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A NOTE ON ONE DECOMPOSITION OF BANACH SPACES

M. V. MARKIN

This note is dedicated to Academician Yu. M. Berezansky in honor of his eightieth anniversary.

Abstract. For a scalar type spectral operator A in complex Banach space X, the decomposition of X into the direct sum

$$X = \ker A \oplus \overline{R(A)},$$

where $\ker A$ is the *kernel* of A and $\overline{R(A)}$ is the closure of its range $R(A)$ is established.

Mathematicians stand on each other's shoulders.
Carl Friedrich Gauss

1. Introduction

We are to prove that, for a *scalar type spectral operator* A in a complex Banach space X, the following *direct sum* decomposition holds:

$$X = \ker A \oplus \overline{R(A)}, \tag{1.1}$$

where $\ker\cdot$ is the *kernel* of an operator and $\overline{R(\cdot)}$ is the closure an operator's range $R(\cdot)$.

This decomposition is a generalization of the well-known fact that, for a *normal operator* in a Hilbert space H,

$$H = \ker A \oplus \overline{R(A)}, \tag{1.2}$$

the direct sum being *orthogonal* in this case (see, e.g., [1, 2, 5, 12]).

Observe that the implications of decomposition (1.1) can be quite instrumental when dealing with the ergodicity of solutions of abstract evolution equations (see, e.g., [8, 7, 13, 9, 10, 11]).

Also note that, according to [14], in a Hilbert space, *scalar type spectral operators* are those similar to *normal* ones.

As is to be expected, abandoning the Hilbert space *inner product techniques*, which makes proving decomposition (1.2) positively effortless, would require a different approach.

2. Preliminaries

Let A be a *scalar type spectral operator* and $E_A(\cdot)$ be its *spectral measure* (the *resolution of the identity*), the operator's spectrum $\sigma(A)$ being the *support* for the latter [3, 6].

For such operators, there has been developed an *operational calculus* for Borel measurable functions on the spectrum of A [3, 6].

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Provided $F(\cdot)$ is such a function, a new *scalar type spectral operator*

$$F(A) = \int\limits_{\sigma(A)} F(\lambda)\, dE_A(\lambda)$$

is defined as follows:

$$F(A)f := \lim_{n\to\infty} F_n(A)f, \quad f \in D(F(A)),$$
$$D(F(A)) := \left\{ f \in X \;\middle|\; \lim_{n\to\infty} F_n(A)f \text{ exists} \right\}$$

($D(\cdot)$ is the *domain* of an operator), where

$$F_n(\cdot) := F(\cdot)\chi_{\{\lambda\in\sigma(A)\,|\,|F(\lambda)|\le n\}}(\cdot), \quad n = 1, 2, \dots,$$

($\chi_\alpha(\cdot)$ is the *characteristic function* of a set α), and

$$F_n(A) := \int\limits_{\sigma(A)} F_n(\lambda)\, dE_A(\lambda), \quad n = 1, 2, \dots,$$

being the integrals of bounded Borel measurable functions on $\sigma(A)$, are *bounded scalar type spectral operators* on X defined in the same manner as for *normal operators* (see, e.g., [1, 2, 5, 12]).

Observe that

$$A = \int\limits_{\sigma(A)} \lambda\, dE_A(\lambda). \tag{2.3}$$

The properties of the *spectral measure*, $E_A(\cdot)$, and the *operational calculus* exhaustively delineated in [3, 6] underly the proof of the succeeding theorem.

3. The decomposition

Theorem. *Let A be a scalar type spectral operator in a complex Banach space X. Then the space X is decomposable into direct sum* (1.1).

Proof. For any $f \in X$, by the properties of the *spectral measure* [3, 6],

$$f = E_A(\{0\})X \oplus E_A(\sigma(A) \setminus \{0\})X.$$

The inclusion

$$E_A(\{0\})X \subseteq \ker A$$

follows directly from representation (2.3) by the properties of the *operational calculus.*

The inverse inclusion

$$\ker A \subseteq E_A(\{0\})X$$

is proved in [6] (Lemma XV.3.1) where, as is easily seen, the requirement of the boundedness of the operator is absolutely superfluous.

Thus,

$$E_A(\{0\})X = \ker A,$$

i.e., $E_A(\{0\})$ is a projection onto $\ker A$.

Let us show now that the operator $E_A(\sigma(A)\setminus\{0\})$ is the projection onto the subspace $\overline{R(A)}$ parallel to $\ker A$.

Let

$$F_n(\lambda) = \begin{cases} 0 & \text{for } \lambda \in \sigma(A),\ |\lambda| \le 1/n \\ \dfrac{1}{\lambda} & \text{for } \lambda \in \sigma(A),\ |\lambda| > 1/n \end{cases}, \ n = 1, 2, \dots .$$

By the properties of the *spectral measure* and *operational calculus* [3, 6], for an arbitrary $f \in X$ and any $n = 1, 2, \dots$,

$$\begin{aligned} E_A(\{\lambda \in \sigma(A) | 0 < |\lambda| \le 1/n\})f & \\ &= E_A(\sigma(A) \setminus \{0\})f - E_A(\{\lambda \in \sigma(A) | |\lambda| > 1/n\})f \\ &= E_A(\sigma(A) \setminus \{0\})f - \int\limits_{\sigma(A)} \chi_{\{\lambda \in \sigma(A) | |\lambda| > 1/n\}}(\lambda)\, dE_A(\lambda)f \\ &= E_A(\sigma(A) \setminus \{0\})f - AF_n(A)f. \end{aligned}$$

By the strong continuity of the *spectral measure*, for any $f \in X$,

$$E_A(\sigma(A) \setminus \{0\})f = \lim_{n \to \infty} AF_n(A)f \in \overline{R(A)}.$$

Hence,

$$E_A(\sigma(A) \setminus \{0\})X \subseteq \overline{R(A)}. \tag{3.4}$$

On the other hand,

$$\overline{R(A)} \subseteq E_A(\sigma(A) \setminus \{0\})X. \tag{3.5}$$

Indeed, for an arbitrary $g \in R(A)$, there is an $f \in D(A)$ such that $g = Af$ and we have

$$\begin{aligned} g = Af = \int\limits_{\sigma(A)} \lambda\, dE_A(\lambda)f = \int\limits_{\sigma(A) \setminus \{0\}} \lambda\, dE_A(\lambda)f = E_A(\sigma(A) \setminus \{0\})Af \\ \in E_A(\sigma(A) \setminus \{0\})X. \end{aligned}$$

Inclusions (3.4) and (3.5) imply that

$$E_A(\sigma(A) \setminus \{0\})X = \overline{R(A)}.$$

Note that the inclusions

$$\ker A \subseteq E_A(\{0\})X \quad \text{and} \quad E_A(\sigma(A) \setminus \{0\})X \subseteq \overline{R(A)}$$

hold true for any *spectral operator* A in a complex Banach space X [3, 6] without it being of *scalar type*. □

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Fresno, CA, USA
E-mail address: mmarkin@comcast.net

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