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BOUNDARY TRIPLETS AND KREIN TYPE RESOLVENT FORMULA FOR SYMMETRIC OPERATORS WITH UNEQUAL DEFECT NUMBERS

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Abstract. Let $\mathfrak{H}$ be a Hilbert space and let A be a symmetric operator in $\mathfrak{H}$ with arbitrary (not necessarily equal) deficiency indices $n_\pm(A)$. We introduce a new concept of a D-boundary triplet for A^*, which may be considered as a natural generalization of the known concept of a boundary triplet (boundary value space) for an operator with equal deficiency indices. With a D-triplet for A^* we associate two Weyl functions $M_+(\cdot)$ and $M_-(\cdot)$. It is proved that the functions $M_\pm(\cdot)$ posses a number of properties similar to those of the known Weyl functions (Q-functions) for the case $n_+(A)=n_-(A)$. We show that every D-triplet for A^* gives rise to Krein type formulas for generalized resolvents of the operator A with arbitrary deficiency indices. The resolvent formulas describe the set of all generalized resolvents by means of two pairs of operator functions which belongs to the Nevanlinna type class $\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$. This class has been earlier introduced by the author.

1. Introduction

Let $\mathfrak{H}$ be a Hilbert space and let A be a symmetric densely defined operator in $\mathfrak{H}$ with the domain $\mathcal{D}(A)$ and defect numbers $n_\pm(A)$. Recall [9] that a triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$, where $\mathcal{H}$ is a Hilbert space and Γ_0, Γ_1 are operators from $\mathcal{D}(A^*)$ to $\mathcal{H}$, is called a boundary triplet for A^*, if the mapping $\Gamma:=(\Gamma_0\ \ \Gamma_1)^\top$ is surjective and the following abstract Green's identity holds

$$(1.1)\qquad (A^*f,g)-(f,A^*g)=(\Gamma_1 f,\Gamma_0 g)_{\mathcal{H}}-(\Gamma_0 f,\Gamma_1 g)_{\mathcal{H}},\quad f,g\in\mathcal{D}(A^*).$$

In [2] an abstract Weyl function was associated to a boundary triplet. Namely, the operator function $M_\Pi(\cdot)$ defined by

$$(1.2)\qquad \Gamma_1 f_\lambda=M_\Pi(\lambda)\Gamma_0 f_\lambda,\quad f_\lambda\in\mathfrak{N}_\lambda(A):=\operatorname{Ker}(A^*-\lambda),\quad \lambda\in\mathbb{C}_+\cup\mathbb{C}_-$$

is called a Weyl function corresponding to the triplet Π. Furthermore, the above definitions was extended in [16, 3] to nondensely defined operators A.

The concept of a boundary triplet and the Weyl function is a convenient tool in the extension theory and its applications (see [9], [2]–[5], [16] and references therein). A motivation for this concept goes back to differential operators, for which the identity (1.1) turns into the classical Green's–Lagrange's identity, while the function $M_\Pi(\cdot)$ coincides with the classical Weyl–Titchmarsh function [23, 4]. Note, however, that every boundary triplet $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^* satisfies the equality $n_+(A)=n_-(A)=\dim\mathcal{H}$. Therefore the method of boundary triplets can be applied only to an operator A with equal defect numbers.

In the present paper a new concept of a D-boundary triplet is introduced. This concept makes it possible to generalize to the case $n_+(A)\neq n_-(A)$ the notion of a boundary

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triplet. Next by using the technique of D-triplets we obtain the Krein–Naimark type formula for generalized resolvents of a symmetric operator with not necessarily equal defect numbers.

Assume that $A^* \subset \mathfrak{H}^2$ is an adjoint linear relation to a not necessarily densely defined symmetric operator A with arbitrary defect numbers. A D-triplet for A^* is defined as follows. Let $\mathcal{H}_1$ be a subspace in a Hilbert space $\mathcal{H}_0$, let $\mathcal{H}_2 = \mathcal{H}_0 \ominus \mathcal{H}_1$, and let P_j be orthoprojectors in $\mathcal{H}_0$ onto $\mathcal{H}_j,\ j \in \{0,1\}$. Then a collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$, where Γ_j are operators from A^* to $\mathcal{H}_j,\ j \in \{0,1\}$, is called a D-boundary triplet for A^*, if the mapping $\Gamma := (\Gamma_0\ \ \Gamma_1)^\top$ is surjective and instead of (1.1) the identity

$$(f', g) - (f, g') = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g}) - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g}) + i(P_2 \Gamma_0 \hat{f}, P_2 \Gamma_0 \hat{g}) \tag{1.3}$$

holds for all $\hat{f} = \{f, f'\}, \hat{g} = \{g, g'\} \in A^*$. In the case $\overline{\mathcal{D}(A)} = \mathfrak{H}$ the operators Γ_0 and Γ_1 may be considered as defined on $\mathcal{D}(A^*)$ and the identity (1.3) takes the form

$$(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g) + i(P_2 \Gamma_0 f, P_2 \Gamma_0 g), \quad f, g \in \mathcal{D}(A^*). \tag{1.4}$$

Such a definition of the D-triplet can be motivated by the following simple example. Assume that

$$l[y] = i\, y^{(3)} \tag{1.5}$$

is a differential expression of the third order on the semiaxis $[0, \infty)$. Let H be a separable Hilbert space and let $\mathfrak{H} := L_2([0, \infty); H)$ be the space of vector-functions $y(\cdot) : [0, \infty) \to H$ such that $\int_0^\infty ||y(t)||^2\, dt < \infty$. The expression (1.5) generates a minimal operator A in $\mathfrak{H}$ with the defect numbers $n_+(A) = 2 \dim H,\ n_-(A) = \dim H$. Moreover the corresponding maximal operator A^* satisfies the Lagrange's identity

$$(A^* y, z)_{\mathfrak{H}} - (y, A^* z)_{\mathfrak{H}} = (-iy''(0), z(0))_H - (y(0), -iz''(0))_H + i(y'(0), z'(0))_H \tag{1.6}$$

for every $y, z \in \mathcal{D}(A^*)$. Letting now $\Gamma_0 y = \{y(0), y'(0)\}$ and $\Gamma_1 y = -iy''(0)$ we derive the D-triplet $\{H^2 \oplus H, \Gamma_0, \Gamma_1\}$, for which the identity (1.4) coincides with the Lagrange's identity (1.6). Note also that in the case $\dim H = \infty$ one has $n_+(A) = n_-(A) = \infty$. In this case the constructed D-triplet is not a boundary triplet (in the sense of [9]), since $(H^2 =)\mathcal{H}_0 \neq \mathcal{H}_1 (= H)$. This shows that D-triplets can be useful even for the operators with equal defect numbers.

It turns out that the above example is typical for D-triplets. Namely, we prove below that every D-triplet $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^* satisfies the relation $\dim \mathcal{H}_1 = n_-(A) \leq n_+(A) = \dim \mathcal{H}_0$. Moreover if $\mathcal{H}_1 = \mathcal{H}_0$, then $n_-(A) = n_+(A)$ and a D-triplet becomes a boundary triplet for A^*. At the same time in the case $n_-(A) = n_+(A) = \infty$ the subspace $\mathcal{H}_1$ may both coincide and not coincide with $\mathcal{H}_0$.

With a D-triplet $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^* we associate two Weyl functions $M_+(\cdot)$ and $M_-(\cdot)$ defined by

$$\begin{aligned} \Gamma_1 f_\lambda &= M_+(\lambda) \Gamma_0 f_\lambda, \quad f_\lambda \in \mathfrak{N}_\lambda(A), \quad \lambda \in \mathbb{C}_+, \\ (\Gamma_1 + iP_2 \Gamma_0) f_z &= M_-(z) P_1 \Gamma_0 f_z, \quad f_z \in \mathfrak{N}_z(A), \quad z \in \mathbb{C}_-. \end{aligned} \tag{1.7}$$

The operator function $M_+(\lambda)$ is defined on $\mathbb{C}_+$ and takes on values in $[\mathcal{H}_0, \mathcal{H}_1]$, while $M_-(z)$ is defined on $\mathbb{C}_-$ and takes on values in $[\mathcal{H}_1, \mathcal{H}_0]$. Moreover these functions are associated via $M_-^*(z) = M_+(\overline{z}),\ z \in \mathbb{C}_-$. In the case $\mathcal{H}_0 = \mathcal{H}_1 := \mathcal{H}$ the Weyl function $M_\Pi(\cdot)$ defined by (1.2) is connected with $M_\pm(\cdot)$ by

$$M_\Pi(\lambda) = M_+(\lambda), \quad \lambda \in \mathbb{C}_+, \qquad M_\Pi(z) = M_-(z), \quad z \in \mathbb{C}_-.$$

We show that the operator functions $M(\cdot) : \mathbb{C}_+ \cup \mathbb{C}_- \to [\mathcal{H}_1],\ N_+(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_2, \mathcal{H}_1]$ and $N_-(\cdot) : \mathbb{C}_- \to [\mathcal{H}_1, \mathcal{H}_2]$ generated by the block-matrix representations

$$M_+(\lambda) = (M(\lambda)\ \ N_+(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1, \quad M_-(z) = (M(z)\ \ N_-(z))^\top : \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2$$

satisfy the relations similar that for the Weyl function (1.2) (see Proposition 3.17). In particular the function $M(\cdot)$ belongs to the class $R^u[\mathcal{H}_1]$, that is $\operatorname{Im}\lambda \operatorname{Im} M(\lambda) \geq 0,\ 0 \in \rho(\operatorname{Im} M(\lambda))$ and $M^*(\lambda) = M(\overline{\lambda})$ for every $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$.

Similarly [2, 16] a D-triplet $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ specify the parameterization of all closed proper extensions $\widetilde{A} \supset A$ by means of closed linear relations $\theta \subset \mathcal{H}_0 \oplus \mathcal{H}_1$ via

$$\widetilde{A} = \widetilde{A}_\theta := \{\hat{f} \in A^* : \{\Gamma_0 \hat{f}, \Gamma_1 \hat{f}\} \in \theta\}. \tag{1.8}$$

Such a parameterization is especially convenient for the description of extensions of differential operators in terms of boundary conditions. In order to describe by (1.8) various classes of extensions (symmetric, selfadjoint, etc) we use some new classes of linear relations $\theta \subset \mathcal{H}_0 \oplus \mathcal{H}_1$ introduced in [21]. Moreover we describe the spectrum of the extension $\widetilde{A}_\theta$ in terms of the parameter θ and the Weyl functions $M_\pm(\cdot)$ (see Proposition 4.1).

It is well known that the Krein–Naimark formula for generalized resolvents of the operator A with equal defect numbers plays an important role in the extension theory (see for instance [13, 14, 2, 3, 16]). A connection between this formula and boundary triplets was discovered in [2, 16]. Namely, it was shown there that every boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* $(n_+(A) = n_-(A) \leq \infty)$ gives rise to the Krein–Naimark formula

$$\mathbb{R}_\lambda = (A_0 - \lambda)^{-1} - \gamma(\lambda) K_0(\lambda) \big(K_1(\lambda) + M_\Pi(\lambda) K_0(\lambda)\big)^{-1} \gamma^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \tag{1.9}$$

where $A_0 := \operatorname{Ker} \Gamma_0$ is a fixed selfadjoint extension of A, $\gamma(\lambda) := (\Gamma_0 \upharpoonright \mathfrak{N}_\lambda(A))^{-1}$ is a γ-field and $M_\Pi(\lambda)$ is a Weyl function (Q-function) (1.2). The formula (1.9) establishes a bijective correspondence between the set of all generalized resolvents $\mathbb{R}_\lambda$ of A and the class $\widetilde{R}(\mathcal{H})$ of all Nevanlinna pairs of operator functions (Nevanlinna families of linear relations) $\{K_0(\lambda), K_1(\lambda)\} := \tau(\lambda)$ in $\mathcal{H}$ [13, 5, 6, 7].

The main result of this paper is the Krein–Naimark type formulas for a symmetric operator A with not necessarily equal defect numbers. Namely, with every D-triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^* we associate two formulas for generalized resolvents

$$\mathbb{R}_\lambda = (A_0 - \lambda)^{-1} - \gamma_+(\lambda) K_0(\lambda) \big(K_1(\lambda) + M_+(\lambda) K_0(\lambda)\big)^{-1} \gamma_-^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+, \tag{1.10}$$

$$\mathbb{R}_\lambda = (A_0^* - \lambda)^{-1} - \gamma_-(\lambda) N_1(\lambda) \big(N_0(\lambda) + M_-(\lambda) N_1(\lambda)\big)^{-1} \gamma_+^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_- \tag{1.11}$$

where $A_0 := \operatorname{Ker} \Gamma_0$ is a maximal symmetric extension of A with $n_-(A_0) = 0$, $\gamma_+(\lambda) := (\Gamma_0 \upharpoonright \mathfrak{N}_\lambda(A))^{-1},\ \lambda \in \mathbb{C}_+$ and $\gamma_-(z) := (P_1 \Gamma_0 \upharpoonright \mathfrak{N}_z(A))^{-1},\ z \in \mathbb{C}_-$ are γ-fields and $M_\pm(\cdot)$ are the corresponding Weyl functions (1.7). The formulas (1.10) and (1.11) describe the set of all generalized resolvents $\mathbb{R}_\lambda$ separately on $\mathbb{C}_+$ and $\mathbb{C}_-$. The part of the parameter here is carried out by two pairs of operator functions $\{K_0(\cdot), K_1(\cdot)\}$ and $\{N_1(\cdot), N_0(\cdot)\}$ which belong to the Nevanlinna type class $\widetilde{R}\,(\mathcal{H}_0, \mathcal{H}_1)$. The definition of this class and the investigations of its properties is contained in our paper [21]. Here we only note that in the special case $\mathcal{H}_1 = \mathcal{H}_0 := \mathcal{H}$ the class $\widetilde{R}\,(\mathcal{H}_0, \mathcal{H}_1)$ coincides with the class $\widetilde{R}(\mathcal{H})$ of Nevanlinna pairs (Nevanlinna families) in $\mathcal{H}$, so that $\widetilde{R}(\mathcal{H}, \mathcal{H}) = \widetilde{R}(\mathcal{H})$.

It is known that the main problem in the derivation of the formula (1.9) is the construction of a generalized resolvent $\mathbb{R}_\lambda$ by a parameter $\tau(\lambda) = \{K_0(\lambda), K_1(\lambda)\}$. The classical approach to this problem is essentially based on the Krein formula for canonical resolvents and the Naimark theorem. The new approach (coupling method) in the case $n_+(A) = n_-(A)$ has been recently proposed in [5, 6, 7]. This method is essentially based on the realization of $\tau(\cdot)$ as the Weyl function. Note that in the case $\tau(\lambda) \notin R^u[\mathcal{H}]$ such an approach requires the use of boundary relations and Weyl families of linear relations instead boundary triplets and Weyl functions respectively (see [6, 7]).

Our construction of $\mathbb{R}_\lambda$ by a parameter $\tau(\lambda)$ is inspired by the papers [5, 6, 7]. At the same time, unlike these papers our approach makes it possible to derive resolvents formula for an operator A with arbitrary defect numbers keeping the framework of boundary triplets (D-triplets) and the Weyl functions. This approach is based on some kind of the dilation theorem [21], which in the case $\mathcal{H}_1 = \mathcal{H}_0 := \mathcal{H}$ can be reformulated in the form of the following well known assertion [25, 8]: for every family $\tau(\lambda) \in \widetilde{R}(\mathcal{H})$ there exist a Hilbert space $\mathfrak{H}_1$ and a selfadjoint linear relation $\widetilde{\theta}$ in $\mathcal{H} \oplus \mathfrak{H}_1$ such that

$$P_{\mathcal{H}}(\widetilde{\theta} - \lambda)^{-1} \upharpoonright \mathcal{H} = -(\tau(\lambda) + \lambda)^{-1}, \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{1.12}$$

Using the dilation theorem we realize a parameter $\tau(\lambda)$ by (1.12) and construct in the explicit form a boundary triplet $\widetilde{\Pi}$ in the exit space $\mathfrak{H} \oplus \mathfrak{H}_1$. Next we show that the extension $\widetilde{A} := \widetilde{A}_{\widetilde{\theta}}$ (in the triplet $\widetilde{\Pi}$, see (1.8)) generates the desired generalized resolvent $\mathbb{R}_\lambda := P_{\mathfrak{H}}(\widetilde{A}_{\widetilde{\theta}} - \lambda)^{-1} | \mathfrak{H}$. Observe that in view of (1.12) the properties of the parameter $\widetilde{\theta}$ and, consequently, of the corresponding generalized resolvent $\mathbb{R}_\lambda = P_{\mathfrak{H}}(\widetilde{A}_{\widetilde{\theta}} - \lambda)^{-1} | \mathfrak{H}$ can be formulated in terms of the parameter $\tau(\lambda)$. Therefore we suppose that our method will be useful in some classical problems (expansion in eigenfunctions, moment and interpolation problems), where the parameter $\tau(\lambda)$ is used for the description of all solutions.

Observe also that our approach enables to obtain formula for generalized resolvents in Straus form [25, 26] directly from the formula (1.10). Moreover we establish a simple connection between these two formulas and find a geometric interpretation of the spectral parameter $\tau(\lambda)$ by means of abstract "boundary conditions". These results may be considered as a generalization to the case $n_+(A) \neq n_-(A)$ the corresponding results from [2, 5, 16] obtained for a symmetric operator with equal defect numbers.

Note in conclusion that our investigations here have also been inspired by the works of M. M. Malamud and the author [18, 19, 20] devoted to the theory of boundary triplets and the corresponding formulas for generalized resolvents of an isometric operator. In these works a concept of a boundary triplet of a symmetric operator has been extended to the case of an isometric operator V and the corresponding formulas for generalized resolvents as well as for the resolvents matrices of an operator V have been obtained. Note, that formulas (1.10), (1.11) are similar to that obtained in [18, 19, 20] for generalized resolvents of an isometric operator V.

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2. Preliminaries

2.1. **Notations.** The following notations will be used throughout the paper: $\mathfrak{H}$, $\mathcal{H}$ denote Hilbert spaces; $[\mathcal{H}_1, \mathcal{H}_2]$ is the set of all bounded linear operators defined on $\mathcal{H}_1$ with values in $\mathcal{H}_2$; $[\mathcal{H}] := [\mathcal{H}, \mathcal{H}]$; $A \upharpoonright L$ is the restriction of an operator A onto the linear manifold L; P_L is the orthogonal projector in $\mathfrak{H}$ onto the subspace $L \subset \mathfrak{H}$; $\mathbb{C}_+$ ($\mathbb{C}_-$) is the upper (lower) half-plain of the complex plain.

For a Hilbert space $\mathfrak{H}$ we denote by $\dim \mathfrak{H}$ its dimension. Moreover we write $\dim \mathfrak{H} < \infty$, if $\mathfrak{H}$ is finite-dimensional and $\dim \mathfrak{H} = \infty$, if $\mathfrak{H}$ is an infinite-dimensional not necessarily separable Hilbert space.

Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be Hilbert spaces. A linear manifold $T \subset \mathcal{H}_0 \oplus \mathcal{H}_1$ is called a linear relation in $\mathcal{H}_0 \oplus \mathcal{H}_1$ (from $\mathcal{H}_0$ to $\mathcal{H}_1$). We denote by $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ ($\widetilde{\mathcal{C}}(\mathcal{H})$) the set of all closed linear relations(closed subspaces) in $\mathcal{H}_0 \oplus \mathcal{H}_1$ (in $\mathcal{H} \oplus \mathcal{H}$). For a linear relation $T \subset \mathcal{H}_0 \oplus \mathcal{H}_1$ we denote by $\mathcal{D}(T)$, $\mathcal{R}(T)$, Ker$T$ and $T(0)$ the domain, the range, the kernel and the multivalued part of T respectively.

If T is a relation in $\mathcal{H}_0 \oplus \mathcal{H}_1$, then the inverse T^{-1} and adjoint T^* relations are defined as

$$T^{-1} = \{\{f', f\} : \{f, f'\} \in T\}, \quad T^{-1} \subset \mathcal{H}_1 \oplus \mathcal{H}_0,$$

and

$$T^* = \{\{g, g'\} \in \mathcal{H}_1 \oplus \mathcal{H}_0 : (f', g) = (f, g'), \quad \{f, f'\} \in T\}, \quad T^* \in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0).$$

A closed linear operator T from $\mathcal{H}_0$ to $\mathcal{H}_1$ is identified with its graph $\mathrm{gr}T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$.

In the case $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we write:

$0 \in \rho(T)$ if $\operatorname{Ker} T = \{0\}$ and $\mathcal{R}(T) = \mathcal{H}_1$, which is equivalent to the condition $T^{-1} \in [\mathcal{H}_1, \mathcal{H}_0]$;

$0 \in \hat{\rho}(T)$ if $\operatorname{Ker} T = \{0\}$ and $\overline{\mathcal{R}(T)} = \mathcal{R}(T) \neq \mathcal{H}_1$;

$0 \in \sigma_c(T)$ if $\operatorname{Ker} T = \{0\}$ and $\overline{\mathcal{R}(T)} = \mathcal{H}_1 \neq \mathcal{R}(T)$;

$0 \in \sigma_p(T)$ if $\operatorname{Ker} T \neq \{0\}$; $0 \in \sigma_r(T)$ if $\operatorname{Ker} T = \{0\}$ and $\overline{\mathcal{R}(T)} \neq \mathcal{H}_1$.

For a linear relation $T \in \widetilde{\mathcal{C}}(\mathcal{H})$ we denote by $\rho(T) = \{\lambda \in \mathbb{C} : \ 0 \in \rho(T - \lambda)\}$ and $\hat{\rho}(T) = \{\lambda \in \mathbb{C} : \ 0 \in \hat{\rho}(T - \lambda)\}$ the resolvent set and the set of regular type points of T respectively. Next, $\sigma(T) = \mathbb{C} \backslash \rho(T)$ stands for the spectrum of T. The spectrum $\sigma(T)$ admits the following classification:

$\sigma_c(T) = \{\lambda \in \mathbb{C} : 0 \in \sigma_c(T - \lambda)\}$ is the continuous spectrum;

$\sigma_p(T) = \{\lambda \in \mathbb{C} : 0 \in \sigma_p(T - \lambda)\}$ is the point spectrum;

$\sigma_r(T) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_c(T)) = \{\lambda \in \mathbb{C} : 0 \in \sigma_r(T - \lambda)\}$ is the residual spectrum.

For a linear relation $T \in \widetilde{\mathcal{C}}(\mathcal{H})$ and for every $\lambda \in \mathbb{C}$ let us introduce the lineal $\mathfrak{M}_\lambda(T) := \mathcal{R}(T - \lambda)$ and the closed subspaces

$$\mathfrak{N}_\lambda(T) := \mathcal{H} \ominus \mathfrak{M}_{\overline{\lambda}}(T) = \operatorname{Ker}(T^* - \lambda), \quad \hat{\mathfrak{N}}_\lambda(T) := \{\{f, \lambda f\} : f \in \mathfrak{N}_\lambda(T)\} \in \widetilde{\mathcal{C}}(\mathcal{H}).$$

It is clear that $\mathfrak{N}_\lambda(T) = \{f \in \mathcal{H} : \{f, \lambda f\} \in T^*\}$, so that $\hat{\mathfrak{N}}_\lambda(T) \subset T^*$. Moreover if $\overline{\lambda} \in \hat{\rho}(T)$ then $\mathfrak{N}_\lambda(T)$ is the defect subspace of a linear relation T.

2.2. **Linear relations and holomorphic functions.** Let H, $\mathcal{H}_0$ $\mathcal{H}_1$ be Hilbert spaces and let $K = (K_0 \ \ K_1)^\top \in [H, \mathcal{H}_0 \oplus \mathcal{H}_1]$. For a (not necessary closed) linear relation $\theta \subset \mathcal{H}_0 \oplus \mathcal{H}_1$ we write $\theta = \{K_0, K_1; H\}$ if $\operatorname{Ker} K = \{0\}$ (that is $\operatorname{Ker} K_0 \cap \operatorname{Ker} K_1 = \{0\}$) and

$$\theta = KH = \{\{K_0 h, K_1 h\} : h \in H\}.$$

Similarly let $C = (C_0 \ \ C_1) \in [\mathcal{H}_0 \oplus \mathcal{H}_1, H]$. For a linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we write $\theta = \{(C_0, C_1); H\}$ if $\mathcal{R}(C) = H$ and

$$\theta = \operatorname{Ker} C = \{\{h_0, h_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1 : C_0 h_0 + C_1 h_1 = 0\}.$$

It is clear that every linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ admits both representations $\theta = \{K_0, K_1; H\}$ and $\theta = \{(C_0, C_1); H'\}$. Moreover the equalities $\dim H = \dim \theta$, $\dim H' = \operatorname{codim} \theta$ are valid.

Let $\mathcal{D}$ be an open set in $\mathbb{C}$ and let $K_0(\cdot) : \mathcal{D} \to [H, \mathcal{H}_0]$, $K_1(\cdot) : \mathcal{D} \to [H, \mathcal{H}_1]$ be a pair of holomorphic operator functions. Such a pair will be called admissible if $\operatorname{Ker} K_0(\lambda) \cap \operatorname{Ker} K_1(\lambda) = \{0\}$, $\lambda \in \mathcal{D}$.

Definition 2.1. Let $\{K_0(\cdot), K_1(\cdot)\}$ and $\{K_0'(\cdot), K_1'(\cdot)\}$ be two admissible pairs of holomorphic operator functions, $K_j : \mathcal{D} \to [H, \mathcal{H}_j]$, $K_j' : \mathcal{D} \to [H', \mathcal{H}_j]$, $j \in \{0, 1\}$. Two such pairs are said to be equivalent if $K_0'(\lambda) = K_0(\lambda)\varphi(\lambda)$ and $K_1'(\lambda) = K_1(\lambda)\varphi(\lambda)$ for some holomorphic operator function $\varphi(\cdot) : \mathcal{D} \to [H', H]$ such that $0 \in \rho(\varphi(\lambda))$ for every $\lambda \in \mathcal{D}$.

Definition 2.2. A function $\tau(\cdot)$, defined on an open set $\mathcal{D} \subset \mathbb{C}$ with values in $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ is called holomorphic on $\mathcal{D}$ if there exist a Hilbert space H and an admissible pair of holomorphic operator functions $K_j(\cdot) : \mathcal{D} \to [H, \mathcal{H}_j]$, $j \in \{0, 1\}$ such that

$$\tau(\lambda) = \{K_0(\lambda), K_1(\lambda); H\} = \{\{K_0(\lambda)h, K_1(\lambda)h\} : h \in H\}, \quad \lambda \in \mathcal{D}. \tag{2.1}$$

It is clear that two pairs $\{K_0(\cdot), K_1(\cdot)\}$ and $\{K_0'(\cdot), K_1'(\cdot)\}$ define by (2.1) the same holomorphic function $\tau(\cdot)$, if and only if they are equivalent. Therefore we will identify (by means of (2.1)) a holomorphic $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$-valued function $\tau(\cdot)$ and the corresponding class of equivalent admissible pairs $\{K_0(\cdot), K_1(\cdot)\}$.

2.3. **Nevanlinna type families and the dilation theorem.** In this subsection some definitions and results from our paper [21] are specified.

Let $\mathcal{H}_1$ be a subspace in a Hilbert space $\mathcal{H}_0$ and let $\mathcal{H}_2 = \mathcal{H}_0 \ominus \mathcal{H}_1$. Denote by P_j the orthoprojector in $\mathcal{H}_0$ onto $\mathcal{H}_j,\ j \in \{1, 2\}$ and introduce the operators

(2.2)
$$J_{01} = \begin{pmatrix} P_2 & -iI_{\mathcal{H}_1} \\ iP_1 & 0 \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1, \quad J_{10} = \begin{pmatrix} 0 & -iP_1 \\ iI_{\mathcal{H}_1} & P_2 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_0 \to \mathcal{H}_1 \oplus \mathcal{H}_0,$$

(2.3)
$$U_{01} = \begin{pmatrix} P_1 & 0 \\ iP_2 & I_{\mathcal{H}_1} \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_0, \qquad U_{10} = \begin{pmatrix} I_{\mathcal{H}_1} & -iP_2 \\ 0 & P_1 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_0 \to \mathcal{H}_0 \oplus \mathcal{H}_1.$$

It is easily seen that J_{01} and J_{10} are signature operators, i.e., $J_{01} = (J_{01})^* = (J_{01})^{-1}$ and $J_{10} = (J_{10})^* = (J_{10})^{-1}$. Furthermore U_{01} and U_{10} are unitary operators connected by the equality $U_{10} = (U_{01})^{-1}$.

For a linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we put

$$\theta^\times = J_{01}(\theta^\perp) = (J_{01}\theta)^\perp, \quad \theta^\times \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1). \tag{2.4}$$

It is clear that $\theta^\times$ is the set of all vectors $\hat{k} = \{k_0, k_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1$ such that

$$(k_1, h_0) - (k_0, h_1) + i(P_2 k_0, P_2 h_0) = 0, \quad \{h_0, h_1\} \in \theta. \tag{2.5}$$

If $\mathcal{H}_1 = \mathcal{H}_0 := \mathcal{H}$, then a linear relation $\theta^\times \in \widetilde{\mathcal{C}}(\mathcal{H})$ coincides with θ^*. Moreover it was shown in [21], Proposition 3.1 that in the general case $\mathcal{H}_1 \subset \mathcal{H}_0$ a relation $\theta^\times$ possesses a number of properties similar that of θ^*.

Definition 2.3. ([21]). Let $\mathcal{H}_1$ be a subspace in a Hilbert space $\mathcal{H}_0$, let θ be a closed linear relation in $\mathcal{H}_0 \oplus \mathcal{H}_1$ and let

$$\varphi_\theta(\hat{h}) = 2\text{Im}(h_1, h_0) + ||P_2 h_0||^2, \quad \hat{h} = \{h_0, h_1\} \in \theta.$$

The relation θ belongs to the class:

1) $\text{Dis}_0(\mathcal{H}_0, \mathcal{H}_1)$ $(\text{Ac}_0(\mathcal{H}_0, \mathcal{H}_1))$, if $\varphi_\theta(\hat{h}) \geq 0$ $(\varphi_\theta(\hat{h}) \leq 0)$ for all $\hat{h} \in \theta$;
2) $\text{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$, if $\theta \subset \theta^\times$ or equivalently if $\varphi_\theta(\hat{h}) = 0,\ \hat{h} \in \theta$;
3) $\text{Self}(\mathcal{H}_0, \mathcal{H}_1)$, if $\theta = \theta^\times$.

Moreover a linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ belongs to one of the classes $\text{Dis}(\mathcal{H}_0, \mathcal{H}_1)$, $\text{Ac}(\mathcal{H}_0, \mathcal{H}_1)$ or $\text{Sym}(\mathcal{H}_0, \mathcal{H}_1)$ if it belongs to the class $\text{Dis}_0(\mathcal{H}_0, \mathcal{H}_1)$, $\text{Ac}_0(\mathcal{H}_0, \mathcal{H}_1)$ or $\text{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$ respectively and there are not extensions $\widetilde{\theta} \supset \theta,\ \widetilde{\theta} \neq \theta$ in the corresponding class.

Let as before $\mathcal{H}_1$ be a subspace in a Hilbert space $\mathcal{H}_0$ and let $\mathcal{H}_2 = \mathcal{H}_0 \ominus \mathcal{H}_1$. In the next definition the concept of a Nevanlinna type class of holomorphic $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$-valued functions (families of linear relations) is introduced.

Definition 2.4. ([21]). A holomorphic $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$-valued function

$$\tau_+(\cdot) : \mathbb{C}_+ \to \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$$

belongs to the Nevanlinna type class $\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$, if $-\tau_+(\lambda)\in \mathrm{Ac}(\mathcal{H}_0,\mathcal{H}_1)$ for every $\lambda\in\mathbb{C}_+$.

A pair of holomorphic functions $\tau_+(\cdot):\mathbb{C}_+\to\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ and $\tau_-(\cdot):\mathbb{C}_-\to\widetilde{\mathcal{C}}(\mathcal{H}_1,\mathcal{H}_0)$ belongs to the class $\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ if $\tau_+(\cdot)\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ and $\tau_-(\overline{\lambda})=\tau_+^*(\lambda)$ for every $\lambda\in\mathbb{C}_+$. In what follows such a pair of functions $\tau_+(\cdot)$ and $\tau_-(\cdot)$ will be denoted by $\tau=\{\tau_+,\tau_-\}$.

A pair of functions $\tau=\{\tau_+(\cdot),\tau_-(\cdot)\}\in\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ is referred to the class $\widetilde{R}^0(\mathcal{H}_0,\mathcal{H}_1)$ if $\tau_+(\lambda)=\tau_+,\ \lambda\in\mathbb{C}_+;\quad \tau_-(z)=\tau_-,\ z\in\mathbb{C}_-$ (ı,e., the functions $\tau_+(\cdot)$ and $\tau_-(\cdot)$ are constant on their domains) and $-\tau_+\in\mathrm{Self}(\mathcal{H}_0,\mathcal{H}_1)$.

It follows from Definition 2.2 that functions $\tau_+(\cdot)$ and $\tau_-(\cdot)$ admit the representations

$$\tau_+(\lambda)=\{K_0(\lambda),K_1(\lambda);H_+\},\quad \lambda\in\mathbb{C}_+, \tag{2.6}$$

$$\tau_-(z)=\{N_1(z),N_0(z);H_-\},\quad z\in\mathbb{C}_- \tag{2.7}$$

where H_+,H_- are auxiliary Hilbert spaces and $K_j(\cdot):\mathbb{C}_+\to[H_+,\mathcal{H}_j]$, $N_j(\cdot):\mathbb{C}_-\to[H_-,\mathcal{H}_j]$ $j\in\{0,1\}$ are holomorphic operator functions. The description of the classes $\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ and $\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ in terms of the corresponding pairs $\{K_0(\cdot),K_1(\cdot)\}$ and $\{N_1(\cdot),N_0(\cdot)\}$ was obtained in [21], Proposition 4.3.

Definition 2.5. ([21]). Let $\mathcal{H}_0$, $\mathfrak{H}_1$ be Hilbert spaces and let $\mathcal{H}_1$ be a subspace in $\mathcal{H}_0$. A linear relation $\widetilde{\theta}\in Self(\mathcal{H}_0\oplus\mathfrak{H}_1,\mathcal{H}_1\oplus\mathfrak{H}_1)$ is called a dilation of a $\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$-valued function $\tau_+(\cdot):\mathbb{C}_+\to\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$, if there exist representations $\widetilde{\theta}=\{\widetilde{K}_0,\widetilde{K}_1;\mathcal{H}_1\oplus\mathfrak{H}_1\}$ and $\tau_+(\lambda)=\{K_0(\lambda),K_1(\lambda);\mathcal{H}_1\}$, $\lambda\in\mathbb{C}_+$ with the following properties:

i) the operators $\widetilde{K}_0\in[\mathcal{H}_1\oplus\mathfrak{H}_1,\mathcal{H}_0\oplus\mathfrak{H}_1]$ and $\widetilde{K}_1\in[\mathcal{H}_1\oplus\mathfrak{H}_1]$ have the block-matrix representations

$$\widetilde{K}_0=\begin{pmatrix}K_1&K_2\\K_3&K_4\end{pmatrix}:\mathcal{H}_1\oplus\mathfrak{H}_1\to\mathcal{H}_0\oplus\mathfrak{H}_1,\quad \widetilde{K}_1=\begin{pmatrix}N_1&N_2\\N_3&N_4\end{pmatrix}:\mathcal{H}_1\oplus\mathfrak{H}_1\to\mathcal{H}_1\oplus\mathfrak{H}_1 \tag{2.8}$$

such that $0\in\rho(N_4-\lambda K_4)$, $\lambda\in\mathbb{C}_+$;

ii) the equalities

$$K_0(\lambda)=-K_1+K_2(N_4-\lambda K_4)^{-1}(N_3-\lambda K_3),\quad \lambda\in\mathbb{C}_+, \tag{2.9}$$

$$K_1(\lambda)=N_1-N_2(N_4-\lambda K_4)^{-1}(N_3-\lambda K_3),\quad \lambda\in\mathbb{C}_+ \tag{2.10}$$

are valid.

A function $\tau_+(\cdot):\mathbb{C}_+\to\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ is called a compression of a linear relation $\widetilde{\theta}\in\mathrm{Self}(\mathcal{H}_0,\mathcal{H}_1)(\mathcal{H}_0\oplus\mathfrak{H}_1,\mathcal{H}_1\oplus\mathfrak{H}_1)$, if $\widetilde{\theta}$ is a dilation of $\tau_+(\cdot)$.

One can easily verify that in the case $\mathcal{H}_0=\mathcal{H}_1:=\mathcal{H}$ a dilation $\widetilde{\theta}=\widetilde{\theta}^*\in\widetilde{\mathcal{C}}(\mathcal{H}\oplus\mathfrak{H}_1)$ and the corresponding compression $\tau_+(\lambda):\mathbb{C}_+\to\widetilde{\mathcal{C}}(\mathcal{H})$ are connected via (c.f. (1.12))

$$-(\tau_+(\lambda)+\lambda)^{-1}=P_{\mathcal{H}}(\widetilde{\theta}-\lambda)^{-1}\upharpoonright\mathcal{H},\quad \lambda\in\mathbb{C}_+.$$

The following dilation theorem was proved in [21].

Theorem 2.6. *If $\widetilde{\theta}\in\mathrm{Self}(\mathcal{H}_0,\mathcal{H}_1)(\mathcal{H}_0\oplus\mathfrak{H}_1,\mathcal{H}_1\oplus\mathfrak{H}_1)$, then there exists a unique compression $\tau_+(\cdot)$ of $\widetilde{\theta}$ and $\tau_+(\cdot)\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$.*

Conversely for every function $\tau_+(\cdot)\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ there exist a Hilbert space $\mathfrak{H}_1$ and a linear relation $\widetilde{\theta}\in\mathrm{Self}(\mathcal{H}_0,\mathcal{H}_1)(\mathcal{H}_0\oplus\mathfrak{H}_1,\mathcal{H}_1\oplus\mathfrak{H}_1)$ such that $\widetilde{\theta}$ is a dilation of $\tau_+(\cdot)$.

3. Boundary D-triplets and Weyl functions

3.1. **Boundary triplets and Weyl functions for dual pairs.** In this subsection we recall some definitions and results from [15, 17], concerning dual pairs of linear relations. These results will be systematically used in what follows.

Definition 3.1. A pair $\{A, B\}$ of closed linear relations A, B in $\mathfrak{H}$ is called a dual pair, if

$$(f', g) = (f, g'), \quad \hat{f} = \{f, f'\} \in A, \ \hat{g} = \{g, g'\} \in B$$

or equivalently if $A \subset B^* (\iff B \subset A^*)$.

A linear relation $\widetilde{A} \in \widetilde{\mathcal{C}}(\mathfrak{H})$ is called a proper extension of a dual pair $\{A, B\}$ if $A \subset \widetilde{A} \subset B^*$. The set of all proper extensions of a dual pair $\{A, B\}$ is denoted by Ext $\{A, B\}$.

Definition 3.2. Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be Hilbert spaces and let $\Gamma^B = (\Gamma_0^B \ \ \Gamma_1^B)^\top : B^* \to \mathcal{H}_0 \oplus \mathcal{H}_1$ and $\Gamma^A = (\Gamma_0^A \ \ \Gamma_1^A)^\top : A^* \to \mathcal{H}_1 \oplus \mathcal{H}_0$ be linear maps. A collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma^B, \Gamma^A\}$ is called a boundary triplet for a dual pair $\{A, B\}$ if:

(i) $\Gamma^B B^* = \mathcal{H}_0 \oplus \mathcal{H}_1, \ \ \Gamma^A A^* = \mathcal{H}_1 \oplus \mathcal{H}_0$;

(ii) the following Green identity holds

(3.1)
$$(f', g) - (f, g') = (\Gamma_1^B \hat{f}, \Gamma_0^A \hat{g}) - (\Gamma_0^B \hat{f}, \Gamma_1^A \hat{g}), \quad \hat{f} = \{f, f'\} \in B^*, \quad \hat{g} = \{g, g'\} \in A^*.$$

Proposition 3.3. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma^B, \Gamma^A\}$ be a boundary triplet for a dual pair $\{A, B\}$. Then the following statements are valid:*

1) $\operatorname{Ker} \Gamma^B = A$, $\operatorname{Ker} \Gamma^A = B$ *and the operators* Γ^B *and* Γ^A *are bounded, that is* $\Gamma^B \in [B^*, \mathcal{H}_0 \oplus \mathcal{H}_1]$, $\Gamma^A \in [A^*, \mathcal{H}_1 \oplus \mathcal{H}_0]$*;*

2) *the collection* $\Pi^* = \{\mathcal{H}_1 \oplus \mathcal{H}_0, \Gamma^A, \Gamma^B\}$ *forms a boundary triplet for the dual pair* $\{B, A\}$*;*

3) *the equality*

$$\widetilde{A} = \widetilde{A}_\theta := \{\hat{f} \in B^* : \{\Gamma_0^B \hat{f}, \Gamma_1^B \hat{f}\} \in \theta\} \tag{3.2}$$

establishes a bijective correspondence between the set of all proper extensions $\widetilde{A} \in$ Ext $\{A, B\}$ *and the set of linear relations* $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$*. Moreover it follows from* (3.2) *that* $\widetilde{A} = \widetilde{A}_\theta \iff \Gamma^B \widetilde{A} = \theta$*;*

4) *If* $\widetilde{A} = \widetilde{A}_\theta \in \operatorname{Ext}\{A, B\}$ *(in the triplet* Π*), then* $\widetilde{A}^* \in \operatorname{Ext}\{B, A\}$ *and* $\widetilde{A}^* = \widetilde{A}_{\theta^*}$ *(in the triplet* Π^**).*

The proper extensions $A_0 := \operatorname{Ker} \Gamma_0^B$ and $A_1 := \operatorname{Ker} \Gamma_1^B$ are naturally associated to a boundary triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma^B, \Gamma^A\}$ for a dual pair $\{A, B\}$. It follows from Proposition 3.3, 4) that $A_0^* = \operatorname{Ker} \Gamma_0^A$ and $A_1^* = \operatorname{Ker} \Gamma_1^A$.

Associated to the boundary triplet Π are holomorphic operator functions (γ-fields) $\hat{\gamma}_\Pi(\lambda) : \rho(A_0) \to [\mathcal{H}_0, \hat{\mathfrak{N}}_\lambda(B)]$ and $\gamma_\Pi(\lambda) : \rho(A_0) \to [\mathcal{H}_0, \mathfrak{N}_\lambda(B)]$ given by

$$\hat{\gamma}_\Pi(\lambda) := (\Gamma_0^B \upharpoonright \hat{\mathfrak{N}}_\lambda(B))^{-1}, \quad \gamma_\Pi(\lambda) := \pi_1 \hat{\gamma}_\Pi(\lambda), \quad \lambda \in \rho(A_0) \tag{3.3}$$

(here π_1 is the orthoprojection in $\mathfrak{H} \oplus \mathfrak{H}$ onto $\mathfrak{H} \oplus \{0\}$). Similarly the holomorphic operator functions $\hat{\gamma}_{\Pi^*}(z) : \rho(A_0^*) \to [\mathcal{H}_1, \hat{\mathfrak{N}}_z(B)]$ and $\gamma_{\Pi^*}(z) : \rho(A_0^*) \to [\mathcal{H}_1, \mathfrak{N}_z(B)]$,

$$\hat{\gamma}_{\Pi^*}(z) := (\Gamma_0^A \upharpoonright \hat{\mathfrak{N}}_z(A))^{-1}, \quad \gamma_\Pi(z) := \pi_1 \hat{\gamma}_{\Pi^*}(z), \quad z \in \rho(A_0^*) \tag{3.4}$$

are associated to the boundary triplet Π^*. It is proved in [17] that the following relations hold

$$\begin{aligned} \gamma_\Pi(\mu) &= \gamma_\Pi(\lambda) + (\mu - \lambda)(A_0 - \mu)^{-1} \gamma_\Pi(\lambda), \qquad \lambda, \ \mu \in \rho(A_0), \\ \gamma_{\Pi^*}(\omega) &= \gamma_{\Pi^*}(z) + (\omega - z)(A_0^* - \omega)^{-1} \gamma_{\Pi^*}(z), \quad \ z, \omega \in \rho(A_0^*). \end{aligned} \tag{3.5}$$

Definition 3.4. ([17]). The holomorphic operator function $M_\Pi(\cdot) : \rho(A_0) \to [\mathcal{H}_0, \mathcal{H}_1]$ defined by the equality

$$\Gamma_1^B \hat{f}_\lambda = M_\Pi(\lambda) \Gamma_0^B \hat{f}_\lambda, \quad \hat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} \in \hat{\mathfrak{N}}_\lambda(B), \quad \lambda \in \rho(A_0) \tag{3.6}$$

is called the Weyl function corresponding to a boundary triplet Π.

It follows from Definition 3.4 that the equality

$$\Gamma_1^A \hat{f}_z = M_{\Pi^*}(z)\Gamma_0^A \hat{f}_z, \quad \hat{f}_z = \{f_z, zf_z\} \in \hat{\mathfrak{N}}_z(A), \quad z \in \rho(A_0^*) \tag{3.7}$$

defines the Weyl function $M_{\Pi^*}(\cdot) : \rho(A_0^*) \to [\mathcal{H}_1, \mathcal{H}_0]$ corresponding to the triplet Π^*.

It is shown in [17] that the functions $M_\Pi(\cdot)$ and $M_{\Pi^*}(\cdot)$ satisfy the relations

$$M_\Pi(\mu) - M_\Pi(\lambda) = (\mu - \lambda)\gamma_{\Pi^*}^*(\overline{\lambda})\gamma_\Pi(\mu), \quad \mu,\ \lambda \in \rho(A_0), \tag{3.8}$$

$$M_{\Pi^*}(\omega) - M_{\Pi^*}(z) = (\omega - z)\gamma_\Pi^*(\overline{z})\gamma_{\Pi^*}(\omega), \quad \omega,\ z \in \rho(A_0^*), \tag{3.9}$$

$$M_{\Pi^*}(\overline{\lambda}) = M_\Pi^*(\lambda), \quad \lambda \in \rho(A_0). \tag{3.10}$$

Theorem 3.5. ([17, 10]). *Suppose that $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma^B, \Gamma^A\}$ is a boundary triplet for a dual pair $\{A, B\}$, $M_\Pi(\lambda)$ is the corresponding Weyl function, $A_0 := \operatorname{Ker}\Gamma_0^B$, $\theta = \{K_0, K_1; H\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ and $\widetilde{A} = \widetilde{A}_\theta \in \operatorname{Ext}\{A, B\}$. Then $\lambda \in \rho(\widetilde{A}) \cap \rho(A_0)$ if and only if $0 \in \rho(K_1 - M_\Pi(\lambda)K_0)$ and the following formula for the canonical resolvents holds true*

$$(\widetilde{A}_\theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma_\Pi(\lambda)K_0(K_1 - M_\Pi(\lambda)K_0)^{-1}\gamma_{\Pi^*}^*(\overline{\lambda}), \quad \lambda \in \rho(\widetilde{A}_\theta) \cap \rho(A_0).$$

3.2. **Boundary D-triplets.** Let $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$ be a closed symmetric linear relation with arbitrary defect numbers $n_\pm(A) = \dim \mathfrak{N}_\lambda(A)$, $\lambda \in \mathbb{C}_\pm$. Denote by Ext_A the set of all proper extensions of A, i.e., the set of all linear relations $\widetilde{A} \in \widetilde{\mathcal{C}}(\mathfrak{H})$ such that $A \subset \widetilde{A} \subset A^*$. It is clear that $\{A, A\}$ is a dual pair in $\mathfrak{H}$ and $\operatorname{Ext}\{A, A\} = \operatorname{Ext}_A$.

Definition 3.6. ([16, 17]). Extensions $\widetilde{A}_1$ and $\widetilde{A}_2 \in \operatorname{Ext}_A$ are called transversal if $\widetilde{A}_1 \cap \widetilde{A}_2 = A$ and $\widetilde{A}_1 + \widetilde{A}_2 = A^*$.

Let $\mathcal{H}_0$ be a Hilbert space, let $\mathcal{H}_1$ be a subspace in $\mathcal{H}_0$ and let $\mathcal{H}_2 = \mathcal{H}_0 \ominus \mathcal{H}_1$. Denote by P_j the orthoprojector in $\mathcal{H}_0$ onto $\mathcal{H}_j$, $j \in \{1, 2\}$.

Definition 3.7. A collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$, where Γ_j are linear mappings from A^* to $\mathcal{H}_j$ $(j \in \{0, 1\})$, will be called a D-boundary triplet (D-triplet) for A^*, if $\Gamma = (\Gamma_0 \ \ \Gamma_1)^\top : A^* \to \mathcal{H}_0 \oplus \mathcal{H}_1$ is a surjective linear mapping from A^* onto $\mathcal{H}_0 \oplus \mathcal{H}_1$ and such that Green's identity

$$(f', g) - (f, g') = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g}) - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g}) + i(P_2\Gamma_0 \hat{f}, P_2\Gamma_0 \hat{g}) \tag{3.11}$$

holds for all $\hat{f} = \{f, f'\}, \hat{g} = \{g, g'\} \in A^*$.

Using the operator J_{01} (see (2.2)) one can rewrite the identity (3.11) as

$$(f', g) - (f, g') = i(J_{01}\Gamma\hat{f}, \Gamma\hat{g}), \quad \hat{f}, \hat{g} \in A^*. \tag{3.12}$$

Lemma 3.8. *Suppose that $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a D-triplet for A^*, U_{01} is the unitary operator* (2.3) *and $\hat{\Gamma}^A = (\hat{\Gamma}_0^A \ \ \hat{\Gamma}_1^A)^\top : A^* \to \mathcal{H}_0 \oplus \mathcal{H}_1$, $\Gamma^A = (\Gamma_0^A \ \ \Gamma_1^A)^\top : A^* \to \mathcal{H}_1 \oplus \mathcal{H}_0$ are the operators given by $\hat{\Gamma}^A = \Gamma$, $\Gamma^A = U_{01}\Gamma$. Then*

$$\hat{\Gamma}_0^A = \Gamma_0, \quad \hat{\Gamma}_1^A = \Gamma_1, \quad \Gamma_0^A = P_1\Gamma_0, \quad \Gamma_1^A = \Gamma_1 + iP_2\Gamma_0 \tag{3.13}$$

and the collection $\hat{\Pi} = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \hat{\Gamma}^A, \Gamma^A\}$ is a boundary triplet for the dual pair $\{A, A\}$ in the sense of Definition 3.2.

Proof. It follows from (3.11) that for every $\hat{f} = \{f, f'\}, \hat{g} = \{g, g'\} \in A^*$

$$(f', g) - (f, g') = (\Gamma_1\hat{f}, P_1\Gamma_0\hat{g}) - (\Gamma_0\hat{f}, \Gamma_1\hat{g} + iP_2\Gamma_0\hat{g}) = (\hat{\Gamma}_1^A\hat{f}, \Gamma_0^A\hat{g}) - (\hat{\Gamma}_0^A\hat{f}, \Gamma_1^A\hat{g}).$$

This yields the identity (3.1) for the triplet $\hat{\Pi}$. The mapping Γ^A is surjective because $\Gamma^A A^* = U_{01}\Gamma A^* = U_{01}(\mathcal{H}_0 \oplus \mathcal{H}_1) = \mathcal{H}_1 \oplus \mathcal{H}_0$. □

Proposition 3.9. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a D-triplet for a linear relation A^*. Then:*

1) $\operatorname{Ker}\Gamma_0 \cap \operatorname{Ker}\Gamma_1 = A$ *and Γ_i is a bounded operator from A^* to $\mathcal{H}_i$, $i \in \{0,1\}$;*

2) *every proper extension $\widetilde{A} \in \operatorname{Ext}_A$ can be parametrized by a linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$. Namely the equality*

$$\widetilde{A} = \widetilde{A}_\theta := \{\hat{f} \in A^* : \{\Gamma_0 \hat{f}, \Gamma_1 \hat{f}\} \in \theta\} \tag{3.14}$$

establishes a bijective correspondence between the set of all proper extensions $\widetilde{A} \in \operatorname{Ext}_A$ and the set of all linear relations $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$. Moreover in view of (3.14) the equality $\widetilde{A} = \widetilde{A}_\theta$ means that $\theta = \Gamma\widetilde{A} = \{\{\Gamma_0\hat{f}, \Gamma_1\hat{f}\} : \hat{f} \in \widetilde{A}\}$;

3) *$\widetilde{A}_{\theta'} \subset \widetilde{A}_{\theta''}$ if and only if $\theta' \subset \theta''$. In this case $\dim \widetilde{A}_{\theta''}/\widetilde{A}_{\theta'} = \dim \theta''/\theta'$;*

4) *$(\widetilde{A}_\theta)^* = \widetilde{A}_{\theta^\times}$;*

5) *an extension $\widetilde{A}_\theta \in \operatorname{Ext}_A$ is maximal dissipative, maximal accumulative, maximal symmetric or selfadjoint if and only if θ belongs to the class $\operatorname{Dis}(\mathcal{H}_0, \mathcal{H}_1)$, $\operatorname{Ac}(\mathcal{H}_0, \mathcal{H}_1)$, $\operatorname{Sym}(\mathcal{H}_0, \mathcal{H}_1)$ or $\operatorname{Self}(\mathcal{H}_0, \mathcal{H}_1)$ respectively;*

6) *extensions $\widetilde{A}_{\theta'}$ and $\widetilde{A}_{\theta''}$ are transversal if and only if $\theta' \dotplus \theta'' = \mathcal{H}_0 \oplus \mathcal{H}_1$.*

Proof. The statements 1), 2) arise from the statements 1), 3) of Proposition 3.3 applied to the boundary triplet $\hat{\Pi}$ for the dual pair $\{A, A\}$ (see Lemma 3.8).The statement 3) follows from the statements 1), 2). The statement 4) is implied by (3.11) and the definition (2.5) of $\theta^\times$.

5) According to (3.11) and Definition 2.3 an extension $\widetilde{A}_\theta$ is dissipative, accumulative or symmetric if and only if θ belongs to the class $\operatorname{Dis}_0(\mathcal{H}_0, \mathcal{H}_1)$, $\operatorname{Ac}_0(\mathcal{H}_0, \mathcal{H}_1)$ or $\operatorname{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$ respectively. This and the statement 3) yield the desired statement.

6) This statement is implied by the equalities $\mathcal{R}(\Gamma) = \mathcal{H}_0 \oplus \mathcal{H}_1$ and $\operatorname{Ker}\Gamma = A$. □

In what follows we will systematically use two proper extensions

$$A_0 := \operatorname{Ker}\Gamma_0 = \{\hat{f} \in A^* : \Gamma_0\hat{f} = 0\}, \quad A_1 := \operatorname{Ker}\Gamma_1 = \{\hat{f} \in A^* : \Gamma_1\hat{f} = 0\}, \tag{3.15}$$

which are naturally associated to a D-triplet $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$. It follows from (3.15) that $A_0 = \widetilde{A}_{\theta_0}$ and $A_1 = \widetilde{A}_{\theta_1}$ where

$$\theta_0 = \{0\} \oplus \mathcal{H}_1 = \{0_{\mathcal{H}_1,\mathcal{H}_0}, I_{\mathcal{H}_1}; \mathcal{H}_1\}, \quad \theta_1 = \mathcal{H}_0 \oplus \{0\} = 0_{\mathcal{H}_0,\mathcal{H}_1}. \tag{3.16}$$

Corollary 3.10. *Extensions $\widetilde{A}_\theta$ and A_0 are transversal if and only if $\theta \in [\mathcal{H}_0, \mathcal{H}_1]$.*

Proof. The desired statement is implied by Proposition 3.9, 6) and the obvious equivalence $\theta \dotplus \theta_0 = \mathcal{H}_0 \oplus \mathcal{H}_1 \iff \theta \in [\mathcal{H}_0, \mathcal{H}_1]$. □

Proposition 3.11. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a D-triplet for a linear relation A^*, $A_i = \operatorname{Ker}\Gamma_i$, $i \in \{0,1\}$ and $\mathcal{H}_2 = \mathcal{H}_0 \ominus \mathcal{H}_1$. Then A_0 and A_1^* are maximal symmetric extensions of A and*

$$\begin{gathered} \dim \mathcal{H}_1 = n_-(A) \le n_+(A) = \dim \mathcal{H}_0, \\ n_-(A_0) = n_-(A_1^*) = 0, \quad n_+(A_0) = n_+(A_1^*) = \dim \mathcal{H}_2. \end{gathered} \tag{3.17}$$

Conversely let $A \subset A^*$, $n_-(A) \le n_+(A)$ and let A_0 be an extension of A such that $A_0 \subset A_0^*$ and $n_-(A_0) = 0$. Then there exists a D-triplet for A^* such that $A_0 = \operatorname{Ker}\Gamma_0$.

Proof. It follows from (3.16) and (2.5) that

$$\theta_0^\times = \mathcal{H}_2 \oplus \mathcal{H}_1, \quad \theta_1^\times = \mathcal{H}_1 \oplus \{0\} = \{I_{\mathcal{H}_1,\mathcal{H}_0}, 0; \mathcal{H}_1\}. \tag{3.18}$$

Therefore by Proposition 3.4 from [21] the linear relations θ_0 and $\theta_1^\times$ belong to the set $\operatorname{Sym}(\mathcal{H}_0, \mathcal{H}_1) \cap \operatorname{Ac}(\mathcal{H}_0, \mathcal{H}_1)$. This and the statements 4), 5) of Proposition 3.9 imply that A_0 and A_1^* are maximal symmetric and $n_-(A_0) = n_-(A_1^*) = 0$.

Furthermore Proposition 3.9, 3) yields

$$n_-(A) = \dim A_0/A = \dim \theta_0 = \dim \mathcal{H}_1, \quad n_+(A) = \dim A_0^*/A = \dim \theta_0^\times = \dim \mathcal{H}_0,$$
$$n_+(A_0) = \dim A_0^*/A_0 = \dim \theta_0^\times/\theta_0 = \dim \mathcal{H}_2,$$
$$n_+(A_1^*) = \dim A_1/A_1^* = \dim \theta_1/\theta_1^\times = \dim \mathcal{H}_2$$

which leads to (3.17).

Next we prove the converse statement. Let $U = \{\{f_0' - i\, f_0, f_0' + i\, f_0\} : \{f_0, f_0'\} \in A_0\}$ be the Cayley transform of A_0. Clearly, U is an isometric operator with $\mathcal{D}(U) = \mathfrak{H}$ and such that $U\mathfrak{N}_{-i}(A) \subset \mathfrak{N}_i(A)$. To construct the desired D-triplet for A^* we put $\mathcal{H}_0 = \mathfrak{N}_i(A)$, $\mathcal{H}_1 = U\mathfrak{N}_{-i}(A)$, $\mathcal{H}_2 = \mathcal{H}_0 \ominus \mathcal{H}_1 = \mathfrak{N}_i(A_0)$ and make use of the decomposition

$$A^* = A \oplus \hat{\mathfrak{N}}_i(A) \oplus \hat{\mathfrak{N}}_{-i}(A). \tag{3.19}$$

Define the operators $\Gamma_j : A^* \to \mathcal{H}_j,\ j \in \{0,1\}$ by setting

$$\Gamma_0 \hat{f} = -i(P_1 f_i + \sqrt{2} P_2 f_i - U f_{-i}), \quad \Gamma_1 \hat{f} = P_1 f_i + U f_{-i} \tag{3.20}$$

where $\hat{f} \in A^*$ and according to (3.19)

$$\hat{f} = \{f_A, f_A'\} + \{f_i, i f_i\} + \{f_{-i}, -i f_{-i}\} = \{f_A + f_i + f_{-i}, f_A' + i f_i - i f_{-i}\},$$
$$\{f_A, f_A'\} \in A, \quad f_{\pm i} \in \mathfrak{N}_{\pm i}.$$

It follows from (3.20) that for every $\hat{f}, \hat{g} \in A^*$

$$\begin{aligned}(\Gamma_1 \hat{f}, \Gamma_0 \hat{g}) - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g}) + i(P_2 \Gamma_0 \hat{f}, P_2 \Gamma_0 \hat{g}) &= 2i[(P_1 f_i, P_1 g_i) - (f_{-i}, g_{-i}) + (P_2 f_i, P_2 g_i)] \\ &= 2i[(f_i, g_i) - (f_{-i}, g_{-i})] = (f', g) - (f, g').\end{aligned}$$

Hence the identity (3.11) for the operators (3.20) is valid. Since the surjectivity of the mapping Γ is obvious, the collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ with operators (3.20) is a D-triplet for A^*. Moreover the equality $\operatorname{Ker} \Gamma_0 = A_0$ is directly implied by (3.20). □

The description of all D-triplets for a given linear relation A is contained in the following

Proposition 3.12. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a D-triplet for a linear relation A^* and let J_{01} be the operator* (2.2). *Then the equality*

$$\begin{pmatrix} \widetilde{\Gamma}_0 \\ \widetilde{\Gamma}_1 \end{pmatrix} = \begin{pmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} \tag{3.21}$$

establishes a bijective correspondence between all D-triplets $\widetilde{\Pi} = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for A^* and all J_{01}-unitary operators $X = (X_{ij})_{i,j=0}^1 \in [\mathcal{H}_0 \oplus \mathcal{H}_1]$.

Proof. Let $\widetilde{\Pi} = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ be a D-triplet for A^*. Then by [17] there exists an automorphism $X = (X_{ij})_{i,j=0}^1 \in [\mathcal{H}_0 \oplus \mathcal{H}_1]$ such that (3.21) holds. Moreover in view of (3.12)

$$X^* J_{01} X = J_{01} \tag{3.22}$$

that is X is a J_{01}-unitary operator.

Conversely, let an operator $\widetilde{\Gamma} = (\widetilde{\Gamma}_0 \quad \widetilde{\Gamma}_1)^\top$ be given by (3.21). Then the Green's identity for the triplet $\widetilde{\Pi}$ is implied by (3.12) and (3.22). Moreover the mapping $\widetilde{\Gamma}$ is surjective, because so are $\Gamma = (\Gamma_0 \quad \Gamma_1)^\top$ and X. □

In the next proposition we complement the result of Proposition 3.11

Proposition 3.13. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a D-triplet for A^*. Then the extensions $A_i (:= \operatorname{Ker} \Gamma_i),\ i \in \{0, 1\}$ are transversal.*

Conversely, let $\widetilde{A}_0, \widetilde{A}_1 \in \operatorname{Ext}_A$ be a pair of transversal extensions such that $\widetilde{A}_0 \subset \widetilde{A}_0^*,\ \widetilde{A}_1^* \subset \widetilde{A}_1$ and $n_-(\widetilde{A}_0) = n_-(\widetilde{A}_1^*) = 0$. Then there exists a D-triplet for A^* such that $\widetilde{A}_i = A_i (:= \operatorname{Ker} \Gamma_i),\ i \in \{0, 1\}$.

Proof. The transversality of the extensions A_0 and A_1 is implied by Proposition 3.9,6) and the equality $\theta_0 \oplus \theta_1 = \mathcal{H}_0 \oplus \mathcal{H}_1$ (here θ_0 and θ_1 ere defined by (3.16)).

Conversely, let extensions $\widetilde{A}_0, \widetilde{A}_1 \in \operatorname{Ext}_A$ satisfy the stated conditions. Then by Proposition 3.11 there is a D-triplet $\Pi' = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0', \Gamma_1'\}$ such that $\widetilde{A}_0 = \operatorname{Ker} \Gamma_0'$. Since $\widetilde{A}_1$ and $\widetilde{A}_0$ are transversal, it follows from Corollary 3.10 that $\widetilde{A}_1 = \widetilde{A}_B = \operatorname{Ker} (\Gamma_1' - B\Gamma_0'),\ B \in [\mathcal{H}_0, \mathcal{H}_1]$. Let

$$B = (B_1 \quad B_2) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \tag{3.23}$$

be the block matrix representation of the operator B. Then $\widetilde{A}_1^* = \widetilde{A}_{B^\times}$ where $B^\times$ is a linear relation defined by

$$B^\times = \{(I_{\mathcal{H}_1} \quad - iB_2^*)^\top, B_1^*; \mathcal{H}_1\}$$

(see [21], formula (3.7)). Moreover $B^\times \in \operatorname{Sym}(\mathcal{H}_0, \mathcal{H}_1)$ and Proposition 3.4 in [21] yields

$$2\operatorname{Im} B_1 - B_2 B_2^* = 0. \tag{3.24}$$

Let now $X_1 \in [\mathcal{H}_0]$ and $X \in [\mathcal{H}_0 \oplus \mathcal{H}_1]$ be operators given by

$$X_1 = \begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ iB_2^* & I_{\mathcal{H}_2} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2, \quad X = \begin{pmatrix} X_1 & 0 \\ -B & I_{\mathcal{H}_1} \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1. \tag{3.25}$$

The immediate calculation with taking into account of (3.24) shows that the operator X satisfies the equality (3.22). Moreover since $0 \in \rho(X_1)$, it follows that $0 \in \rho(X)$. Hence the operator X is J_{01}-unitary and by Proposition 3.12 the collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ with $\Gamma_0 = X_1 \Gamma_0',\ \Gamma_1 = -B\Gamma_0' + \Gamma_1'$ is a D-triplet for A^*. It is easily seen that for this triplet $\operatorname{Ker} \Gamma_0 = \widetilde{A}_0$ and $\operatorname{Ker} \Gamma_1 = \widetilde{A}_1$. □

Remark 3.14. i) Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a D-triplet for a linear relation A^*. Then by Proposition 3.11 $n_-(A) \le n_+(A),\ \ \mathbb{C}_+ \in \rho(A_0)$ and the following equivalences hold

$$\mathcal{H}_0 = \mathcal{H}_1 \iff A_0 = A_0^* \iff A_1 = A_1^* \iff \mathbb{C}_+ \ne \rho(A_0). \tag{3.26}$$

If at least on of the conditions (3.26) is fulfilled, then $n_-(A) = n_+(A)$ and the identity (3.11) takes the form

$$(f', g) - (f, g') = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g}) - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g}), \quad \hat{f} = \{f, f'\}, \quad \hat{g} = \{g, g'\} \in A^*.$$

In this case the collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ ($\mathcal{H} := \mathcal{H}_0 = \mathcal{H}_1$) is a boundary triplet (boundary value space) for A^* in the sense of [9, 16]. Observe also that in the case $n_+(A) < \infty$ formula (3.17) yields the equivalence $\mathcal{H}_0 = \mathcal{H}_1 \iff n_-(A) = n_+(A)$. Therefore if $n_-(A) = n_+(A) < \infty$, then there are not other D-triplets for A^* besides boundary triplets. At the same time if $n_-(A) = n_+(A) = \infty$, then the subspace $\mathcal{H}_1$ may coincide or not coincide with $\mathcal{H}_0$. In what follows we do not exclude both this cases from our considerations.

ii) In the case of a linear relation A with equal defect numbers, the statement of Proposition 3.11 and the formulas (3.20) go back to [9, 16]. Observe also that in Proposition 3.13 we generalize similar construction of a boundary triplet from [16].

3.3. **γ-fields and Weyl functions.** In this subsection we introduce γ-fields and Weyl functions associated to a boundary D-triplet and investigate their properties.

Assume that $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ is a D-triplet for A^*, $\hat\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\hat\Gamma^A,\Gamma^A\}$ is a boundary triplet (3.13) for the dual pair $\{A,A\}$ and $\gamma_{\hat\Pi}(\cdot),\gamma_{\hat\Pi^*}(\cdot)$ are γ-fields (3.3), (3.4). Since $\operatorname{Ker}\hat\Gamma_0^A=A_0(=\operatorname{Ker}\Gamma_0)$ and $\mathbb{C}_+\subset\rho(A_0)$, the functions $\gamma_{\hat\Pi}$ and $\gamma_{\hat\Pi^*}$ are defined at least on $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively. This allows us to introduce the holomorphic operator-functions (γ-fields) $\hat\gamma_+(\cdot):\mathbb{C}_+\to[\mathcal{H}_0,\mathfrak{H}^2]$, $\gamma_+(\cdot):\mathbb{C}_+\to[\mathcal{H}_0,\mathfrak{H}]$ and $\hat\gamma_-(\cdot):\mathbb{C}_-\to[\mathcal{H}_1,\mathfrak{H}^2]$, $\gamma_-(\cdot):\mathbb{C}_-\to[\mathcal{H}_1,\mathfrak{H}]$ by setting

$$\begin{aligned}\hat\gamma_+(\lambda)&=\hat\gamma_{\hat\Pi}(\lambda),\quad \gamma_+(\lambda)=\gamma_{\hat\Pi}(\lambda),\quad \lambda\in\mathbb{C}_+,\\ \hat\gamma_-(z)&=\hat\gamma_{\hat\Pi^*}(z),\quad \gamma_-(z)=\gamma_{\hat\Pi^*}(z),\quad z\in\mathbb{C}_-.\end{aligned}\tag{3.27}$$

It follows from (3.13) that

$$\hat\gamma_+(\lambda)=(\Gamma_0\restriction\hat{\mathfrak{N}}_\lambda(A))^{-1},\qquad \gamma_+(\lambda)=\pi_1\hat\gamma_+(\lambda),\qquad \lambda\in\mathbb{C}_+,\tag{3.28}$$

$$\hat\gamma_-(z)=(P_1\Gamma_0\restriction\hat{\mathfrak{N}}_z(A))^{-1},\qquad \gamma_-(z)=\pi_1\hat\gamma_-(z),\qquad z\in\mathbb{C}_-\tag{3.29}$$

where π_1 is the orthoprojection in $\mathfrak{H}\oplus\mathfrak{H}$ onto $\mathfrak{H}\oplus\{0\}$.

Let $\lambda\in\mathbb{C}_+$. According to the decomposition $\mathcal{H}_0=\mathcal{H}_1\oplus\mathcal{H}_2$ the operator $\gamma_+(\lambda)$ admits the block-matrix representation

$$\gamma_+(\lambda)=(\gamma(\lambda)\ \ \delta_+(\lambda)):\mathcal{H}_1\oplus\mathcal{H}_2\to\mathfrak{H}.\tag{3.30}$$

By means of (3.30) we define the operator-functions $\gamma(\cdot):\mathbb{C}_+\to[\mathcal{H}_1,\mathfrak{H}]$ and $\delta_+(\cdot):\mathbb{C}_+\to[\mathcal{H}_2,\mathfrak{H}]$.

Proposition 3.15. *1) The operator-functions $\gamma_+(\cdot)$ and $\gamma_-(\cdot)$ satisfy the following relations*

$$\gamma_+(\mu)=\gamma_+(\lambda)+(\mu-\lambda)(A_0-\mu)^{-1}\gamma_+(\lambda),\qquad \lambda,\mu\in\mathbb{C}_+,\tag{3.31}$$

$$\gamma_-(\omega)=\gamma_-(z)+(\omega-z)(A_0^*-\omega)^{-1}\gamma_-(z),\qquad z,\omega\in\mathbb{C}_-,\tag{3.32}$$

$$\gamma_-(z)P_1=\gamma_+(\lambda)+(z-\lambda)(A_0^*-z)^{-1}\gamma_+(\lambda),\qquad \lambda\in\mathbb{C}_+,\qquad z\in\mathbb{C}_-.\tag{3.33}$$

2) For every $\lambda\in\mathbb{C}_+$ the operator $\sqrt{2\mathrm{Im}\lambda}\,\delta_+(\lambda)$ is an isometry from $\mathcal{H}_2$ onto $\mathfrak{N}_\lambda(A_0)$ and

$$(\mu-\overline{\lambda})\delta_+^*(\lambda)\delta_+(\mu)=iI_{\mathcal{H}_2},\quad \lambda,\mu\in\mathbb{C}_+.\tag{3.34}$$

Proof. 1) Formulas (3.31), (3.32) are implied by (3.27) and relations (3.5) for the functions $\gamma_{\hat\Pi}(\cdot)$ and $\gamma_{\hat\Pi^*}(\cdot)$.

Let further $\lambda\in\mathbb{C}_+$, $z\in\mathbb{C}_-$ and $h\in\mathcal{H}_0$. Letting

$$g:=\gamma_+(\lambda)h+(z-\lambda)(A_0^*-z)^{-1}\gamma_+(\lambda)h\tag{3.35}$$

we derive

$$\{g,zg\}=\{\gamma_+(\lambda)h,\lambda\gamma_+(\lambda)h\}+(z-\lambda)\left\{(A_0^*-z)^{-1}\gamma_+(\lambda)h,(I+z(A_0^*-z)^{-1})\gamma_+(\lambda)h\right\}.\tag{3.36}$$

Applying the operator $P_1\Gamma_0$ to (3.36) and taking into account that

$$\{(A_0^*-z)^{-1}\gamma_+(\lambda)h,(I+z(A_0^*-z)^{-1})\gamma_+(\lambda)h\}\in A_0^*=\operatorname{Ker}P_1\Gamma_0$$

we arrive at the relation

$$P_1\Gamma_0\{g,zg\}=P_1\Gamma_0\{\gamma_+(\lambda)h,\ \lambda\gamma_+(\lambda)h\}=P_1h.$$

It means that $g=\gamma_-(z)P_1h$. Combining this equality with (3.35) we arrive at (3.33).

2) Since $A_0=\operatorname{Ker}\Gamma_0$, the identity (3.11) yields

$$\begin{aligned}(f_0'-\overline{\lambda}f_0,g_\lambda)&=(\Gamma_1\hat f_0,\Gamma_0\hat g_\lambda),\\ \hat f_0=\{f_0,f_0'\}\in A_0,\quad \hat g_\lambda&=\{g_\lambda,\lambda g_\lambda\}\in\hat{\mathfrak{N}}_\lambda(A),\quad \lambda\in\mathbb{C}_+.\end{aligned}\tag{3.37}$$

It follows from (3.37) and the equality $\Gamma_1 A_0 = \mathcal{H}_1$ that $g_\lambda \in \mathfrak{N}_\lambda(A_0) \Longleftrightarrow \Gamma_0\{g_\lambda, \lambda g_\lambda\} \in \mathcal{H}_2$. This and (3.28) show that $\delta_+(\lambda)\mathcal{H}_2 (= \gamma_+(\lambda)\mathcal{H}_2) = \mathfrak{N}_\lambda(A_0),\ \lambda \in \mathbb{C}_+$.

Let now $\lambda, \mu \in \mathbb{C}_+,\ h_2, h_2' \in \mathcal{H}_2$ and let $\hat{f}_\mu = \hat{\gamma}_+(\mu)h_2 = \{\delta_+(\mu)h_2, \mu\delta_+(\mu)h_2\}$, $\hat{g}_\lambda = \hat{\gamma}_+(\lambda)h_2' = \{\delta_+(\lambda)h_2', \lambda\delta_+(\lambda)h_2'\}$. Then $\Gamma_0\hat{f}_\mu = h_2, \Gamma_0\hat{g}_\lambda = h_2'$ and by (3.11)

$$(\mu - \overline{\lambda})(\delta_+(\mu)h_2, \delta_+(\lambda)h_2') = i(h_2, h_2'), \quad h_2, h_2' \in \mathcal{H}_2.$$

Hence (3.34) is valid. Letting in (3.34) $\mu = \lambda$, one obtains $2\mathrm{Im}\lambda\, \delta_+^*(\lambda)\delta_+(\lambda) = I_{\mathcal{H}_2}$. Therefore the operator $\sqrt{2\mathrm{Im}\lambda}\, \delta_+(\lambda)$ is an isometry from $\mathcal{H}_2$ onto $\mathfrak{N}_\lambda(A_0)$. □

The same arguments as for the functions $\gamma_\pm$ allows us to introduce the following definition.

Definition 3.16. The holomorphic operator-functions $M_+(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_0, \mathcal{H}_1]$ and $M_-(\cdot) : \mathbb{C}_- \to [\mathcal{H}_1, \mathcal{H}_0]$ given by

$$M_+(\lambda) = M_{\hat{\Pi}}(\lambda), \quad \lambda \in \mathbb{C}_+, \qquad M_-(z) = M_{\hat{\Pi}^*}(z), \quad z \in \mathbb{C}_- \tag{3.38}$$

will be called the Weyl functions corresponding to the D-triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^*.

It follows from (3.6), (3.7) and (3.13) that the functions $M_+(\cdot)$ and $M_-(\cdot)$ are defined by

$$\Gamma_1 \upharpoonright \hat{\mathfrak{N}}_\lambda(A) = M_+(\lambda)\Gamma_0 \upharpoonright \hat{\mathfrak{N}}_\lambda(A), \qquad \lambda \in \mathbb{C}_+, \tag{3.39}$$

$$(\Gamma_1 + iP_2\Gamma_0) \upharpoonright \hat{\mathfrak{N}}_z(A) = M_-(z)P_1\Gamma_0 \upharpoonright \hat{\mathfrak{N}}_z(A), \quad z \in \mathbb{C}_-. \tag{3.40}$$

Let $\lambda \in \mathbb{C}_+,\ z \in \mathbb{C}_-$ and let

$$M_+(\lambda) = (M(\lambda)\ \ N_+(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1, \quad M_-(z) = (M(z)\ \ N_-(z))^\top : \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2 \tag{3.41}$$

be the block-matrix representation of the operators $M_+(\lambda)$ and $M_-(z)$. Formulas (3.41) define operator-functions $M(\cdot) : \mathbb{C}_+ \cup \mathbb{C}_- \to [\mathcal{H}_1],\ \ N_+(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_2, \mathcal{H}_1]$ and $N_-(\cdot) : \mathbb{C}_- \to [\mathcal{H}_1, \mathcal{H}_2]$.

Proposition 3.17. *The Weyl functions satisfy the identities*

$$M(\mu) - M^*(\lambda) = (\mu - \overline{\lambda})\, \gamma^*(\lambda)\gamma(\mu), \qquad \mu, \lambda \in \mathbb{C}_+, \tag{3.42}$$

$$N_+(\mu) = (\mu - \overline{\lambda})\, \gamma^*(\lambda)\delta_+(\mu), \qquad \mu, \lambda \in \mathbb{C}_+, \tag{3.43}$$

$$M(\omega) - M^*(z) + iN_-^*(z)N_-(\omega) = (\omega - \overline{z})\, \gamma_-^*(z)\gamma_-(\omega), \qquad \omega, z \in \mathbb{C}_-, \tag{3.44}$$

$$M_+(\lambda) - M_-^*(z) = (\lambda - \overline{z})\, \gamma_-^*(z)\gamma_+(\lambda), \qquad \lambda \in \mathbb{C}_+, \quad z \in \mathbb{C}_-. \tag{3.45}$$

Moreover the following relations hold:

$$M_-(\overline{\lambda}) = M_+^*(\lambda), \quad M(\overline{\lambda}) = M^*(\lambda), \quad N_-(\overline{\lambda}) = N_+^*(\lambda), \quad \lambda \in \mathbb{C}_+. \tag{3.46}$$

Proof. Let $\lambda, \mu \in \mathbb{C}_+,\ h_1, h_1' \in \mathcal{H}_1,\ h_2 \in \mathcal{H}_2$ and let

$$\hat{f}_\mu = \hat{\gamma}_+(\mu)h_1 = \{\gamma(\mu)h_1, \mu\,\gamma(\mu)h_1\}, \quad \hat{g}_\lambda = \hat{\gamma}_+(\lambda)h_1' = \{\gamma(\lambda)h_1', \lambda\gamma(\lambda)h_1'\},$$
$$\hat{\varphi}_\mu = \hat{\gamma}_+(\mu)h_2 = \{\delta_+(\mu)h_2, \mu\,\delta_+(\mu)h_2\}.$$

Then

$$\Gamma_0\hat{f}_\mu = h_1, \quad \Gamma_1\hat{f}_\mu = M(\mu)h_1, \quad \Gamma_0\hat{g}_\lambda = h_1', \quad \Gamma_1\hat{g}_\lambda = M(\lambda)h_1',$$
$$\Gamma_0\hat{\varphi}_\mu = h_2, \quad \Gamma_1\hat{\varphi}_\mu = N_+(\mu)h_2$$

and the identity (3.11) yields

$$(\mu - \overline{\lambda})\,(\gamma(\mu)h_1, \gamma(\lambda)h_1') = (M(\mu)h_1, h_1') - (M^*(\lambda)h_1, h_1'), \quad h_1, h_1' \in \mathcal{H}_1$$
$$(\mu - \overline{\lambda})\,(\delta_+(\mu)h_2, \gamma(\lambda)h_1') = (N_+(\mu)h_2, h_1'), \quad h_1' \in \mathcal{H}_1, \quad h_2 \in \mathcal{H}_2.$$

This leads to (3.42), (3.43).

Let now $z,\omega\in\mathbb{C}_-,\ h_1,h_1'\in\mathcal{H}_1$ and let

$$\hat{f}_\omega=\hat{\gamma}_-(\omega)h_1=\{\gamma_-(\omega)h_1,\,\omega\gamma_-(\omega)h_1\},\quad \hat{g}_z=\hat{\gamma}_-(z)h_1'=\{\gamma_-(z)h_1',\,z\gamma_-(z)h_1'\}.$$

Then $P_1\Gamma_0\hat{f}_\omega=h_1,\ P_1\Gamma_0\hat{g}_z=h_1'$ and by (3.40)

$$\Gamma_1\hat{f}_\omega=M(\omega)h_1,\quad P_2\Gamma_0\hat{f}_\omega=-iN_-(\omega)h_1,\quad \Gamma_1\hat{g}_z=M(z)h_1',\quad P_2\Gamma_0\hat{g}_z=-iN_-(z)h_1'.$$

This and (3.11) imply the equality

$$(\omega-\overline{z})(\gamma_-(\omega)h_1,\gamma_-(z)h_1')=(M(\omega)h_1,h_1')-(M^*(z)h_1,h_1')+i(N_-^*(z)N_-(\omega)h_1,h_1'),$$
$$h_1,h_1'\in\mathcal{H}_1.$$

Hence (3.44) is valid.

Finally, the equalities (3.45) and (3.46) are implied by (3.38) and the relations (3.8), (3.10) for $M_{\hat{\Pi}}(\lambda)$. □

For an operator $T=T^*\in[\mathcal{H}]$ we write $T\gg 0$, if $T\geq\alpha I$ with some $\alpha>0$, and $T\ll 0$, if $-T\gg 0$.

Corollary 3.18. *Let $M_+(\lambda)=(M(\lambda)\quad N_+(\lambda)),\ \lambda\in\mathbb{C}_+$ and $M_-(z)=(M(z)\quad N_-(z))^\top$, $z\in\mathbb{C}_-$ be the Weyl functions corresponding to the D-triplet $\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ for A^*. Then:*

1) $2\mathrm{Im}M(\lambda)-N_+(\lambda)N_+^*(\lambda)\gg 0,\ \ \lambda\in\mathbb{C}_+,\quad 2\mathrm{Im}M(z)+N_-^*(z)N_-(z)\ll 0,\ \ z\in\mathbb{C}_-,$

2) *the function $M(\lambda),\ \lambda\in\mathbb{C}_+\cup\mathbb{C}_-$ belongs to the class $R^u[\mathcal{H}_1]$, i.e., $M(\overline{\lambda})=M^*(\lambda)$ and $\mathrm{Im}\lambda\,\mathrm{Im}M(\lambda)\gg 0,\ \lambda\in\mathbb{C}_+\cup\mathbb{C}_-$.*

Proof. In view of (3.46) the relations 1) are mutually equivalent. Moreover the second of them is implied by (3.44) and the inclusion $0\in\hat{\rho}(\gamma_-(z)),\ z\in\mathbb{C}_-$.

The statement 2) immediately follows from 1). □

Let J_{01},J_{10},U_{01} and U_{10} be the operators (2.2), (2.3) and let

(3.47)
$$X=\begin{pmatrix}X_{00}&X_{01}\\X_{10}&X_{11}\end{pmatrix}:\mathcal{H}_0\oplus\mathcal{H}_1\to\mathcal{H}_0\oplus\mathcal{H}_1,\quad Y=\begin{pmatrix}Y_{00}&Y_{01}\\Y_{10}&Y_{11}\end{pmatrix}:\mathcal{H}_1\oplus\mathcal{H}_0\to\mathcal{H}_1\oplus\mathcal{H}_0$$

be operators associated by $Y=U_{01}XU_{10}(\iff X=U_{10}YU_{01})$. Clearly, the operator X is J_{01}-unitary if and only if Y is J_{10}-unitary.

In the next proposition we provide the connection between Weyl functions corresponding to different D-triplets.

Proposition 3.19. *Let $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ and $\widetilde{\Pi}=\{\mathcal{H}_0\oplus\mathcal{H}_1,\widetilde{\Gamma}_0,\widetilde{\Gamma}_1\}$ be D-triplets for A^* associated by (3.21) with the J_{01}-unitary operator $X=(X_{ij})_{i,j=0}^1$ and let $M_\pm(\cdot),\ \widetilde{M}_\pm(\cdot)$ be the corresponding Weyl functions. Then*

$$\widetilde{M}_+(\lambda)=(X_{10}+X_{11}M_+(\lambda))(X_{00}+X_{01}M_+(\lambda))^{-1},\quad \lambda\in\mathbb{C}_+,\tag{3.48}$$

$$\widetilde{M}_-(z)=(Y_{10}+Y_{11}M_-(z))(Y_{00}+Y_{01}M_-(z))^{-1},\qquad z\in\mathbb{C}_-\tag{3.49}$$

where $Y=(Y_{ij})_{i,j=0}^1$ is a J_{10} unitary operator given by $Y=U_{01}XU_{10}$.

Proof. It follows from (3.39) and (3.40) that for every D-triplet $\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ the corresponding Weyl functions may be written as

$$\mathrm{gr}M_+(\lambda)=\Gamma\hat{\mathfrak{N}}_\lambda(A),\quad \lambda\in\mathbb{C}_+,\qquad \mathrm{gr}M_-(z)=U_{01}\Gamma\hat{\mathfrak{N}}_z(A),\quad z\in\mathbb{C}_-.\tag{3.50}$$

Now assume that the assumptions of the proposition are satisfied. Since $\widetilde{\Gamma} = X\Gamma$, (3.50) yields

$$\text{gr}\widetilde{M}_+(\lambda) = X\text{gr}M_+(\lambda), \quad \lambda \in \mathbb{C}_+, \qquad \text{gr}\widetilde{M}_-(z) = Y\text{gr}M_-(z), \quad z \in \mathbb{C}_-. \tag{3.51}$$

Let the operators X and Y have the block-matrix representation (3.47). Then in view of (3.51)

$$\text{gr}\widetilde{M}_+(\lambda) = \{X_{00} + X_{01}M_+(\lambda),\ X_{10} + X_{11}M_+(\lambda);\ \mathcal{H}_0\}, \tag{3.52}$$

$$\text{gr}\widetilde{M}_-(z) = \{Y_{00} + Y_{01}M_-(z),\ Y_{10} + Y_{11}M_-(z);\ \mathcal{H}_1\}. \tag{3.53}$$

Moreover since $\mathcal{R}(X_{00} + X_{01}M_+(\lambda)) = \mathcal{D}(\widetilde{M}_+(\lambda)) = \mathcal{H}_0$ and $\text{Ker}\,(X_{00} + X_{01}M_+(\lambda)) = \text{Ker}\,(X_{00} + X_{01}M_+(\lambda)) \cap \text{Ker}\,(X_{10} + X_{11}M_+(\lambda)) = \{0\}$, it follows that $0 \in \rho(X_{00} + X_{01}M_+(\lambda))$. This and (3.52) show that $\widetilde{M}_+(\lambda)$ is of the form (3.48). Similar arguments applied to (3.53) lead to the equality (3.49) for $\widetilde{M}_-(z)$. □

Remark 3.20. If $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a D-triplet for A^* and $\mathcal{H}_1 \neq \mathcal{H}_0$, then $\rho(A_0) = \mathbb{C}_+$, $\rho(A_0^*) = \mathbb{C}_-$ and, therefore, the operator-functions $\gamma_\pm$ and $M_\pm$ are defined on the same natural domains as the corresponding functions $\gamma_{\hat{\Pi}}$, $\gamma_{\hat{\Pi}^*}$, $M_{\hat{\Pi}}$ and $M_{\hat{\Pi}^*}$ (compare (3.27), (3.38) and (3.3),(3.6)).

On the other hand if $\mathcal{H}_1 = \mathcal{H}_0 := \mathcal{H}$, then according to Remark 3.14 $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* and $A_0 = A_0^*$, so that $(\mathbb{C}_+ \cup \mathbb{C}_-) \subset \rho(A_0)$. In this case $M_\pm(\lambda) = M(\lambda)$, $\lambda \in \mathbb{C}_\pm$ and the operator-functions

$$\gamma(\lambda) := \begin{cases} \gamma_+(\lambda), & \lambda \in \mathbb{C}_+ \\ \gamma_-(\lambda), & \lambda \in \mathbb{C}_- \end{cases}$$

and $M(\lambda)$ are γ-field and the Weyl function for Π respectively [2, 16]. Moreover in this case all relations in Proposition 3.15 and 3.17 may be written in the well known form [13, 14, 2, 16]

$$\gamma(\mu) = \gamma(\lambda) + (\mu - \lambda)(A_0 - \mu)^{-1}\gamma(\lambda), \quad M(\mu) - M^*(\lambda) = (\mu - \overline{\lambda})\,\gamma^*(\lambda)\gamma(\mu), \quad \mu, \lambda \in \rho(A_0)$$

Note also that in the case $\mathcal{H}_1 = \mathcal{H}_0 := \mathcal{H}$ the operator J_{01} takes the form

$$J_{01} := J = \begin{pmatrix} 0 & -iI_{\mathcal{H}} \\ iI_{\mathcal{H}} & 0 \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$$

and formulas (3.21), (3.22) describe all boundary triplets (in the sense of [9, 16]) for A^*. Moreover in this case $X = Y$ and the equalities (3.48), (3.49) imply that the connection between the Weyl functions for boundary triplets Π and $\widetilde{\Pi}$ associated via (3.21) is given by

$$\widetilde{M}(\lambda) = (X_{10} + X_{11}M(\lambda))(X_{00} + X_{01}M(\lambda))^{-1}, \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-.$$

These results were obtained in [3, 16].

4. Spectrum of proper extensions and formulas for generalized resolvents

4.1. Spectrum of proper extensions and formulas for canonical resolvents. In this subsection we describe the spectrum of a proper extension $\widetilde{A} = \widetilde{A}_\theta$ in terms of the boundary parameter θ and the corresponding Weyl function and derive Krein–Naimark type formula for canonical resolvents. Note that these results are simple consequences of similar type results obtained in [17] for dual pairs of linear relations.

Proposition 4.1. 1) *Let* $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *be a D-triplet for* A^* *with the Weyl functions* $M_\pm(\cdot)$ *and let* $\widetilde{A} = \widetilde{A}_\theta \in \mathrm{Ext}_A$*, where* $\theta = \{K_0, K_1; H\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$*. Assume also that* $K_0 = (K_{01} \;\; K_{02})^\top : H \to \mathcal{H}_1 \oplus \mathcal{H}_2$ *is the block-matrix representation of* K_0*,* $N_1 := K_{01},\ N_0 := (K_1 \;\; i\,K_{02})^\top : H \to \mathcal{H}_1 \oplus \mathcal{H}_2$ *and let*

$$T_+(\lambda) := K_1 - M_+(\lambda)K_0 \in [H, \mathcal{H}_1], \quad \lambda \in \mathbb{C}_+,$$
$$T_-(\lambda) := N_0 - M_-(\lambda)N_1 \in [H, \mathcal{H}_0], \quad \lambda \in \mathbb{C}_-.$$

Then the following relations hold:

(4.1)
$$\lambda \in \rho(\widetilde{A}) \Leftrightarrow 0 \in \rho(T_\pm(\lambda)), \quad \lambda \in \sigma_j(\widetilde{A}) \Leftrightarrow 0 \in \sigma_j(T_\pm(\lambda)), \quad j = p, c, r, \quad \lambda \in \mathbb{C}_\pm;$$
(4.2)
$$\lambda \in \hat{\rho}(\widetilde{A}) \iff 0 \in \hat{\rho}(T_\pm(\lambda)), \quad \lambda \in \mathbb{C}_\pm;$$
(4.3)
$$\overline{\mathcal{R}(\widetilde{A} - \lambda)} = \mathcal{R}(\widetilde{A} - \lambda) \iff \overline{\mathcal{R}(T_\pm(\lambda))} = \mathcal{R}(T_\pm(\lambda)), \quad \lambda \in \mathbb{C}_\pm;$$
(4.4)
$$\dim \mathrm{Ker}\,(\widetilde{A} - \lambda) = \dim \mathrm{Ker}\, T_\pm(\lambda), \quad \mathrm{codim}\, \mathcal{R}(\widetilde{A} - \lambda) = \mathrm{codim}\, \mathcal{R}(T_\pm(\lambda)), \quad \lambda \in \mathbb{C}_\pm.$$

2) *Let* H *be a Hilbert space and let an extension* $\widetilde{A} \in \mathrm{Ext}_A$ *be defined by an abstract boundary condition*

$$\hat{f} = \{f, f'\} \in \widetilde{A} \iff C_0\Gamma_0\hat{f} + C_1\Gamma_1\hat{f} = 0 \tag{4.5}$$

where $C_0 \in [\mathcal{H}_0, H],\ C_1 \in [\mathcal{H}_1, H]$ *and the range of the operator* $C = (C_0 \;\; C_1) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to H$ *coincides with* H*. Moreover assume that* $C_0 = (C_{01} \;\; C_{02}) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to H$ *is the block-matrix representation of* C_0*,* $\widetilde{C}_1 := C_{01},\ \widetilde{C}_0 := (C_1 \;\; -i\,C_{02}) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to H$ *and let*

$$S_+(\lambda) := C_0 + C_1 M_+(\lambda) \in [\mathcal{H}_0, H], \quad \lambda \in \mathbb{C}_+,$$
$$S_-(\lambda) := \widetilde{C}_1 + \widetilde{C}_0 M_-(\lambda) \in [\mathcal{H}_1, H], \quad \lambda \in \mathbb{C}_-.$$

Then the relations (4.1)–(4.4) *hold with* $S_\pm(\lambda)$ *instead of* $T_\pm(\lambda)$*.*

Proof. 1) Let $\hat{\Pi} = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \hat{\Gamma}^A, \Gamma^A\}$ be a boundary triplet (3.13) for the dual pair $\{A, A\}$. Since $\hat{\Gamma}^A = \Gamma$, it follows that in this triplet $\widetilde{A} = \widetilde{A}_\theta$. Furthermore consider a boundary triplet $\hat{\Pi}^* = \{\mathcal{H}_1 \oplus \mathcal{H}_0, \Gamma^A, \hat{\Gamma}^A\}$ for $\{A, A\}$ and let $\widetilde{A} = \widetilde{A}_{\theta_-}$ in the triplet $\hat{\Pi}^*$. Then by Lemma 3.8

$$\theta_- = \Gamma^A \widetilde{A} = U_{01}\Gamma\widetilde{A} = U_{01}\theta, \quad \theta_- \in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0) \tag{4.6}$$

and (2.3) yields $\theta_- = \{N_1, N_0; H\}$ where

$$N_1 = P_1 K_0 = K_{01}, \quad N_0 = K_1 + i\,P_2 K_0 = (K_1 \;\; i\,K_{02})^\top : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_0. \tag{4.7}$$

Note also that in view of (3.38) the Weyl functions for the triplets $\hat{\Pi}$ and $\hat{\Pi}^*$ coincide with $M_+(\cdot)$ and $M_-(\cdot)$ respectively. Now the relations (4.1)–(4.4) are implied by [17], Proposition 5.2 (see also (2.8)–(2.10) in [17]).

2) The relation (4.5) means that in the triplet $\hat{\Pi}$ the extension $\widetilde{A}$ is parameterized as $\widetilde{A} = \widetilde{A}_\theta$ with $\theta = \mathrm{Ker}\, C = ((C_0, C_1); H)$. Let θ_- be a linear relation (4.6) and let $U_{01},\ U_{10}$ be operators (2.3). Then the following equivalences are obvious

$$\hat{h} \in \theta_- \iff U_{10}\hat{h} \in \theta \iff CU_{10}\hat{h} = 0 \iff \hat{h} \in \mathrm{Ker}\,(CU_{10}).$$

Hence $\theta_- = \operatorname{Ker} \widetilde{C}$ where

$$\widetilde{C} = CU_{10} = (C_0 \ \ C_1)\begin{pmatrix} I_{\mathcal{H}_1} & -iP_2 \\ 0 & P_1 \end{pmatrix} = (C_{01} \quad C_1 P_1 - iC_{02}P_2).$$

Letting $\widetilde{C}_1 = C_{01}$ and $\widetilde{C}_0 = (C_1 \quad -iC_{02}) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to H$ one obtains $\widetilde{C} = (\widetilde{C}_1 \ \ \widetilde{C}_0) : \mathcal{H}_1 \oplus \mathcal{H}_0 \to H$. This means that $\theta_- = \{(\widetilde{C}_1, \widetilde{C}_0); H\}$ so that for the triplet $\hat{\Pi}^*$ the extension $\widetilde{A}$ is defined as

$$\hat{f} = \{f, f'\} \in \widetilde{A} \iff \widetilde{C}_1 \Gamma_0^A \hat{f} + \widetilde{C}_0 \Gamma_1^A \hat{f} = 0.$$

Now the desired statement follows from [17], Corollary 5.3. □

Theorem 4.2. *Suppose that $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a D-triplet for A^*, $M_\pm(\cdot)$ are the corresponding Weyl functions, $A_0 = \operatorname{Ker} \Gamma_0$ and $\widetilde{A} = \widetilde{A}_\theta \in \operatorname{Ext}_A$ with $\theta = \{K_0, K_1; H\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$. Moreover let $K_0 = (K_{01} \quad K_{02})^\top : H \to \mathcal{H}_1 \oplus \mathcal{H}_2$ be the block-matrix representation of K_0 and let $N_1 := K_{01}$, $N_0 := (K_1 \quad i\,K_{02})^\top : H \to \mathcal{H}_1 \oplus \mathcal{H}_2$. Then the following formulas for canonical resolvents hold:*

$$(A_\theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma_+(\lambda) K_0 \big(K_1 - M_+(\lambda) K_0\big)^{-1} \gamma_-^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+, \tag{4.8}$$

$$(A_\theta - z)^{-1} = (A_0^* - z)^{-1} + \gamma_-(z) N_1 \big(N_0 - M_-(z) N_1\big)^{-1} \gamma_+^*(\overline{z}), \quad z \in \mathbb{C}_-. \tag{4.9}$$

Proof. As in the proof of Proposition 4.1 consider the boundary triplets $\hat{\Pi}$ and $\hat{\Pi}^*$ for the dual pair $\{A, A\}$ defined in Lemma 3.8. It was shown in Proposition 4.1 that $\widetilde{A} = \widetilde{A}_\theta$ in the triplet $\hat{\Pi}$ and $\widetilde{A} = \widetilde{A}_{\theta_-}$ in the triplet $\hat{\Pi}^*$, where $\theta_- = \{N_1, N_0; H\}$ and the operators N_j, $j \in \{0, 1\}$ are defined by (4.7). Note also that in view of (3.13) $\operatorname{Ker} \hat{\Gamma}_0^A = A_0 (= \operatorname{Ker} \Gamma_0)$. Applying now Theorem 3.5 to the triplets $\hat{\Pi}$ and $\hat{\Pi}^*$ and taking into account of (3.27) and (3.38), we arrive at the formulas (4.8), (4.9). □

4.2. **Formulas for generalized resolvents.** First recall the following definition.

Definition 4.3. An operator function $\mathbb{R}_\lambda : \mathbb{C}_+ \cup \mathbb{C}_- \to \mathfrak{H}$ is called a generalized resolvent of a symmetric linear relation $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$, if there exist a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ and a linear relation $\widetilde{A} = \widetilde{A}^* \in \widetilde{C}(\widetilde{\mathfrak{H}})$ such that $A \subset \widetilde{A}$ and $\mathbb{R}_\lambda = P_{\mathfrak{H}} (\widetilde{A} - \lambda)^{-1} | \mathfrak{H}$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$.

The next theorem plays together with Theorem 2.6 a fundamental role in our derivation of formulas for generalized resolvents.

Theorem 4.4. *Suppose that A is a closed symmetric linear relation in $\mathfrak{H}$ with defect numbers $n_-(A) \le n_+(A) \le \infty$, $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a D-triplet for A^*, $A_0 = \operatorname{Ker} \Gamma_0$ and $\gamma_\pm(\cdot)$, $M_\pm(\cdot)$ are the corresponding γ-fields and Weyl functions respectively. Furthermore, let $\mathfrak{H}_1$ be a Hilbert space, $\widetilde{\mathfrak{H}} := \mathfrak{H} \oplus \mathfrak{H}_1$ and let G_0, $G_1 \in [\mathfrak{H}_1^2, \mathfrak{H}_1]$ be operators given by*

$$G_0 \hat{h}_1 = h_1, \quad G_1 \hat{h}_1 = h_1', \quad \hat{h}_1 = \{h_1, h_1'\} \in \mathfrak{H}_1^2. \tag{4.10}$$

Then

1) *the adjoint linear relation of A in the space $\widetilde{\mathfrak{H}}$ is*

$$A^*_{\widetilde{\mathfrak{H}}} = A^* \oplus \mathfrak{H}_1^2; \tag{4.11}$$

2) *the operators*

$$\widetilde{\Gamma}_0 = \begin{pmatrix} \Gamma_0 & 0 \\ 0 & G_0 \end{pmatrix} \in [A^* \oplus \mathfrak{H}_1^2, \mathcal{H}_0 \oplus \mathfrak{H}_1], \quad \widetilde{\Gamma}_1 = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & G_1 \end{pmatrix} \in [A^* \oplus \mathfrak{H}_1^2, \mathcal{H}_1 \oplus \mathfrak{H}_1] \tag{4.12}$$

form a D-triplet $\widetilde{\Pi} = \{(\mathcal{H}_0 \oplus \mathfrak{H}_1) \oplus (\mathcal{H}_1 \oplus \mathfrak{H}_1), \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for $A^*_{\widetilde{\mathfrak{H}}}$ with

$$\widetilde{A}_0 (= \operatorname{Ker} \widetilde{\Gamma}_0) = A_0 \oplus (\{0\} \oplus \mathfrak{H}_1). \tag{4.13}$$

Moreover the corresponding γ-fields and the Weyl functions for the triplet $\widetilde{\Pi}$ are defined by

$$\widetilde{\gamma}_+(\lambda) = \begin{pmatrix} \gamma_+(\lambda) & 0 \\ 0 & I_{\mathfrak{H}_1} \end{pmatrix}, \quad \lambda \in \mathbb{C}_+, \quad \widetilde{\gamma}_-(z) = \begin{pmatrix} \gamma_-(z) & 0 \\ 0 & I_{\mathfrak{H}_1} \end{pmatrix}, \quad z \in \mathbb{C}_-, \tag{4.14}$$

$$\widetilde{M}_+(\lambda) = \begin{pmatrix} M_+(\lambda) & 0 \\ 0 & \lambda I_{\mathfrak{H}_1} \end{pmatrix}, \quad \lambda \in \mathbb{C}_+, \quad \widetilde{M}_-(z) = \begin{pmatrix} M_-(z) & 0 \\ 0 & z I_{\mathfrak{H}_1} \end{pmatrix}, \quad z \in \mathbb{C}_-; \tag{4.15}$$

3) *if an extension $\widetilde{A} = \widetilde{A}^* \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ of A is parameterized in the D-triplet $\widetilde{\Pi}$ as $\widetilde{A} = \widetilde{A}_{\widetilde{\theta}}$ with $\widetilde{\theta} \in \mathrm{Self}(\mathcal{H}_0, \mathcal{H}_1)(\mathcal{H}_0 \oplus \mathfrak{H}_1, \mathcal{H}_1 \oplus \mathfrak{H}_1)$ and $\tau_+(\lambda) = \{K_0(\lambda), K_1(\lambda); H_+\} \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ is a compression of $\widetilde{\theta}$, then $0 \in \rho(K_1(\lambda) + M_+(\lambda)K_0(\lambda))$, $\lambda \in \mathbb{C}_+$ and*

$$P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H} = (A_0 - \lambda)^{-1} - \gamma_+(\lambda)K_0(\lambda)\big(K_1(\lambda) + M_+(\lambda)K_0(\lambda)\big)^{-1}\gamma_-^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+. \tag{4.16}$$

Proof. The statement 1) is obvious.

2) Let $\widetilde{\mathcal{H}}_0 := \mathcal{H}_0 \oplus \mathfrak{H}_1$ be a Hilbert space, let $\widetilde{\mathcal{H}}_1 := \mathcal{H}_1 \oplus \mathfrak{H}_1$ be a subspace in $\widetilde{\mathcal{H}}_0$ and let $\widetilde{\mathcal{H}}_2 := \widetilde{\mathcal{H}}_0 \ominus \widetilde{\mathcal{H}}_1$. It is clear that $\widetilde{\mathcal{H}}_2 = \mathcal{H}_2 \oplus \{0\}$.

In view of the decomposition (4.11) every vector $\hat{f} \in A^*_{\widetilde{\mathfrak{H}}}$ admits the representation $\hat{f} = \{g + h_1, g' + h_1'\}$ where $\{g, g'\} \in A^*$ and $h_1, h_1' \in \mathfrak{H}_1$. This and (4.12) imply the equalities

$$\widetilde{\Gamma}_0\hat{f} = \{\Gamma_0\{g, g'\}, h_1\} \in \mathcal{H}_0 \oplus \mathfrak{H}_1, \qquad \widetilde{\Gamma}_1\hat{f} = \{\Gamma_1\{g, g'\}, h_1'\} \in \mathcal{H}_1 \oplus \mathfrak{H}_1, \tag{4.17}$$

$$P_{\widetilde{\mathcal{H}}_1}\widetilde{\Gamma}_0\hat{f} = \{P_1\Gamma_0\{g, g'\}, h_1\} \in \mathcal{H}_1 \oplus \mathfrak{H}_1, \quad P_{\widetilde{\mathcal{H}}_2}\widetilde{\Gamma}_0\hat{f} = \{P_2\Gamma_0\{g, g'\}, 0\} \in \mathcal{H}_2 \oplus \{0\}. \tag{4.18}$$

Now the immediate checking shows that $\widetilde{\Pi} := \{\widetilde{\mathcal{H}}_0 \oplus \widetilde{\mathcal{H}}_1, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is a D-triplet for $A^*_{\widetilde{\mathfrak{H}}}$.

Next assume that $\hat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} \in A^*_{\widetilde{\mathfrak{H}}}$ for some $\lambda \in \mathbb{C}$. Then $\hat{f}_\lambda = \{g_\lambda + h_1, \lambda g_\lambda + \lambda h_1\}$ where $\{g_\lambda, \lambda g_\lambda\} \in \hat{\mathfrak{N}}_\lambda(A)$ and $h_1 \in \mathfrak{H}_1$. Hence in view of (4.17) and (4.18) one has

$$\widetilde{\Gamma}_0\hat{f}_\lambda = \{\Gamma_0\{g_\lambda, \lambda g_\lambda\}, h_1\}, \ \ \widetilde{\Gamma}_1\hat{f}_\lambda = \{\Gamma_1\{g_\lambda, \lambda g_\lambda\}, \lambda h_1\}, \ \ P_{\widetilde{\mathcal{H}}_1}\widetilde{\Gamma}_0\hat{f}_\lambda = \{P_1\Gamma_0\{g_\lambda, \lambda g_\lambda\}, h_1\}$$

which leads to (4.14) and (4.15).

3) According to Theorem 2.6 there exists a representation $\widetilde{\theta} = \{\widetilde{K}_0, \widetilde{K}_1; \mathcal{H}_1 \oplus \mathfrak{H}_1\}$ such that the operators $\widetilde{K}_0 \in [\mathcal{H}_1 \oplus \mathfrak{H}_1, \mathcal{H}_0 \oplus \mathfrak{H}_1]$ and $\widetilde{K}_1 \in [\mathcal{H}_1 \oplus \mathfrak{H}_1]$ have the block-matrix representations (2.8) with $0 \in \rho(N_4 - \lambda K_4)$. Moreover the compression $\tau_+(\lambda)$ of $\widetilde{\theta}$ admits the representation $\tau_+(\lambda) := \{K_0(\lambda), K_1(\lambda); \mathcal{H}_1\}$, $\lambda \in \mathbb{C}_+$ where $K_0(\lambda)$ and $K_1(\lambda)$ are operator functions (2.9) and (2.10) respectively.

Applying formula (4.8) to the extension $\widetilde{A} = \widetilde{A}_{\widetilde{\theta}}$ and taking into account (4.13) and (4.14) we get the equality

$$P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H} = (A_0 - \lambda)^{-1} + \gamma_+(\lambda)P_{\mathcal{H}_0}\widetilde{K}_0\big(\widetilde{K}_1 - \widetilde{M}_+(\lambda)\widetilde{K}_0\big)^{-1} \upharpoonright \mathcal{H}_1\,\gamma_-^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+. \tag{4.19}$$

It follows from (2.8) and (4.15) that

$$\widetilde{K}_1 - \widetilde{M}_+(\lambda)\widetilde{K}_0 = \begin{pmatrix} N_1 - M_+(\lambda)K_1 & N_2 - M_+(\lambda)K_2 \\ N_3 - \lambda K_3 & N_4 - \lambda K_4 \end{pmatrix}.$$

Let us put

$$F(\lambda) := (N_1 - M_+(\lambda)K_1) - (N_2 - M_+(\lambda)K_2)(N_4 - \lambda K_4)^{-1}(N_3 - \lambda K_3).$$

Since $0 \in \rho(\widetilde{K}_1 - \widetilde{M}_+(\lambda)\widetilde{K}_0) \cap \rho(N_4 - \lambda K_4)$, the inclusion $0 \in \rho(F(\lambda))$ is valid. Moreover in view of (2.9) and (2.10)

$$F(\lambda) = K_1(\lambda) + M_+(\lambda)K_0(\lambda), \quad \lambda \in \mathbb{C}_+,$$

so that $0 \in \rho(K_1(\lambda) + M_+(\lambda)K_0(\lambda))$, $\lambda \in \mathbb{C}_+$. Using now the Frobenius formula we derive

$$(\widetilde{K}_1 - \widetilde{M}_+(\lambda)\widetilde{K}_0)^{-1} = \begin{pmatrix} F^{-1}(\lambda) & * \\ -(N_4 - \lambda K_4)^{-1}(N_3 - \lambda K_3)F^{-1}(\lambda) & * \end{pmatrix}.$$

This and the first equality in (2.8) imply

$$\begin{aligned} &P_{\mathcal{H}_0}\widetilde{K}_0\big(\widetilde{K}_1 - \widetilde{M}_+(\lambda)\widetilde{K}_0\big)^{-1} \upharpoonright \mathcal{H}_1 \\ &\quad = K_1F^{-1}(\lambda) + K_2[-(N_4 - \lambda K_4)^{-1}(N_3 - \lambda K_3)F^{-1}(\lambda)] \\ &\quad = [K_1 - K_2(N_4 - \lambda K_4)^{-1}(N_3 - \lambda K_3)](K_1(\lambda) + M_+(\lambda)K_0(\lambda))^{-1} \\ &\quad = -K_0(\lambda)(K_1(\lambda) + M_+(\lambda)K_0(\lambda))^{-1}. \end{aligned}$$

Combining this formula with (4.19) we arrive at (4.16). □

Now we are ready to derive the Krein–Naymark type formulas for generalized resolvents of a symmetric operator with unequal defect numbers. These formulas give a parameterization of all generalized resolvents $\mathbb{R}_\lambda$ by means of pairs of functions $\tau = \{\tau_+(\cdot), \tau_-(\cdot)\} \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$. Note, that our proof is based on Theorem 2.6, which enables to represent a function $\tau_+(\lambda) \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ as a compression of an exit space linear relation $\widetilde{\theta} \in \mathrm{Self}(\mathcal{H}_0, \mathcal{H}_1)(\mathcal{H}_0 \oplus \mathfrak{H}_1, \mathcal{H}_1 \oplus \mathfrak{H}_1)$. Next by using Theorem 4.4 we construct the generalized resolvent $\mathbb{R}_\lambda$, corresponding to the given parameter τ.

Theorem 4.5. *Assume that A is a closed symmetric linear relation in $\mathfrak{H}$ with defect numbers $n_-(A) \le n_+(A) \le \infty$, $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a D-triplet for A^*, $A_0 = \operatorname{Ker}\Gamma_0$ and $M_\pm(\cdot)$ are the corresponding Weyl functions. Then the formulas*

$$\mathbb{R}_\lambda = (A_0 - \lambda)^{-1} - \gamma_+(\lambda)\,K_0(\lambda)\,\big(K_1(\lambda) + M_+(\lambda)K_0(\lambda)\big)^{-1}\gamma_-^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+, \tag{4.20}$$

$$\mathbb{R}_\lambda = (A_0^* - \lambda)^{-1} - \gamma_-(\lambda)\,N_1(\lambda)\,\big(N_0(\lambda) + M_-(\lambda)N_1(\lambda)\big)^{-1}\gamma_+^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_- \tag{4.21}$$

establish a bijective correspondence between all generalized resolvents $\mathbb{R}_\lambda$ of A and all pairs of functions $\tau = \{\tau_+(\cdot), \tau_-(\cdot)\} \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$, defined by (2.6) *and* (2.7)*. Moreover, $\mathbb{R}_\lambda$ is a canonical resolvent if and only if $\tau = \{\tau_+, \tau_-\} \in \widetilde{R}^0(\mathcal{H}_0, \mathcal{H}_1)$, which is possible only in the case* $\dim \mathcal{H}_1 = \dim \mathcal{H}_0(\iff n_-(A) = n_+(A))$.

Proof. First we show that the formula (4.20) establishes a bijective correspondence between all generalized resolvents $\mathbb{R}_\lambda$, $\lambda \in \mathbb{C}_+$ and all $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$-valued functions $\tau_+(\cdot) \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ given by (2.6).

Let $\mathbb{R}_\lambda = P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H}$, $\lambda \in \mathbb{C}_+$ where $\widetilde{A} \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ is a selfadjoint extension of A in the exit space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$. Then by Theorem 4.4 there exists a function $\tau_+(\lambda) = \{K_0(\lambda), K_1(\lambda); H_+\} \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ such that (4.20) holds.

Conversely assume that a function $\tau_+(\lambda) \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ is given by (2.6) and let $\mathbb{R}_\lambda$ be an operator function (4.20) with $K_0(\lambda)$ and $K_1(\lambda)$ taken from (2.6). Next we show that $\mathbb{R}_\lambda$ is a generalized resolvent of A.

According to Theorem 2.6 there exist a Hilbert space $\mathfrak{H}_1$ and a linear relation $\widetilde{\theta} \in Self(\mathcal{H}_0 \oplus \mathfrak{H}_1, \mathcal{H}_1 \oplus \mathfrak{H}_1)$ such that $\widetilde{\theta}$ is a dilation of $\tau_+(\lambda)$, i.e., $\tau_+(\lambda)$ is a compression of $\widetilde{\theta}$. Let $\widetilde{\mathfrak{H}} := \mathfrak{H} \oplus \mathfrak{H}_1$ and let $\widetilde{\Pi} = \{(\mathcal{H}_0 \oplus \mathfrak{H}_1) \oplus (\mathcal{H}_1 \oplus \mathfrak{H}_1), \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ be a D-triplet (4.12) for $\widetilde{A}^*_{\widetilde{\mathfrak{H}}}$. Consider an extension $\widetilde{A} = \widetilde{A}^* \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ of A given by $\widetilde{A} = \widetilde{A}_{\widetilde{\theta}}$ (in the triplet $\widetilde{\Pi}$). It follows from Theorem 4.4, 3) that the right hand side of the equality (4.20) is equal $P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1}|\mathfrak{H}$, $\lambda \in \mathbb{C}_+$. Hence $\mathbb{R}_\lambda = P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1}|\mathfrak{H}$, $\lambda \in \mathbb{C}_+$ which was had to be proved.

To prove the formula (4.21) note that (4.20) and (4.21) are equivalent to

$$\mathbb{R}_\lambda = (A_0 - \lambda)^{-1} - \gamma_+(\lambda)\left(\tau_+(\lambda) + M_+(\lambda)\right)^{-1}\gamma_-^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+, \tag{4.22}$$

$$\mathbb{R}_\lambda = (A_0^* - \lambda)^{-1} - \gamma_-(\lambda)\left(\tau_-(\lambda) + M_-(\lambda)\right)^{-1}\gamma_+^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_- \tag{4.23}$$

where $\tau_+(\lambda) = \{K_0(\lambda), K_1(\lambda); H_+\}$ and $\tau_-(\lambda) = \{N_1(\lambda), N_0(\lambda); H_-\}$, Now the formula (4.23) (and, consequently, (4.21)) is implied by (4.22) and the equalities $\mathbb{R}_\lambda = (\mathbb{R}_{\overline{\lambda}})^*$, $\tau_-(\lambda) = (\tau_+(\overline{\lambda}))^*$, $\lambda \in \mathbb{C}_-$.

Finally the last statement of the theorem follows from Proposition 3.9, 5). □

In the next corollary we obtain as a consequence of the formula (4.20) the description of all generalized resolvents in Srtraus form. Moreover we find here a geometric interpretation of the parameter $\tau(\lambda)$ in (4.20) by means of abstract "boundary conditions".

Corollary 4.6. *Let the assumptions of Theorem 4.5 by satisfied. Then:*

1) *the formula for generalized resolvents (in the Straus form)*

$$\mathbb{R}_\lambda = (\widetilde{A}(\lambda) - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \tag{4.24}$$

establishes a bijective correspondence between the generalized resolvents $\mathbb{R}_\lambda$ of A and $\widetilde{\mathcal{C}}(\mathfrak{H})$-valued functions $\widetilde{A}(\lambda)$, which are holomorphic on $\mathbb{C}_+ \cup \mathbb{C}_-$, take on values in Ext_A and satisfy the conditions

$$\mathrm{Im}\lambda\, \mathrm{Im}\widetilde{A}(\lambda) \le 0, \quad (\widetilde{A}(\lambda))^* = \widetilde{A}(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{4.25}$$

Moreover the connection between formulas (4.20) *and* (4.24) *is given by $\widetilde{A}(\lambda) = \widetilde{A}_{-\tau_+(\lambda)}$ where $\tau_+(\lambda) := -\Gamma\widetilde{A}(\lambda)$, $\lambda \in \mathbb{C}_+$.*

2) *for every generalized resolvent $\mathbb{R}_\lambda$ of A there exist a Hilbert space $\widetilde{H}_+$ and a pair of holomorphic operator-functions $C_j(\lambda) : \mathbb{C}_+ \to [\mathcal{H}_j, \widetilde{H}_+]$, $j \in \{0,1\}$ such that:*

i)

$$2\mathrm{Im}(C_1(\lambda)C_{01}^*(\lambda)) - C_{02}(\lambda)C_{02}^*(\lambda) \le 0, \quad 0 \in \rho(C_0(\lambda) + i\, C_1(\lambda)P_1), \quad \lambda \in \mathbb{C}_+ \tag{4.26}$$

where $C_{0j}(\lambda) : \mathbb{C}_+ \to [\mathcal{H}_j, \widetilde{H}_+]$, $j \in \{1,2\}$ are operator functions generated by the block-matrix representation $C_0(\lambda) = (C_{01}(\lambda)\ \ C_{02}(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \widetilde{H}_+$;

ii) *for every $g \in \mathfrak{H}$ the vector-function $f = \mathbb{R}_\lambda g$, $\lambda \in \mathbb{C}_+$ is a solution of the following boundary-value problem with the spectral parameters $C_0(\lambda)$ and $C_1(\lambda)$ in the "boundary condition"*

$$\hat{f} = \{f, f'\} \in A^*, \quad f' - \lambda f = g, \quad C_0(\lambda)\Gamma_0\hat{f} + C_1(\lambda)\Gamma_1\hat{f} = 0, \quad \lambda \in \mathbb{C}_+. \tag{4.27}$$

Moreover the connection between (4.20) *and* (4.27) *is given by $\tau_+(\lambda) = \{(C_0(\lambda),\ -C_1(\lambda)); \widetilde{H}_+\}$, $\lambda \in \mathbb{C}_+$.*

Proof. 1) It is known (see the proof of Theorem 4.2 in [10]) that the equality

$$\widetilde{A}(\lambda) = \widetilde{A}_{-\tau_+(\lambda)}, \quad \lambda \in \mathbb{C}_+ \tag{4.28}$$

establishes a bijective correspondence between $\widetilde{\mathcal{C}}(\mathfrak{H})$-valued functions $\widetilde{A}(\lambda)$, holomorphic on $\mathbb{C}_+$ with values in Ext_A and $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$-valued holomorphic functions $\tau_+(\lambda) : \mathbb{C}_+ \to \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$. This and Proposition 3.9, 5) imply, that the equality (4.28) defines a bijective correspondence between holomorphic functions $\widetilde{A}(\cdot) : \mathbb{C}_+ \to \mathrm{Ext}_A$ such that $\mathrm{Im}\widetilde{A}(\lambda) \le 0$ and $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$-valued functions $\tau_+(\cdot) \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$.

Assume now that $\widetilde{A}(\cdot) : \mathbb{C}_+ \cup \mathbb{C}_- \to \mathrm{Ext}_A$ is a holomorphic $\widetilde{\mathcal{C}}(\mathfrak{H})$-valued function satisfying (4.25). Then by (4.28) $\widetilde{A}(\lambda) = \widetilde{A}_{-\tau_+(\lambda)}$, $\lambda \in \mathbb{C}_+$ where $\tau_+(\lambda) \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$. Let (2.6) be a representation of $\tau_+(\lambda)$. Applying formula (4.8) for canonical resolvents to $\widetilde{A}_{-\tau_+(\lambda)}$ for a fixed $\lambda \in \mathbb{C}_+$ we arrive at (4.22) with $\mathbb{R}_\lambda$ replaced by $(\widetilde{A}(\lambda) - \lambda)^{-1}$. On the other hand according to Theorem 4.5 the function $\tau_+(\lambda) = \{K_0(\lambda), K_1(\lambda); H_+\}$ generates a generalized resolvent $\mathbb{R}_\lambda$ by formula (4.22). Hence the equality $(\widetilde{A}(\lambda)-\lambda)^{-1} = \mathbb{R}_\lambda$ holds for every $\lambda \in \mathbb{C}_+$. This and the relations $\mathbb{R}_\lambda = (\mathbb{R}_{\overline{\lambda}})^*, (\widetilde{A}(\lambda))^* = \widetilde{A}(\overline{\lambda})$ yield the equality (4.24) for every $\lambda \in \mathbb{C}_-$. The inverse statement (i.e., the construction of $\widetilde{A}(\lambda)$ by $\mathbb{R}_\lambda$) can be proved similarly.

2) Let $\mathbb{R}_\lambda$ be a generalized resolvent of A. Then by the statement 1) $\mathbb{R}_\lambda = (\widetilde{A}(\lambda)-\lambda)^{-1}$, where $\widetilde{A}(\lambda)$ satisfies (4.28) with $\tau_+(\cdot) \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$. It follows from [10], Lemma 2.2 that the function $-\tau_+(\cdot)$ admits the representation $-\tau_+(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \widetilde{H}_+\},\ \lambda \in \mathbb{C}_+$ with holomorphic operator functions $C_j(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_j, \widetilde{H}_+],\ j \in \{0, 1\}$. Moreover since $-\tau_+(\lambda) \in \mathrm{Ac}(\mathcal{H}_0, \mathcal{H}_1)$ for every $\lambda \in \mathbb{C}_+$, Proposition 3.4 in [21] imply the relations (4.26) for the operator functions $C_0(\lambda)$ and $C_1(\lambda)$. This and the obvious equivalence $\hat{f} \in \widetilde{A}(\lambda) \iff C_0(\lambda)\Gamma_0\hat{f} + C_1(\lambda)\Gamma_1\hat{f} = 0$ yield the desired statement. □

Remark 4.7. 1) Let in Theorem 4.5 $\mathcal{H}_0 = \mathcal{H}_1 := \mathcal{H}$, so that $n_-(A) = n_+(A)$ and Π is a boundary triplet for A^*. Then the parameter $\tau(\cdot) = \{K_0(\lambda), K_1(\lambda)\}$ belongs to the class $\widetilde{R}(\mathcal{H})$ and formulas for generalized resolvents (4.20), (4.21) take the classical Krein–Naimark form

$$\mathbb{R}_\lambda = (A_0 - \lambda)^{-1} - \gamma(\lambda)K_0(\lambda)\big(K_1(\lambda) + M(\lambda)K_0(\lambda)\big)^{-1}\gamma^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \tag{4.29}$$

The description of all generalized resolvents was originally given by M. G. Krein [11, 12] and M. A. Naimark [22]. The general version of the formula (4.29) for a densely defined operator A with equal defect numbers was obtained in [24] (see also [13]) and for nondensely defined operators it was given in [14]. Another proof of this formula as well as its connection with boundary triplets was discovered in [2, 16].

Observe that classical proof of (4.29) is based on the formula (4.24) and the Krein formula for canonical resolvents. Principally another approach (coupling method) for the construction of the generalized resolvent was proposed in [5, 6, 7], where the parameter $\tau(\lambda)$ was realized as a Weyl family for some boundary relation in the exit space. Our approach was inspired by this method.

2) A description of all generalized resolvents of a symmetric operator with arbitrary defect numbers in the form (4.24) has been obtained by A. V. Shtraus [25, 26] (see also [8]). In [1] this description for a densely defined operator A with $n_+(A) = n_-(A)$ has been rewritten similar that in (4.27). A connection of these results with boundary triplets has been discovered in [2, 5, 16]. Our Corollary 4.6 is a generalization of results of [1, 2, 16] to the case $n_+(A) \neq n_-(A)$.

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