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A LOCALLY CONVEX QUOTIENT CONE

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ABSTRACT. We define a quotient locally convex cone and verify some topological properties of it. We show that the extra conditions are necessary.

1. Introduction

A *cone* (cf. [1] and [2]) is a set $\mathcal{P}$ endowed with an addition $(a,b) \to a+b$ and a scalar multiplication $(\alpha, a) \to \alpha a$ for $a, b \in \mathcal{P}$ and real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is, $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a + \beta a$, and $\alpha(a+b) = \alpha a + \alpha b$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$. We have $1a = a$ and $0a = 0$ for all $a \in \mathcal{P}$. The *cancellation law*, stating that $a+c = b+c$, implies $a = b$, however, is not required in general. It holds if and only if the cone $\mathcal{P}$ may be embedded into a real vector space. In addition we assume that $\mathcal{P}$ carries a preorder, i.e., a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a+c \leq b+c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$.

Let $\mathcal{P}$ be a preordered cone. A subset $\mathcal{V}$ of $\mathcal{P}$ is called an *(abstract) 0-neighborhood system*, if the following properties hold:

(1) $0 < v$ for all $v \in \mathcal{V}$;
(2) for all $u, v \in \mathcal{V}$ there is $w \in \mathcal{V}$ with $w \leq u$ and $w \leq v$;
(3) $u + v \in \mathcal{V}$ and $\alpha v \in \mathcal{V}$ whenever $u, v \in \mathcal{V}$ and $\alpha > 0$.

That is $\mathcal{V}$ is a subcone without 0 that is directed downward. Each element v of $\mathcal{V}$ defines an *upper*, respectively *lower*, neighborhood for the elements of $\mathcal{P}$ by

$$v(a) = \{b \in \mathcal{P} \mid b \leq a + v\}, \quad \text{respectively} \quad (a)v = \{b \in \mathcal{P} \mid a \leq b + v\},$$

creating the upper, respectively lower, topologies on $\mathcal{P}$. Their common refinement is called *symmetric topology* which we show the neighborhoods in this topology as $v(a) \cap (a)v$ or $v(a)v$ for $a \in \mathcal{P}$ and $v \in \mathcal{V}$.

An ordered cone $\mathcal{P}$ is called a *full locally convex cone* if it contains an (abstract) 0-neighborhood system $\mathcal{V}$ such that all its elements are *bounded below*, i.e., for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \rho v$ for some $\rho > 0$. We show a full locally convex cone as a pair $(\mathcal{P}, \mathcal{V})$. A locally convex cone is a pair $(\mathcal{Q}, \mathcal{V})$ where $\mathcal{Q}$ is a subcone of $\mathcal{P}$ not necessarily containing the (abstract) 0-neighborhood system.

For every $a \in \mathcal{P}$, the *closure* of a is defined to be the set $\bar{a} = \{\, b \in \mathcal{P} \mid b \leq a + v \ \text{ for all } v \in \mathcal{V}\} = \bigcap_{v \in \mathcal{V}} v(a)$. A locally convex cone is called *separated* if $\bar{a} = \bar{b}$ implies $a = b$, i.e., if different elements have different closures.

The extended scalar field $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ of real numbers with the usual order is an example of an ordered cone. We consider $\alpha + (+\infty) = +\infty$ and $\alpha.(+\infty) = +\infty$ for all $\alpha > 0$ and $0.(+\infty) = 0$. Endowed with $\mathcal{V} = \{\epsilon \in \mathbb{R} \mid \epsilon > 0\}$, $(\bar{\mathbb{R}}, \mathcal{V})$ is a full locally convex cone.

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For cones $\mathcal{P}$ and $\mathcal{P}'$ a mapping $T : \mathcal{P} \to \mathcal{P}'$ is called a linear operator if $T(a+b) = T(a)+T(b)$ and $T(\alpha a) = \alpha T(a)$ hold for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. T is said to be *monotone* if $a \leq b$ implies $T(a) \leq T(b)$. If $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{P}', \mathcal{V}')$ are locally convex cones, T is called *uniformly continuous* (*u-continuous*) if for every $v' \in \mathcal{V}'$ there is a $v \in \mathcal{V}$ such that $T(a) \leq T(b) + v'$ whenever $a \leq b + v$ for $a, b \in \mathcal{P}$. If $(\mathcal{P}, \mathcal{V})$ is a full locally convex cone and T is monotone, T is u-continuous if and only if for every $v' \in \mathcal{V}'$, there is a $v \in \mathcal{V}$ such that $T(v) \leq v'$. A linear functional on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \bar{\mathbb{R}}$. μ is u-continuous if there is a $v \in \mathcal{V}$ such that $\mu(a) \leq \mu(b)+1$ whenever $a \leq b+v$ for $a, b \in \mathcal{P}$.

The u-continuous linear operators from a locally convex cone $(\mathcal{P}, \mathcal{V})$ to another locally convex cone $(\mathcal{P}', \mathcal{V}')$ form again a cone, and the cone of u-continuous functionals from $(\mathcal{P}, \mathcal{V})$ is called the dual of $\mathcal{P}$ and denoted by $\mathcal{P}^*$. The polar v° of every neighborhood $v \in \mathcal{V}$ is the set of all $\mu \in \mathcal{P}^*$ such that $\mu(a) \leq \mu(b) + 1$ whenever $a \leq b + v$ for $a, b \in \mathcal{P}$. The union of all polars yields $\mathcal{P}^*$ and by endowing $\mathcal{P}^*$ with the topology $w(\mathcal{P}^*, \mathcal{P})$, v° is $w(\mathcal{P}^*, \mathcal{P})$ compact and convex (cf. [2], II.2.4).

In this paper we present a quotient locally convex cone. We introduce a quotient cone, an order and an (abstract) 0-neighborhood system for it. With this (abstract) 0-neighborhood system the quotient cone is a locally convex cone. We verify some topological properties for this locally convex cone.

2. A locally convex quotient cone

Let $\mathcal{P}$ be a cone and $\mathcal{Q}$ be a subcone of $\mathcal{P}$. We consider the relation $\sim$ on $\mathcal{P}$ as $x \sim y$ if and only if $x + \mathcal{Q} = y + \mathcal{Q}$. It is clear that $\sim$ is an equivalence relation on $\mathcal{P}$. This equivalence relation means that $x \sim y$ if both $x = y + p$ and $y = x + q$ for some $p, q \in \mathcal{P}$. The equivalence class $\tilde{x}$ is a subset of $\hat{x} = x + \mathcal{Q}$ in general and if $\mathcal{Q}$ is a vector space, then $\tilde{x} = \hat{x}$. The set $\{\hat{x} : x \in \mathcal{P}\}$ with the usual addition $\hat{x} + \hat{y} = (x + y)\hat{}$ and the scalar multiplication $\alpha\hat{x} = (\alpha x)\hat{}$ for $x, y \in \mathcal{P}$ and $\alpha > 0$ is a cone. The scalar multiplication is completed with $0.\hat{x} = \hat{0}$ for each $x \in \mathcal{P}$. We denote this cone by $\mathcal{P}/\mathcal{Q}$ and we call it the *quotient* of $\mathcal{P}$ on $\mathcal{Q}$. The zero element in $\mathcal{P}/\mathcal{Q}$ is clearly $\hat{0} = \mathcal{Q}$. The mapping $k(x) = \hat{x}$ is a linear mapping, which is called the *canonical mapping* of $\mathcal{P}$ onto $\mathcal{P}/\mathcal{Q}$.

If we denote by $\mathrm{Conv}(\mathcal{P})$ the set of all non-empty convex subsets of the cone $\mathcal{P}$, with the addition and scalar multiplication defined by

$$A + B = \{a + b|\ a \in A,\ b \in B\}, \quad A, B \in \mathrm{Conv}(\mathcal{P}) \ ,$$

$$\alpha A = \{\alpha a|\ a \in A\}, \quad A \in \mathrm{Conv}(\mathcal{P}), \quad \alpha \geq 0,$$

$\mathrm{Conv}(\mathcal{P})$ is again a cone. The quotient cone $\mathcal{P}/\mathcal{Q}$ is in fact a subcone of $\mathrm{Conv}(\mathcal{P})$ (cf. [2], I. Example 1.4).

If $\mathcal{P}'$ is another cone, each linear mapping t from $\mathcal{P}$ into $\mathcal{P}'$ vanishing on $\mathcal{Q}$ has the decomposition $t = u \circ k$, where u is a linear mapping of $\mathcal{P}/\mathcal{Q}$ into $\mathcal{P}'$; $u(\hat{x})$ is the common value of $t(x)$ for $x \in \hat{x}$. It is easily verified that the mapping $t \to u$ is a one to one correspondence between the cone of all linear mappings of $\mathcal{P}$ into $\mathcal{P}'$ that vanish on $\mathcal{Q}$ and the cone of all linear mappings of $\mathcal{P}/\mathcal{Q}$ into $\mathcal{P}'$.

Remark 2.1. It is well-known that any linear mapping t of the vector space E into the vector space F has the decomposition $t = u \circ k$, where u is a one to one linear mapping of $E/t^{-1}(0)$ into F and k the canonical mapping of E onto $E/t^{-1}(0)$. This is not the case for cones, that is, $u : \mathcal{P}/t^{-1}(0) \to \mathcal{P}'$ is not necessarily one to one if $\mathcal{P}$ and $\mathcal{P}'$ are cones. See the following example.

Example 2.2. Let $\mathcal{P} = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0\}$, $\mathcal{P}' = \mathbb{R}^+ = [0, +\infty)$, and $t : \mathcal{P} \to \mathcal{P}'$ defined by $t(x, y) = x + y$. Then we have $t^{-1}(0) = \{(0, 0)\}$, $t(z, 0) = t(0, z) = z$ and $u(\ (z, 0)\hat{}\ ) = u(\ (0, z)\hat{}\ ) = z$, but $(z, 0)\hat{} = (z, 0) + t^{-1}(0) \neq (0, z) + t^{-1}(0) = (0, z)\hat{}$ if $z \neq 0$.

If $\leq$ is a preorder on $\mathcal{P}$, we can consider a preorder on $\mathcal{P}/\mathcal{Q}$ defined by $\hat{x} \leq \hat{y}$ if for each $x \in \hat{x}$ there is a $y \in \hat{y}$ such that $x \leq y$. If $\mathcal{V}$ is an (abstract) 0-neighborhood system for $\mathcal{P}$, then $\hat{\mathcal{V}} = \{k(v) = \hat{v} : v \in \mathcal{V}\}$ is an (abstract) 0-neighborhood system for $\mathcal{P}/\mathcal{Q}$ and is called the *quotient (abstract) 0-neighborhood system.* $\hat{x} \leq \hat{y} + \hat{v} = (y+v)\hat{}$ means that for each $q \in \mathcal{Q}$ there is a $q' \in \mathcal{Q}$ such that $x + q \leq y + q' + v$. This implies in particular $x \leq y + q' + v$ for some $q' \in \mathcal{Q}$. Now if $(\mathcal{P}, \mathcal{V})$ is a locally convex cone and $\mathcal{F}$ is the full cone and $\mathcal{P}$ is a subcone of $\mathcal{F}$, $(\mathcal{P}/\mathcal{Q}, \hat{\mathcal{V}})$ becomes a locally convex cone such that its full cone is $\mathcal{F}/\mathcal{Q}$, and we call it a *quotient locally convex cone.*

Lemma 2.3. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone, $\mathcal{Q} \subseteq \mathcal{P}$ a subcone, and $(\mathcal{P}/\mathcal{Q}, \hat{\mathcal{V}})$ the quotient locally convex cone. For every $a \in \mathcal{P}$ and $v \in \mathcal{V}$,*

(i) $(v(a))\hat{} \subseteq \hat{v}(\hat{a})$ *and* $((a)v)\hat{} \subseteq (\hat{a})\hat{v}$.

(ii) *If for every $q \in \mathcal{Q}$ and $v \in \mathcal{V}$ there is an invertible element $p \in \mathcal{Q}$ such that $p + q \leq v$, then $\hat{v'}(\hat{a}) \subseteq (v(a))\hat{}$ and $(\hat{a})\hat{v'} \subseteq ((a)v)\hat{}$ for some $v' \in \mathcal{V}$.*

Proof. (i) is obvious. For (ii) set $v' = \frac{v}{2}$. Let $\hat{b} \in \hat{v'}(\hat{a})$. There is a $q \in \mathcal{Q}$ such that $b \leq a + q + v'$. Suppose p is an invertible element of $\mathcal{Q}$ such that $p + q \leq v'$. We have $b + p \leq a + 2v'$ and so $(b+p)\hat{} \leq \hat{a} + (2v')\hat{}$ and $b + p \in v(a)$. Since p is invertible, we see that $\hat{b} = (b+p)\hat{}$, i.e., $\hat{b} \in (v(a))\hat{}$. □

An element a of a locally convex cone $(\mathcal{P}, \mathcal{V})$ is called bounded if it is also *upper bounded*, i.e., for every $v \in \mathcal{V}$ there is a $\rho > 0$ such that $a \leq \rho v$. Since every invertible element of a locally convex cone is bounded, the condition (ii) implies that every element of a subcone $\mathcal{Q}$ with this property is bounded (similar to the elements of the topological vector spaces). Every subcone $\mathcal{Q}$ of locally convex cone $\mathcal{P}$ which is also a vector space has this property (for instance $\mathcal{P} = \bar{\mathbb{R}}$ and $\mathcal{Q} = \mathbb{R}$).

Theorem 2.4. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone, $(\mathcal{P}/\mathcal{Q}, \hat{\mathcal{V}})$ a quotient cone and k be the canonical mapping. Then*

(1) *the map k is monotone and u-continuous;*

(2) *if $\mathcal{Q}$ satisfies the conditions (ii) of Lemma 2.3, then k is an open mapping under the lower, upper and symmetric topologies;*

(3) *if for each $v \in \mathcal{V}$ and each $q \in \mathcal{Q}$, $q \leq v$, then k is also a closed mapping under the lower, upper and symmetric topologies.*

An open (closed) mapping is a mapping such that the image of the open (closed) set under this mapping is open (closed).

Proof. (1) It is obvious that k is monotone and we have from (i) of Lemma 2.3 that k is u-continuous. (2) If $\mathcal{Q}$ satisfies the condition (ii) of Lemma 2.3, k is an open mapping under the upper, lower and symmetric topologies. (3) We consider the state of the upper topology. Let $A \subseteq \mathcal{P}$ be a closed set under the upper topology. We show that $\hat{A}$ is closed. Let $\hat{x} \in \bar{\hat{A}}$, $\hat{v} \in \hat{\mathcal{V}}$ and $0 < \lambda < 1$. We have $q \leq \lambda v$ for all $q \in \mathcal{Q}$. Let $v' \leq (1-\lambda)v$. There is an $a \in A$ such that $\hat{a} \in \hat{v'}(\hat{x})$, i.e., $\hat{a} \leq \hat{x} + \hat{v'}$, then $a \leq x + q + v'$ for some $q \in \mathcal{Q}$. Hence $a \leq x + v$, i.e., $a \in v(x)$ which implies that $x \in \bar{A} = A$ and so $\hat{x} \in \hat{A}$. □

The condition (3) of Theorem 2.4 implies the condition (ii) of Lemma 2.3, since we can put $p = 0 \in \mathcal{Q}$ to be an invertible element. So with this condition the canonical mapping k is also open under the lower, upper and symmetric topologies. In the following examples we show that without this condition, it is not necessary that k be an open or closed mapping.

Example 2.5. Let $\mathcal{P} = \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, $\mathcal{Q} = \bar{\mathbb{R}}^+ = [0, +\infty]$ and $\mathcal{V} = \{\epsilon \in \mathbb{R} \mid \epsilon > 0\}$. Let $A = \{+\infty\}$. For every $\epsilon > 0$ we have $(+\infty)\epsilon = \epsilon(+\infty)\epsilon = A$. That is the set A is open and closed under the lower and symmetric topologies. But $\hat{A} = \{+\hat{\infty}\} = \{\{\infty\}\}$, $\hat{\epsilon} = [\epsilon, +\infty]$ and $(+\hat{\infty})\hat{\epsilon} = \hat{\epsilon}(+\hat{\infty})\hat{\epsilon} = \mathcal{P}/\mathcal{Q}$, that is the set $\hat{A}$ is not open under the lower and symmetric topologies. Also for each $\epsilon > 0$ the intersection of $\hat{\epsilon}(\hat{1})$ and $(\hat{1})\hat{\epsilon}$, i.e.,

$\hat{\epsilon}(\hat{1})\hat{\epsilon}$ with $\hat{A}$ is not empty and then $\hat{1}$ is in the closure of the set $\hat{A}$ relative to the upper, lower and symmetric topologies. But $\hat{1} \notin \hat{A}$, i.e., $\hat{A}$ is not closed under the lower and symmetric topologies.

Example 2.6. Let $\mathcal{P}$ and $\mathcal{V}$ be as above and $\mathcal{Q} = \{0, +\infty\}$. The set $B = (-\infty, -1)$ is an open set under the upper topology. We have $\hat{B} = \{\{r, +\infty\} \mid r < -1\}$ and for every $\epsilon > 0$, $\hat{\epsilon} = \{\epsilon, +\infty\}$. If we choose an element of $\hat{B}$, e.g., $\hat{-2} = \{-2, +\infty\}$, then $\hat{+\infty} \in \hat{\epsilon}(\hat{-2})$, but $\hat{+\infty} \notin \hat{B}$, i.e., the set $\hat{B}$ is not open under the upper topology.

The set $C = [1, +\infty)$ is closed under the upper topology. $\hat{C} = \{\{r, +\infty\} | r \geq 1\}$ is not closed under the upper topology in $\mathcal{P}/\mathcal{Q}$, because $\{+\infty\}$ is in the closure of $\hat{C}$ under the upper topology, but $\{+\infty\} \notin \hat{C}$.

Theorem 2.7. *Let $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{P}', \mathcal{V}')$ be two locally convex cones and $\mathcal{Q}$ be a subcone of $\mathcal{P}$. Let $t : \mathcal{P} \to \mathcal{P}'$ be a linear mapping which vanishes on $\mathcal{Q}$, $k : \mathcal{P} \to \mathcal{P}/\mathcal{Q}$ be the canonical mapping and $u : \mathcal{P}/\mathcal{Q} \to \mathcal{P}'$ be a linear mapping such that $t = u \circ k$. Then t is u-continuous if and only if u is u-continuous.*

Proof. Let t be u-continuous and let $v' \in \mathcal{V}'$. There is a $v \in \mathcal{V}$ such that $a \leq b + v$ implies $t(a) \leq t(b) + v'$. If $\hat{a} \leq \hat{b} + \hat{v}$, then there is a $q \in \mathcal{Q}$ such that $a \leq b + q + v$. Since $t(q) = 0$, we have $t(a) \leq t(b) + v'$, that is, $u(\hat{a}) \leq u(\hat{b}) + v'$, which implies that u is u-continuous. Now let u be u-continuous and let $v' \in \mathcal{V}'$ be arbitrary. There is a $\hat{v} \in \hat{\mathcal{V}}$ such that $\hat{a} \leq \hat{b} + \hat{v}$ implies $u(\hat{a}) \leq u(\hat{b}) + v'$. Now put $v + 0 \in \hat{v}$. If $a \leq b + v$, then $\hat{a} \leq \hat{b} + \hat{v}$ and so $u(\hat{a}) \leq u(\hat{b}) + v'$. This means that $t(a) \leq t(b) + v'$ and t is u-continuous. □

Corollary 2.8. *If $(\mathcal{P}, \mathcal{V})$ is a locally convex cone and $\mathcal{Q}$ is a subcone of $\mathcal{P}$ such that $(\mathcal{P}/\mathcal{Q}, \hat{\mathcal{V}})$ is a quotient cone, then there is a one to one correspondence between $\hat{v}^\circ$ and the set of all elements of v° that vanish on $\mathcal{Q}$.*

Proof. The proof is easy with the use of Theorem 2.7. □

Let $(\mathcal{Q}, \mathcal{V})$ be a locally convex cone. We have that $(\mathcal{Q}, \mathcal{V})$ is separated if and only if the upper topology on $\mathcal{Q}$ is T_0 if and only if the lower topology on $\mathcal{Q}$ is T_0 if and only if the symmetric topology on $\mathcal{Q}$ is T_0 if and only if the symmetric topology on $\mathcal{Q}$ is Hausdorff (cf. [2] I, 3.9).

We note that $\mathcal{Q}$ can not be Hausdorff under the upper or the lower topologies.

Proposition 2.9. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone and $\mathcal{Q}$ be a subcone of $\mathcal{P}$. If $\mathcal{P}/\mathcal{Q}$ is Hausdorff under the symmetric topology (separated), then $\mathcal{Q}$ is closed under the symmetric topology.*

Proof. If $a \notin \mathcal{Q}$ then $\hat{a} \neq \hat{0} = \mathcal{Q}$. Since $\mathcal{P}/\mathcal{Q}$ is T_0 under the upper topology, there is a $v \in \mathcal{V}$ such that $\hat{a} \notin \hat{v}(\hat{0})$ or $\hat{0} \notin \hat{v}(\hat{a})$, i.e., $a \notin v(q)$ or $q \notin v(a)$ for all $q \in \mathcal{Q}$ and so $q \notin (a)v$ or $q \notin v(a)$ and then $q \notin v(a)v$, i.e., $\mathcal{Q} \cap v(a)v = \emptyset$ and then $a \notin \bar{\mathcal{Q}}$ (the closure is taken with respect to the symmetric topology). □

Remark 2.10. (a) We have a converse to the above proposition for locally convex spaces (cf. [4] I, 2.3), but it is not true for locally convex cones, even if the locally convex cones are separated. Consider the following two examples.

(1) If $\mathcal{V}$ is a 0-neighborhood base for $\mathbb{R}$ under the usual topology, $(\mathrm{Conv}(\mathbb{R}), \mathcal{V})$ with $\subseteq$ is a locally convex cone that is not separated (since $A = (-1, 1)$ and $B = [-1, 1]$ but $\bar{A} = \bar{B} = [-1, 1]$). In this case $\mathcal{Q} = \{\{q\} \mid q \in \mathbb{R}\}$ is a subcone of $\mathrm{Conv}(\mathbb{R})$ that is closed under the symmetric topology, because if $A \in \bar{\mathcal{Q}}$, then $V(A)V \cap \mathcal{Q} \neq \emptyset$ for all $V \in \mathcal{V}$, i.e., there is a $\{q\} \in \mathcal{Q}$ such that $\{q\} \subseteq A + V$ and $A \subseteq \{q\} + V$ for all $V \in \mathcal{V}$. Now since $\mathbb{R}$ is Hausdorff with the usual topology, $A \subseteq \bar{A} = \overline{\{q\}} = \{q\}$ and then $A = \{q\} \in \mathcal{Q}$.

But $\mathcal{P}/\mathcal{Q}$ is not separated, because if $A=(-1,1)$ and $B=[-1,1]$, then $\hat{A}=\{(-1+r,1+r)\mid r\in\mathbb{R}\}$ and $\hat{B}=\{[-1+r,1+r]\mid r\in\mathbb{R}\}$ but $\hat{A}\neq\hat{B}$ and $\hat{A}\in\hat{V}(\hat{B})$ and $\hat{B}\in\hat{V}(\hat{A})$ for all $V\in V$, i.e., $\mathcal{P}/\mathcal{Q}$ is not T_0 under the upper topology and so is not Hausdorff under the symmetric topology.

(2) If we consider $\mathcal{P}=\mathbb{R}$ with the usual order and $\mathcal{V}=\{\epsilon\in\mathbb{R}\mid\epsilon>o\}$ as an (abstract) 0-neighborhood system, which is separated, and $\mathcal{Q}=\mathbb{R}^+=\{r\in\mathbb{R}|r\geq 0\}$, which is closed under the symmetric topology, then $\mathcal{P}/\mathcal{Q}$ is not a separated locally convex cone, since for example $\hat{1}=[1,+\infty)$, $\hat{2}=[2,+\infty)$ and $\hat{\epsilon}(\hat{1})\cap\hat{\epsilon}(\hat{2})=\mathcal{P}/\mathcal{Q}$ for all $\epsilon>0$.

(b) We saw in (a) that it is not necessary that $\mathcal{P}/\mathcal{Q}$ be Hausdorff under the symmetric topology and so it is not necessary for one-point sets to be closed under this topology. In spite of this, if $\mathcal{Q}$ is closed in $\mathcal{P}$, the one-point set $\hat{\mathcal{Q}}=\{\mathcal{Q}\}=\{\hat{0}\}$ is closed, because if $\hat{x}\notin\hat{\mathcal{Q}}$, then $x\notin\mathcal{Q}$ and so $v(x)v\cap\mathcal{Q}=\emptyset$ for some $v\in\mathcal{V}$, i.e., $q\notin v(x)v$ for all $q\in\mathcal{Q}$ and then $\mathcal{Q}\notin\hat{v}(\hat{x})\hat{v}$. That is, $\hat{v}(\hat{x})\hat{v}\cap\{\mathcal{Q}\}=\emptyset$ and then $\hat{x}\notin\{\hat{\mathcal{Q}}\}$.

Proposition 2.11. *If $\mathcal{P}$ is Hausdorff under the symmetric topology and $\mathcal{Q}$ satisfies the condition (3) of Theorem 2.4, then $\mathcal{P}/\mathcal{Q}$ is Hausdorff under the symmetric topology.*

Proof. Let $\bar{\hat{x}}=\bar{\hat{y}}$ for $\hat{x},\hat{y}\in\mathcal{P}/\mathcal{Q}$, $\hat{v}\in\hat{\mathcal{V}}$ and $0<\lambda<1$. In this case $q\leq\lambda v$ for all $q\in\mathcal{Q}$. We have $\hat{x}\leq\hat{y}+((1-\lambda)v)\hat{}$. Hence $x\leq y+q+(1-\lambda)v$ for some $q\in\mathcal{Q}$. We have $x\leq y+v$ and so $x\in\bar{y}$. Similarly $y\in\bar{x}$. That is $\bar{x}=\bar{y}$, which implies $x=y$ and so $\hat{x}=\hat{y}$. □

Finally we note that no nonzero subspace of a topological vector space satisfies the condition (3) of Theorem 2.4, but in locally convex cones this may happen frequently. We give examples with this condition satisfied.

Example 2.12. (1) Let $\mathcal{P}=\mathrm{Conv}(\mathbb{R})$. For $A,B\in\mathcal{P}$ by considering $A\leq B$ if for every $a\in A$ there is some $b\in B$ such that $a\leq b$, the set $\mathcal{V}=\{(\epsilon,+\infty)\mid\epsilon>0\}$ is as an (abstract) 0-neighborhood system for $\mathcal{P}$. The set $\mathcal{Q}=\{\{r\}|r\in\mathbb{R}\}$ is a subcone of $\mathcal{P}$ and satisfies the condition (3) of Theorem 2.4.

(2) Let $\mathcal{P}=\mathbb{R}^2$ with the following preorder:

$$(x,y)\leq(x',y')\quad\text{if and only if}\quad x\leq x'.$$

With $\mathcal{V}=\{(\epsilon,0)|\epsilon>0\}$ as an (abstract) 0-neighborhood system for $\mathcal{P}$, $(\mathcal{P},\mathcal{V})$ is a locally convex cone. $\mathcal{Q}=\{(x,y)|x\leq 0\}$ is a subcone of $\mathcal{P}$ which has satisfies the mentioned condition.

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