Methods of Functional Analysis and Topology
Vol. 12 (2006), no. 3, pp. 234–242

∇-FREDHOLM OPERATORS IN BANACH–KANTOROVICH SPACES

K. K. KUDAYBERGENOV

Abstract. The paper is devoted to studying ∇-Fredholm operators in Banach–Kantorovich spaces over a ring of measurable functions. We show that a bounded linear operator acting in Banach–Kantorovich space is ∇-Fredholm if and only if it can be represented as a sum of an invertible operator and a cyclically compact operator.

1. Introduction

It is well-known that one of the important notions in the theory of operator equations in Banach spaces is that of a Fredholm operator. In 1943 by M. S. Nikolsky it was proved that a bounded linear operator acting in Banach space is Fredholm if and only if it can be represented as a sum of an invertible operator and a compact operator (see [1]). In this paper we considered the ∇-Fredholm operators acting in a Banach–Kantorovich space over a ring of measurable functions. It is known [2] that every Banach–Kantorovich space over a ring measurable functions can be represented as a measurable bundle of Banach spaces. Cyclically compact sets and operators in lattice-normed spaces were introduced by Kusraev in [3] and [4], respectively. In [5] (see also [6]) a general form of cyclically compact operators in Kaplansky–Hilbert module, as well as a variant of Fredholm alternative for cyclically compact operators, are also given. In [7] it was proved that every cyclically compact operator acting in Banach–Kantorovich space over a ring measurable functions can be represented as a measurable bundle of compact operators acting in Banach spaces. For different aspects of cyclical compactness, see [8–11]. In [12] there was given a structure of modules over the ring of measurable functions, which is represented as a measurable bundle of finite dimensional spaces. Using this representation we show that every ∇-Fredholm operator acting in Banach–Kantorovich space can be represented as a measurable bundle of Fredholm operators acting in Banach spaces and prove a vector version of Nikolsky theorem for a bounded linear operators acting in Banach–Kantorovich spaces.

2. Preliminaries

Let (Ω, Σ, μ) be a measurable space with a finite measure and $L^0 = L^0(\Omega)$ be the algebra of equivalence classes of all complex measurable functions on (Ω, Σ, μ).

A complex linear space E is said to be normed by L^0 if there is a map $\|\cdot\| : E \longrightarrow L^0$ such that for any $x, y \in E, \lambda \in \mathbb{C}$, the following conditions are fulfilled: $\|x\| \geq 0$; $\|x\| = 0 \Longleftrightarrow x = 0$; $\|\lambda x\| = |\lambda|\|x\|$; $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(E, \|\cdot\|)$ is called a lattice-normed space over L^0. A lattice-normed space E is called d-decomposable if for any $x \in E$ with $\|x\| = \lambda_1 + \lambda_2$, $\lambda_1, \lambda_2 \in L^0$, $\lambda_1\lambda_2 = 0$ there exist $x_1, x_2 \in E$ such that $x = x_1 + x_2$ and $\|x_i\| = \lambda_i$, $i = 1, 2$. A net (x_α) in E is

---

2000 *Mathematics Subject Classification.* 47A13.

Key words and phrases. Banach–Kantorovich space, cyclically compact operator, ∇-Fredholm operator, measurable bundle.

(bo)-converging to $x \in E$, if $\|x_\alpha - x\| \xrightarrow{(o)} 0$ in L^0 (note that the order convergence in L^0 coincides with convergence almost everywhere).

A lattice-normed space E which is d-decomposable and complete with respect to (bo)-convergence is called a *Banach–Kantorovich space* (BKS).

It is known that every BKS E over L^0 is a module over L^0 and $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in L^0$, $x \in E$ (see [2], [4]).

We shall consider a map $X : \omega \in \Omega \to (X(\omega), \|\cdot\|_{X(\omega)})$, where $(X(\omega), \|\cdot\|_{X(\omega)})$ is a Banach space for all $\omega \in \Omega$. A function u is called a section of X if it is defined on Ω almost everywhere and takes a value $u(\omega) \in X(\omega)$ for $\omega \in \mathrm{dom}(u)$, where $\mathrm{dom}(u)$ is the domain of u.

Let L be some set of sections.

Definition 2.1. [2] (see also [6]). A pair (X, L) is called *a measurable Banach bundle* (MBB) over Ω, if

a) $\lambda_1 c_1 + \lambda_2 c_2 \in L$ for all $\lambda_1, \lambda_2 \in \mathbb{C}$, $c_1, c_2 \in L$, where
$\lambda_1 c_1 + \lambda_2 c_2 : \omega \in \mathrm{dom}(c_1) \cap \mathrm{dom}(c_2) \to \lambda_1 c_1(\omega) + \lambda_2 c_2(\omega)$;

b) the function $\|c\| : \omega \in \mathrm{dom}(c) \to \|c(\omega)\|_{X(\omega)}$ is measurable for all $c \in L$;

c) for all $\omega \in \Omega$ the set $\{c(\omega) : c \in L, \omega \in \mathrm{dom}(c)\}$ is dense in $X(\omega)$.

A section s is called simple if there exists $c_i \in L, A_i \in \Sigma, i = \overline{1, n}$, such that $s(\omega) = \sum\limits_{i=1}^{n} \chi_{A_i}(\omega) c_i(\omega)$. A section u is called measurable if there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of simple sections such that $\|s_n(\omega) - u(\omega)\|_{X(\omega)} \to 0$ for almost all $\omega \in \Omega$.

We denote by $\mathcal{M}(\Omega, X)$ the set of all measurable sections and $L^0(\Omega, X)$ denotes the factorization of this set with respect to equality almost everywhere. By $\hat{u}$ we denote the class from $L^0(\Omega, X)$, containing section $u \in \mathcal{M}(\Omega, X)$. A function $\omega \to \|u(\omega)\|_{X(\omega)}$ is measurable for all $u \in \mathcal{M}(\Omega, X)$. By $\|\hat{u}\|$ we denote the element in L^0, containing the function $\|u(\omega)\|_{X(\omega)}$.

It is known [2] that $(L^0(\Omega, X), \|\cdot\|)$ is BKS over L^0.

We denote by $\mathcal{L}^\infty(\Omega)$ the set of all bounded complex measurable functions on Ω and $L^\infty(\Omega) = \{f \in L^0 : \exists \lambda \in \mathbb{R}, \lambda > 0, |f| \leq \lambda \mathbf{1}\}$, where $\mathbf{1}$ is unit in L^0. Let

$$\mathcal{L}^\infty(\Omega, X) = \{u \in \mathcal{M}(\Omega, X) : \|u(\omega)\|_{X(\omega)} \in \mathcal{L}^\infty(\Omega)\}$$

and $L^\infty(\Omega, X) = \{\hat{u} \in L^0(\Omega, X) : \|\hat{u}\| \in L^\infty(\Omega)\}$.

The sets $\mathcal{M}(\Omega, X)$ and $\mathcal{L}^\infty(\Omega, X)$ are often identified with $L^0(\Omega, X)$ and $L^\infty(\Omega, X)$, by identifying a measurable section u and the corresponding equivalence class $\hat{u}$.

We consider a lifting $p : L^\infty(\Omega) \to \mathcal{L}^\infty(\Omega)$ (see [2]).

Definition 2.2. [2] (see also [6]). The map $\rho_X : L^\infty(\Omega, X) \to \mathcal{L}^\infty(\Omega, X)$ is called a vector valued lifting on $L^\infty(\Omega, X)$ (associated with p), if:

a) $\rho_X(\hat{u}) \in \hat{u}, \mathrm{dom}(\rho_X(\hat{u})) = \Omega$ for all $\hat{u} \in L^\infty(\Omega, X)$;

b) $\|\rho_X(\hat{u})(\omega)\|_{X(\omega)} = p(\|\hat{u}\|)(\omega)$ for all $\hat{u} \in L^\infty(\Omega, X)$;

c) $\rho_X(\hat{u} + \hat{v}) = \rho_X(\hat{u}) + \rho_X(\hat{v})$ for all $\hat{u}, \hat{v} \in L^\infty(\Omega, X)$;

d) $\rho_X(e\hat{u}) = p(e)\rho_X(\hat{u})$ for all $\hat{u} \in L^\infty(\Omega, X)$ and $e \in L^\infty(\Omega)$;

e) the set $\{\rho_X(\hat{u})(\omega) : \hat{u} \in L^\infty(\Omega, X)\}$ is dense in $X(\omega)$ for all $\omega \in \Omega$.

It is known [2, Theorem 4.4.1] that for any BKS E over L^0 there is a MBB (X, L) such that E is isometrically isomorphic to $L^0(\Omega, X)$ and on $L^\infty(\Omega, X)$ there exists a vector valued lifting such that $\{\rho_X(\hat{u})(\omega) : \hat{u} \in L^\infty(\Omega, X)\} = X(\omega)$ for all $\omega \in \Omega$.

Let ∇ be the Boolean algebra of idempotents in L^0. If $(u_\alpha)_{\alpha\in A} \subset L^0(\Omega, X)$ and $(\pi_\alpha)_{\alpha\in A}$ is a partition of the unit in ∇, then the series $\sum\limits_{\alpha} \pi_\alpha u_\alpha$ (bo)-converges in $L^0(\Omega, X)$ and its sum is called the mixing of $(u_\alpha)_{\alpha\in A}$ with respect to $(\pi_\alpha)_{\alpha\in A}$. We denote this sum by $\mathrm{mix}(\pi_\alpha u_\alpha)$. A subset $K \subset L^0(\Omega, X)$ is called cyclic, if $\mathrm{mix}(\pi_\alpha u_\alpha) \in K$ for each $(u_\alpha)_{\alpha\in A} \subset K$ and any partition of the unit $(\pi_\alpha)_{\alpha\in A}$ in ∇. For every directed set A

denote by $\nabla(A)$ the set of all partitions of the unit in ∇, which are indexed by elements of the set A. More precisely,

$$\nabla(A) = \{\nu : A \to \nabla : (\forall \alpha, \beta \in A)(\alpha \neq \beta \to \nu(\alpha) \wedge \nu(\beta) = 0) \wedge \bigvee_{\alpha \in A} \nu(\alpha) = 1\}.$$

For $\nu_1, \nu_2 \in \nabla(A)$ we put $\nu_1 \leq \nu_2 \leftrightarrow \forall \alpha, \beta \in A$, $(\nu_1(\alpha) \wedge \nu_2(\beta) \neq 0 \to \alpha \leq \beta)$. Then $\nabla(A)$ is a directed set. Let $(u_\alpha)_{\alpha \in A}$ be a net in $L^0(\Omega, X)$. For every $\nu \in \nabla(A)$ we put $u_\nu = \text{mix}(\nu(\alpha) u_\alpha)$ and obtain a new net $(u_\nu)_{\nu \in \nabla(A)}$. Every subnet of the net $(u_\nu)_{\nu \in \nabla(A)}$ is called cyclic subnet of the original net $(u_\alpha)_{\alpha \in A}$.

Definition 2.3. [4] (see also [5], [6]). A subset $K \subset L^0(\Omega, X)$ is called *cyclically compact,* if K is cyclic and every net in K has a cyclic subnet that (bo)-converges to some point of K. A subset in $L^0(\Omega, X)$ is called *relatively cyclically compact* if it is contained in a cyclically compact set.

Let X and Y be MBBs over Ω with vector valued liftings ρ_X and ρ_Y on $L^\infty(\Omega, X)$ and $L^\infty(\Omega, Y)$, respectively. A linear operator $T : L^0(\Omega, X) \to L^0(\Omega, Y)$ is called L^0-bounded, if there exists an element $c \in L^0$ such that $\|T(x)\| \leq c\|x\|$ for any $x \in L^0(\Omega, X)$. Every L^0-bounded linear operator $T : L^0(\Omega, X) \to L^0(\Omega, Y)$ is L^0-linear, i. e., $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $\alpha, \beta \in L^0, x, y \in L^0(\Omega, X)$ (see [4]).

A linear operator T is called cyclically compact, if for every bounded set B in $L^0(\Omega, X)$ the set $T(B)$ is relatively cyclically compact in $L^0(\Omega, Y)$. For a L^0-bounded linear operator T we put $\|T\| = \sup\{\|T(x)\| : \|x\| \leq \mathbf{1}\}$.

It is known [7] (see also [6, p. 530]), that for any L^0-bounded (cyclically compact) linear operator $T : L^0(\Omega, X) \to L^0(\Omega, Y)$ there is a family of bounded (compact) linear operators $\{T_\omega : X(\omega) \longrightarrow Y(\omega)\}$ such that for any $x \in L^0(\Omega, X)$ the following equality holds: $T(x)(\omega) = T_\omega(x(\omega))$ for almost all $\omega \in \Omega$. If $\|T\| \in L^\infty(\Omega)$, then $\rho_Y(T(x))(\omega) = T_\omega(\rho_X(x)(\omega))$ for all $x \in L^\infty(\Omega, X)$, $\omega \in \Omega$.

Conversely, if $\{T_\omega : X(\omega) \longrightarrow Y(\omega)\}$ is a family of bounded (compact) linear operators such that $T_\omega(x(\omega)) \in \mathcal{M}(\Omega, Y)$ for any $x \in \mathcal{M}(\Omega, X)$, then the operator $T : L^0(\Omega, X) \longrightarrow L^0(\Omega, Y)$ defined by $T(\hat{u}) = \widehat{T_\omega(u(\omega))}$ is L^0-bounded (cyclically compact).

Let $L^0(\Omega, X)^*$ be the dual space of $L^0(\Omega, X)$, i. e., the set of all L^0-bounded linear functionals from $L^0(\Omega, X)$ into L^0. For every $f \in L^0(\Omega, X)^*$ with $\|f\| \in L^\infty(\Omega)$ we put $f_\omega(\rho_X(x)(\omega)) = p(f(x))(\omega)$, $x \in L^\infty(\Omega, X)$, $\omega \in \Omega$. Then $f_\omega \in X(\omega)^{'}$ for every $\omega \in \Omega$, where $X(\omega)^{'}$ is the dual space of $X(\omega)$. Let $X^{'}(\omega) = \{f_\omega : f \in L^0(\Omega, X)^*, \|f\| \in L^\infty(\Omega)\}$, $X^{'} : \omega \to X^{'}(\omega)$, $L^{'} = \{\omega \to f_\omega : f \in L^0(\Omega, X)^*, \|f\| \in L^\infty(\Omega)\}$.

It is known [2, Theorem 4.4.7] that $(X^{'}, L^{'})$ is a MBB with vector valued lifting; $X^{'}(\omega)$ is a closed subspace in $X(\omega)^{'}$ for all $\omega \in \Omega$; $L^0(\Omega, X^{'})$ is isometrically isomorphic to $L^0(\Omega, X)^*$.

A module E over L^0 is said to be finite dimensional (or finitely generated), if there are $x_1, x_2, \ldots, x_n$ in E such that for any $x \in E$ there exist $\lambda_i \in L^0$ $(i = \overline{1, n})$ with $x = \lambda_1 x_1 + \cdots + \lambda_n x_n$. The elements $x_1, x_2, \ldots, x_n$ are called generators of E. We denote by $d(E)$ the minimal number of generators of E.

A module E over L^0 is called σ-finite-dimensional, if there exists a partition $(\pi_\alpha)_{\alpha \in A}$ of the unit in ∇ such that $\pi_\alpha E$ is finitely generated for any α. A finite-dimensional module E over L^0 is called homogeneous of type n, if for every nonzero $e \in \nabla$ we have $n = d(eE)$.

A family $\{x_1, x_2, \ldots, x_n\}$ in E is called ∇-linearly independent, if for all $\pi \in \nabla$ and $\lambda_1, \ldots, \lambda_n \in L^0$, from $\pi \sum\limits_{k=1}^{n} \lambda_k x_k = 0$ it follows that $\pi\lambda_1 = \cdots = \pi\lambda_n = 0$.

A module E is homogeneous of type n if and only if there exist generators $\{x_1, \ldots, x_n\}$ in E, consisting of ∇-linearly independent elements (see [12], Proposition 6). Such generators form a ∇-*basis* of E.

An L^0-bounded linear operator $T : L^0(\Omega, X) \to L^0(\Omega, Y)$ is called finite dimensional or finitely generated (σ-finite dimensional, homogeneous of type n) if $R(T) = \{T(x) : x \in L^0(\Omega, X)\}$ is a finite dimensional (respectively σ-finite dimensional, homogeneous of type n) submodule in $L^0(\Omega, Y)$.

Any σ-finite dimensional operator $T : L^0(\Omega, X) \to L^0(\Omega, Y)$ can be represented as $T = \sum\limits_{\alpha\in A} \pi_\alpha T_\alpha$, where $(\pi_\alpha)_{\alpha\in A}$ is a partition of the unit in ∇, $T_\alpha : L^0(\Omega, X) \to L^0(\Omega, Y)$ are homogeneous operators of finite type for all α. If T is finite dimensional, then $(\pi_\alpha)_{\alpha\in A}$ is a finite partition of the unit in ∇.

Any cyclically compact operator $T : X \to Y$ is L^0-bounded. Since the unit ball in a σ-finite dimensional module over L^0 is a cyclically compact set ([12], Corollary 2), any σ-finite dimensional operator is cyclically compact.

Now we give a definition of ∇-Fredholm operators, which was introduced by Kusraev [4] (see also [5], [6]). Let $T : L^0(\Omega, X) \to L^0(\Omega, Y)$ be a L^0-bounded linear operator.

We consider the homogeneous equations

$$T(x) = 0, \quad T^*(g) = 0$$

and, respectively, the main equation

$$T(x) = y$$

and the conjugate equation

$$T^*(g) = f.$$

An operator T is called ∇-*Fredholm,* if there exists a partition of unity $(\pi_n)_{n\in\mathbb{N}\cup\{0\}}$ in ∇ such that the following conditions are fulfilled:

1) The homogeneous equation $\pi_0 T(x) = 0$ has the only zero solution. The homogeneous conjugate equation $\pi_0 T^*(g) = 0$ has the only zero solution. The equation $\pi_0 T(x) = \pi_0 y$ is solvable and has a unique solution for a given arbitrary $y \in L^0(\Omega, Y)$. The conjugate equation $\pi_0 T^*(g) = f$ is solvable and has a unique solution for a given arbitrary $f \in L^0(\Omega, X)^*$.

2) For every $n \in \mathbb{N}$ the homogeneous equation $\pi_n T(x) = 0$ has n ∇-linearly independent solutions $x_{1,n}, \ldots, x_{n,n}$ and the homogeneous conjugate equation $\pi_n T^*(g) = 0$ has n ∇-linearly independent solutions $g_{1,n}, \ldots, g_{n,n}$.

3) The equation $T(x) = y$ is solvable if and only if $\pi_n g_{i,n}(y) = 0$ $(n \in \mathbb{N}, i \le n)$. The conjugate equation $T^*(g) = f$ is solvable if and only if $\pi_n f_{i,n}(x) = 0$ $(n \in \mathbb{N}, i \le n)$.

4) The general solution x of the equation $T(x) = y$ has the form

$$x = \sum_{n=1}^{\infty} \pi_n (x_n + \sum_{i=1}^{n} c_{i,n} x_{i,n}),$$

where x_n is a particular solution of the equation $\pi_n T(x) = \pi_n y$ and $\{c_{i,n}\}_{n\in\mathbb{N}, i\le n}$ are arbitrary elements in L^0.

The general solution g of the conjugate equation $T^*(g) = f$ has the form

$$g = \sum_{n=1}^{\infty} \pi_n (g_n + \sum_{i=1}^{n} c_{i,n} g_{i,n}),$$

where g_n is a particular solution of the equation $\pi_n T^*(g) = \pi_n f$, and $\{c_{i,n}\}_{n\in\mathbb{N}, i\le n}$ are arbitrary elements in L^0.

3. Measurable bundles of Fredholm operators

Proposition 3.1. *Let $T : L^0(\Omega, X) \to L^0(\Omega, Y)$ be a L^0-bounded linear operator. If $T_\omega : X(\omega) \to Y(\omega)$ are Fredholm operators and $\dim\ker T_\omega = n$ for almost all $\omega \in \Omega$, then*

1) *$R(T)$ is (bo)-closed in $L^0(\Omega, Y)$ and $R(T^*)$ is (bo)-closed in $L^0(\Omega, X)^*$;*

2) *$R(T) = {}^\perp\ker T^*$, where ${}^\perp\ker T^* = \{y \in L^0(\Omega, Y) : f(y) = 0, \forall f \in \ker T^*\}$;*

3) *$R(T^*) = (\ker T)^\perp$, where $(\ker T)^\perp = \{f \in L^0(\Omega, X)^* : f(x) = 0, \forall x \in \ker T\}$;*

4) *$\ker T$ and $\ker T^*$ are homogeneous of type n.*

Proof. Replacing T with $\dfrac{T}{\mathbf{1}+\|T\|}$, we may assume that $\|T\| \in L^\infty(\Omega)$. Since T_ω is a Fredholm operator for almost all $\omega \in \Omega$, we see that $R(T_\omega)$ is closed in $Y(\omega)$ for almost all $\omega \in \Omega$. Therefore [13, Theorem 2] implies 1), 2), and 3).

4) Put $N(\omega) = \{\rho_X(x)(\omega) : x \in \ker T \bigcap L^\infty(\Omega, X)\}$. Let $x \in \ker T \bigcap L^\infty(\Omega, X)$. We have $T_\omega(\rho_X(x)(\omega)) = \rho_Y(T(x))(\omega) = \rho_Y(0)(\omega) = 0$. Thus $N(\omega) \subset \ker T_\omega$. Therefore $\dim N(\omega) \le n$. By [12, Theorem 1] $\ker T$ is a finitely generated submodule in $L^0(\Omega, X)$ and $d(\ker T) \le n$. Then by [12, Proposition 3] there exist a (bo)-closed submodule M in $L^0(\Omega, X)$ such that $L^0(\Omega, X) = \ker T \oplus M$.

Consider an operator $S : M \to R(T)$ defined by $S(x) = T(x)$, $x \in M$. Then $\ker S = \{0\}$ and $R(S) = R(T)$. Since $R(T)$ is (bo)-closed in $L^0(\Omega, Y)$, we see that $R(T)$ is a BKS over L^0. By [14, Theorem 2] the operator $S^{-1} : R(T) \to M$ is L^0-bounded. Without loss of generality we may assume that $\|S^{-1}\| \in L^\infty(\Omega)$.

Now show that $\ker T_\omega = N(\omega)$ for all $\omega \in \Omega$. We take $x_\omega \in \ker T_\omega$ and $x \in L^\infty(\Omega, X)$ such that $\rho_X(x)(\omega) = x_\omega$. Then $x = x_1 + x_2$, where $x_1 \in \ker T$, $x_2 \in M$. Since $\rho_X(x)(\omega) = \rho_X(x_1)(\omega) + \rho_X(x_2)(\omega)$ we get

$$T_\omega(\rho_X(x)(\omega)) = T_\omega(\rho_X(x_1)(\omega)) + T_\omega(\rho_X(x_2)(\omega)). \tag{1}$$

Because $x_1 \in \ker T$, we have $T_\omega(\rho_X(x_1))(\omega) = \rho_Y(T(x_1))(\omega) = 0$. From $\rho_X(x)(\omega) \in \ker T_\omega$ it follows that $T_\omega(\rho_X(x))(\omega) = 0$. Therefore by (1) we get $T_\omega(\rho_X(x_2))(\omega) = 0$. Since $x_2 \in M$ we have

$$\|x_2\| = \|S^{-1}(S(x_2))\| \le \|S^{-1}\|\|S(x_2)\| = \|S^{-1}\|\|T(x_2)\|.$$

Thus

$$\|\rho_X(x_2)(\omega)\|_{X(\omega)} \le p(\|S^{-1}\|)(\omega)\|T_\omega(\rho_X(x_2)(\omega)\|_{Y(\omega)} = 0.$$

Therefore $\rho_X(x_2)(\omega) = 0$. Hence $x_\omega = \rho_X(x)(\omega) = \rho_X(x_1)(\omega)$. Since $x_1 \in \ker T$ we get $x_\omega \in N(\omega)$. Therefore $\ker T_\omega = N(\omega)$. Since $\ker T_\omega = n$ for almost all $\omega \in \Omega$ by [12, Theorem 1] it follows that $\ker T$ is homogeneous of type n.

Now we shall show that $\ker T^*$ is homogeneous of type n. By a similar argument as in the case of the operator T we have that $\ker T^*$ is a finitely generated module and $d(\ker T^*) \le n$.

Let $S = T|_{L^\infty(\Omega,X)}$, $f \in L^\infty(\Omega, Y)^*$ and $x \in L^\infty(\Omega, X)$. Then $S^*(f)(x) = f(S(x)) = f(T(x)) = T^*(f)(x)$. Thus $T^*|_{L^\infty(\Omega,Y)^*} = S^*$ and $d(\ker T^*) \ge d(\ker S^*)$.

We show that $d(\ker S^*) \ge n$. Without loss of generality we may assume that $\ker S^*$ is homogeneous of type m. Let $\{\psi_1, \ldots, \psi_m\}$ be a ∇-basis in $\ker S^*$. By [12, Proposition 2] there exist $\{z_1, \ldots, z_m\} \subset L^0(\Omega, X)$ such that $z_i(\psi_j) = \delta_{i,j}1$, where $\delta_{i,j}$ the Kronecker symbol. Without loss of generality we can assume that $\|z_k\| \in L^\infty(\Omega)$ for all $k = \overline{1, m}$.

For $y \in L^\infty(\Omega, Y)$ denote $\overline{y} = y - \sum\limits_{i=1}^{m} \psi_i(y) z_i$. Then $\psi_k(\overline{y}) = \psi_k(y) - \sum\limits_{i=1}^{m} \psi_i(y)\psi_k(z_i) = \psi_k(y) - \psi_k(y) = 0$, $k = \overline{1, m}$. Thus $\overline{y}$ belongs to ${}^\perp \ker S^*$. Therefore by [13, Theorem 2] the point $\overline{y}$ belongs $R(S) = {}^\perp \ker S^*$. Thus there exists $\overline{x} \in L^\infty(\Omega, X)$ such that $S(\overline{x}) = \overline{y}$. Applying the lifting ρ_Y to $\overline{y} = y - \sum\limits_{i=1}^{m} \psi_i(y) z_i$ we have that any $y(\omega) \in Y(\omega)$ can be represented in the form $y(\omega) = \overline{y}(\omega) + \sum\limits_{i=1}^{m} \alpha_i(\omega) z_i(\omega)$, where $\overline{y}(\omega) \in R(T_\omega)$, $\alpha_i(\omega) \in \mathbb{C}$, $z_i(\omega) = \rho_Y(z_i)(\omega)$, $i = \overline{1, m}$. Since T_ω is a Fredholm operator and $\dim \ker T_\omega = n$ there exists a subspace $M(\omega) \subset Y(\omega)$ such that $\dim M(\omega) = n$ and $Y(\omega) = R(T_\omega) \oplus M(\omega)$. Therefore, $\{z_1(\omega), \ldots, z_m(\omega)\}$ is contains n linearly independent elements. Thus $m \ge n$. Therefore $m = n$ and $\ker T^*$ is homogeneous of type n. $\square$

The following result shows that a measurable bundle of Fredholm operators generates a ∇-Fredholm operator.

Theorem 3.2. *Let $T : L^0(\Omega, X) \to L^0(\Omega, Y)$ be a L^0-bounded linear operator. If T_ω is a Fredholm operator for almost all $\omega \in \Omega$, then T is a ∇-Fredholm operator.*

Proof. Let T_ω be a Fredholm operator for almost all $\omega \in \Omega$. Then $\dim \ker T_\omega < \infty$ for almost all $\omega \in \Omega$. By [12, Theorem 1] there exists a partition of the unit $(\pi_n)_{n\in\mathbb{N}}$ in ∇ such that

$$d(\ker \pi_n T) = d(\ker \pi_n T^*) = \begin{cases} 0, & \text{if} \quad \pi_n = 0, \\ n, & \text{if} \quad \pi_n \neq 0, \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Case 1. $\pi_0 = \mathbf{1}$. Then $\ker T = \{0\}$ and $\ker T^* = \{0\}$. By Proposition 3.1 we have $R(T) = {}^\perp\{0\} = L^0(\Omega, Y)$ and $R(T^*) = \{0\}^\perp = L^0(\Omega, X)^*$. Hence, $\ker T = \{0\}$, $\ker T^* = \{0\}$, $R(T) = L^0(\Omega, Y)$ and $R(T^*) = L^0(\Omega, X)^*$. This means that T is a ∇-Fredholm operator.

Case 2. $\pi_0 \neq \mathbf{1}$. In this case there exists $n \geq 1$ such that $\pi_n \neq 0$. Without loss of generality we may assume that $\pi_n = \mathbf{1}$ for some $n \in \mathbb{N}$. Then by Proposition 3.1, $\ker T$ and $\ker T^*$ are homogeneous of type n.

Let $x_1, \ldots, x_n$ and $g_1, \ldots, g_n$ be ∇-bases in $\ker T$ and $\ker T^*$, respectively. By Proposition 3.1 we have that the equation $T(x) = y$ (respectively $T^*(g) = f$) is solvable if and only if $g_k(y) = 0$ (respectively $f(x_k) = 0$) for all $k = \overline{1, n}$.

Now fix some solution x^* of the main equation. Let x be an arbitrary solution of the main equation. Then $x - x^* \in \ker T$. Since $d(\ker T) = n$ there are $c_1, c_2, \ldots, c_n \in L^0$ such that $x = x^* + c_1x_1 + c_2x_2 + \cdots + c_nx_n$. The general form of the solution of the conjugate equation is established by similar arguments. □

4. Nikolsky theorem for a linear operators in Banach–Kantorovich spaces

Let T be a ∇-Fredholm operator and $\ker T$ be homogeneous of type n. Let $\{e_1, \ldots, e_n\}$ and $\{\psi_1, \ldots, \psi_n\}$ be ∇-bases in $\ker T$ and $\ker T^*$, respectively. We take $\{f_1, \ldots, f_n\}$ from $L^0(\Omega, X)^*$ and $\{z_1, \ldots, z_n\}$ from $L^0(\Omega, X)$ such that $f_i(e_j) = \delta_{i,j}\mathbf{1}$ and $\psi_i(z_j) = \delta_{i,j}\mathbf{1}$ (see [12, Proposition 2]).

We consider a finitely generated operator $K : L^0(\Omega, X) \to L^0(\Omega, Y)$ defined by

$$K(x) = \sum_{i=1}^{n} f_i(x)z_i, \quad x \in L^0(\Omega, X). \tag{2}$$

Proposition 4.1. *Let $T : L^0(\Omega, X) \to L^0(\Omega, Y)$ be a ∇-Fredholm operator and $\ker T$ be homogeneous of type n. Then the operator $B = T + K$ is invertible and B^{-1} is L^0-bounded, where K defined by* (2).

Proof. We show that $\ker B = \{0\}$ and $R(B) = L^0(\Omega, Y)$.

1) $\ker B = \{0\}$. Take $x \in \ker B$. This means that

$$T(x) = -\sum_{i=1}^{n} f_i(x)z_i. \tag{3}$$

Since $\psi_i(z_j) = \delta_{i,j}\mathbf{1}$ we get $\psi_k(T(x)) = -\sum\limits_{i=1}^{n} f_i(x)\psi_k(z_i) = -f_k(x)$. On the other hand, $\psi_k(T(x)) = T^*(\psi_k)(x) = 0(x) = 0$. Thus $f_k(x) = 0, k = \overline{1, n}$. Hence (3) has the form $T(x) = 0$, thus $x = \sum\limits_{m=1}^{n} \xi_m e_m$, where $\xi_m \in L^0$, $m = \overline{1, n}$. From $f_k(x) = 0, k = \overline{1, n}$ we obtain $0 = f_k(x) = f_k(\sum\limits_{m=1}^{n} \xi_m e_m) = \sum\limits_{m=1}^{n} \xi_m f_k(e_m) = \xi_m$. Therefore $\xi_m = 0$ for all m. Thus $x = 0$. Hence $\ker B = \{0\}$.

2) $R(B) = L^0(\Omega, Y)$. Let $y \in L^0(\Omega, Y)$.
Put

$$\overline{y} = y - \sum_{i=1}^{n} \psi_i(y) z_i. \tag{4}$$

Since $\psi_k(\overline{y}) = \psi_k(y) - \sum\limits_{i=1}^{n} \psi_i(y)\psi_k(z_i) = \psi_k(y) - \psi_k(y) = 0, k = \overline{1,n}$, and T is ∇-Fredholm we have $\overline{y} \in R(T)$. Take $\overline{x} \in L^0(\Omega, X)$ such that $T(\overline{x}) = \overline{y}$. Put $x = \overline{x} + \sum\limits_{i=1}^{n} [\psi_i(y) - f_i(\overline{x})] e_i$. By (4) and using the identities $T(x) = T(\overline{x})$, $K(\overline{x}) = \sum\limits_{i=1}^{n} f_i(\overline{x}) z_i$, $K(e_i) = z_i$ we get $B(x) = T(x) + K(x) = T(\overline{x}) + \sum\limits_{i=1}^{n} f_i(\overline{x}) z_i + \sum\limits_{i=1}^{n} [\psi_i(y) - f_i(\overline{x})] z_i = \overline{y} + \sum\limits_{i=1}^{n} \psi_i(y) z_i = y$.

Therefore $\ker B = \{0\}$, $R(B) = L^0(\Omega, Y)$ and by [14, Theorem 2] we have that B^{-1} is an L^0-bounded operator. □

The following result is a vector version of the Nikolsky theorem for linear operators on Banach–Kantorovich spaces.

Theorem 4.2. *For an L^0-bounded linear operator $T : L^0(\Omega, X) \to L^0(\Omega, Y)$, the following conditions are equivalent:*

1) *T_ω is a Fredholm operator for almost all $\omega \in \Omega$;*

2) *T is a ∇-Fredholm operator*;

3) *there are operators A, K from $L^0(\Omega, X)$ to $L^0(\Omega, Y)$ such that A is invertible, K is σ-finite-dimensional and $T = A + K$;*

4) *there are operators A, K from $L^0(\Omega, X)$ to $L^0(\Omega, Y)$ such that A is invertible, K is cyclically compact and $T = A + K$.*

Proof. Implication 1) ⇒ 2) follows from Theorem 3.2.

2) ⇒ 3). Let T be a ∇-Fredholm operator and $(\pi_n)_{n \in \mathbb{N} \cup \{0\}}$ a partition of the unit in ∇ such that $d(\pi_n \ker T) = n$, $n \in \mathbb{N} \cup \{0\}$.

Case 1. $\pi_0 = \mathbf{1}$. In this case $\ker T = \{0\}$. Since T is ∇-Fredholm we get $R(T) = L^0(\Omega, Y)$. By [14, Theorem 2], the operator T^{-1} is L^0-bounded. Therefore in this case we put $A = T, K = 0$.

Case 2. $\pi_0 \neq \mathbf{1}$. For ∇-Fredholm operators $\pi_n T$, $n \in \mathbb{N}$, by Proposition 4.1 there are finitely generated operators $K_n : \pi_n L^0(\Omega, X) \to \pi_n L^0(\Omega, Y)$ such that $A_n = \pi_n T + K_n$ is an invertible operator from $\pi_n L^0(\Omega, X)$ on $\pi_n L^0(\Omega, Y)$. Since K_n is L^0-bounded for all $n \in \mathbb{N}$ and $L^0(\Omega, Y)$ is a BKS over L^0, there exists $K(x) = \sum\limits_{n=1}^{\infty} \pi_n K_n(\pi_n x)$ for all $x \in L^0(\Omega, X)$. Then K is a σ-finite-dimensional operator, $A = T + K$ is invertible and $T = A + (-K)$.

3) ⇒ 4) is trivial, because every σ-finite-dimensional operator is cyclically compact.

4) ⇒ 1). We need following.

Proposition 4.3. *If an L^0-bounded linear operator $T : L^0(\Omega, X) \to L^0(\Omega, Y)$ is an invertible, then T_ω is invertible for almost all $\omega \in \Omega$.*

Proof. Let T be invertible and $U : L^0(\Omega, Y) \to L^0(\Omega, X)$ be the inverse of T. We take a partition $(\pi_n)_{n \in \mathbb{N}}$ of unit in ∇ such that $\pi_n \|U\| \in L^\infty(\Omega)$ and $\pi_n \|T\| \in L^\infty(\Omega)$ for any $n \in \mathbb{N}$. Then $\pi_n U(y) \in L^\infty(\Omega, X)$ and $\pi_n T(x) \in L^\infty(\Omega, Y)$ for all $x \in L^\infty(\Omega, X)$, $y \in L^\infty(\Omega, Y)$. Denote $\Omega_n = \{\omega \in \Omega : p(\pi_n)(\omega) = 1\}$ and $\Omega_0 = \bigcup_{n=1}^{\infty} \Omega_n$. For $\omega \in \Omega_n$ we put $U_\omega(\rho_Y(y)(\omega)) = \rho_X(U(\pi_n y))(\omega)$ for all $y \in L^\infty(\Omega, Y)$. For $y \in L^\infty(\Omega, Y)$ we have

$$\begin{aligned} \|U_\omega(\rho_Y(y)(\omega))\|_{X(\omega)} &= \|\rho_X(U(\pi_n y))(\omega)\|_{X(\omega)} = p(\|U(\pi_n y)\|)(\omega) \\ &\leq p(\pi_n \|U\| \|y\|)(\omega) = p(\pi_n \|U\|)(\omega) \|\rho_Y(y)(\omega)\|_{Y(\omega)}. \end{aligned}$$

Hence, U_ω is a bounded operator for any $\omega \in \Omega_0$.

Since

$$U_\omega(T_\omega(\rho_X(x)(\omega))) = \rho_X(x)(\omega) \quad \text{and} \quad T_\omega(U_\omega(\rho_Y(y)(\omega))) = \rho_Y(y)(\omega)$$

for any $x \in L^\infty(\Omega, X)$, $y \in L^\infty(\Omega, Y)$, we have that U_ω is an inverse of T_ω. □

Suppose that $T = A + K$, where A is invertible, K is a cyclically compact operator. Then A_ω is invertible (Proposition 4.3) and K_ω is a compact operator [7, Theorem 4] for almost all $\omega \in \Omega$. By Nikolsky's classical theorem, $T_\omega = A_\omega + K_\omega$ is a Fredholm operator for almost all $\omega \in \Omega$. □

Corollary 4.4. *Let X, Y and Z be* MBBs *over Ω with vector valued liftings. If $U : L^0(\Omega, X) \to L^0(\Omega, Y)$ and $V : L^0(\Omega, Y) \to L^o(\Omega, Z)$ are ∇-Fredholm operators then UV is a ∇-Fredholm operator.*

Theorem 4.5. *Let $U : L^0(\Omega, X) \to L^0(\Omega, X)$ be a L^0-bounded linear operator such that U^m is cyclically compact for some $m \in \mathbb{N}$. Then $I - U$ is a ∇-Fredholm operator.*

Proof. Without loss of generality we may assume that $\|U\| = \mathbf{1}$. Let $\{U_\omega : \omega \in \Omega\}$ be a measurable bundle of operator U. For any $x \in L^\infty(\Omega, X)$ it follows that

$$\rho_X(U^m(x))(\omega) = \rho_X(U(U^{m-1}(x))(\omega) = \cdots = U_\omega^m(\rho_X(x)(\omega)).$$

Therefore, the family $\{U_\omega^m : \omega \in \Omega\}$ is a measurable bundle of cyclically compact operators U^m. Since U^m is cyclically compact by [7, Theorem 4] it follows that U_ω^m is a compact operator for almost all $\omega \in \Omega$. Therefore $I_\omega - U_\omega$ is a Fredholm operator for almost all $\omega \in \Omega$. By Theorem 4.2 we have that $I - U$ is a ∇-Fredholm operator. □

Remark. In [4] (see also [5], [6]) Kusraev proves that $I - U$ is a ∇-Fredholm operator if U is a cyclically compact operator.

Example. Let $L^{2,0}(\Omega^2)$ be the set of complex-valued measurable functions f on Ω^2 such that

$$\int_\Omega |f(s,\omega)|^2 \, d\mu(s) \in L^0$$

exists.

For $f \in L^{2,0}(\Omega^2)$ denote $\|f\|(\omega) = \sqrt{\int_\Omega |f(s,\omega)|^2 \, d\mu(s)}$. Then $(L^{2,0}(\Omega^2), \|\cdot\|)$ is a BKS over L^0. Let $k(t,s,\omega)$ be a complex-valued measurable function on Ω^3 such that $\int_\Omega \int_\Omega |k(t,s,\omega)|^2 \, d\mu(s) \, d\mu(t)$ exists.

Consider an operator $T : L^{2,0}(\Omega^2) \to L^{2,0}(\Omega^2)$ defined by

$$T(f)(t,\omega) = \int_\Omega k(t,s,\omega) f(s,\omega) \, d\mu(s), \quad f \in L^{2,0}(\Omega^2).$$

For any $\omega \in \Omega$ we put $k_\omega(t,s) = k(t,s,\omega)$. Then for almost all $\omega \in \Omega$ the function $k_\omega(t,s)$ belongs to $L^2(\Omega^2)$. For almost all $\omega \in \Omega$ the operator $T_\omega : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$T_\omega(f_\omega)(t) = \int_\Omega k_\omega(t,s) f_\omega(s) \, d\mu(s), \quad f_\omega \in L^2(\Omega).$$

It is well-known that T_ω is a compact operator for almost all $\omega \in \Omega$. For $f \in L^{2,0}(\Omega^2)$ we have

$$T(f)(t,\omega) = \int_\Omega k(t,\omega,s) f(s,\omega) \, d\mu(s) = \int_\Omega k_\omega(t,s) f_\omega(s) \, ds = T_\omega(f_\omega)(t)$$

for almost all $(t,\omega) \in \Omega^2$, where $f_\omega(s) = f(s,\omega)$. This means that $\{T_\omega : \omega \in \Omega\}$ is a measurable bundle of compact operators. Therefore, by [7, Theorem 3] the operator T is cyclically compact. By Theorem 4.5 we have that $I - T$ is a ∇-Fredholm operator.

References

1. L. V. Kantorovich, G. P. Akilov, *Functional Analysis*, Nauka, Moscow, 1977. (Russian)
2. A. E. Gutman, *Banach bundles in the theory of lattice-normed spaces.* II. *Measurable Banach bundles*, Siberian Adv. Math. **3** (1993), no. 3, 1–55.
3. A. G. Kusraev, *Boolean valued analysis of duality between universally complete modules*, Dokl. Akad. Nauk SSSR **267** (1982), no. 5, 1049–1052.
4. A. G. Kusraev, *Vector Duality and Its Applications*, Nauka, Novosibirsk, 1985. (Russian)
5. A. G. Kusraev, *Cyclically compact operators in Banach spaces*, Vladikavkaz. Math. J. **2** (2000), no. 1, 10–23.
6. A. G. Kusraev, *Dominated Operators*, Nauka, Moscow, 2003. (Russian)
7. I. G. Ganiev, K. K. Kudaybergenov, *Measurable bundles of compact operators*, Methods Funct. Anal. Topology **7** (2001), no. 4, 1–5.
8. A. G. Kusraev and S. S. Kutateladze, *Nonstandard methods for Kantorovich space*, Siberian Adv. Math. **2** (1992), no. 2, 114–152.
9. S. S. Kutateladze, *Cyclic monads and their applications*, Siberian Math. J. **27** (1986), no. 1, 100–110.
10. S. S. Kutateladze, *Monads of ultrafilters and extensional filters*, Siberian Math. J. **30** (1989), no. 1, 129–133.
11. A. E. Gutman, E. Yu. Emel'yanov, A. G. Kusraev, and S. S. Kutateladze, *Nonstandard Analysis and Vector Lattices*, Sobolev Institute Press, Novosibirsk, 1999.
12. I. G. Ganiev and K. K. Kudaybergenov, *A finite dimensional module over a ring of measurable functions*, Uzb. Math. J. (2004), no. 4, 3–9.
13. K. K. Kudaybergenov, *Measurable bundle of operators with closed ranges*, Uzb. Math. J. (2005), no. 3, 54–62.
14. I. G. Ganiev and K. K. Kudaybergenov, *Banach theorem about of inverse operators in the Banach–Kantorovich spaces*, Vladikavkaz. Math. J. **6** (2004), no. 3, 21–25.

Institute of Mathematics, Uzbek Academy of Sciences, 29 F. Khodjaev, Tashkent, 700125, Uzbekistan

E-mail address: karim2006@mail.ru

Received 07/04/2005