Methods of Functional Analysis and Topology
Vol. 12 (2006), no. 3, pp. 220–233

BORG-TYPE THEOREMS FOR GENERALIZED JACOBI MATRICES AND TRACE FORMULAS

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Abstract. The paper deals with two types of inverse spectral problems for the class of generalized Jacobi matrices introduced in [9]. Following the scheme proposed in [5], we deduce analogs of the Hochstadt–Lieberman theorem and the Borg theorem. Properties of a Weyl function of the generalized Jacobi matrix are systematically used to prove the uniqueness theorems. Trace formulas for the generalized Jacobi matrix are also derived.

1. Introduction

In this paper we consider two types of inverse spectral problems for a class of generalized Jacobi matrices associated with finite sequences of polynomials and numbers. Recall from [9] that a sequence of monic real polynomials p_j of degree k_j $(j = 0, \dots, N)$, a sequence of positive numbers $\{b_j\}_{j=0}^{N-1}$, and a set of numbers $\varepsilon_j = \pm 1$ $(j = 0, \dots, N)$ define the following tridiagonal block matrix

$$H = \begin{pmatrix} A_0 & \widetilde{B}_0 & & \mathbf{0} \\ B_0 & \ddots & \ddots & \\ & \ddots & \ddots & \widetilde{B}_{N-1} \\ \mathbf{0} & & B_{N-1} & A_N \end{pmatrix}, \tag{1.1}$$

where A_j is the companion matrix for the polynomial p_j (see [17]), the rectangular matrices B_j and $\widetilde{B}_j$ are defined by the numbers k_j, k_{j+1}, ε_j, ε_{j+1}, b_j (see (2.1) below). The matrix H is called a generalized Jacobi matrix associated with the sequences of polynomials $\{\varepsilon_j p_j\}_{j=0}^{N}$ and numbers $\{b_j\}_{j=0}^{N-1}$. If each polynomial p_j has degree $k_j = 1$ and $\varepsilon_j = 1$ $(j = 0, \dots, N)$ then the matrix H is a classical Jacobi matrix, that is, a tridiagonal real symmetric matrix. In general case, the generalized Jacobi matrix H defines a simple symmetric operator in a space with an indefinite inner product. As is known (see for instance [17]), a matrix defines a simple operator whenever its minimal polynomial coincides with the characteristic polynomial. So, denote by $\sigma(H)$ the set of all eigenvalues of the matrix H with the convention that each eigenvalue is repeated as often as its algebraic multiplicity. The spectral theory of generalized Jacobi matrices associated with sequences of polynomials and positive numbers was studied in [9] (see also [20]).

The first type of inverse spectral problems in question originates from the following theorem of H. Hochstadt and B. Lieberman [16]: if the potential $q(x)$ of a Sturm–Liouville problem is given over the interval $(1/2, 1)$ then knowing the spectrum alone is sufficient for determining $q(x)$ on the interval $(0, 1/2)$. A discrete version of the Hochstadt–Lieberman

2000 *Mathematics Subject Classification.* Primary 15A29, 39A12; Secondary 47B36, 47B50.

Key words and phrases. Generalized Jacobi matrix, difference equation, inverse spectral problem, trace formula, Weyl function.

theorem has been obtained by H. Hochstadt [15]. Namely, it is proved that the classical Jacobi matrix

$$J = \begin{pmatrix} a_0 & b_0 & & \mathbf{0} \\ b_0 & \ddots & \ddots & \\ & \ddots & \ddots & b_{N-1} \\ \mathbf{0} & & b_{N-1} & a_N \end{pmatrix} \quad (a_j \in \mathbb{R},\ b_j > 0)$$

is uniquely determined by its entries $a_0, \dots, a_{[(N-1)/2]}, b_0, \dots, b_{[(N-2)/2]}$ (where $[x]$ denotes the greatest integer less than or equal to x) and its spectrum $\sigma(J)$. Moreover, B. Simon and F. Gesztesy [5] have extended the result in the following way: if $c_0 = a_0, c_1 = b_0, \dots, c_{2N-1} = b_{N-1}, c_{2N} = a_N$ then any $j \in \{1, \dots, N+1\}$ eigenvalues and $c_j, \dots, c_{2N}$ uniquely determine $c_0, \dots, c_{j-1}$, that is, the matrix J. In the present paper a similar result for generalized Jacobi matrices is proved. To formulate this in brief, let us consider a single sequence of entries $C_0 = A_0$, $C_1 = \widetilde{B}_0, \dots, C_{2N-1} = \widetilde{B}_{N-1}$, $C_{2N} = A_N$ of the generalized Jacobi matrix H. Roughly speaking, we will show that *the subset of the spectrum $\sigma(H)$ and $C_j, \dots, C_{2N}$ uniquely determine $C_0, \dots, C_{j-1}$*. It is natural to refer to such problems as inverse problems with mixed given data.

The second type of inverse spectral problems to be studied in the paper goes back to Borg's famous theorem [3] (see also [21]) that the spectra of two boundary value problems of a regular Schrödinger operator on $(0,1)$ uniquely determine the potential. Analogs of this result for a finite classical Jacobi matrix were considered in [1], [13], [14] (see also [4], where a mechanical interpretation of such analogs is given). A straightforward analog says that the eigenvalues $\lambda_0, \dots, \lambda_N$ of the matrix J and the eigenvalues $\lambda_0(\tau), \dots, \lambda_N(\tau)$ of its rank one perturbation $J(\tau)$ ($\neq J$, $\tau \in \mathbb{R}$), having the form

$$J(\tau) = J + \tau\,(\cdot, e)\,e, \quad e = (\delta_{0i})_{i=0}^N,$$

uniquely determine J and τ. A version of the Borg theorem for generalized Jacobi matrices to be proved in the paper is following: *the spectrum $\sigma(H)$ of the generalized Jacobi matrix H and the spectrum $\sigma(H(\tau))$ of its rank one perturbation uniquely determine H and τ.*

In the present paper we prove the uniqueness results by using the scheme proposed in [5]. The scheme is based on properties of the Weyl function of H (see [11], [12], [23]). More precisely, we essentially use a continued fraction expansion of the Weyl function. With this machinery available, we obtain trace formulas for generalized Jacobi matrices. In addition, we stress that our results can be viewed as uniqueness theorems for the underlying second order difference equation with nonlinear dependence on the spectral parameter,

$$\widetilde{b}_{j-1} y_{j-1} + b_j y_{j+1} = p_j(\lambda) y_j \quad (j = 0, \dots, N). \tag{1.2}$$

It should be remarked that three-term recurrence relations similar to (1.2) occur in indefinite moment problems [9] and the Pade approximation theory [22] and the theory of formal orthogonal polynomials [28].

A standard approach via orthogonal polynomials to inverse spectral problems for classical Jacobi matrices is given in [2]. An adaptation of this approach to generalized Jacobi matrices was presented by Kreĭn and Langer (see [20]). The relation between classical Jacobi matrices and boundary problems for difference equation and inverse problems were considered in Atkinson's book [1]. We remark in conclusion that using transformation operators technique M. M. Malamud has recently obtained analogs of the Borg theorem and the Hochstadt–Liemerman theorem for matrix Sturm–Liouville operators [25] as well as for systems of ordinary differential equations on a finite interval [24].

The main results of the present paper were announced in [8].

2. Generalized Jacobi matrices and underlying difference equations

We begin with a precise definition of a generalized Jacobi matrix associated with sequences of polynomials and positive numbers. Let $p(\lambda) = p_k\lambda^k + \cdots + p_1\lambda + p_0$ be a monic real scalar polynomial of degree k. Let us associate to the polynomial p its symmetrization operator E_p and the companion matrix L_p given by

$$E_p = \begin{pmatrix} p_1 & \dots & p_k \\ \vdots & \cdot & \\ p_k & & \mathbf{0} \end{pmatrix}, \quad L_p = \begin{pmatrix} 0 & \dots & 0 & -p_0 \\ 1 & & \mathbf{0} & -p_1 \\ & \ddots & & \vdots \\ \mathbf{0} & & 1 & -p_{k-1} \end{pmatrix}.$$

Definition 2.1. ([9]). Let p_j be real monic polynomials of degree k_j,

$$p_j(\lambda) = \lambda^{k_j} + p^{(j)}_{k_j-1}\lambda^{k_j-1} + \cdots + p^{(j)}_1\lambda + p^{(j)}_0,$$

and let $\varepsilon_j = \pm 1$ $(j = 0, \dots, N)$, $b_j > 0$, $\widetilde{b}_j := \varepsilon_j\varepsilon_{j+1}b_j$ $(j = 0, \dots, N-1)$. The tridiagonal block matrix H of the form (1.1), where $A_j = L_{p_j}$ and $k_{j+1} \times k_j$ matrices B_j and $k_j \times k_{j+1}$ matrices $\widetilde{B}_j$ are given by

$$B_j = \begin{pmatrix} 0 & \dots & b_j \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix}, \quad \widetilde{B}_j = \begin{pmatrix} 0 & \dots & \widetilde{b}_j \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix} \quad (j = 0, \dots, N-1) \tag{2.1}$$

will be called a *generalized Jacobi matrix* associated with the sequences of polynomials $\{\varepsilon_j p_j\}_{j=0}^N$ *and numbers* $\{b_j\}_{j=0}^{N-1}$.

Let $n + 1 = \sum_{j=0}^{N} k_j$ be the total number of rows in H. Define an $(n+1)\times(n+1)$ matrix G by the equality

$$G = \operatorname{diag}(G_0, \dots, G_N), \quad G_j = \varepsilon_j E^{-1}_{p_j} \quad (j = 0, \dots N).$$

We denote by $\ell^2_{[0,n]}(G)$ the space of $(n+1)$ vectors with the inner product

$$\langle x, y\rangle = (Gx, y)_{\ell^2_{[0,n]}} \quad (x, y \in \ell^2_{[0,n]}).$$

Let us set $n_0 = 0$, $n_j = \sum_{i=0}^{j-1} k_i$ $(j = 1, \dots, N+1)$. It will be convenient to define a standard basis in $\ell^2_{[0,n]}(G)$ by the equalities

$$e_{j,k} = \{\delta_{l,n_j+k}\}_{l=0}^n \quad (j = 0, \dots, N; k = 0, \dots, k_j - 1), \quad e := e_{0,0}.$$

Proposition 2.2. ([9]). *A generalized Jacobi matrix H defines a simple symmetric operator in $\ell^2_{[0,n]}(G)$, that is,*

$$\langle Hx, y\rangle = \langle x, Hy\rangle, \quad x, y \in \ell^2_{[0,n]}(G).$$

Setting $\widetilde{b}_{-1} = b_N = 1$, define polynomials of the first kind (cf. [20]), $P_j(\lambda)$, as solutions $u_j = P_j(\lambda)$ of the second order difference equation

$$\widetilde{b}_{j-1}u_{j-1} - p_j(\lambda)u_j + b_ju_{j+1} = 0 \quad (j = 0, \dots, N) \tag{2.2}$$

with the initial conditions

$$u_{-1} = 0, \quad u_0 = 1. \tag{2.3}$$

It follows from (2.2) and (2.3) that P_j is a polynomial of degree n_j with the leading coefficient $(b_0 \dots b_{j-1})^{-1}$. Moreover, denoting by $H_{[j,m]}$ the submatrix of H corresponding to the basis vectors $\{e_{i,k}\}_{k=0,\ldots,k_i-1}^{i=j,\ldots,m}$ $(0 \le j \le m \le N)$, we get the following connection between the polynomials of the first kind and the shortened Jacobi matrices $H_{[0,j]}$, $j \in \{0,\ldots,N\}$.

Proposition 2.3. ([9]). *The polynomials P_j can be found by the formulas*

$$P_{j+1}(\lambda) = (b_0 \ldots b_j)^{-1} \det(\lambda - H_{[0,j]}) \quad (j = 0,\ldots,N). \tag{2.4}$$

Let us extend the system of polynomials $\{P_j(\lambda)\}_{j=0}^N$ in the following way:

$$P_{j,k}(\lambda) = \lambda^k P_j(\lambda) \quad (j = 0,\ldots,N;\ k = 0,\ldots,k_j - 1). \tag{2.5}$$

The system (2.5) gives us the possibility to rewrite the Cauchy problem (2.2), (2.3) in the matrix form

$$\mathbf{P}(\lambda)(\lambda - H) = (0,\ldots,0, b_N P_{N+1}(\lambda)), \tag{2.6}$$

where $\mathbf{P}(\lambda) = (P_{0,0}(\lambda),\ldots,P_{0,k_0-1}(\lambda),\ldots,P_{N,0}(\lambda),\ldots,P_{N,n_N-1}(\lambda))$. Another important system of polynomials $\{\psi_j\}_{j=-1}^{N+1}$ is defined as a solution of the difference equation (2.2) by choosing the initialization

$$u_{N+1} = 0, \quad u_N = 1. \tag{2.7}$$

The polynomials ψ_j are just like the polynomials P_j but run from the other end and, so, the following statement holds.

Proposition 2.4. *The polynomials ψ_j can be found by the formulas*

$$\psi_j(\lambda) = (\widetilde{b}_{N-1} \ldots \widetilde{b}_j)^{-1} \det(\lambda - H_{[j+1,N]}) \quad (j = -1,\ldots,N-1). \tag{2.8}$$

Proof. The formulas (2.8) immediately follow from the relations (2.2), (2.7) by applying the Laplace theorem to $\det(\lambda - H_{[j+1,N]})$. □

Proposition 2.5. ([9]). *The polynomials ψ_j (P_j) and ψ_{j+1} (P_{j+1}) have no common zeros.*

Proof. Suppose that the polynomials ψ_j and ψ_{j+1} have a common zero λ_0, i.e., $\psi_j(\lambda_0) = \psi_{j+1}(\lambda_0) = 0$. Then, due to (2.2), we have that $\psi_i(\lambda_0) = 0$ $(i = j,\ldots,N+1)$. This is contrary to (2.7). The contradiction proves the desired assertion. The proof for the polynomials P_j and P_{j+1} is in line with the foregoing. □

Further, extending the system $\{\psi_j\}_{j=0}^N$ by the equalities

$$\psi_{j,k}(\lambda) = \lambda^k \psi_j(\lambda) \quad (j = 0,\ldots,N;\ k = 0,\ldots,k_j - 1),$$

one gets the following form of the Cauchy problem (2.2), (2.7):

$$\Psi(\lambda)(\lambda - H) = (\underbrace{0,\ldots,0}_{k_0-1}, \psi_{-1}(\lambda), 0,\ldots,0) =: \Phi(\lambda), \tag{2.9}$$

where $\mathbf{\Psi}(\lambda) = (\psi_{0,0}(\lambda),\ldots,\psi_{0,k_0-1}(\lambda),\ldots,\psi_{N,0}(\lambda),\ldots,\psi_{N,n_N-1}(\lambda))$.

To conclude this section, we remark that by virtue of (2.6) (or (2.9)) the spectrum of H coincides with the spectrum of the second order difference equation (2.2) with the boundary conditions

$$u_{-1} = u_{N+1} = 0. \tag{2.10}$$

So, the spectral problem $Hu = \lambda u$ is a linearized form of the boundary problem (2.2), (2.10) with nonlinear dependence on the spectral parameter.

3. Boundary triplets and abstract Weyl functions

Let $\mathfrak{H} = \ell^2_{[0,n]}(G)$ be the Pontryagin space introduced in the previous section. Let S be a nondensely defined symmetric operator in the space $\mathfrak{H}$ such that its graph is represented as follows:

$$\operatorname{gr} S = \{\{f, Hf\} : f \langle\perp\rangle e, f \in \mathfrak{H}\}, \tag{3.1}$$

where H is a generalized Jacobi matrix associated with sequences of polynomials and numbers. In the sequel, we will need some elementary facts from the theory of linear relations. If S_1 and S_2 are linear relations in $\mathfrak{H}$, that is, linear subspaces of $\mathfrak{H}^2$, then a linear combination $\alpha_1 S_1 + \alpha_2 S_2$ is defined in the following manner:

$$\alpha_1 S_1 + \alpha_2 S_2 = \{\{f, \alpha_1 f' + \alpha_2 g'\} : \{f, f'\} \in S_1, \{f, g'\} \in S_2\}.$$

The inverse linear relation of S is the linear relation $S^{-1} = \{\{f', f\} : \{f, f'\} \in S\}$. The linear relation $I = \{\{f, f\} : f \in \mathfrak{H}\}$ is called the identity relation. The resolvent set $\rho(S)$ (or the set of regular points) of the linear relation S consists of all points $\lambda \in \mathbb{C}$ such that the linear relation $(S - \lambda I)^{-1}$ is a bounded linear operator. Denote by S^+ the adjoint linear relation in $\mathfrak{H}$ defined as follows:

$$S^+ = \{\{f, f'\} \in \mathfrak{H}^2 : \langle f, g'\rangle_{\mathfrak{H}} = \langle f', g\rangle_{\mathfrak{H}} \ \ \{g, g'\} \in S\}.$$

By the above definition, it is easy to see that

$$S^+ = \{\{f, Hf + ce\} : f \in \mathfrak{H}, c \in \mathbb{C}\}.$$

It is convenient to identify an operator with its graph. So, we have the inclusion $S \subset S^+$. The defect subspace $\mathfrak{N}_\lambda$ of the operator S is given by

$$\mathfrak{N}_\lambda = \ker(S^+ - \lambda I) = \{f \in \mathfrak{H} : \{f, \lambda f\} \in S^+\}, \quad \widehat{\mathfrak{N}}_\lambda = \{\{f_\lambda, \lambda f_\lambda\} : f_\lambda \in \mathfrak{N}_\lambda\}.$$

The numbers $n_+ := \dim \mathfrak{N}_i$ and $n_- := \dim \mathfrak{N}_{-i}$ are called defect numbers (see [12]). It can easily be checked that $n_+ = n_-$.

Let us recall from [11] definitions of a boundary triplet and a Weyl function of a nondensely defined symmetric operator S in a Pontryagin space with equal defect numbers $n_+ = n_- < \infty$ (see [23] for the definite case).

Definition 3.1. ([11]). A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of a Hilbert space $\mathcal{H}$ and two linear mappings Γ_i $(i = 0, 1)$ from S^+ to $\mathcal{H}$ is called a boundary triplet for S^+ if the mapping $\Gamma : \widehat{f} \mapsto \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\}$ from S^+ into $\mathcal{H} \oplus \mathcal{H}$ is surjective and the following Green's identity holds for every $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in S^+$:

$$\langle f', g\rangle_{\mathfrak{H}} - \langle f, g'\rangle_{\mathfrak{H}} = \left(\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g}\right)_{\mathcal{H}} - \left(\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g}\right)_{\mathcal{H}}.$$

Proposition 3.2. ([11]). *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^+. Then*

1) *The linear relation $S_0 = \ker \Gamma_0$ is a selfadjoint extension of the operator S, i.e., $S_0^+ = S_0$ and $S_0 \supset S$;*

2) *For any linear bounded selfadjoint operator B from $\mathcal{H}$ to $\mathcal{H}$, the linear relation $\widetilde{S} = \ker(\Gamma_1 + B\Gamma_0)$ is a selfadjoint extension of the operator S, i.e., $\widetilde{S}^+ = \widetilde{S}$ and $\widetilde{S} \supset S$.*

Definition 3.3. ([11]). The operator-valued function m defined on $\rho(S_0)$ by the equality

$$m(\lambda)\Gamma_0 \widehat{f}_\lambda = \Gamma_1 \widehat{f}_\lambda \quad (\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda) \tag{3.2}$$

is said to be an abstract Weyl function for the operator S corresponding to the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$.

Remark 3.4. The concept of a boundary triplet for a symmetric operator in a Hilbert space was introduced by V. M. Bruk and A. N. Kochubei (see [6]). The abstract Weyl function and its intimate relationship to the extension theory were presented by V. A. Derkach and M. M. Malamud in [11], [12], [23].

Proposition 3.5. *Let S be a symmetric operator in $\mathfrak{H} = \ell^2_{[0,n]}(G)$ defined by* (3.1)*. Then the triplet $\Pi = \{\mathbb{C}, \Gamma_0, \Gamma_1\}$ of the space $\mathbb{C}$ and the mappings*

$$\Gamma_0 \widehat{f} = -c, \quad \Gamma_1 \widehat{f} = \langle f, e\rangle \quad \widehat{f} = \{f, Hf + ce\} \in S^+ \tag{3.3}$$

is a boundary triplet for the linear relation S^+. Moreover, the corresponding abstract Weyl function admits the following representations:

$$m(\lambda) = \left\langle (H-\lambda)^{-1}e, e\right\rangle = -\varepsilon_0 \frac{\psi_0(\lambda)}{\psi_{-1}(\lambda)}. \tag{3.4}$$

Proof. The surjectivity of the mapping $\Gamma : \widehat{f} \mapsto \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\}$ from S^+ into $\mathbb{C}\oplus\mathbb{C}$ is obvious. Since

$$\langle Hf + ce, g\rangle - \langle f, Hg + de\rangle = -\langle f, e\rangle \overline{d} + c\overline{\langle g, e\rangle},$$

the corresponding abstract Green's identity holds. Thus, Π is a boundary triplet for S^+. It is easy to check that $f_\lambda = (H-\lambda)^{-1}e \in \mathfrak{N}_\lambda$ and $\{(H-\lambda)^{-1}e, \lambda(H-\lambda)^{-1}e\} = \{(H-\lambda)^{-1}e, H(H-\lambda)^{-1}e - e\} \in \widehat{\mathfrak{N}}_\lambda$. Then, according to (3.2) and (3.3), one gets the first equality in (3.4). In view of (2.9), we have

$$H^\top \Psi^\top(\lambda) = \lambda \Psi^\top(\lambda) - \Phi^\top(\lambda). \tag{3.5}$$

Since H is a symmetric operator in $\ell^2_{[0,n]}(G)$, $H = G^{-1}H^\top G$ and, so, (3.5) can be rewritten as follows:

$$HG^{-1}\Psi^\top(\lambda) = \lambda G^{-1}\Psi^\top(\lambda) - G^{-1}\Phi^\top(\lambda),$$

where $G^{-1}\Phi^\top(\lambda) = \varepsilon_0 \left(\psi_{-1,0}(\lambda) 0 \dots 0\right)^\top$. Therefore, one has $G^{-1}\Psi(\lambda) \in \mathfrak{N}_\lambda$ and $\{G^{-1}\Psi(\lambda), \lambda G^{-1}\Psi(\lambda)\} = \{G^{-1}\Psi(\lambda), HG^{-1}\Psi(\lambda) + \varepsilon_0\psi_{-1}e\} \in \widehat{\mathfrak{N}}_\lambda$. Combining (3.2) with (3.3), we claim that

$$m(\lambda)(-\varepsilon_0\psi_{-1}(\lambda)) = \left\langle G^{-1}\Psi(\lambda), e\right\rangle = \psi_0(\lambda).$$

The latter relation implies the second equality in (3.4). □

From now on, following [9] (see also [5]), the Weyl function of S possessing the form (3.4) will be called the m-function of the generalized Jacobi matrix H.

Before closing this section, we consider the linear relation $H(\tau) = \ker(\Gamma_1 + \frac{1}{\tau}\Gamma_0)$ to be used later; here Γ_0 and Γ_1 are defined by (3.3). From Proposition 3.2 we infer that $H(\tau)$ is a selfadjoint extension of S. Moreover, one obviously gets

$$\ker(\Gamma_1 + \frac{1}{\tau}\Gamma_0) = H + \tau \langle \cdot, e\rangle e = H(\tau). \tag{3.6}$$

It is clear that $H(\tau)$ is a generalized Jacobi matrix associated with sequences of polynomials and numbers. Moreover, $H(\tau)$ is a rank one perturbation of H and, so, its spectrum can be characterized by the m-function of H. The following assertion giving a criterion for a point λ to be an eigenvalue of $H(\tau)$ in terms of the m-function of H is an essential ingredient in recovering the generalized Jacobi matrices from two spectra.

Proposition 3.6. *A complex number $\lambda_j(\tau)$ is an eigenvalue of $H(\tau)$ of multiplicity r_j iff $\lambda_j(\tau)$ is a root of the equation $\tau m(\lambda) + 1 = 0$ of multiplicity r_j.*

Let us sketch the proof. If H is replaced with $H(\tau)$ in the formulas (2.2), (2.7) and (2.8) then, using the second equality in (3.4), one proves the assertion. For a detailed proof of a much more general statement we refer the reader to [10, p. 11].

4. Recovery of a generalized Jacobi matrix from its Weyl function

The aim of this section is twofold. Firstly, we are concerned with a procedure to reconstruct the generalized Jacobi matrix from its Weyl functions. Secondly, a bond between the Weyl functions of the submatrices of H is established.

It is appropriate to begin with a definition of an important class of the Weyl functions. First, however, let us denote by $G_{[j,m]}$ the submatrix of G corresponding to the basis vectors $\{e_{i,k}\}_{k=0,\dots,k_i-1}^{i=j,\dots,m}$ $(0\le j\le m\le N)$.

Definition 4.1. ([9]). Define the function $m_+(\lambda,j)$ by the equality

$$m_+(\lambda,j)=\left\langle (H_{[j,N]}-\lambda)^{-1}e_{j,0},e_{j,0}\right\rangle_{\ell^2_{[n_j,n]}(G_{[j,N]})}\quad (j=0,\dots,N). \tag{4.1}$$

Remark 4.2. The vectors $e_{j,k}$ in (4.1) are considered as elements of the space

$$\ell^2_{[n_j,n]}=\text{span}\,\{e_{i,k}|i=j,\dots,N;k=0,1,\dots,k_i-1\}.$$

In view of Proposition 3.5, $m(\lambda)=m_+(\lambda,0)$ and $m_+(\lambda,N)=-\frac{\varepsilon_N}{p_N(\lambda)}$. Clearly, $m_+(\lambda,j)$ is the m-function of the generalized Jacobi matrix $H_{[j,N]}$. So, by virtue of the formula (3.4) and Proposition 2.4, one has

$$m_+(\lambda,j)=-\varepsilon_j\frac{\psi_j(\lambda)}{\widetilde{b}_{j-1}\psi_{j-1}(\lambda)}=-\varepsilon_j\frac{\det(\lambda-H_{[j+1,N]})}{\det(\lambda-H_{[j,N]})}\quad (j=0,\dots,N). \tag{4.2}$$

Combining (4.2) with (2.2), (2.7), we get the Riccati equation

$$\varepsilon_j\frac{1}{m_+(\lambda,j)}+\varepsilon_jb_j^2m_+(\lambda,j+1)=-p_j(\lambda)\quad (j=0,\dots,N-1). \tag{4.3}$$

Proposition 4.3. ([9]). *The function $m_+(\lambda,j)$ uniquely determines $H_{[j,N]}$.*

Proof. Due to (4.3), $m_+(\lambda,j)$ admits the following representation as a finite continued fraction:

$$m_+(\lambda,j)=-\cfrac{\varepsilon_j}{p_j(\lambda)-}\,\cfrac{\varepsilon_j\varepsilon_{j+1}b_j^2}{p_{j+1}(\lambda)-}\quad\cdots\quad-\cfrac{\varepsilon_{N-1}\varepsilon_Nb_{N-1}^2}{p_N(\lambda)}. \tag{4.4}$$

Hence, sequences of the polynomials $\{p_i\}_{i=j}^N$ and the numbers $\{\varepsilon_i\}_{i=j}^N$, $\{b_i\}_{i=j}^{N-1}$ to recover $H_{[j,N]}$ are determined. □

Remark 4.4. A continued fraction of the form (4.4) is called a P-fraction. The P-fractions were presented by A. Magnus in [22]. According to [22] (see also [7]), every real proper rational function $\varphi(\not\equiv 0)$ admits P-fraction expansion (4.4). Therefore, such a function φ turns out to be an m-function of a unique generalized Jacobi matrix associated with the sequences of polynomials and numbers up to a constant factor (see [9]).

Definition 4.5. Define the function $m_-(\lambda,j)$ by the equality

$$m_-(\lambda,j)=\left\langle (H_{[0,j-1]}-\lambda)^{-1}e_{j-1,0},e_{j-1,0}\right\rangle_{\ell^2_{[0,n_j-1]}(G_{[0,j-1]})}\quad (j=1,\dots,N+1). \tag{4.5}$$

Remark 4.6. The vectors $e_{j,k}$ in (4.5) are considered as elements of the space

$$\ell^2_{[0,n_{j-1}]}=\text{span}\,\{e_{i,k}|i=0,\dots,j-1;k=0,1,\dots,k_i-1\}.$$

Remark 4.7. Let S_{j-1} be a nondensely defined symmetric operator in $\ell^2_{[0,n_j-1]}(G_{[0,j-1]})$ having the graph

$$\text{gr}\,S_{j-1}=\{\{f,H_{[0,j-1]}f\}:f\,\langle\perp\rangle\,,f\in\ell^2_{[0,n_j-1]}(G_{[0,j-1]})\}.$$

Then, by the same reasoning as in Section 2, the function $m_-(\lambda,j)$ is the Weyl function for S_{j-1} corresponding to the boundary triplet $\Pi=\{\mathbb{C},\Gamma_0,\Gamma_1\}$, where

$$\Gamma_0\widehat{f}=-c,\quad \Gamma_1\widehat{f}=\langle f,e_{j-1,0}\rangle\quad \widehat{f}=\{f,H_{[0,j-1]}f+ce_{j-1,0}\}\in S_{j-1}^+.$$

Precisely the same argument as we have just used for $m_+(\lambda, j)$ yields

$$(4.6)\qquad \begin{aligned} m_-(\lambda, j) &= -\varepsilon_{j-1}\frac{P_{j-1}(\lambda)}{b_{j-1}P_j(\lambda)} = -\varepsilon_{j-1}\frac{\det(\lambda - H_{[0,j-2]})}{\det(\lambda - H_{[0,j-1]})} \quad (j = 1, \dots, N+1),\\ \varepsilon_j \frac{1}{m_-(\lambda, j+1)} &+ \varepsilon_j b_{j-1}^2 m_-(\lambda, j) = -p_j(\lambda) \quad (j = 1, \dots, N). \end{aligned}$$

Similarly, $m_-(\lambda, j)$ has the following P-fraction expansion:

$$(4.7)\qquad m_-(\lambda, j) = -\frac{\varepsilon_{j-1}}{p_{j-1}(\lambda) -}\,\frac{\varepsilon_{j-1}\varepsilon_j b_j^2}{p_{j-2}(\lambda) -}\ \cdots\ -\frac{\varepsilon_1\varepsilon_0 b_0^2}{p_0(\lambda)}.$$

In analogy with Proposition 4.3, (4.7) leads to the following uniqueness result.

Proposition 4.8. *The function $m_-(\lambda, j)$ uniquely determines $H_{[0,j-1]}$.*

The following inverse theorem is a slight generalization of the assertion stated in Remark 4.4.

Theorem 4.9. *Let $\phi(\not\equiv 0)$ be a real-valued rational function. Then there exist a symmetric operator S having the form* (3.1) *and a boundary triplet Π such that the corresponding Weyl function is proportional to ϕ.*

Proof. Observe that if ϕ is a real proper rational function the statement is implied by Remark 4.4 and Proposition 3.5. Now, let $\phi = g/h$ be a real improper rational function (i.e. $\deg g \geq \deg h$). Then one of the rational functions

$$(4.8)\qquad \phi_1(\lambda) = -\frac{h(\lambda)}{g(\lambda)}, \quad \phi_2(\lambda) = \frac{g(\lambda)}{h(\lambda)} - c \quad (c = \lim_{x\to+\infty} f(x))$$

is always proper. We already know that the theorem is true for that function. So, one can construct a symmetric operator S and a boundary triplet Π such that the corresponding Weyl function is proportional to the proper rational function in (4.8). It remains to note that if $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for S^+ then the triplets

$$\Pi_1 = \{\mathcal{H}, -\Gamma_1, \Gamma_0\}, \quad \Pi_2 = \{\mathcal{H}, \Gamma_0, \Gamma_1 - c\Gamma_0\}, \quad c \in \mathbb{R}\setminus\{0\},$$

are also boundary triplets for S^+. □

The following theorem reveals the an implicit relation between $m_+(\lambda, j)$ and $m_-(\lambda, j)$.

Theorem 4.10. *Let λ_k be an eigenvalue of H and let r_k be its multiplicity. Then for any $j \in \{1, \dots, N\}$ the following equalities hold:*

$$(P_j(\lambda)\psi_{j-1}(\lambda) - P_{j-1}(\lambda)\psi_j(\lambda))^{(i)}\Big|_{\lambda=\lambda_k} = 0 \quad (i = 0, \dots, r_k - 1).$$

Proof. The determinant $\det(\lambda - H)$, by applying the Laplace theorem to its first n_j rows, is reduced to

$$(4.9)\quad \det(\lambda - H_{[0,j-1]})\det(\lambda - H_{[j,N]}) - b_{j-1}\widetilde{b}_{j-1}\det(\lambda - H_{[0,j-2]})\det(\lambda - H_{[j+1,N]}).$$

Due to (2.4), (2.8) and (4.9), $\det(\lambda - H)$ has the representation

$$(4.10)\qquad \det(\lambda - H) = b_0 \dots b_{j-1}\widetilde{b}_{j-1}\dots\widetilde{b}_{N-1}\left(P_j(\lambda)\psi_{j-1}(\lambda) - P_{j-1}(\lambda)\psi_j(\lambda)\right).$$

Now, the theorem follows from (4.10) and the fact that λ_k is a root of $\det(\lambda - H)$ possessing multiplicity r_k. □

5. Uniqueness results for problems with mixed given data

Here we extend the scheme proposed in [5] to prove analogs of the Hochstadt–Lieberman theorem in the case of the matrices under discussion.

To prepare for the proof, let us examine some elementary properties of rational functions. Recall that a rational function f is a ratio of two relatively prime polynomials g and h. By a monic rational function we mean a rational function $f = g/h$, where the polynomials g and h are monic. Let $f_1 = g_1/h_1$ and $f_2 = g_2/h_2$ be two rational functions. We will say that f_1 and f_2 agree on some set $\Lambda_d = \{\lambda_0, \dots, \lambda_{d-1}\} = \{\underbrace{\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_1}_{r_1}, \dots, \underbrace{\widetilde{\lambda}_k, \dots \widetilde{\lambda}_k}_{r_k}\} \in \mathbb{C}^d$ (here $\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_k$ are all the different numbers of Λ_d and $r_1, \dots, r_k$ are the multiplicities of $\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_k$ in Λ_d, respectively) if the following equalities hold:

$$\Big(g_1(\lambda)h_2(\lambda) - g_2(\lambda)h_1(\lambda)\Big)^{(j)}\Big|_{\lambda=\widetilde{\lambda}_i} = 0 \quad (j = 0, \dots, r_i - 1, i = 1, \dots, k).$$

The subsequent lemmas give sufficient conditions for two rational functions to be equal.

Lemma 5.1. *Assume that* $f_1 = \dfrac{g_1}{h_1}$ *and* $f_2 = \dfrac{g_2}{h_2}$, *where* $\deg g_1 = \deg g_2$ *and* $\deg h_1 = \deg h_2$. *Let* $d = \deg g_1 + \deg h_1$.

1) *If* f_1 *and* f_2 *agree on* Λ_{d+1} *then* $f_1 \equiv f_2$*;*
2) *If* f_1 *and* f_2 *are both monic and they agree on* Λ_d *then* $f_1 \equiv f_2$.

Proof. To prove the first statement, consider the polynomial $R(\lambda) = g_1(\lambda)h_2(\lambda) - g_2(\lambda)h_1(\lambda)$. By the assumption, the degree of R is not greater than d. Since f_1 and f_2 agree on Λ_{d+1}, $R(\lambda)$ is divisible by $\widetilde{R}(\lambda) = (\lambda - \widetilde{\lambda}_1)^{r_1} \dots (\lambda - \widetilde{\lambda}_k)^{r_k}$. So, we have $\deg R \le d < d + 1 = \deg \widetilde{R}$. The latter implies $R(\lambda) \equiv 0$. This proves the desired assertion. The second statement can be proved in the same way as the first one. □

Lemma 5.2. *Let* $f = \dfrac{g}{h}$ *be a rational function. Then, under the assumption of Lemma 5.1, one has*

1) *If* f_1 *and* f_2 *agree with* f *on* Λ_{d+1} *then* $f_1 \equiv f_2$*;*
2) *If* f_1 *and* f_2 *are both monic and they agree with* f *on* Λ_d *then* $f_1 \equiv f_2$.

Proof. Let us consider the polynomials $R_1(\lambda) = g(\lambda)h_1(\lambda) - g_1(\lambda)h(\lambda)$ and $R_2(\lambda) = g(\lambda)h_2(\lambda) - g_2(\lambda)h(\lambda)$. Since f_1 and f_2 agree with f on Λ_{d+1} (or Λ_d), we see that

$$R_1^{(j)}(\lambda)\Big|_{\lambda=\widetilde{\lambda}_i} = 0, \quad R_2^{(j)}(\lambda)\Big|_{\lambda=\widetilde{\lambda}_i} = 0 \quad (j = 0, \dots, r_i - 1, i = 1, \dots, k). \tag{5.1}$$

Fix $i \in \{1, \dots, k\}$. First, assume that $h(\widetilde{\lambda}_i) \ne 0$. Using (5.1) with $j = 0$, one concludes that $h_1(\widetilde{\lambda}_i) \ne 0$ and $h_2(\widetilde{\lambda}_i) \ne 0$. So, the identities (5.1) yield

$$\Big(f_1 - f_2\Big)^{(j)}\Big|_{\lambda=\widetilde{\lambda}_i} = 0. \tag{5.2}$$

The equality (5.2) can be rewritten as follows:

$$\Big(g_1(\lambda)h_2(\lambda) - g_2(\lambda)h_1(\lambda)\Big)^{(j)}\Big|_{\lambda=\widetilde{\lambda}_i} = 0. \tag{5.3}$$

On the other hand, if $h(\widetilde{\lambda}_i) = 0$ then $g(\widetilde{\lambda}_i) \ne 0$. Arguing for the functions $f^{-1}, f_1^{-1}, f_2^{-1}$ as above, we again derive (5.3). Finally, Lemma 5.1 completes the proof. □

Now, we are ready to prove the following theorem on unique recovery of a generalized Jacobi matrix from given mixed data.

Theorem 5.3. *Let H be a generalized Jacobi matrix and let the order $n+1$ of H and the natural number $t_l := 2\left(n+1-\sum_{i=l}^{N} k_i\right) - k_{l-1}$ $(l \in \mathbb{N})$ be given.*

1) If $t_l \le n+1$ then $\widetilde{B}_{l-1}, A_l, \dots, \widetilde{B}_{N-1}, A_N$ and any t_l numbers belonging to $\sigma(H)$ uniquely determine $A_0, \widetilde{B}_0, \dots, \widetilde{B}_{l-2}, A_{l-1}$;

2) If $t_l \le n$ then $A_l, \widetilde{B}_l, \dots, \widetilde{B}_{N-1}, A_N$ and any t_l+1 numbers belonging to $\sigma(H)$ uniquely determine $A_0, \widetilde{B}_0, \dots, A_{l-1}, \widetilde{B}_{l-1}$.

Proof. According to (4.6) and Proposition 2.5, the sum of the degrees of the denominator and the numerator of $m_-(\lambda, l)$ is equal to t_l.

1) Let Λ_{t_l} be a given subset of $\sigma(H)$. To be definite, assume that $\varepsilon_N = 1$. Knowing the matrices $\widetilde{B}_{l-1}, A_l, \dots, A_N$ enables us to calculate ε_l and $m_+(\lambda, l)$. Taking into account (4.2), (4.6) and Theorem 4.10, we may conclude that the monic rational function $-\varepsilon_{l-1} m_-(\lambda, l)$ agrees with $-(\varepsilon_l b_{l-1}\widetilde{b}_{l-1} m_+(\lambda, l))^{-1}$ on Λ_{t_l}. So, by part 2) of Lemma 5.2, $\varepsilon_{l-1} m_-(\lambda, l)$ is uniquely determined. Finally, it follows from Proposition 4.8 that $m_-(\lambda, l)$ uniquely determines $H_{[0,l-1]}$, that is, the matrices $A_0, \widetilde{B}_0, \dots, \widetilde{B}_{l-2}, A_{l-1}$.

2) On account of (4.2), (4.6) and Theorem 4.10, the functions $\varepsilon_{l-1} b_{l-1}\widetilde{b}_{l-1} m_-(\lambda, l)$ and $(\varepsilon_l m_+(\lambda, l))^{-1}$ agree on the given subset Λ_{t_l+1} of $\sigma(H)$. Therefore, a uniqueness of $\varepsilon_{l-1} b_{l-1}\widetilde{b}_{l-1} m_-(\lambda, l)$ is implied by part 1) of Lemma 5.2. Due to the formula (4.6), $\varepsilon_{l-1} b_{l-1}\widetilde{b}_{l-1} m_-(\lambda, l)$ admits the following asymptotic expansion:

$$\varepsilon_{l-1} b_{l-1}\widetilde{b}_{l-1} m_-(\lambda, l) = -\frac{b_{l-1}\widetilde{b}_{l-1}}{\lambda^{k_{j-1}}} + o\left(\frac{1}{\lambda^{k_{j-1}+1}}\right), \quad \lambda = iy, \quad y \to +\infty.$$

Now, we use the above asymptotic expansion to determine $\widetilde{B}_{l-1}$. In view of Proposition 4.8, $\varepsilon_{l-1} m_-(\lambda, l)$ uniquely determines the generalized Jacobi matrix $H_{[0,l-1]}$ or, equivalently, the matrices $A_0, \widetilde{B}_0, \dots, \widetilde{B}_{l-2}, A_{l-1}$. □

Example 5.4. Let us consider the generalized Jacobi matrix

$$H = \begin{pmatrix} A_0 & \widetilde{B}_0 & 0 & 0 \\ B_0 & A_1 & \widetilde{B}_1 & 0 \\ 0 & B_1 & A_2 & \widetilde{B}_2 \\ 0 & 0 & B_2 & A_3 \end{pmatrix}$$

having the order $n+1$. Given are the entries $\widetilde{B}_1, A_2, \widetilde{B}_2, A_3$ of H and the spectrum $\sigma(H) = \{\lambda_0, \dots, \lambda_n\}$ of H. Then $l = 2$, $k_0 = n+1-k_1-k_2-k_3$, and $t_2 = 2k_0 + k_1$. Assume that $t_2 > n+1$, i.e. $k_0 > k_2 + k_3$. Since the assumptions of Theorem 5.3 are not satisfied, we want to study the uniqueness of the generalized Jacobi matrix with the given data. To reconstruct the matrices A_0, $\widetilde{B}_0$, A_1, one needs to represent $\det(\lambda - H)$ in the following form (see (4.6), (4.10) and Proposition 4.8):

$$\det(\lambda - H) = \phi_1(\lambda)\psi_1(\lambda) - \phi_2(\lambda)\psi_2(\lambda). \tag{5.4}$$

Here ϕ_1 and ϕ_2 are real relatively prime polynomials of degree k_0+k_1 and k_0, respectively; ψ_1 and ψ_2 are given by (2.8) ($\deg\psi_1 = k_2+k_3$, $\deg\psi_2 = k_1+k_2+k_3$). Additionally assuming $k_0 \ge k_1+k_2+k_3$, we see that for any polynomial ϕ such that $\deg\phi \le \max\{k_0 - k_2 - k_3, k_0 - k_1 - k_2 - k_3\}$ the characteristic polynomial of H is decomposed as follows:

$$\det(\lambda - H) = (\phi_1(\lambda) + \phi(\lambda)\psi_2(\lambda))\psi_1(\lambda) - (\phi_2(\lambda) + \phi(\lambda)\psi_1(\lambda))\psi_2(\lambda),$$

where $\phi_1(\lambda) + \phi(\lambda)\psi_2(\lambda)$ and $\phi_2(\lambda) + \phi(\lambda)\psi_1(\lambda)$ are real polynomials of degree $k_0 + k_1$ and k_0, respectively. It suffices to choose ϕ in the following way:

$$\phi_1(\lambda_j) + \phi(\lambda_j)\psi_2(\lambda_j) \ne 0 \quad (j = 0, \dots, n) \tag{5.5}$$

for the polynomials $\phi_1(\lambda)+\phi(\lambda)\psi_2(\lambda)$ and $\phi_2(\lambda)+\phi(\lambda)\psi_1(\lambda)$ to be relatively prime. In view of Proposition 2.5, $\psi_2(\lambda_j)\neq 0$ $(j=0,\dots,n)$. Therefore, the polynomial $\phi\equiv c$ satisfies (5.5) for sufficiently large $c\in\mathbb{R}$. This shows that the decomposition (5.4) is not unique. However, every representation (5.4) gives rise to the generalized Jacobi matrix with the prescribed data: $\sigma(H)$ and $\widetilde{B}_1, A_2, \widetilde{B}_2, A_3$.

6. Trace formulas

Our goal in this section is to derive trace formulas for generalized Jacobi matrices. The trace formula for a selfadjoint operator in a Hilbert space was first studied in details by M. G. Kreĭn (see [19]). P. Jonas [18] has obtained the trace formula for some class of nonselfadjoint operators. Notice that the trace formulas considered in the present paper are particular cases of the Jonas trace formula. However, the proof below is unrelated to that in [18]. Namely, in our considerations we essentially use the structure of the matrices in question and, moreover, an explicit form of the trace formulas is given.

Theorem 6.1. *Let H be a generalized Jacobi matrix. Then the following formulas hold:*

$$\operatorname{tr} A_0^l = \operatorname{tr} H^l - \operatorname{tr} H^l_{[1,N]} \quad (1\le l<k_0+k_1) \tag{6.1}$$

$$\operatorname{tr} A_0^{k_0+k_1} + (k_0+k_1)b_0\widetilde{b}_0 = \operatorname{tr} H^{k_0+k_1} - \operatorname{tr} H^{k_0+k_1}_{[1,N]}. \tag{6.2}$$

Proof. Let us consider the function Q given by

$$Q(\lambda) := \ln(-\varepsilon_0\lambda^{k_0}m(\lambda)), \tag{6.3}$$

where ln denotes the principal branch of logarithm, i.e., $\ln e^{i\phi}=\ln r+i\phi$ $(0<\phi<2\pi)$. In view of the monodromy theorem, Q is well defined in the exterior of the circle with a sufficiently large radius and the center at the origin. To obtain (6.1) and (6.2), we find the asymptotic expansion of Q near $\lambda=\infty$ in two ways. Firstly, rewrite the Riccati equation (4.3) as follows:

$$m(\lambda) = -\varepsilon_0\frac{1}{p_0(\lambda)+\varepsilon_0 b_0^2 m_+(\lambda,1)}, \tag{6.4}$$

where $m_+(\lambda,1)$ is the m-function of $H_{[1,N]}$. Substituting (6.4) into (6.3), one has

$$Q(\lambda) = \ln\left(\frac{\lambda^{k_0}}{p_0(\lambda)+\varepsilon_0 b_0^2 m_+(\lambda,1)}\right) = -\ln\frac{p_0(\lambda)}{\lambda^{k_0}} - \ln\left(1+\varepsilon_0 b_0^2\frac{m_+(\lambda,1)}{p_0(\lambda)}\right). \tag{6.5}$$

Since p_0 is a monic polynomial and $m_+(\lambda,1)$ is defined by (3.4), we see that

$$\frac{m_+(\lambda,1)}{p_0(\lambda)} = -\frac{\varepsilon_1}{\lambda^{k_0+k_1}} + o\left(\frac{1}{\lambda^{k_0+k_1}}\right) \quad (|\lambda|\to+\infty). \tag{6.6}$$

Let $\mu_1,\dots,\mu_{k_0}$ be all the roots of p_0 (or, equivalently, the eigenvalues of A_0). Then (6.5) can be rewritten as

$$Q(\lambda) = -\sum_{j=1}^{k_0}\ln\left(1-\frac{\mu_j}{\lambda}\right) - \ln\left(1+\varepsilon_0 b_0^2\frac{m_+(\lambda,1)}{p_0(\lambda)}\right). \tag{6.7}$$

Now, the Maclaurin formula for $y(x)=\ln(1+x)$ and (6.6), (6.7) yield

$$Q(\lambda) \sim \frac{\sum_{j=1}^{k_0}\mu_j}{\lambda} + \dots + \frac{\sum_{j=1}^{k_0}\mu_j^{k_0+k_1-1}}{(k_0+k_1-1)\lambda^{k_0+k_1-1}} + \frac{\sum_{j=1}^{k_0}\mu_j^{k_0+k_1} + (k_0+k_1)\varepsilon_0\varepsilon_1 b_0^2}{(k_0+k_1)\lambda^{k_0+k_1}} \tag{6.8}$$

as $|\lambda|\to+\infty$. On the other hand, by using (3.4) and (2.8), the m-function is represented in the form

$$m(\lambda) = -\varepsilon_0\frac{\prod_{i=k_0}^{n}(\lambda-\nu_i)}{\prod_{i=0}^{n}(\lambda-\lambda_i)}, \tag{6.9}$$

where $\{\nu_i\}_{i=k_0}^n$ and $\{\lambda_j\}_{j=0}^n$ are the eigenvalues of $H_{[1,N]}$ and H, respectively. Combining (6.9) with (6.3), one arrives at

$$Q(\lambda)=\sum_{i=k_0}^{n}\ln\left(1-\frac{\nu_i}{\lambda}\right)-\sum_{j=0}^{n}\ln\left(1-\frac{\lambda_j}{\lambda}\right).$$

As above, the well-known asymptotic expansion of $\ln(1+x)$ near $x=0$ leads to

$$Q(\lambda)\sim\frac{\sum_{j=0}^n\lambda_j-\sum_{i=k_0}^n\nu_i}{\lambda}+\cdots+\frac{\sum_{j=0}^n\lambda_j^{k_0+k_1}-\sum_{i=k_0}^n\nu_i^{k_0+k_1}}{(k_0+k_1)\lambda^{k_0+k_1}}\tag{6.10}$$

as $|\lambda|\to+\infty$. Since for any $\alpha,\beta,\gamma\in\mathbb{N}$ the following equalities hold:

$$\sum_{j=1}^{k_0}\mu_j^\alpha=\operatorname{tr}A_0^\alpha,\quad\sum_{j=0}^{n}\lambda_j^\beta=\operatorname{tr}H^\beta,\quad\sum_{j=k_0}^{n}\nu_j^\gamma=\operatorname{tr}H_{[1,N]}^\gamma,$$

comparing (6.10) with (6.8) yields (6.1) and (6.2). □

The trace formulas enable us to derive explicit formulas for the entries of A_0, $\widetilde{B}_0$, B_0. Suppose that the spectra $\sigma(H)$ and $\sigma(H_{[1,N]})$ are known. To reconstruct A_0, it is sufficient to know the coefficients of its characteristic polynomial $p_0(\lambda)=\lambda^{k_0}+p_{k_0-1}^{(0)}\lambda^{k_0-1}+\cdots+p_1^{(0)}\lambda+p_0^{(0)}$. Let $\mu_1,\dots,\mu_{k_0}$ be all the roots of p_0. According to (6.1), one calculates the following sums

$$\mu_1^l+\cdots+\mu_{k_0}^l=\operatorname{tr}A_0^l\quad(1\le l\le k_0).$$

Further, Newton's identities (see [26, the formula (1.2.14)]) imply

$$\operatorname{tr}A_0^l+p_{k_0-1}^{(0)}\operatorname{tr}A_0^{l-1}+\cdots+p_{k_0-l+1}^{(0)}\operatorname{tr}A_0+lp_{k_0-l}^{(0)}=0\quad(1\le l\le k_0).$$

Expressing $p_k^{(0)}$ from the latter system, one arrives at the following recurrence relations:

$$p_{k_0-1}^{(0)}=-\operatorname{tr}A_0,\dots,p_0^{(0)}=-\frac{1}{k_0}(\operatorname{tr}A_0^{k_0}+p_{k_0-1}^{(0)}\operatorname{tr}A_0^{k_0-1}+\cdots+p_1^{(0)}\operatorname{tr}A_0).\tag{6.11}$$

So, A_0 has just been reconstructed. Finally, $b_0\widetilde{b}_0$ can be obtained from (6.2). Since $b_0>0$ and $|b_0|=|\widetilde{b}_0|$, $b_0=|b_0\widetilde{b}_0|^{1/2}$. Therefore, B_0 and $\widetilde{B}_0$ have been determined.

Generally speaking, the spectra of H and $H_{[1,N]}$ uniquely determine H. Besides, any two disjoint sets $\{\lambda_i\}_{i=0}^n$ and $\{\nu_i\}_{i=k_0}^n$ of complex numbers turn out to be the spectra of a unique generalized Jacobi matrix H associated with sequences of polynomials and numbers and its submatrix $H_{[1,N]}$, respectively, iff these sets are both symmetric with respect to $\mathbb{R}$ (see [9]). Also, S. M. Malamud has recently obtained necessary and sufficient conditions for the sequences $\{\lambda_i\}_{i=0}^n$ and $\{\nu_i\}_{i=1}^n$ to be the spectra of a normal $(n+1)\times(n+1)$-matrix and its submatrix of order n, respectively (see [26], [27]).

7. Reconstruction of a generalized Jacobi matrix from two spectra

The generalized Jacobi matrix $H(\tau)$ defined by (3.6) is a rank one perturbation of H. In view of Proposition 3.6, the spectrum $\sigma(H(\tau))$ can be found by H and τ. Here we consider the inverse problem of recovering H and τ from two spectra $\sigma(H)$ and $\sigma(H(\tau))$. Mimicking the scheme considered in [5], it is appropriate to begin with the following uniqueness result.

Theorem 7.1. *The spectrum $\sigma(H)=\{\lambda_0,\dots,\lambda_n\}$, any $n-k_0+1$ numbers belonging to $\sigma(H(\tau))=\{\lambda_0(\tau),\dots,\lambda_n(\tau)\}$, and $\tau\in\mathbb{R}\setminus\{0\}$ uniquely determine H.*

Proof. According to (3.4), the m-function m of H is represented as follows:

$$m(\lambda) = -\varepsilon_0 \frac{g(\lambda)}{h(\lambda)},$$

where g and h are monic polynomials of degree $n-k_0+1$ and $n+1$, respectively. Notice that $h(\lambda) = \prod_{j=0}^{n}(\lambda-\lambda_j)$ is known. Without loss of generality, it can be assumed that $\varepsilon_0 = 1$. Let $\widetilde{\lambda}_1(\tau), \dots, \widetilde{\lambda}_k(\tau)$ be a set of all the different numbers belonging to a given subset of $\sigma(H(\tau))$ and let $r_1, \dots, r_k$ be its multiplicities, respectively. So, $r_1 + \dots + r_k = n - k_0 + 1$. Proposition 3.6 gives the possibility to calculate the following collection of jets of g at the points $\widetilde{\lambda}_j(\tau)$

$$g(\widetilde{\lambda}_1(\tau)), \dots, g^{(r_1-1)}(\widetilde{\lambda}_1(\tau)), \dots, g(\widetilde{\lambda}_k(\tau)), \dots, g^{(r_k-1)}(\widetilde{\lambda}_k(\tau)). \tag{7.1}$$

As is known, the jets (7.1) uniquely determine a monic polynomial g of degree $n-k_0+1$ by means of the Hermite-Lagrange interpolation formula. Hence, the given data uniquely determine m. It remains to apply Proposition 3.6. □

Now, we are in a position to formulate the straightforward analog of the Borg result.

Theorem 7.2. *The spectra $\sigma(H)$ and $\sigma(H(\tau))(\neq \sigma(H))$ uniquely determine H and $\tau \in \mathbb{R}\setminus\{0\}$.*

Proof. In view of definition (3.6), the matrix $H(\tau)$ differs from H in the only one entry. Namely, $A_0(\tau)$ is given by

$$A_0(\tau) = \begin{pmatrix} 0 & \dots & 0 & -p_0^{(0)}+\tau \\ 1 & & \mathbf{0} & -p_1^{(0)} \\ & \ddots & & \vdots \\ \mathbf{0} & & 1 & -p_{k_0-1}^{(0)} \end{pmatrix}.$$

So, the formula (6.11) implies

$$\operatorname{tr} A_0 = \operatorname{tr} A_0(\tau), \dots, \operatorname{tr} A_0^{k_0-1} = \operatorname{tr} A_0^{k_0-1}(\tau), \quad \operatorname{tr} A_0^{k_0} = \operatorname{tr} A_0^{k_0}(\tau) + k_0\tau. \tag{7.2}$$

Taking into account (6.1) for the matrices H and $H(\tau)$, (7.2) yields

$$\operatorname{tr} H - \operatorname{tr} H(\tau) = 0, \dots, \operatorname{tr} H^{k_0-1} - \operatorname{tr} H^{k_0-1}(\tau) = 0, \quad \operatorname{tr} H^{k_0} - \operatorname{tr} H^{k_0}(\tau) = k_0\tau. \tag{7.3}$$

The identities (7.3) allow us to determine k_0 as a number of the first nonvanishing relation. So, τ is obtained from (7.3). Now, the previous theorem completes the proof. □

Acknowledgments. I express my gratitude to V. A. Derkach for guidance in the work and many useful discussions. I am also grateful to M. M. Malamud for his attention to this work and valuable remarks.

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Received 07/11/2005