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# A NOTE ON ONE DECOMPOSITION OF BANACH SPACES

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This note is dedicated to Academician Yu. M. Berezansky in honor of his eightieth anniversary.

ABSTRACT. For a scalar type spectral operator A in complex Banach space X, the decomposition of X into the direct sum

$$X = \ker A \oplus \overline{R(A)},$$

where ker A is the kernel of A and  $\overline{R(A)}$  is the closure of its range R(A) is established.

Mathematicians stand on each other's shoulders. Carl Friedrich Gauss

## 1. INTRODUCTION

We are to prove that, for a scalar type spectral operator A in a complex Banach space X, the following direct sum decomposition holds:

(1.1) 
$$X = \ker A \oplus \overline{R(A)},$$

where ker  $\cdot$  is the *kernel* of an operator and  $\overline{R(\cdot)}$  is the closure an operator's range  $R(\cdot)$ .

This decomposition is a generalization of the well-known fact that, for a *normal operator* in a Hilbert space H,

(1.2) 
$$H = \ker A \oplus R(A),$$

the direct sum being *orthogonal* in this case (see, e.g., [1, 2, 5, 12]).

Observe that the implications of decomposition (1.1) can be quite instrumental when dealing with the ergodicity of solutions of abstract evolution equations (see, e.g., [8, 7, 13, 9, 10, 11]).

Also note that, according to [14], in a Hilbert space, *scalar type spectral operators* are those similar to *normal* ones.

As is to be expected, abandoning the Hilbert space *inner product techniques*, which makes proving decomposition (1.2) positively effortless, would require a different approach.

### 2. Preliminaries

Let A be a scalar type spectral operator and  $E_A(\cdot)$  be its spectral measure (the resolution of the identity), the operator's spectrum  $\sigma(A)$  being the support for the latter [3, 6].

For such operators, there has been developed an *operational calculus* for Borel measurable functions on the spectrum of A [3, 6].

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Provided  $F(\cdot)$  is such a function, a new scalar type spectral operator

$$F(A) = \int_{\sigma(A)} F(\lambda) \, dE_A(\lambda)$$

is defined as follows:

$$F(A)f := \lim_{n \to \infty} F_n(A)f, \quad f \in D(F(A)),$$
$$D(F(A)) := \left\{ f \in X \mid \lim_{n \to \infty} F_n(A)f \text{ exists} \right\}$$

 $(D(\cdot))$  is the *domain* of an operator), where

$$F_n(\cdot) := F(\cdot)\chi_{\{\lambda \in \sigma(A) \mid |F(\lambda)| \le n\}}(\cdot), \quad n = 1, 2, \dots$$

 $(\chi_{\alpha}(\cdot))$  is the *characteristic function* of a set  $\alpha$ ), and

$$F_n(A) := \int_{\sigma(A)} F_n(\lambda) \, dE_A(\lambda), \quad n = 1, 2, \dots,$$

being the integrals of bounded Borel measurable functions on  $\sigma(A)$ , are bounded scalar type spectral operators on X defined in the same manner as for normal operators (see, e.g., [1, 2, 5, 12]).

Observe that

(2.3) 
$$A = \int_{\sigma(A)} \lambda \, dE_A(\lambda).$$

The properties of the spectral measure,  $E_A(\cdot)$ , and the operational calculus exhaustively delineated in [3, 6] underly the proof of the succeeding theorem.

#### 3. The decomposition

**Theorem.** Let A be a scalar type spectral operator in a complex Banach space X. Then the space X is decomposable into direct sum (1.1).

*Proof.* For any  $f \in X$ , by the properties of the spectral measure [3, 6],

$$f = E_A(\{0\})X \oplus E_A(\sigma(A) \setminus \{0\})X.$$

The inclusion

## $E_A(\{0\})X \subseteq \ker A$

follows directly from representation (2.3) by the properties of the operational calculus.

The inverse inclusion

$$\ker A \subseteq E_A(\{0\})X$$

is proved in [6] (Lemma XV.3.1) where, as is easily seen, the requirement of the boundedness of the operator is absolutely superfluous.

Thus,

$$E_A(\{0\})X = \ker A_A$$

i.e.,  $E_A(\{0\})$  is a projection onto ker A.

Let us show now that the operator  $E_A(\sigma(A) \setminus \{0\})$  is the projection onto the subspace  $\overline{R(A)}$  parallel to ker A.

Let

$$F_n(\lambda) = \begin{cases} 0 & \text{for } \lambda \in \sigma(A), \, |\lambda| \le 1/n \\ \frac{1}{\lambda} & \text{for } \lambda \in \sigma(A), \, |\lambda| > 1/n \end{cases}, \, n = 1, 2, \dots .$$

By the properties of the spectral measure and operational calculus [3, 6], for an arbitrary  $f \in X$  and any n = 1, 2, ...,

$$E_A(\{\lambda \in \sigma(A) | 0 < |\lambda| \le 1/n\})f$$
  
=  $E_A(\sigma(A) \setminus \{0\})f - E_A(\{\lambda \in \sigma(A) | |\lambda| > 1/n\})f$   
=  $E_A(\sigma(A) \setminus \{0\})f - \int_{\sigma(A)} \chi_{\{\lambda \in \sigma(A) | |\lambda| > 1/n\}}(\lambda) dE_A(\lambda)f$   
=  $E_A(\sigma(A) \setminus \{0\})f - AF_n(A)f.$ 

By the strong continuity of the *spectral measure*, for any  $f \in X$ ,

 $E_A(\sigma(A) \setminus \{0\})f = \lim_{n \to \infty} AF_n(A)f \in \overline{R(A)}.$ 

Hence,

$$(3.4) E_A(\sigma(A) \setminus \{0\}) X \subseteq \overline{R(A)}.$$

On the other hand,

(3.5) 
$$\overline{R(A)} \subseteq E_A(\sigma(A) \setminus \{0\})X$$

Indeed, for an arbitrary  $g \in R(A)$ , there is an  $f \in D(A)$  such that g = Af and we have

$$g = Af = \int_{\sigma(A)} \lambda \, dE_A(\lambda)f = \int_{\sigma(A) \setminus \{0\}} \lambda \, dE_A(\lambda)f = E_A(\sigma(A) \setminus \{0\})Af$$
$$\in E_A(\sigma(A) \setminus \{0\})X.$$

Inclusions (3.4) and (3.5) imply that

$$E_A(\sigma(A) \setminus \{0\})X = \overline{R(A)}.$$

Note that the inclusions

$$\ker A \subseteq E_A(\{0\})X \quad \text{and} \quad E_A(\sigma(A) \setminus \{0\})X \subseteq \overline{R(A)}$$

hold true for any spectral operator A in a complex Banach space X [3, 6] without it being of scalar type.  $\Box$ 

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256

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