Methods of Functional Analysis and Topology
Vol. 12 (2006), no. 3, pp. 243–253

FINITE RANK SELF-ADJOINT PERTURBATIONS

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Abstract. Finite rank perturbations of a semi-bounded self-adjoint operator A are studied. Different types of finite rank perturbations (regular, singular, mixed singular) are described from a unique point of view and by the same formula with the help of quasi-boundary value spaces. As an application, a Schrödinger operator with nonlocal point interactions is considered.

1. Introduction

Let A be a semibounded self-adjoint operator acting on a separable Hilbert space $\mathcal{H}$ with an inner product $(\cdot,\cdot)$, and let $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\ker A$ denote the domain, the range, and the null-space of A, respectively. Without loss of generality, we will assume that $A \geq I$.

We recall that a self-adjoint operator $\widetilde{A} \neq A$ acting on $\mathcal{H}$ is called *a finite rank perturbation* of A if the difference $(\widetilde{A}-zI)^{-1}-(A-zI)^{-1}$ is a finite rank operator on $\mathcal{H}$ for at least one point $z \in \mathbb{C}\setminus\mathbb{R}$ [11].

If $\widetilde{A}$ is a finite rank perturbation of A, then the corresponding symmetric operator

$$A_{\rm sym} = A\restriction_{\mathcal{D}} = \widetilde{A}\restriction_{\mathcal{D}}, \quad \mathcal{D} = \{u \in \mathcal{D}(A)\cap\mathcal{D}(\widetilde{A}) \mid Au = \widetilde{A}u\} \tag{1.1}$$

arises naturally (here the symbol $A\restriction_X$ means the restriction of A onto the set X). This operator has finite and equal deficiency numbers.

It is important that the operator $A_{\rm sym}$ can uniquely be recovered from its defect subspace $N = \mathcal{H}\ominus\mathcal{R}(A_{\rm sym})$ and the initial operator A. One has namely,

$$A_{\rm sym} = A\restriction_{\mathcal{D}(A_{\rm sym})}, \quad \mathcal{D}(A_{\rm sym}) = \{u \in \mathcal{D}(A) \mid (Au,\eta) = 0, \ \forall \eta \in N\}. \tag{1.2}$$

Moreover, the choice of an arbitrary finite dimensional subspace N of $\mathcal{H}$ as a defect subspace allows one to determine by (1.2) a closed symmetric operator $A_{\rm sym}$ with finite and equal defect numbers. To stress on this relation, *we will use notation A_N instead of* $A_{\rm sym}$.

Let us denote by $\mathcal{P}(A_N)$ the set of all self-adjoint extensions of A_N. By the Krein's resolvent formula, any $\widetilde{A} \in \mathcal{P}(A_N)$ is a finite rank perturbation of A.

A finite rank perturbation $\widetilde{A}$ of A is called *regular* if $\mathcal{D}(A) = \mathcal{D}(\widetilde{A})$. Otherwise (i.e, $\mathcal{D}(A) \neq \mathcal{D}(\widetilde{A})$), the operator $\widetilde{A}$ is called *a singular perturbation*.

It is convenient to split the class of singular perturbations into two subclasses. Precisely, we will say that a singular perturbation $\widetilde{A}$ is *purely singular* if the symmetric operator $A_{\rm sym} = A_N$ defined by (1.1) is densely defined (i.e., $N \cap \mathcal{D}(A) = \{0\}$) and *mixed singular* if A_N is nondensely defined (i.e., $N \cap \mathcal{D}(A) \neq \{0\}$).

Thus, the class of finite rank perturbations of A can be subdivided into mutually disjoint subclasses of regular, mixed singular, and purely singular perturbations.

2000 *Mathematics Subject Classification.* 47A10, 47A55, 81Q10.
Key words and phrases. Self-adjoint and quasi-adjoint operators, boundary value spaces, singular, mixed singular, and regular perturbations.

The aim of the present paper consists in a development of a unique approach to the study of finite rank perturbations of a self-adjoint operator A that enables one to describe self-adjoint operator realizations of regular, mixed singular, and pure singular perturbations of A from a simple point of view and by the same formula.

It is well known that the method of boundary value spaces (BVS) provides a convenient approach to the description of purely singular perturbations [10], [14].

The description of mixed singular perturbations of A is more complicated because the corresponding symmetric operator A_N is nondensely defined and, hence, the adjoint of A_N does not exist. To overcome this problem, a certain generalization of the concept of BVS is required. The key point here is the replacement of the adjoint operator A_N^* with a suitable object. In [9], [15], the operator A_N and its 'adjoint' are understood as linear relations and a description of all self-adjoint *relations* that are extensions of the graph of A_N was obtained. In [14], a pair of maximal dissipative extensions of A_N and its adjoint (maximal accumulative extension) was used instead of A_N^*. This allows one to describe self-adjoint extensions directly as operators without using linear relations technique.

The approaches mentioned above are general and they can be applied to an arbitrary nondensely defined symmetric operator with arbitrary defect numbers. However, in a particular case where A_N is determined as a restriction of the initial self-adjoint operator A, it is natural to use A for the description of extensions of A_N (see [6], [7], [12]). In [2], developing the ideas proposed recently in [3], [16], the operator A was used for the determination of a quasi-adjoint operator of A_N.

The concept of quasi-adjoint operators allows one to modify the definition of BVS in such a form (quasi-BVS) that would permit to describe regular and mixed singular perturbations of A in just the same way as purely singular perturbations.

In conclusion we remark that the characteristic feature of quasi-BVS extension theory consists in the description of *essentially* self-adjoint extensions of A_N (i.e., those extensions that turn out to be self-adjoint after closure). This property is convenient for the description of self-adjoint differential expressions with complicated boundary conditions.

2. The case of purely singular perturbations

In what follows, without loss of generality, we assume that A is a self-adjoint operator acting on $\mathcal{H}$ and such that $A \geq I$.

Let N be an arbitrary finite dimensional subspace of $\mathcal{H}$ and $N \cap \mathcal{D}(A) = \{0\}$. In this case, the symmetric operator

$$A_N = A \restriction_{\mathcal{D}(A_N)}, \quad \mathcal{D}(A_N) = \{u \in \mathcal{D}(A) \mid (Au, \eta) = 0, \ \forall \eta \in N\} \tag{2.1}$$

is densely defined in $\mathcal{H}$ and the set $\mathcal{P}(A_N)$ of self-adjoint extensions of A_N consists of purely singular perturbations of A.

The set $\mathcal{P}(A_N)$ admits a convenient description in terms of boundary value spaces (see [10], [14] and references therein).

Definition 1. A triple $(\mathfrak{N}, \Gamma_0, \Gamma_1)$, where $\mathfrak{N}$ is an auxiliary Hilbert space and Γ_0, Γ_1 are linear mappings of $\mathcal{D}(A^*_{\mathrm{sym}})$ into $\mathfrak{N}$, is called a boundary value space (BVS) of A_N if the abstract Green identity

$$(A_N^* f, g) - (f, A_N^* g) = (\Gamma_1 f, \Gamma_0 g)_{\mathfrak{N}} - (\Gamma_0 f, \Gamma_1 g)_{\mathfrak{N}}, \quad f, g \in \mathcal{D}(A_N^*) \tag{2.2}$$

is satisfied and the map $(\Gamma_0, \Gamma_1) : \mathcal{D}(A_N^*) \to \mathfrak{N} \oplus \mathfrak{N}$ is surjective.

Here A_N^* is the adjoint of A_N. Its domain has the form $\mathcal{D}(A_N^*) = \mathcal{D}(A) \dot{+} N$ and

$$A_N^* f = A_N^*(u + \eta) = Au, \quad \forall f = u + \eta \in \mathcal{D}(A_N^*) \quad (u \in \mathcal{D}(A), \ \eta \in N). \tag{2.3}$$

A boundary value space for A_N always exists and the dimension of $\mathfrak{N}$ coincides with $\dim N$. Furthermore [10]

$$\mathcal{D}(A_N) = \ker \Gamma_0 \cap \ker \Gamma_1. \tag{2.4}$$

Another important property of a BVS $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ consists in the fact that the formulas

$$A_i := A_N^* \restriction_{\mathcal{D}(A_i)}, \quad \mathcal{D}(A_i) = \ker \Gamma_i \quad (i = 0, 1) \tag{2.5}$$

determine self-adjoint extensions A_1 and A_2 of A_N such that

$$\mathcal{D}(A_1) \cap \mathcal{D}(A_2) = \mathcal{D}(A_N), \quad \mathcal{D}(A_1) + \mathcal{D}(A_2) = \mathcal{D}(A_N^*) \tag{2.6}$$

(the transversality property). In the case where $\dim N < \infty$, equalities of (2.6) are equivalent.

The next theorem describes elements of $\mathcal{P}(A_N)$ with the help of BVS.

Theorem 2.1. ([10]). *Let $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ be a* BVS *of A_N. Then any $\widetilde{A} \in \mathcal{P}(A_N)$ coincides with the restriction of A_N^* onto*

$$\mathcal{D}(\widetilde{A}) = \{f \in \mathcal{D}(A_N^*) \mid (I - U)\Gamma_0 f = i(I + U)\Gamma_1 f\}, \tag{2.7}$$

where U is a unitary operator in $\mathfrak{N}$. Moreover, the correspondence $\widetilde{A} \leftrightarrow U$ is a bijection between the sets of all self-adjoint extensions of A_N and all unitary operators in $\mathfrak{N}$.

In cases where self-adjoint extensions are described by sufficiently complicated boundary conditions (see, e.g., [5], [13],), the representation (2.7) is not always convenient because it contains the same unitary factor U on the both sides. A more natural way is to present $\mathcal{D}(\widetilde{A})$ in (2.7) as follows:

$$\mathcal{D}(\widetilde{A}) = \{f \in \mathcal{D}(A_{\mathrm{sym}}^*) \mid B\Gamma_0 f = C\Gamma_1 f\}, \tag{2.8}$$

where B and C are bounded operators (matrices) in the finite dimensional space $\mathfrak{N}$.

Of course, for arbitrary B and C, formula (2.8) determines an extension $\widetilde{A}$ of A_N not necessarily self-adjoint. In order to preserve the self-adjointness of $\widetilde{A}$ it is necessary to impose certain commutation conditions on B and C (see, e.g., [1], [17], [13]). It is important that these conditions are always satisfied if B is an arbitrary self-adjoint operator on $\mathfrak{N}$ and $C = I$. In other words, the formula

$$\widetilde{A} := A_N^* \restriction_{\mathcal{D}(\widetilde{A})}, \quad \mathcal{D}(\widetilde{A}) = \{f \in \mathcal{D}(A_N^*) \mid B\Gamma_0 f = \Gamma_1 f\} \tag{2.9}$$

determines a self-adjoint extension $\widetilde{A}$ of A_N for any choice of a self-adjoint operator B on $\mathfrak{N}$. However, the set $\mathcal{L}(A_N)$ of self-adjoint extensions of A_N described by (2.9) is only a part of $\mathcal{P}(A_N)$. Precisely,

$$\widetilde{A} \in \mathcal{L}(A_N) \iff \widetilde{A} \in \mathcal{P}(A_N) \quad \text{and} \quad \mathcal{D}(\widetilde{A}) \cap \ker \Gamma_0 = \mathcal{D}(A_N). \tag{2.10}$$

To overcome this inconvenience, we use another approach that enables to remove the factor C in (2.8) but, simultaneously, to preserve the description of all self-adjoint extensions of A_N. The main idea here consists in the use of a family BVS $(\mathfrak{N}, \Gamma_0^R, \Gamma_1)$ instead of a fixed BVS $(\mathfrak{N}, \Gamma_0, \Gamma_1)$, where R is an additional self-adjoint operator parameter.

Let $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ be a fixed BVS of A_N. Then, for any self-adjoint operator R acting on $\mathfrak{N}$, the triple $(\mathfrak{N}, \Gamma_0^R, \Gamma_1)$, where

$$\Gamma_0^R = \Gamma_0 - R\Gamma_1 \tag{2.11}$$

is also a BVS of A_N. Similarly to (2.9) and (2.10), the expression

$$A_{B,R} := A_N^* \restriction_{\mathcal{D}(A_{B,R})}, \quad \mathcal{D}(A_{B,R}) = \{f \in \mathcal{D}(A_N^*) \mid B\Gamma_0^R f = \Gamma_1 f\}, \tag{2.12}$$

where B is an arbitrary self-adjoint operator on $\mathfrak{N}$, determines a subset $\mathcal{L}_R(A_N)$ of the set $\mathcal{P}(A_N)$ of all self-adjoint extensions of A_N and $A_{B,R} \in \mathcal{L}_R(A_N) \iff \mathcal{D}(A_{B,R}) \cap \ker \Gamma_0^R = \mathcal{D}(A_N)$.

Theorem 2.2. *Let R be an arbitrary invertible self-adjoint operator acting on $\mathfrak{N}$. Then*

$$\mathcal{P}(A_N) = \bigcup_{i=0}^{n} \mathcal{L}_{R_i}(A_N) \quad (n = \dim \mathfrak{N} = \dim N)$$

where $R_0 = a_0 R,\ R_1 = a_1 R,\ \ldots,\ R_n = a_n R$ and $a_i > 0$ $(i = 0, \ldots, n)$ are different numbers.

Proof. Let $\widetilde{A}$ be an arbitrary self-adjoint extension of A_N. Then $\mathcal{D}(\widetilde{A}) = \mathcal{D}(\widetilde{A}) \cap \ker \Gamma_1 \dot{+} \mathcal{M}$, where the lineal $\mathcal{M}$ satisfies the condition $0 \le \dim \mathcal{M} \le n$ $(n = \dim N)$.

Denote $\mathfrak{N}' = \Gamma_1 \mathcal{D}(\widetilde{A}) = \Gamma_1 \mathcal{M}$. It is clear that $\dim \mathcal{M} = \dim \mathfrak{N}'$. Thus, $0 \le \dim \mathfrak{N}' \le n$.

Let us assume that there exists $\widetilde{A} \in \mathcal{P}(A_N)$ such that $\widetilde{A} \notin \bigcup_{i=0}^{n} \mathcal{L}_{R_i}(A_N)$. Then, by (2.10),

$$\mathcal{M}_{R_i} = \mathcal{D}(\widetilde{A}) \cap \ker \Gamma_0^{R_i} \supset \mathcal{D}(A_N)$$

for any $i = 0, 1, \ldots, n$. By (2.11), $\Gamma_1 \mathcal{M}_{R_i}$ $(i = 0, \ldots, n)$ are nontrivial subspaces of $\mathfrak{N}'$. Our aim now is to show their linear independence. To do this, we consider the subspace $M = \{< \Gamma_0 f, \Gamma_1 f > | \forall f \in \mathcal{M}\}$ of the space $\mathfrak{N} \dot{+} \mathfrak{N}$. Since the equality $\Gamma_1 f = 0$ $(f \in \mathcal{M})$ implies $f = 0$ and, hence, $\Gamma_0 f = 0$, the subspace M can be represented as follows: $M = \{< \widetilde{R} h, h > | \forall h \in \mathfrak{N}'\}$, where $\widetilde{R} : \mathfrak{N}' \to \mathfrak{N}$.

Let $h_i \in \Gamma_1 \mathcal{M}_{R_i}$ $(i = 0, 1 \ldots n)$ and let

$$h_0 + h_1 + \ldots + h_n = 0. \tag{2.13}$$

By the definition of $\mathcal{M}_{R_i}$ and (2.11), $\widetilde{R} h_i = R_i h_i = a_i R h_i$. But then, applying k times the operator $\widetilde{R}$ to the both parts of (2.13), we get $R^k \sum_{i=0}^{n} a_i^k h_i = 0$. Since R is invertible,

$$a_0^k h_0 + a_1^k h_1 + \ldots + a_n^k h_n = 0$$

for any positive integer k. The latter equality and (2.13) are possible only in the case where $h_0 = \ldots = h_n = 0$. So, we arrive at the conclusion that the $n+1$ *nontrivial* subspaces $\Gamma_1 \mathcal{M}_{R_i}$ of the space $\mathfrak{N}'$ with the dimension $\dim \mathfrak{N}' \le n$ are linearly independent. The obtained contradiction implies that $\mathcal{P}(A_N) = \bigcup_{i=0}^{n} \mathcal{L}_{R_i}(A_N)$. Theorem 2.2 is proved. □

By Theorem 2.2, formula (2.12), where operators B and R play a role of parameters, provides a description of all self-adjoint extensions of A_N. However the correspondence between parameters B, R and self-adjoint extensions of A_N is not one-to-one. For example, the operator A_1 defined by (2.5) belongs to $\mathcal{L}_R(A_N)$ for any R.

3. The case of mixed singular perturbations

3.1. **The concept of quasi-BVS.** In the case of mixed singular perturbations, the operator A_N defined by (2.1) is nondensely defined (i.e., $\mathcal{D}(A) \cap N \neq \{0\}$). Hence, the adjoint A_N^* does not exist as a uniquely defined operator and we need some generalization of the concept of the BVS to describe all self-adjoint extensions of A_N.

Let us suppose that *there exists an integer* $m > 1$ such that $N \cap \mathcal{D}(A^m) = \{0\}$. Then, the direct sum

$$\mathcal{L}_m := \mathcal{D}(A^m) \dot{+} N \tag{3.1}$$

is well defined and we can define on $\mathcal{L}_m$ a quasi-adjoint operator $A_N^{(*)}$ by the formula

$$A_N^{(*)} f = A_N^{(*)}(u + \eta) = Au, \quad \forall f = u + \eta \in \mathcal{L}_m \quad (u \in \mathcal{D}(A^m),\ \eta \in N). \tag{3.2}$$

Formula (3.2) is an analog of formula (2.3) for the adjoint operator A_N^* and we can use $A_N^{(*)}$ as an analog of the adjoint one.

In general, $A_N^{(*)}$ is not closable and it turns out to be closable only if A_N is densely defined.

The concept of quasi-adjoint operators allows to modify Definition 1 and to extend it to the case of nondensely defined symmetric operators.

Definition 2. ([2]). A triple $(\mathfrak{N}, \Gamma_0, \Gamma_1)$, where Γ_i are linear mappings of $\mathcal{L}_m$ in an auxiliary Hilbert space $\mathfrak{N}$, is called a quasi-BVS of A_N if the abstract Green identity

$$(A_N^{(*)}f, g) - (f, A_N^{(*)}g) = (\Gamma_1 f, \Gamma_0 g)_{\mathfrak{N}} - (\Gamma_0 f, \Gamma_1 g)_{\mathfrak{N}}, \quad \forall f, g \in \mathcal{L}_m \tag{3.3}$$

is satisfied and the map $(\Gamma_0, \Gamma_1) : \mathcal{L}_m \to \mathfrak{N} \oplus \mathfrak{N}$ is surjective.

Proposition 3.1. *If A_N is densely defined, then an arbitrary* BVS $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ *of A_N is a quasi-*BVS.

Proof. It suffices to establish the preservation of the surjective property for the restricted mapping $(\Gamma_0, \Gamma_1) \restriction_{\mathcal{L}_m}$. Obviously, (Γ_0, Γ_1) maps $\mathcal{L}_m$ onto $\mathfrak{N} \oplus \mathfrak{N}$ if and only if

$$\mathcal{L}_m + \mathcal{D}(A_N) = \mathcal{D}(A_N^*). \tag{3.4}$$

Let us verify (3.4). Since $\mathcal{D}(A^{m-1})$ is dense in $\mathcal{H}$ and $\dim N < \infty$, the relation $P_N \mathcal{D}(A^{m-1}) = N$ (P_N is the orthoprojector onto N in $\mathcal{H}$) holds for any $m \in \mathbb{N}$. This equality enables one to directly verify (with the use of (2.1)) that $A^{-1}\mathcal{D}(A^{m-1}) + \mathcal{D}(A_N) \supset A^{-1}N$. But then, the representation $\mathcal{D}(A) = \mathcal{D}(A_N) \dot{+} A^{-1}N$ and (2.3) imply

$$\mathcal{L}_m + \mathcal{D}(A_N) = A^{-1}\mathcal{D}(A^{m-1}) + \mathcal{D}(A_N) + N = \mathcal{D}(A) \dot{+} N = \mathcal{D}(A_N^*).$$

Proposition 3.1 is proved. □

Using (3.4), it is easy to verify that any quasi-BVS $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ of a densely defined operator A_N can be extended (by setting $\Gamma_i f = 0$ for all $f \in \mathcal{D}(A_N)$) to a BVS of A_N.

Thus, in the case where A_N is densely defined, the concepts of BVS and quasi-BVS determine the same class of objects.

Proposition 3.2. *If A_N is nondensely defined, then the triple $(N, \Gamma_0^R, \Gamma_1)$, where*

$$\Gamma_0^R(u + \eta) = P_N A u + R\eta, \quad \Gamma_1(u + \eta) = -\eta \quad (u \in \mathcal{D}(A^m), \quad \eta \in N), \tag{3.5}$$

*is a quasi-*BVS *of A_N for any choice of a self-adjoint operator R acting on N.*

Proof. The Green identity (3.3) for Γ_0^R and Γ_1 is a direct consequence of (3.2) and (3.5). To establish that (Γ_0^R, Γ_1) maps $\mathcal{L}_m$ onto $\mathfrak{N} \oplus \mathfrak{N}$, we observe that, for any elements $F_i \in N$, the relation $P_N \mathcal{D}(A^{m-1}) = P_N A \mathcal{D}(A^m) = N$ (proved above) guarantees the existence $f = u + \eta \in \mathcal{L}_m$ such that $P_N A u = F_0 + RF_1$ and $\eta = -F_1$. Comparing these relations with (3.5), we get $\Gamma_0^R f = F_0$ and $\Gamma_1 f = F_1$. Thus, (Γ_0^R, Γ_1) maps $\mathcal{L}_m$ onto $\mathfrak{N} \oplus \mathfrak{N}$ and, hence, $(N, \Gamma_0^R, \Gamma_1)$ is a quasi-BVS of A_N. Proposition 3.2 is proved. □

The next statement shows that the property (2.4) of BVS is extended directly to the case of quasi-BVS.

Lemma 3.1. *Let $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ be a quasi-*BVS *of a symmetric operator A_N. Then the symmetric operator*

$$A'_N = A_N^{(*)} \restriction_{\mathcal{D}(A'_N)}, \quad \mathcal{D}(A'_N) = \ker \Gamma_0 \cap \ker \Gamma_1 \tag{3.6}$$

does not depend on the choice of quasi-BVS and its closure coincides with A_N.

Proof. It is easy to see from (3.2) and (3.3) that $\ker \Gamma_0 \cap \ker \Gamma_1 \subset \mathcal{D}(A_N) \cap \mathcal{D}(A^m)$. Conversely, for a given $u \in \mathcal{D}(A_N) \cap \mathcal{D}(A^m)$, according to Definition 2, we can choose an element $g \in \mathcal{L}_m$ such that $\Gamma_0 g = \Gamma_1 u$, $\Gamma_1 g = \Gamma_0 u$. In this case,

$$0 = (Au, g) - (u, A_N^{(*)}g) = \|\Gamma_1 u\|^2 + \|\Gamma_0 u\|^2.$$

Therefore, $\Gamma_0 u = \Gamma_1 u = 0$. Thus

$$\mathcal{D}(A'_N) = \ker \Gamma_0 \cap \ker \Gamma_1 = \mathcal{D}(A_N) \cap \mathcal{D}(A^m).$$

Hence, $\mathcal{D}(A'_N)$ does not depend on the choice of quasi-BVS $(\mathfrak{N}, \Gamma_0, \Gamma_1)$.

To prove that the closure of A'_N coincides with A_N, it is sufficient to establish that the set $\mathcal{R}(A'_N)$ is dense in $\mathcal{R}(A_N)$. Using the evident representations

$$\mathcal{R}(A_N) = \mathcal{H} \ominus N, \qquad \mathcal{R}(A'_N) = (\mathcal{H} \ominus N) \cap \mathcal{D}(A^{m-1}),$$

we reduce the proof of this assertion to the well-known fact (see, e.g., [14, Section 3, Proposition 2.6]) that $(\mathcal{H} \ominus N) \cap \mathcal{D}(A^{m-1})$ is a dense set in $\mathcal{H} \ominus N$. Lemma 3.1 is proved. □

Let $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ be a quasi-BVS of A_N. We will say that a self-adjoint operator B in $\mathfrak{N}$ is *admissible* with respect to $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ if the equation

$$B\Gamma_0 f = \Gamma_1 f, \quad \forall f \in \mathcal{D}(A_N) \cap \mathcal{L}_m \tag{3.7}$$

has only the trivial solution $\Gamma_0 f = \Gamma_1 f = 0$.

If A_N is densely defined, then $\mathcal{D}(A_N) \cap \mathcal{L}_m = \mathcal{D}(A_N) \cap \mathcal{D}(A^m) = \mathcal{D}(A'_N)$ and, by virtue of (3.6), any self-adjoint operator B in $\mathfrak{N}$ is admissible. Otherwise (A_N is nondensely defined),

$$\mathcal{D}(A_N) \cap \mathcal{L}_m = \mathcal{D}(A'_N) \dot{+} \mathcal{F}, \tag{3.8}$$

where $\dim \mathcal{F} = \dim(N \cap \mathcal{D}(A))$. Vectors $f \in \mathcal{F}$ have the form $f = u + \eta$, where η is an arbitrary element of $N \cap \mathcal{D}(A)$ and $u \in \mathcal{D}(A^m)$ is determined by η with the help of relation $P_N A(u + \eta) = 0$ (this determination is unique modulo $\mathcal{D}(A'_N)$). Comparing (3.8) and Lemma 3.1, we arrive at the conclusion that the condition of admissibility takes away the lineal $\mathcal{F}$ from the set of solutions of (3.7).

Theorem 3.1. *Let $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ be a quasi-*BVS *of A_N and let B be an admissible operator with respect to $(\mathfrak{N}, \Gamma_0, \Gamma_1)$. Then, the closure of the symmetric operator*

$$\widetilde{A}' = A_N^{(*)} \restriction_{\mathcal{D}(\widetilde{A}')}, \quad \mathcal{D}(\widetilde{A}') = \{f \in \mathcal{L}_m \mid B\Gamma_0 f = \Gamma_1 f\} \tag{3.9}$$

is a self-adjoint extension of A_N.

A self-adjoint extension $\widetilde{A}$ of A_N can be represented as the closure of a certain essentially self-adjoint operator $\widetilde{A}'$ defined by (3.9) if and only if $\mathcal{D}(\widetilde{A}) \cap \ker \Gamma_0 = \mathcal{D}(A'_N)$.

Proof. Since B is a self-adjoint operator, formula (3.3) yields that $\widetilde{A}'$ is a symmetric extension of A'_N. Furthermore, there exists a linear subspace $\mathcal{M}$ of $\mathcal{L}_m$ such that $\dim \mathcal{M} = \dim \mathfrak{N} = \dim N$ and

$$\mathcal{D}(\widetilde{A}') = \mathcal{D}(A'_N) \dot{+} \mathcal{M}. \tag{3.10}$$

It follows from the property of admissibility of B and (3.10) that $\mathcal{M} \cap \mathcal{D}(A_N) = 0$. The latter relation and Lemma 3.1 mean that $\widetilde{A}'$ is closable and its closure $\widetilde{A}$ is a symmetric operator defined by the formula

$$\widetilde{A} f = \widetilde{A}(u + m) = A_N u + A_N^{(*)} m, \quad \forall u \in \mathcal{D}(A_N), \quad \forall m \in \mathcal{M} \tag{3.11}$$

on the set $\mathcal{D}(\widetilde{A}) = \mathcal{D}(A_N) \dot{+} \mathcal{M}$.

Since $\dim \mathcal{M} = \dim N$, the defect numbers of $\widetilde{A}$ in the upper (lower) half plane are equal to 0 and, hence, $\widetilde{A}$ is a self-adjoint extension of A_N. Thus, the closure of $\widetilde{A}'$ defined by (3.9) is a self-adjoint extension of A_N.

Conversely, let $\widetilde{A}$ be a self-adjoint extension of A_N. Our aim now is to show that the domain $\mathcal{D}(\widetilde{A})$ of $\widetilde{A}$ can be decomposed as

$$\mathcal{D}(\widetilde{A}) = \mathcal{D}(A_N) \dot{+} \mathcal{M}, \tag{3.12}$$

where $\mathcal{M} \subset \mathcal{L}_m$ and $\dim \mathcal{M} = \dim N$.

Let $\widetilde{A}$ be invertible. Then, using the well-known representation

$$\mathcal{D}(\widetilde{A}) = \{f = u + CP_N Au \mid \forall u \in \mathcal{D}(A)\} \quad (C = \widetilde{A}^{-1} - A^{-1} : N \to N) \tag{3.13}$$

and the fact that $P_N\mathcal{D}(A^{m-1}) = N$, we easy get the decomposition (3.12) by choosing $\mathcal{M}$ as the linear span $< m_1 \dots m_n >$ of elements $m_j = u_j + CP_NAu_j$ $(j = 1 \dots n)$, where $u_j \in \mathcal{D}(A^m)$ are determined in such a way that the elements P_NAu_j $(j = 1 \dots n)$ form a basis of N. Moreover, as follows from (3.13) and (3.2) $\widetilde{A}m_j = \widetilde{A}(u_j + CP_NAu_j) = Au_j = A_N^{(*)}m_j \quad (j = 1 \dots n)$ that justify (3.11).

The determination of $\mathcal{M}$ presented above needs some trivial modification if $\widetilde{A}^{-1}$ does not exist. We indicate the principal steps only.

At first, there exists $a > 0$ such that $\widetilde{A} + aI$ is invertible. Repeating the arguments presented above, we get $\mathcal{D}(\widetilde{A}) = \mathcal{D}(A_N)\dot{+}\mathcal{M}_a$, where

$$\mathcal{M}_a \subset \mathcal{L}_m(a) := \mathcal{D}(A^m)\dot{+}N_a \quad (N_a = \mathcal{H} \ominus \mathcal{R}(A_N + aI))$$

and $\mathcal{M}_a =< g_1 \dots g_n >$, where $g_j = v_j + C_aP_{N_a}(A + aI)v_j$ and $v_j \in \mathcal{D}(A^m)$ are determined in such a way that the elements P_NAv_j $(j = 1 \dots n)$ form a basis of N_a.

Repeating the proof of Proposition 3.1 for the cases where the operators A and $A+aI$ are considered, we arrive at the conclusion that

$$\mathcal{L}_m + \mathcal{D}(A_N) = \mathcal{D}(A) \dot{+} N \quad \text{and} \quad \mathcal{L}_m(a) + \mathcal{D}(A_N) = \mathcal{D}(A) \dot{+} N_a.$$

However, elements of N_a and N are related as follows:

$$\eta_a \in N_a \iff \eta_a = A(A + aI)^{-1}\eta = \eta - a(A + aI)^{-1}\eta, \quad \eta \in N.$$

This means that

$$\mathcal{L}_m + \mathcal{D}(A_N) = \mathcal{L}_m(a) + \mathcal{D}(A_N).$$

The latter equality enables to decompose the basis vectors g_j of $\mathcal{M}_a$ as $g_j = f_j + z_j$ $(f_j \in \mathcal{L}_m,\ z_j \in \mathcal{D}(A_N))$. Set $\mathcal{M} =< f_1 \dots f_n >$. Since the linear span $\mathcal{M}_a$ of g_j has a trivial intersection with $\mathcal{D}(A_N)$, we get $\mathcal{M} \cap \mathcal{D}(A_N) = \{0\}$. But then $\mathcal{D}(\widetilde{A}) = \mathcal{D}(A_N)\dot{+}\mathcal{M}$, which proves equality (3.12).

It follows from (3.2), (3.11), and (3.12) that the symmetric operator $\widetilde{A}' = \widetilde{A} \restriction_{\mathcal{D}(\widetilde{A}) \cap \mathcal{L}_m}$ coincides with the restriction of $A_N^{(*)}$ onto the domain $\mathcal{D}(\widetilde{A}')$ defined by (3.10) and $\widetilde{A}'$ is an essentially self-adjoint restriction of $\widetilde{A}$.

Further, it is easy to see that the domain (3.10) admits the representation (3.9) with a certain operator B acting on $\mathfrak{N}$ if and only $\mathcal{D}(\widetilde{A}) \cap \ker \Gamma_0 = \mathcal{D}(A_N')$. In this case, the admissibility of B follows from the relation $\mathcal{M} \cap \mathcal{D}(A_N) = 0$. The self-adjointness of B is a direct consequence of (3.3) and the property of $\widetilde{A}' = A_N^{(*)} \restriction_{\mathcal{D}(\widetilde{A}) \cap \mathcal{L}_m}$ to be a symmetric operator. Theorem 3.1 is proved. □

Remarks. **1.** If B is not admissible, then the domain $\mathcal{D}(\widetilde{A}')$ defined by (3.9) has a nontrivial intersection with $\mathcal{F}$ and $\widetilde{A}'$ is not closable.

2. Since (3.9) does not determine all self-adjoint extensions of A_N, a situation where any operator B is admissible in (3.9) is possible. Namely, the following simple statement is true.

Proposition 3.3. *If* $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ *is a quasi-*BVS *of* A_N *such that* $\ker \Gamma_0 \supset \mathcal{D}(A_N) \cap \mathcal{L}_m$, *then the closure of* $\widetilde{A}'$ *determined by (3.9) is a self-adjoint extension of* A_N *for any self-adjoint operator* B *acting on* $\mathfrak{N}$.

Proof. If $\ker \Gamma_0 \supset \mathcal{D}(A_N) \cap \mathcal{L}_m$, then the equation $B\Gamma_0 f = \Gamma_1 f$ $(f \in \mathcal{D}(A_N) \cap \mathcal{L}_m)$ has only the trivial solution $\Gamma_0 f = \Gamma_1 f = 0$ and, hence, any self-adjoint operator B is admissible with respect to $(\mathfrak{N}, \Gamma_0, \Gamma_1)$. Proposition 3.3 is proved. □

Example. *A Schrödinger operator with nonlocal point interactions.* Let us consider a Schrödinger operator that is determined with the help of an additive mixed singular perturbation

$$-\frac{d^2}{dx^2} + b_{11} < \cdot, \delta > \delta + b_{12}(\cdot, q)\delta + b_{21} < \cdot, \delta > q + b_{22}(\cdot, q)q, \tag{3.14}$$

where $< \cdot, \delta >$ is the Dirac δ-function, a function $q \in L_2(\mathbb{R})$, and coefficients $b_{ij} \in \mathbb{C}$ form an Hermitian matrix $B = (b_{ij})$. In our case, $A = -d^2/dx^2 + I$, $\mathcal{D}(A) = W_2^2(\mathbb{R})$ and the defect subspace $N \subset L_2(\mathbb{R})$ is the linear span of the functions $\eta_1(x) = A^{-1}\delta = \frac{1}{2}e^{-|x|}$, $\eta_2(x) = A^{-1}q(x)$.

For the sake of simplicity, we assume that the function $q(x)$ coincides with the fundamental solution $\mathfrak{m}_{2k}(x)$ $(k \geq 1)$ of the equation $(-d^2/dx^2 + I)^k \mathfrak{m}_{2k}(x) = \delta$. In this case, $\eta_1 = \mathfrak{m}_2$, $\eta_2 = \mathfrak{m}_{2k+2}$.

Let us fix $m = k+1$, then, according to (3.1),

$$\mathcal{L}_m = W_2^{2k+2}(\mathbb{R}) \dot{+} N \subset W_2^{2k+2}(\mathbb{R}\backslash\{0\}).$$

Assume that $f(x) \in \mathcal{L}_m$. By the description of $\mathcal{L}_m$, the derivative $f'(x)$ of $f(x)$ has right(left)-side limits at the point 0. This is also true for the function $f^{[2k+1]}(x) := \frac{d}{dx}(-\frac{d^2}{dx^2} + I)^k f(x)$ $(x \neq 0)$.

Denote by $f'_s = f'(+0) - f'(-0)$, $f_s^{[2k+1]} = f^{[2k+1]}(+0) - f^{[2k+1]}(-0)$ the jumps of the functions $f'(x)$ and $f^{[2k+1]}(x)$ at the point $x = 0$.

It is easy to see that an arbitrary function $f \in \mathcal{L}_m$ admits the representation

$$f(x) = u(x) - f'_s \mathfrak{m}_2(x) - f_s^{[2k+1]} \mathfrak{m}_{2k+2},$$

where $u(x) \in W_2^{2k+2}(\mathbb{R})$.

By a direct verification, we get that the triple $(\mathbb{C}^2, \Gamma_0, \Gamma_1)$, where

$$\Gamma_0 f(x) = \begin{pmatrix} f(0) \\ (f, \mathfrak{m}_2) \end{pmatrix}, \quad \Gamma_1 f(x) = \begin{pmatrix} f'_s \\ f_s^{[2k+1]} \end{pmatrix}, \quad \forall f(x) \in \mathcal{L}_m$$

is a quasi-BVS of A_N.

In our case, all conditions of Proposition 3.3 are satisfied and, hence, the restriction of $A_N^{(*)}$ $(A_N^{(*)} f(x) = -f''(x) + f(x),\ x \neq 0)$ onto the collection of functions $f \in \mathcal{L}_m$ that are specified by the boundary conditions

$$f'_s = b_{11} f(0) + b_{12}(f, \mathfrak{m}_2), \quad f_s^{[2k+1]} = b_{21} f(0) + b_{22}(f, \mathfrak{m}_2)$$

is an essentially self-adjoint operator on $L_2(\mathbb{R})$. The closure of such an operator has the form $A_q + I$, where A_q is a self-adjoint realization of the heuristic expression (3.14).

The operator A_q can be interpreted as a Schrödinger operator with nonlocal point interaction [4]. Its domain $\mathcal{D}(A_q)$ consists of all functions $f \in W_2^2(\mathbb{R}\backslash\{0\})$ that satisfy the boundary conditions $f_s = 0$, $f'_s = b_{11} f(0) + b_{12}(f, q)$ and the action of $A_q f$ is determined as follows [4]:

$$A_q f = -f''(x) + b_{21} q(x) f(0) + b_{22}(f, q) q(x), \quad x \neq 0.$$

3.2. **Description of mixed singular perturbations.** In the case where A_N is nondensely defined, the set $\mathcal{P}(A_N)$ consists of mixed singular perturbations of A. It is convenient to describe $\mathcal{P}(A_N)$ in terms of quasi-BVS.

Let $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ be a fixed quasi-BVS of A_N. Repeating the proof of Proposition 3.2, we arrive at the conclusion that the triple $(\mathfrak{N}, \Gamma_0^R, \Gamma_1)$, where

$$\Gamma_0^R = \Gamma_0 - R\Gamma_1 \tag{3.15}$$

is also a quasi-BVS of A_N for any self-adjoint operator R acting on $\mathfrak{N}$. (In fact, a family of quasi-BVS of such a type was presented in Proposition 3.2.)

We remark that the family of quasi-BVS $(\mathfrak{N}, \Gamma_0^R, \Gamma_1)$ determined by (3.15) is sufficiently representative. Namely, the next simple statement can be justified similarly to the case of BVS (for details, see the proof of Proposition 1 in [8]).

Proposition 3.4. *If* $(\mathfrak{N}, \widetilde{\Gamma}_0, \Gamma_1)$ *is a quasi-*BVS *of* A_N *with the same auxiliary space* $\mathfrak{N}$ *and the boundary operator* Γ_1 *as in the family* $(\mathfrak{N}, \Gamma_0^R, \Gamma_1)$*, then* $\widetilde{\Gamma}_0 = \Gamma_0^R$ *for such a choice of* R.

Denote by $\mathcal{L}_R(A'_N)$ the set of all essentially self-adjoint extensions $A'_{B,R}$ of A'_N that are defined by the expression

$$A_{B,R} := A_N^{(*)} \restriction_{\mathcal{D}(A'_{B,R})}, \quad \mathcal{D}(A'_{B,R}) = \{f \in \mathcal{D}(A_N^{(*)}) \mid B\Gamma_0^R f = \Gamma_1 f\} \tag{3.16}$$

(here B runs over the collection of operators admissible with respect to $(\mathfrak{N}, \Gamma_0^R, \Gamma_1)$).

Using Theorem 3.1 and repeating the reasoning used in the proof of Theorem 2.2, we arrive at the conclusion that all assertions of Theorem 2.2 remain true for the case of quasi-BVS.

Theorem 3.2. (cf. Theorem 2.2). *Let* R *be an arbitrary invertible self-adjoint operator acting on* $\mathfrak{N}$. *Then*

$$\mathcal{P}(A_N) = \bigcup_{i=0}^{n} \mathcal{L}_{R_i}(A'_N) \quad (n = \dim \mathfrak{N} = \dim N)$$

where $R_0 = a_0 R,\ R_1 = a_1 R,\ \ldots,\ R_n = a_n R$ *and* $a_i > 0$ *(*$i = 0, \ldots, n$*) are different numbers.*

Thus, elements of $\mathcal{P}(A_N)$ can be described with the help of any family of quasi-BVS $(\mathfrak{N}, \Gamma_0^R, \Gamma_1)$ determined by (3.15). Such an approach is maximally adapted for a description of finite rank *additive* perturbations of A [2]. The next section illustrates this point.

In conclusion, we specify and supplement the obtained results for the family of quasi-BVS $(N, \Gamma_0^R, \Gamma_1)$ determined by (3.5).

Theorem 3.3. 1. *A self-adjoint operator* B *acting on* N *is admissible with respect to* $(N, \Gamma_0^R, \Gamma_1)$ *if and only if the equation*

$$BP_N A\eta = (I + BR)\eta, \quad \forall \eta \in N \cap \mathcal{D}(A) \tag{3.17}$$

has the unique solution $\eta = 0$.

2. *If the operator-parameter* R *is chosen in such a way that* $P_N A\eta = R\eta$ *for all* $\eta \in N \cap \mathcal{D}(A)$*, then any self-adjoint operator* B *is admissible with respect to* $(N, \Gamma_0^R, \Gamma_1)$.

3. *Let* B *be an admissible operator with respect to* $(N, \Gamma_0^R, \Gamma_1)$*. Then the closure* $A_{B,R}$ *of*

$$A'_{B,R} := A_N^{(*)} \restriction_{\mathcal{D}(A'_{B,R})}, \quad \mathcal{D}(A'_{B,R}) = \{f \in \mathcal{D}(A_N^{(*)}) \mid B\Gamma_0^R f = \Gamma_1 f\}, \tag{3.18}$$

is invertible (i.e., $0 \in \rho(A_{B,R})$*) if and only if* $\ker(BR + I) = \{0\}$.

4. *If* $R = 0$*, then the formula (3.18), where* B *is an arbitrary self-adjoint operator such that* $\ker(BP_N A - I) \restriction_{N \cap \mathcal{D}(A)} = \{0\}$*, provides a description of all invertible self-adjoint extensions of* A_N.

Proof. Assertion 1 follows directly from (3.5) and the description of elements of $\mathcal{F} \subset \mathcal{D}(A_N) \cap \mathcal{L}_m$. To establish assertion 2, it suffices to observe that $\Gamma_0^R f = P_N A u + R\eta = -P_N A\eta + R\eta$ for all elements $f = u + \eta \in \mathcal{F}$. Thus,

$$\ker \Gamma_0^R \supset \mathcal{F} \iff P_N A\eta = R\eta \quad \text{for all} \quad \eta \in N \cap \mathcal{D}(A).$$

Employing now Proposition 3.3, we complete the proof of assertion 2.

It follows from (3.2), (3.5), and (3.18) that

$$\ker A'_{B,R} = \{0\} \iff \ker(BR + I) = \{0\}.$$

Let $f \in \ker A_{B,R}$. Then there exists a sequence $f_n = u_n + \eta_n \in \mathcal{D}(A'_{B,R})$ such that $f_n \to f$ and $A'_{B,R} f_n = A_N^{(*)}(u_n + \eta_n) = Au_n \to 0 = Af$. Since $0 \in \rho(A)$, we get $u_n \to 0$.

Hence, $\{\eta_n\}$ forms a fundamental sequence and η_n tends to a certain element $\eta \in N$. On the other hand, the description of $\mathcal{D}(A'_{B,R})$ given in (3.18) and (3.5) imposes the following relations on u_n and η_n: $B(P_N Au_n + R\eta_n) = -\eta_n$. Passing on to the limit, we get $(BR + I)\eta = 0$, which is possible only in the case $\eta = 0$. But then $f_n \to 0$, $f = 0$ and, hence, $\ker A_{B,R} = \{0\}$. Thus the condition $\ker(BR + I) = \{0\}$ is equivalent to the invertibility of $A_{B,R}$. Assertion 3 is proved.

Since the restriction $\ker(BR + I) = \{0\}$ vanishes for $R = 0$, the formula (3.18) determines an invertible operator for any choice of the admissible operator B. In this case, the condition of admissibility of B is simplified and takes the form $\ker(BP_N A - I) \restriction_{N\cap\mathcal{D}(A)} = \{0\}$.

To complete the proof of assertion 4, it suffices to show that any invertible self-adjoint extension $\widetilde{A}$ of A_N can be represented by (3.18). This fact immediately follows from the well-known representation ([10])

$$\mathcal{D}(\widetilde{A}) = \{f = u + CP_N Au \mid \forall u \in \mathcal{D}(A)\} \tag{3.19}$$

$(C = \widetilde{A}^{-1} - A^{-1} : N \to N)$. Indeed, in this case, $\mathcal{D}(\widetilde{A}') = \mathcal{D}(\widetilde{A}) \cap \mathcal{L}_m$ is determined by (3.19), where elements u run over the set $\mathcal{D}(A^m)$. Moreover, $\Gamma_0^0 f = P_N Au$, $\Gamma_1 f = -CP_N Au$ and, hence, $B = -C = A^{-1} - \widetilde{A}^{-1}$. Theorem 3.3 is proved. □

4. The case of finite rank regular additive perturbations

Here, we are going to show that the concept of quasi-BVS enables to describe finite rank regular perturbations of A in just the same way as finite rank singular perturbations. To do this, we consider the following n-rank perturbation of A:

$$\widetilde{A} = A + V, \quad V = \sum_{i,j=1}^{n} b_{ij}(\cdot, \psi_j)\psi_i, \tag{4.1}$$

where all ψ_j belong to $\mathcal{H}$ and the coefficients $b_{ij} \in \mathbb{C}$ form an Hermitian matrix $B = (b_{ij})$.

Since V is a bounded self-adjoint operator on $\mathcal{H}$, the operator $\widetilde{A}$ is self-adjoint on the domain $\mathcal{D}(A)$. Thus, $\widetilde{A}$ is a finite rank regular additive perturbation of A.

On the other hand, we can consider $\widetilde{A}$ and A as two different self-adjoint extensions of the symmetric nondensely defined operator (cf. (2.1))

$$A_N = A \restriction_{\mathcal{D}(A_N)}, \quad \mathcal{D}(A_N) = \{u \in \mathcal{D}(A) \mid (u, \psi_j) = (Au, A^{-1}\psi_j) = 0\} \tag{4.2}$$

$(j = 1 \dots n)$. Here N is the linear span of $\eta_j = A^{-1}\psi_j$ (i.e., $N =< \eta_1, \eta_2 \dots \eta_n >$) and $N \subset \mathcal{D}(A)$.

Assume that $N \cap C^\infty(A) = \{0\}$ where $C^\infty(A) = \bigcap_{k=1}^\infty \mathcal{D}(A^k)$. In this case, there exists an integer m such that $\mathcal{D}(A^m) \cap N = \{0\}$ and we can apply the quasi-BVS approach for a description of self-adjoint extensions of A_N.

To do this, without loss of generality, we will assume that the collection $e_j = A^{-1}\psi_j$ $(j = 1 \dots n)$ forms an orthonormal basis in N. In this case, using the natural isomorphism between elements $\eta \in N$ and columns $\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \in \mathbb{C}^n$ $(\eta_j = (\eta, e_j))$, we can rewrite the family of quasi-BVS determined by (3.5) as follows: $(\mathbb{C}^n, \Gamma_0^R, \Gamma_1)$, where

$$\Gamma_0^R(u + \eta) = \begin{pmatrix} (Au, e_1) \\ \vdots \\ (Au, e_n) \end{pmatrix} + R \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}, \quad \Gamma_1(u + \eta) = -\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}.$$

Here, $f = u + \eta \in \mathcal{L}_m = \mathcal{D}(A^m) \dot{+} N$ $(u \in \mathcal{D}(A^m)$, $\eta \in N)$ and $R = (r_{ij})_1^n$ is an arbitrary Hermitian matrix.

The family of quasi-BVS $(\mathbb{C}^n, \Gamma_0^R, \Gamma_1)$ enables to describe all self-adjoint extensions of A_N (see Theorem 3.2). Moreover, by assertion 2 of Theorem 3.3, the choice of entries r_{ij} of R as $r_{ij} = (Ae_j, e_i) = (\psi_j, A^{-1}\psi_i)$ guarantees that the formula $A'_{B,R}(u+\eta) = Au$,

$$\mathcal{D}(A'_{B,R}) = \left\{ u+\eta \;\mid\; B\left[\begin{pmatrix} (Au, e_1) \\ \vdots \\ (Au, e_n) \end{pmatrix} + R\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}\right] = -\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \right\},$$

determines essentially self-adjoint extensions of A_N for an arbitrary Hermitian matrix $B = (b_{ij})$. It is easy to verify (by a direct calculation) that $A'_{B,R}$ is a restriction of the self-adjoint operator $\widetilde{A}$ determined by (4.1) (i.e., $A'_{B,R} = \widetilde{A} \restriction_{\mathcal{D}(A'_{B,R})}$).

Acknowledgments. The authors thank DFG for the financial support of the project 436 UKR 113/79.

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Received 06/03/2006