{\mathcal{G}, (l^\varepsilon_\alpha)\}$ be a $*$-Lie algebra with a canonical set of generators. Suppose that $\{\mathcal{G}, (l^\varepsilon_\alpha)\}$ admits a Fock representation. Under which conditions on the measure space $(S, \mathcal{B}, \mu)$ and on the $*$-sub-algebra

$$\mathcal{C} \subseteq L^\infty(S, \mathcal{B}, \mu)$$

does the current algebra

$$\{l_\alpha^\varepsilon(f) : \varepsilon \in \{+,-,0\}\ ,\ \alpha \in I \quad \text{or} \quad \alpha \in I_0\ ,\ f \in \mathcal{C}\}$$

admit a Fock representation?

3. q-NUMBERS

The following notation will be used throughout this paper (cf. [10]). For $n = 1, 2, \ldots$

$$[n]_q := \frac{q^n - 1}{q - 1}, \qquad [0]_q := 0,$$

$$[n]_q! := \prod_{m=1}^{n} [m]_q, \quad [0]_q! := 1,$$

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \prod_{i=1}^{n-k} \frac{q^{k+i} - 1}{q^i - 1}$$

and we have the q-binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q y^k\, x^{n-k}$$

where $n = 1, 2, \ldots$ and x, y are such that $x\,y = q\,y\,x$. The following known identity will be crucial in establishing the explicit form of the lower bound in the no–go Theorem 1 below.

Lemma 1. *For $q \in (-1, 1)$*

$$\sum_{\lambda=0}^{n} q^{\lambda^2} \binom{n}{\lambda}_q^2 = \binom{2n}{n}_q. \tag{3.1}$$

Proof. Let x, y be such that $x\,y = q\,y\,x$. Then by the q-binomial theorem

$$(x+y)^{2n} = \sum_{k=0}^{n} \binom{2n}{k}_q y^k\, x^{n-k} \tag{3.2}$$

and also

$$\begin{aligned} (x+y)^{2n} = (x+y)^n\,(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k}_q y^k\, x^{n-k} \sum_{l=0}^{n} \binom{n}{l}_q y^l\, x^{n-l} \\ &= \sum_{k,l=0}^{n} \binom{n}{k}_q \binom{n}{l}_q y^k\, x^{n-k}\, y^l\, x^{n-l} \qquad (3.3) \\ &= \sum_{k,l=0}^{n} \binom{n}{k}_q \binom{n}{l}_q y^{k+l}\ q^{(n-k)\,l} x^{2n-k-l}. \end{aligned}$$

Looking at (3.3) for $k + l = n$ and equating coefficients with (3.2) we obtain

$$\binom{2n}{n}_q = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{n-k}_q^2 = \sum_{\lambda=0}^{n} q^{\lambda^2} \binom{n}{\lambda}_q^2$$

which is (3.1). $\square$

4. 1-mode q-commutation relations

In this section we recall some known facts on the q-commutation relations and then we describe the q-analogue of the full oscillator algebra, i.e. the Lie algebra of all polynomials in the 1-mode Boson creation and annihilation operators.

Definition 5. *The q-commutator algebra is the associative $*$-algebra with canonical generators $\{a, a^+, 1\}$ satisfying $(a^+)^* = a$; $1^* = 1$ (identity) and the commutation relations*

$$a\, a^\dagger - q\, a^\dagger\, a = 1. \tag{4.1}$$

Bozeiko, Kummerer and Speicher proved in [10] that this algebra admits a Fock representation if $q \in (-1, 1)$. Bozeiko and Speicher had also proved in [11] that this algebra does not admit a Fock representation if $|q| > 1$.

In the following we sum up some known properties of these operator which will be used throughout the present paper.

Proposition 1. *Then, for $n \geq 1$ and $q \neq \pm 1$ one has*

$$a\,(a^\dagger)^n - q^n\,(a^\dagger)^n\,a = \frac{q^n - 1}{q - 1}\,(a^\dagger)^{n-1} = [n]_q\,(a^\dagger)^{n-1}. \tag{4.2}$$

Proof. The case $n = 1$ is clear. Assuming (4.2) to be true for n we have

$$\begin{aligned}
a\,(a^\dagger)^{n+1} = a\,(a^\dagger)^n\,a^\dagger &= \left(\frac{q^n - 1}{q - 1}\,(a^\dagger)^{n-1} + q^n\,(a^\dagger)^n\,a\right) a^\dagger \\
&= \frac{q^n - 1}{q - 1}\,(a^\dagger)^n + q^n\,(a^\dagger)^n\,a\,a^\dagger \\
&= \frac{q^n - 1}{q - 1}\,(a^\dagger)^n + q^n\,(a^\dagger)^n(1 + q\,a^\dagger\,a) \\
&= \frac{q^n - 1}{q - 1}\,(a^\dagger)^n + q^n\,(a^\dagger)^n + q^{n+1}\,(a^\dagger)^{n+1}\,a \\
&= q^{n+1}\,(a^\dagger)^{n+1}\,a + \frac{q^{n+1} - 1}{q - 1}\,(a^\dagger)^n
\end{aligned}$$

so (4.2) is true for $n + 1$ also. □

Proposition 2. *Let $a, a^\dagger$ and q be as in Proposition 1 and let $n, k \geq 1$. Then*

$$a^n\,(a^\dagger)^k - q^{nk}\,(a^\dagger)^k\,a^n = \sum_{\lambda=1}^{n} \phi_\lambda(n, k; q)\,(a^\dagger)^{k-\lambda}\,a^{n-\lambda} \tag{4.3}$$

where

$$\begin{aligned}
\phi_\lambda(n, k; q) &= \begin{cases} q^{(n-\lambda)(k-\lambda)} \displaystyle\prod_{\rho=0}^{\lambda-1} \frac{q^{k-\rho} - 1}{q - 1} \left(\delta_{n,\lambda} + (1 - \delta_{n,\lambda}) \prod_{\rho'=0}^{n-\lambda-1} \frac{q^{n-\rho'} - 1}{q^{n-\lambda-\rho'} - 1}\right), \\ \hfill \text{if } \lambda \leq n \text{ and } \lambda \leq k \\ 0, \hfill \text{if } \lambda > n \text{ or } \lambda > k \end{cases} \\
&= \begin{cases} q^{(n-\lambda)(k-\lambda)} \dfrac{[k]_q!}{[k-\lambda]_q!} \left(\delta_{n,\lambda} + (1 - \delta_{n,\lambda}) \dbinom{n}{\lambda}_q\right), & \text{if } \lambda \leq n \text{ and } \lambda \leq k \\ 0, & \text{if } \lambda > n \text{ or } \lambda > k \end{cases}
\end{aligned}$$

where $\delta_{n,\lambda}$ is Kronecker's delta and we assume, without loss of generality, that $k \geq n$ (otherwise, just take the adjoint of (4.3)).

Proof. We will use induction on n and treat k as a constant. The case $n = 1$ is just Corollary 1. Assume (4.3) to be true for n. Then

$$\begin{aligned}
a^{n+1}\,(a^\dagger)^k &= a\,a^n\,(a^\dagger)^k \\
&= a\left(q^{nk}\,(a^\dagger)^k\,a^n + \sum_{\lambda=1}^{n}\phi_\lambda(n,k;q)\,(a^\dagger)^{k-\lambda}\,a^{n-\lambda}\right) \\
&= q^{nk}\,a\,(a^\dagger)^k\,a^n + \sum_{\lambda=1}^{n}\phi_\lambda(n,k;q)\,a\,(a^\dagger)^{k-\lambda}\,a^{n-\lambda} \\
&= q^{nk}\left(q^k\,(a^\dagger)^k\,a + \frac{q^k-1}{q-1}\,(a^\dagger)^{k-1}\right)a^n \\
&+ \sum_{\lambda=1}^{n}\phi_\lambda(n,k;q)\left(q^{k-\lambda}\,(a^\dagger)^{k-\lambda}\,a + \frac{q^{k-\lambda}-1}{q-1}\,(a^\dagger)^{k-\lambda-1}\right)a^{n-\lambda} \\
&= q^{(n+1)k}\,(a^\dagger)^k\,a^{n+1} + q^{nk}\,\frac{q^k-1}{q-1}\,(a^\dagger)^{k-1}\,a^n \\
&+ \sum_{\lambda=1}^{n}\phi_\lambda(n,k;q)\,q^{k-\lambda}\,(a^\dagger)^{k-\lambda}\,a^{n-\lambda+1} \\
&+ \sum_{\lambda=1}^{n}\phi_\lambda(n,k;q)\,\frac{q^{k-\lambda}-1}{q-1}\,(a^\dagger)^{k-\lambda-1}\,a^{n-\lambda} \\
&= q^{(n+1)k}\,(a^\dagger)^k\,a^{n+1} + \left(q^{nk}\,\frac{q^k-1}{q-1} + \phi_1(n,k;q)\,q^{k-1}\right)(a^\dagger)^{k-1}\,a^n \\
&+ \sum_{\lambda=2}^{n}\phi_\lambda(n,k;q)\,q^{k-\lambda}\,(a^\dagger)^{k-\lambda}\,a^{n-\lambda+1} \\
&+ \sum_{\lambda=1}^{n}\phi_\lambda(n,k;q)\,\frac{q^{k-\lambda}-1}{q-1}\,(a^\dagger)^{k-\lambda-1}\,a^{n-\lambda} \\
&= q^{(n+1)k}\,(a^\dagger)^k\,a^{n+1} + \left(q^{nk}\,\frac{q^k-1}{q-1} + \phi_1(n,k;q)\,q^{k-1}\right)(a^\dagger)^{k-1}\,a^n \\
&+ \sum_{\lambda=2}^{n}\phi_\lambda(n,k;q)\,q^{k-\lambda}\,(a^\dagger)^{k-\lambda}\,a^{n-\lambda+1} \\
&+ \sum_{\lambda=2}^{n+1}\phi_{\lambda-1}(n,k;q)\,\frac{q^{k-\lambda+1}-1}{q-1}\,(a^\dagger)^{k-\lambda}\,a^{n+1-\lambda} \\
&= q^{(n+1)k}\,(a^\dagger)^k\,a^{n+1} + \left(q^{nk}\,\frac{q^k-1}{q-1} + \phi_1(n,k;q)\,q^{k-1}\right)(a^\dagger)^{k-1}\,a^n \\
&+ \sum_{\lambda=2}^{n}\left(\phi_\lambda(n,k;q)\,q^{k-\lambda} + \phi_{\lambda-1}(n,k;q)\,\frac{q^{k-\lambda+1}-1}{q-1}\right)(a^\dagger)^{k-\lambda}\,a^{n+1-\lambda} \\
&+ \phi_n(n,k;q)\,\frac{q^{k-n}-1}{q-1}\,(a^\dagger)^{k-n-1}.
\end{aligned}$$

It thus suffices to show that

$$\phi_1(n+1,k;q) = q^{nk}\,\frac{q^k-1}{q-1} + \phi_1(n,k;q)\,q^{k-1}, \tag{4.4}$$

$$\phi_\lambda(n+1,k;q) = \phi_\lambda(n,k;q)\,q^{k-\lambda} + \phi_{\lambda-1}(n,k;q)\,\frac{q^{k-\lambda+1}-1}{q-1} \quad \text{for } \lambda = 2,3,\ldots,n, \tag{4.5}$$

$$\phi_{n+1}(n+1,k;q)=\phi_n(n,k;q)\,\frac{q^{k-n}-1}{q-1}. \tag{4.6}$$

To prove (4.4) we have

$$\begin{aligned}
q^{nk}\,\frac{q^k-1}{q-1}+\phi_1(n,k;q)\,q^{k-1} &= q^{nk}\,\frac{q^k-1}{q-1}+q^{(n-1)(k-1)}\,\frac{q^k-1}{q-1}\,\frac{q^n-1}{q-1}\,q^{k-1}\\
&= q^{nk}\,\frac{q^k-1}{q-1}+q^{n(k-1)}\,\frac{q^k-1}{q-1}\,\frac{q^n-1}{q-1}\\
&= q^{n(k-1)}\,\frac{q^k-1}{q-1}\left(q^n+\frac{q^n-1}{q-1}\right)\\
&= q^{n(k-1)}\,\frac{q^k-1}{q-1}\,\frac{q^{n+1}-1}{q-1}=\phi_1(n+1,k;q).
\end{aligned}$$

To prove (4.6) we have

$$\phi_n(n,k;q)\,\frac{q^{k-n}-1}{q-1}=\prod_{\rho=0}^{n-1}\frac{q^{k-\rho}-1}{q-1}\,\frac{q^{k-n}-1}{q-1}=\prod_{\rho=0}^{n}\frac{q^{k-\rho}-1}{q-1}=\phi_{n+1}(n+1,k;q).$$

Finally, we prove (4.5) by distinguishing cases $\lambda=n$, $\lambda=n-1$, and $2\le\lambda\le n-2$. Starting with the right hand side of (4.5), for $\lambda=n$ we have

$$\begin{aligned}
&\phi_n(n,k;q)\,q^{k-n}+\phi_{n-1}(n,k;q)\,\frac{q^{k-n+1}-1}{q-1}\\
&=\prod_{\rho=0}^{n-1}\frac{q^{k-\rho}-1}{q-1}\,q^{k-n}+q^{k-n+1}\prod_{\rho=0}^{n-2}\frac{q^{k-\rho}-1}{q-1}\,\frac{q^n-1}{q-1}\,\frac{q^{k-n+1}-1}{q-1}\\
&=q^{k-n}\prod_{\rho=0}^{n-1}\frac{q^{k-\rho}-1}{q-1}+q^{k-n+1}\,\frac{q^n-1}{q-1}\prod_{\rho=0}^{n-1}\frac{q^{k-\rho}-1}{q-1}\\
&=\left(q^{k-n}+q^{k-n+1}\,\frac{q^n-1}{q-1}\right)\prod_{\rho=0}^{n-1}\frac{q^{k-\rho}-1}{q-1}\\
&=q^{k-n}\,\frac{q^{n+1}-1}{q-1}\prod_{\rho=0}^{n-1}\frac{q^{k-\rho}-1}{q-1}=\phi_n(n+1,k;q).
\end{aligned}$$

Similarly, for $\lambda=n-1$ we have

$$\begin{aligned}
&\phi_{n-1}(n,k;q)\,q^{k-n+1}+\phi_{n-2}(n,k;q)\,\frac{q^{k-n+2}-1}{q-1}\\
&=q^{k-n+1}\prod_{\rho=0}^{n-2}\frac{q^{k-\rho}-1}{q-1}\,\frac{q^n-1}{q-1}\,q^{k-n+1}\\
&+q^{2(k-n+2)}\prod_{\rho=0}^{n-3}\frac{q^{k-\rho}-1}{q-1}\,\prod_{\rho'=0}^{1}\frac{q^{n-\rho'}-1}{q^{2-\rho'}-1}\,\frac{q^{k-n+2}-1}{q-1}\\
&=q^{2(k-n+1)}\prod_{\rho=0}^{n-2}\frac{q^{k-\rho}-1}{q-1}\,\frac{q^n-1}{q-1}\\
&+q^{2(k-n+2)}\prod_{\rho=0}^{n-2}\frac{q^{k-\rho}-1}{q-1}\,\frac{q^n-1}{q^2-1}\,\frac{q^{n-1}-1}{q-1}
\end{aligned}$$

$$= q^{2(k-n+1)} \prod_{\rho=0}^{n-2} \frac{q^{k-\rho}-1}{q-1} \frac{q^n-1}{q^2-1} \frac{q^{n-1}-1}{q-1} \left(\frac{q^2-1}{q^{n-1}-1} + q^2 \right)$$
$$= q^{2(k-n+1)} \prod_{\rho=0}^{n-2} \frac{q^{k-\rho}-1}{q-1} \frac{q^n-1}{q^2-1} \frac{q^{n-1}-1}{q-1} \frac{q^{n+1}-1}{q^{n-1}-1}$$
$$= q^{2(k-n+1)} \prod_{\rho=0}^{n-2} \frac{q^{k-\rho}-1}{q-1} \frac{q^{n+1}-1}{q^2-1} \frac{q^n-1}{q-1} = \phi_{n-1}(n+1,k;q).$$

Finally, for $2 \le \lambda \le n-2$ the right hand side of (4.5) becomes

$$\phi_\lambda(n,k;q)\, q^{k-\lambda} + \phi_{\lambda-1}(n,k;q)\, \frac{q^{k-\lambda+1}-1}{q-1}$$
$$= q^{(n-\lambda)(k-\lambda)} \prod_{\rho=0}^{\lambda-1} \frac{q^{k-\rho}-1}{q-1} \prod_{\rho'=0}^{n-\lambda-1} \frac{q^{n-\rho'}-1}{q^{n-\lambda-\rho'}-1}\, q^{k-\lambda}$$
$$+ q^{(n-\lambda+1)(k-\lambda+1)} \prod_{\rho=0}^{\lambda-2} \frac{q^{k-\rho}-1}{q-1} \prod_{\rho'=0}^{n-\lambda} \frac{q^{n-\rho'}-1}{q^{n-\lambda+1-\rho'}-1} \frac{q^{k-\lambda+1}-1}{q-1}$$
$$= q^{(n-\lambda+1)(k-\lambda)} \prod_{\rho=0}^{\lambda-1} \frac{q^{k-\rho}-1}{q-1}$$
$$\times \left(\prod_{\rho'=0}^{n-\lambda-1} \frac{q^{n-\rho'}-1}{q^{n-\lambda-\rho'}-1} + q^{n-\lambda+1} \prod_{\rho'=0}^{n-\lambda} \frac{q^{n-\rho'}-1}{q^{n+1-\lambda-\rho'}-1} \right)$$
$$= q^{(n-\lambda+1)(k-\lambda)} \prod_{\rho=0}^{\lambda-1} \frac{q^{k-\rho}-1}{q-1}$$
$$\times \left(\prod_{\rho'=1}^{n-\lambda} \frac{q^{n+1-\rho'}-1}{q^{n+1-\lambda-\rho'}-1} + q^{n-\lambda+1} \frac{\prod_{\rho''=1}^{n-\lambda+1}(q^{n+1-\rho''}-1)}{\prod_{\rho'=0}^{n-\lambda}(q^{n+1-\lambda-\rho'}-1)} \right)$$
$$= q^{(n-\lambda+1)(k-\lambda)} \prod_{\rho=0}^{\lambda-1} \frac{q^{k-\rho}-1}{q-1} \prod_{\rho'=0}^{n-\lambda} \frac{q^{n+1-\rho'}-1}{q^{n+1-\lambda-\rho'}-1}$$

$$\times \left(\frac{q^{n+1-\lambda}-1}{q^{n+1}-1} + q^{n-\lambda+1} \frac{q^\lambda-1}{q^{n+1}-1} \right)$$
$$= q^{(n-\lambda+1)(k-\lambda)} \prod_{\rho=0}^{\lambda-1} \frac{q^{k-\rho}-1}{q-1} \prod_{\rho'=0}^{n-\lambda} \frac{q^{n+1-\rho'}-1}{q^{n+1-\lambda-\rho'}-1} \cdot 1 = \phi_\lambda(n+1,k;q).$$

□

Proposition 3. *Let $0 \neq q \neq \pm 1$. For all $n,k,N,K \ge 0$*

$$\begin{aligned} a^{\dagger n}\, a^k\, a^{\dagger N}\, a^K - q^{kN-nK}\, a^{\dagger N}\, a^K\, a^{\dagger n}\, a^k &= \sum_{\lambda=1}^{n} \phi_\lambda(k,N;q)\, a^{\dagger n+N-\lambda}\, a^{k+K-\lambda} \\ &\quad - q^{kN-nK} \sum_{\lambda=1}^{K} \phi_\lambda(K,n;q)\, a^{\dagger n+N-\lambda}\, a^{k+K-\lambda}. \end{aligned} \tag{4.7}$$

Proof. To prove (4.7) we notice that

$$\begin{aligned} a^{\dagger n}\, a^k\, a^{\dagger N}\, a^K &= a^{\dagger n}\left(q^{kN}\, a^{\dagger N}\, a^k + \sum_{\lambda=1}^{k} \phi_\lambda(k,N;q)\, a^{\dagger N-\lambda}\, a^{k-\lambda}\right) a^K \\ &= q^{kN}\, a^{\dagger n}\, a^{\dagger N}\, a^k\, a^K + \sum_{\lambda=1}^{k} \phi_\lambda(k,N;q)\, a^{\dagger n}\, a^{\dagger N-\lambda}\, a^{k-\lambda}\, a^K. \end{aligned} \tag{4.8}$$

But

$$\begin{aligned} a^{\dagger n} a^{\dagger N} a^k a^K &= a^{\dagger N} a^{\dagger n} a^K a^k \\ &= a^{\dagger N}\left(\frac{1}{q^{nK}}\, a^K\, a^{\dagger n} - \frac{1}{q^{nK}} \sum_{\lambda=1}^{K} \phi_\lambda(K,n;q)\, a^{\dagger n-\lambda}\, a^{K-\lambda}\right) a^k \\ &= q^{-nK}\, a^{\dagger N}\, a^K\, a^{\dagger n}\, a^k - q^{-nK} \sum_{\lambda=1}^{K} \phi_\lambda(K,n;q)\, a^{\dagger N}\, a^{\dagger n-\lambda}\, a^{K-\lambda}\, a^k \end{aligned} \tag{4.9}$$

and (4.7) follows by substituting (4.9) into the right hand side of (4.8). □

5. Current algebras over the q-commutation relations

In this section we introduce the q-deformed white noise and we prove that, when one tries to introduce higher powers of white noise then, as in the Boson case, ill defined powers of the δ-function appear. We then apply the same renormalization used in the Boson case (see Proposition 4 below) and this allows to deduce the q-deformed version of the Lie algebra of renormalized higher powers of white noise.

We fix our measure space to be

$$(S, \mathcal{B}, \mu) = (\mathbb{R}, dx)$$

and the algebra $\mathcal{C} \subseteq L^\infty(\mathbb{R})$ to be the $*$-algebra of finitely valued Borel measurable step functions with bounded support.

The current algebra extension of the relation (4.1) is

$$a_f a_g^+ - q a_g^+ a_f = \langle f, g\rangle, \quad f, g \in \mathcal{C} \tag{5.1}$$

where $\langle f, g\rangle$ denotes the scalar product in $L^2(\mathbb{R}, dx)$. In white noise notations this becomes

$$a_t\, a_s^\dagger - q\, a_s^\dagger\, a_t = \delta(t-s). \tag{5.2}$$

Corollary 1. *For all $t, s \in \mathbb{R}_+$ and $n \geq 1$, one has*

$$a_t\, (a_s^\dagger)^n - q^n\, (a_s^\dagger)^n\, a_t = \frac{q^n - 1}{q-1}\, (a_s^\dagger)^{n-1}\, \delta(t-s) = [n]_q\, (a_s^\dagger)^{n-1}\, \delta(t-s). \tag{5.3}$$

Proof. Fix $f = g$ in (5.1). defining $\delta := \|f\|$ and

$$b^\dagger = \frac{a_f^\dagger}{\delta^{1/2}}, \quad b = \frac{a_f}{\delta^{1/2}}. \tag{5.4}$$

Then b, b^+ satisfy (4.1) hence (4.2) holds with b replacing a because of Proposition 4.2. By polarization and passing to white noise notations, we obtain (5.3). □

Corollary 2. *For all $t, s \in \mathbb{R}_+$ and $n \geq 1$ the rule*

$$a_t\, a_s^\dagger - q\, a_s^\dagger\, a_t = \delta(t-s) \tag{5.5}$$

implies that

$$a_t^n\,(a_s^\dagger)^k - q^{nk}\,(a_s^\dagger)^k\,a_t^n = \sum_{\lambda=1}^{n} \phi_\lambda(n,k;q)\,(a_s^\dagger)^{k-\lambda}\,a_t^{n-\lambda}\,\delta^\lambda(t-s). \tag{5.6}$$

Proof. The proof of (5.6) is similar to that of Corollary 1 by applying Proposition 2 to $b^\dagger$ and b defined by (5.4). □

Proposition 4. *For any complex-valued Schwartz function f define the symbols*

$$B_k^n(f) := \int_{\mathbb{R}} f(s)\,{a_s^\dagger}^n\,a_s^k\,ds \tag{5.7}$$

with involution

$$(B_k^n(f))^* = B_n^k(\bar{f}) \tag{5.8}$$

and

$$B_0^0(f) = \int_{\mathbb{R}} f(s)\,ds\,\cdot\,1 \tag{5.9}$$

where 1 is the identity operator. Then, using for $\lambda \geq 1$ the renormalization rule

$$\delta^\lambda(t-s) := c^{\lambda-1}\delta(t-s) \tag{5.10}$$

where $c > 0$ is a constant, for any $n,k,N,K \in \{0,1,2,\ldots\}$, one has

$$\begin{aligned} &B_k^n(f)\,B_K^N(g) - q^{kN-nK}\,B_K^N(g)\,B_k^n(f) \\ &\quad= \sum_{\lambda=1}^{k} c^{\lambda-1}\,\phi_\lambda(k,N;q)\,B_{k+K-\lambda}^{n+N-\lambda}(f\,g) \\ &\quad\quad - q^{kN-nK}\sum_{\lambda=1}^{K} c^{\lambda-1}\,\phi_\lambda(K,n;q)\,B_{k+K-\lambda}^{n+N-\lambda}(f\,g) \end{aligned} \tag{5.11}$$

where for $k=0$ and/or $K=0$ the corresponding sums on the right hand side of (5.11) are interpreted as zero.

Proof. The proof follows by replacing a and $a^\dagger$ by $\frac{a_t}{\delta(t-s)^{1/2}}$ and $\frac{a_s^\dagger}{\delta(t-s)^{1/2}}$ and then multiplying both sides of (4.7) by $f(t)\,g(s)$ followed by a formal integration of the resulting identity (i.e. taking $\int\int \cdots\,ds\,dt$). □

6. The no–go theorem

Theorem 1. *Let $q \in (-1,1), q \neq 0$ and for a fixed interval $I \subset \mathbb{R}$ and $n,k \geq 0$ let*

$$B_k^n := B_k^n(\chi_I)$$

with $B_0^0 = \mu(I)\cdot 1$ the Lebesgue measure of I. Let also the "vacuum vector" Φ be such that $B_k^n\,\Phi = 0$ whenever $k \neq 0$ and let $\langle\, x\,\rangle := \langle \Phi, x\,\Phi\rangle$ denote the "vacuum expectation" of an operator x. We assume that $\langle \Phi, \Phi\rangle = 1$. Define

$$A(n,q;I) := \begin{bmatrix} \langle B_{2n}^0 B_0^{2n}\rangle & \langle B_{2n}^0 (B_0^n)^2\rangle \\ \\ \langle B_{2n}^0 (B_0^n)^2\rangle & \langle (B_n^0)^2 (B_0^n)^2\rangle \end{bmatrix}.$$

Then, for any choice of n and q, the matrix $A(n,q;I)$ is not positive semi-definite whenever $c\,\mu(I) \leq 1$ where c is the renormalization constant.

Proof. Using Proposition 4 we have

$$B_{2n}^0\, B_0^{2n}\Phi = q^{4n^2}\, B_{2n}^0\, B_0^{2n}\,\Phi + \sum_{\lambda=1}^{2n} c^{\lambda-1}\,\phi_\lambda(2n,2n;q)\,B_{2n-\lambda}^{2n-\lambda}\,\Phi$$
$$= c^{2n-1}\,\phi_{2n}(2n,2n;q)\,B_0^0\,\Phi = \mu(I)\,c^{2n-1}\prod_{\rho=0}^{2n-1}\frac{q^{2n-\rho}-1}{q-1}\,\Phi.$$

Thus

$$\langle B_{2n}^0 B_0^{2n}\rangle = \mu(I)\,c^{2n-1}\prod_{\rho=0}^{2n-1}\frac{q^{2n-\rho}-1}{q-1}. \tag{6.1}$$

Similarly,

$$B_n^0\, B_0^{2n}\Phi = q^{2n^2}\, B_0^{2n}\, B_n^0\,\Phi + \sum_{\lambda=1}^{n} c^{\lambda-1}\,\phi_\lambda(n,2n;q)\,B_{n-\lambda}^{2n-\lambda}\,\Phi$$
$$= c^{n-1}\,\phi_n(n,2n;q)\,B_0^n\,\Phi = c^{n-1}\prod_{\rho=0}^{n-1}\frac{q^{2n-\rho}-1}{q-1}\,B_0^n\,\Phi.$$

Thus

$$B_n^0\, B_n^0\, B_0^{2n}\Phi = c^{n-1}\prod_{\rho=0}^{n-1}\frac{q^{2n-\rho}-1}{q-1}\,B_n^0\,B_0^n\,\Phi$$
$$= c^{2n-2}\,\mu(I)\prod_{\rho=0}^{n-1}\frac{q^{2n-\rho}-1}{q-1}\prod_{\rho=0}^{n-1}\frac{q^{n-\rho}-1}{q-1}\,\Phi$$

and so

$$\langle B_{2n}^0 (B_0^n)^2\rangle = c^{2n-2}\mu(I)\prod_{\rho=0}^{n-1}\frac{q^{2n-\rho}-1}{q-1}\prod_{\rho=0}^{n-1}\frac{q^{n-\rho}-1}{q-1}. \tag{6.2}$$

Finally,

$$B_n^0\, B_0^n \quad = \quad q^{n^2}\, B_0^n\, B_n^0 + \sum_{\lambda=1}^{n} c^{\lambda-1}\,\phi_\lambda(n,n;q)\,B_{n-\lambda}^{n-\lambda} \tag{6.3}$$

implies

$$B_n^0\,(B_0^n)^2 = q^{n^2}\, B_0^n\, B_n^0\, B_0^n + \sum_{\lambda=1}^{n} c^{\lambda-1}\,\phi_\lambda(n,n;q)\,B_{n-\lambda}^{n-\lambda}\,B_0^n$$
$$= q^{n^2}\, B_0^n\left(q^{n^2}\, B_0^n\, B_n^0 + \sum_{\lambda=1}^{n} c^{\lambda-1}\,\phi_\lambda(n,n;q)\,B_{n-\lambda}^{n-\lambda}\right)$$
$$+ \sum_{\lambda=1}^{n} c^{\lambda-1}\,\phi_\lambda(n,n;q)$$
$$\times\left(q^{(n-\lambda)\,n}\,B_0^n\,B_{n-\lambda}^{n-\lambda} + \sum_{\lambda'=1}^{n-\lambda} c^{\lambda'-1}\,\phi_{\lambda'}(n-\lambda,n;q)\,B_{n-\lambda-\lambda'}^{2n-\lambda-\lambda'}\right)$$

and so

$$B_n^0\,(B_0^n)^2\,\Phi = (q^{n^2}\,\mu(I)\,c^{n-1}\,\phi_n(n,n;q) + \mu(I)\,c^{n-1}\,\phi_n(n,n;q) + c^{n-2}\sum_{\lambda+\lambda'=n}\phi_\lambda(n,n;q)\,\phi_{\lambda'}(n-\lambda,n;q))\,B_0^n\,\Phi.$$

Thus

$$(B_n^0)^2(B_0^n)^2\Phi = (q^{n^2}\,\mu(I)\,c^{n-1}\,\phi_n(n,n;q) + \mu(I)\,c^{n-1}\,\phi_n(n,n;q) + c^{n-2}\sum_{\lambda+\lambda'=n}\phi_\lambda(n,n;q)\,\phi_{\lambda'}(n-\lambda,n;q))\,B_n^0\,B_0^n\,\Phi$$

which using (6.3) implies

$$\begin{aligned}\langle (B_n^0)^2(B_0^n)^2\rangle &= \mu(I)^2\,c^{2n-2}\,(1+q^{n^2})\left(\prod_{\rho=0}^{n-1}\frac{q^{n-\rho}-1}{q-1}\right)^2\\ &\quad + \mu(I)\,c^{2n-3}\prod_{\rho=0}^{n-1}\frac{q^{n-\rho}-1}{q-1}\sum_{\lambda=1}^{n-1}\phi_\lambda(n,n;q)\,\phi_{n-\lambda}(n-\lambda,n;q).\end{aligned}\tag{6.4}$$

Thus, by (6.1)–(6.4),

$$A(n,q;I) = \begin{bmatrix} \mu(I)\,c^{2n-1}\,[2n]_q! & c^{2n-2}\mu(I)\,[2n]_q! \\ c^{2n-2}\mu(I)\,[2n]_q! & \begin{array}{c}\mu(I)^2\,c^{2n-2}\,(1+q^{n^2})\,([n]_q!)^2 \\ +\mu(I)\,c^{2n-3}\,([n]_q!)^2\sum_{\lambda=1}^{n-1}q^{(n-\lambda)^2}\binom{n}{\lambda}_q^2\end{array}\end{bmatrix}$$

since

$$\begin{aligned}\sum_{\lambda=1}^{n-1}\phi_\lambda(n,n;q)\,\phi_{n-\lambda}(n-\lambda,n;q) &= \sum_{\lambda=1}^{n-1} q^{(n-\lambda)^2}\,\frac{[n]_q!}{[n-\lambda]_q!}\binom{n}{\lambda}_q\frac{[n]_q!}{[\lambda]_q!}\\ &= \sum_{\lambda=1}^{n-1} q^{(n-\lambda)^2}\binom{n}{\lambda}_q^2\,[n]_q!\,.\end{aligned}$$

$A(n,q;I)$ is a symmetric matrix, so it is positive semi-definite if and only if its minors are non-negative. The minor determinants of $A(n,q;I)$ are

$$d_1 = \mu(I)\,c^{2n-1}\,[2n]_q!$$

which is non-negative for all I and

$$d_2 = \mu(I)^2\,c^{4n-4}\,[2n]_q!\left(c\,\mu(I)\,(1+q^{n^2})\,([n]_q!)^2 + ([n]_q!)^2\sum_{\lambda=1}^{n-1}q^{(n-\lambda)^2}\binom{n}{\lambda}_q^2 - [2n]_q!\right)$$

which is bigger or equal to zero if and only if

$$c\,\mu(I)\,(1+q^{n^2})\,([n]_q!)^2 + ([n]_q!)^2\sum_{\lambda=1}^{n-1}q^{(n-\lambda)^2}\binom{n}{\lambda}_q^2 - [2n]_q! \geq 0$$

or equivalently

$$c\,\mu(I) \geq \frac{[2n]_q! - ([n]_q!)^2\sum_{\lambda=1}^{n-1}q^{(n-\lambda)^2}\binom{n}{\lambda}_q^2}{(1+q^{n^2})\,([n]_q!)^2} = 1$$

since by Lemma 1

$$[2n]_q! - ([n]_q!)^2 \sum_{\lambda=1}^{n-1} q^{(n-\lambda)^2} \binom{n}{\lambda}_q^2 = [2n]_q! - ([n]_q!)^2 \left(\sum_{\lambda=0}^{n} q^{\lambda^2} \binom{n}{\lambda}_q^2 - 1 - q^{n^2} \right)$$
$$= [2n]_q! - ([n]_q!)^2 \left(\binom{2n}{n}_q - 1 - q^{n^2} \right) = [2n]_q! - ([n]_q!)^2 \left(\frac{[2n]_q!}{([n]_q!)^2} - 1 - q^{n^2} \right)$$
$$= (1 + q^{n^2}) ([n]_q!)^2.$$

Thus

$$d_2 \geq 0 \Leftrightarrow \mu(I) \geq \frac{1}{c} \tag{6.5}$$

which cannot be true for arbitrarily small I. □

7. The Fermion case $q = -1$

The commutation relations for the renormalized powers of Fermion white noise can be obtained either directly in analogy to Section 2 , or as a limiting case of the commutation relations obtained in Section 2. If the second approach is used we find that

$$\phi_\lambda(n, k; -1) := \lim_{q \to -1} \phi_\lambda(n, k; q) = \begin{cases} 0, & \text{if } \lambda > 1 \\ \frac{(1-(-1)^n)\,(1-(-1)^k)}{4}, & \text{if } \lambda = 1 \end{cases}$$

where $\phi_\lambda(n, k; q)$ is as in Proposition 2, and (5.6), (5.11) become

$$\begin{aligned} &a_t^n\,(a_s^\dagger)^k - (-1)^{nk}\,(a_s^\dagger)^k\,a_t^n \\ &\qquad = \frac{(1-(-1)^n)\,(1-(-1)^k)}{4}\,(a_s^\dagger)^{k-1}\,a_t^{n-1}\,\delta(t-s) \end{aligned} \tag{7.1}$$

and

$$B_k^n(f)\,B_K^N(g) - (-1)^{kN-nK}\,B_K^N(g)\,B_k^n(f) = x(n, k; N, K)\,B_{k+K-1}^{n+N-1}(f\,g) \tag{7.2}$$

where

$$x(n, k; N, K) := \left(\frac{(1-(-1)^k)\,(1-(-1)^N)}{4} + (-1)^{kN-nK+1}\,\frac{(1-(-1)^n)\,(1-(-1)^K)}{4} \right).$$

Notice that (7.1) and (7.2) do not contain the renormalization constant $c > 0$ and no higher powers of the delta function appear. In fact, the need for renormalization does not arise in the Fermion case. Moreover

$$A(n, -1; I) = \lim_{q \to -1} A(n, q; I) = 0$$

which is trivially positive semi-definite for all I.

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