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ON SIMILARITY OF CONVOLUTION VOLTERRA OPERATORS IN SOBOLEV SPACES

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Abstract. Necessary and sufficient conditions for a convolution Volterra operator to be similar in a Sobolev space to the operator J^α are obtained. A criterion of similarity is obtained as well.

1. Introduction

Consider a convolution Volterra operator on $W_p^n[0,1]$ ($n \in \mathbb{Z}_+ \setminus \{0\}$, $1 \le p \le +\infty$) defined by

$$K: \quad f \to \int_0^x k(x-t)f(t)\,dt, \quad f \in W_p^n[0,1] \tag{1}$$

with a summable kernel $k \in L_1[0,1]$. The simplest Volterra operators of the form (1) are the operator of integration J,

$$J: \quad f \to \int_0^x f(t)\,dt, \tag{2}$$

and its positive powers J^α,

$$J^\alpha: \quad f \to \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)\,dt, \quad \alpha \in \mathbb{R}_+. \tag{3}$$

Recall that two bounded operators A and B on a Banach space X are called similar if there exists a bounded operator T on X with a bounded inverse T^{-1} (an automorphism on X) such as $B = TAT^{-1}$.

The first result on similarity of an operator K of the form (1) to the operator of integration J as well as to its integer powers J^n ($n \in \mathbb{Z}_+ \setminus \{0\}$) has been obtained by G. K. Kalish [10] and L. A. Sakhnovich [20]. These results have been specified in several parers (see [5]–[17]).

Later, using techniques from complex analysis R. Frankfurt and J. Rovnyak [4], [5] have obtained sufficient conditions for a convolution Volterra operator K to be similar in $L_2[0,1]$ to the operator J^α of the form (3) with an arbitrary $\alpha \in \mathbb{R}_+ \setminus \{0\}$.

Using another approach, the result from [5] has been brightened up by M. M. Malamud in [14]. He has also obtained (see [15]) sufficient conditions for a nonconvolution Volterra operator K to be similar in $L_p[0,1]$ to the operator (3).

In this paper, following M. M. Malamud [14], we obtain necessary and sufficient conditions for a convolution Volterra operator K to be similar in $W_p^n[0,1]$ ($n \in \mathbb{Z}_+ \setminus \{0\}$) to the operator J^α of the form (3) with an arbitrary $\alpha \in \mathbb{R}_+ \setminus \{0\}$.

For example, we show that the operator $J + J^\alpha$ is similar to the operator J in $W_p^n[0,1]$ for $n \ge 3$ if and only if $\alpha > n - \frac{1}{p}$.

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Notations. Let $W_p^n[0,1]$ be the Sobolev space, i.e., if f has $n-1$ absolutely continuous derivatives and $f^{(n)} \in L_p[0,1]$, then $f \in W_p^n[0,1]$.

By definition, put $C_0^\infty[0,1] = \{f \in C^\infty[0,1] : f^{(j)}(0) = 0,\ \ j \in \mathbb{Z}_+\}$. We denote by $W_{p,0}^n[0,1]$ the closure of the lineal $C_0^\infty[0,1]$ in $W_p^n[0,1]$.

By $L_p^s[0,1]$ denote the Liouville space, i.e., $f \in L_p^s[0,1]$ if f has generalized fractional derivative $f^{(s-[s])}$ of the order $s-[s]$ and $f^{(s-[s])} \in W_p^{[s]}[0,1]$. The functions $f \in L_p^s[0,1]$ are characterized (see [3]) by the integral representation

$$f(x) = \sum_{m=1}^{[s]+1} c_m \frac{x^{s-m}}{\Gamma(s-m+1)} + \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} g(t)\, dt \tag{4}$$

where $c_m = f^{(s-m)}(0)$, and $g(x) = f^{(s)}(x)$.

By definition, put $L_{p,0}^s[0,1] := \{f \in L_p^s[0,1] \colon f^{(s-m)}(0) = 0, 1 \le m \le [s]+1\}$ and $L_{p,0}^0[0,1] := L_p^0[0,1] := L_p[0,1]$.

2. Similarity of the operators K and J

Theorem 2.1. *Let K be an operator of the form (1) with the kernel $k(\cdot)$ and let the following conditions be satisfied:*

1. $k(0) = 1$;
2 a). $k(\cdot) \in W_p^{n-1}[0,1]$, *if* $n \ge 3$;
2 b). $k(\cdot) \in W_1^2[0,1]$, *if* $n \in \{1,2\}$.

Then the operator K is similar to the operator of integration J in the spaces $W_p^n[0,1]$, $p \in [1,+\infty]$, $n \in \mathbb{Z}_+$.

Proof. The theorem is obtained by induction on n.

The case $n=0$ was investigated by M. M. Malamud in [13] (see also [17]). Namely, he has constructed the bounded and boundedly invertible operator V of the form

$$V: \quad f \to \phi(x) \left(f(x) + \int_0^x N(x-t) f(t)\, dt \right)$$

that intertwines the operators K and J in the spaces $L_p[0,1]$.

a) Suppose $k(\cdot) \in W_p^{m-1}[0,1]$. Then by the inductive assumption, the operator K is similar to the operator J_{m-1} of integration in the spaces $W_p^{m-1}[0,1]$.

This implies that there exists a bounded and boundedly invertible operator V_{m-1} on $W_p^{m-1}[0,1]$ that intertwines the operators K and J_{m-1}, that is, $KV_{m-1} = V_{m-1}J_{m-1}$.

It is clear that $W_p^m[0,1] = J(W_p^{m-1}[0,1]) \dot{+} \mathbb{C}$.

Let us define an operator V_m by

$$V_m: \quad \begin{pmatrix} f_0 \\ c \end{pmatrix} \to \int_0^x V_{m-1}\left(\frac{d}{dx} f_0\right) dx + c, \qquad f_0 \in JW_p^{m-1}[0,1], \qquad c \in \mathbb{C}. \tag{5}$$

It follows from $f_0 \in JW_p^{m-1}[0,1]$ that $\frac{d}{dx} f_0 \in W_p^{m-1}[0,1]$. The operator V_{m-1} is bounded on $W_p^{m-1}[0,1]$. Thus we have $V_{m-1}\left(\frac{d}{dx} f_0\right) \in W_p^{m-1}[0,1]$. Therefore, $\int_0^x V_{m-1}\left(\frac{d}{dx} f_0\right) dx \in W_p^m[0,1]$. Hence, the operator V_m is bounded on the spaces $W_p^m[0,1]$.

Then using the commutative property of convolution and the equality $KV_{m-1} = V_{m-1}J_{m-1}$, we get

$$
\begin{aligned}
KV_m f = KV_m \begin{pmatrix} f_0 \\ c \end{pmatrix} &= K\left(\int_0^x V_{m-1}\left(\frac{d}{dt}f_0\right)dt + c\right) = \int_0^x KV_{m-1}\left(\frac{d}{ds}f_0(s)\right)ds \\
&+ c\int_0^x k(x-t)\,dt = \int_0^x V_{m-1}J_{m-1}\left(\frac{d}{ds}f_0(s)\right)ds + c\int_0^x k(x-t)\,dt \\
&= \int_0^x V_{m-1}f_0(s)\,ds + c\int_0^x k(x-t)\,dt = JV_{m-1}f_0 + cJk.
\end{aligned} \tag{6}
$$

Since $f(x) = x \in JW_p^{m-1}[0,1]$ and $V_{m-1}\mathbf{1} = k(x)$ (see [2]), and taking (5) into account we obtain

$$
\begin{aligned}
V_m J_m f = V_m J_m \begin{pmatrix} f_0 \\ c \end{pmatrix} &= V_m \begin{pmatrix} J_m f_0 + cx \\ 0 \end{pmatrix} = JV_{m-1}\frac{d}{dx}(J_m f_0 + cx) + 0 \\
&= JV_{m-1}f_0 + cJV_{m-1}\mathbf{1} = JV_{m-1}f_0 + c\int_0^x k(t)\,dt = JV_{m-1}f_0 + cJk.
\end{aligned} \tag{7}
$$

Combining (6) with (7) we get $KV_m = V_m J_m$, that is, the operator V_m intertwines the operators K and J_m in the spaces $W_p^m[0,1]$. Further, note that the operator V_m of the form (5) is boundedly invertible in $W_p^m[0,1]$. Indeed,

$$
V_m^{-1}: \quad \begin{pmatrix} f_0 \\ c \end{pmatrix} \to \int_0^x (V_{m-1})^{-1}\left(\frac{d}{dx}f_0\right)dx + c, \quad f_0 \in JW_p^{m-1}[0,1], \quad c \in \mathbb{C}. \tag{8}
$$

Therefore the operators K and J are similar in $W_p^m[0,1]$. This proves the theorem. □

Example 2.2. Consider an operator

$$
F_{\alpha\beta}: \quad f \to \int_0^x \left[1 + (x-t)^\alpha \ln^\beta \frac{a}{x-t}\right] f(t)\,dt, \quad a > 1 \tag{9}
$$

acting on $W_p^n[0,1]$. The operator $F_{\alpha\beta}$ is similar to the operator J of integration in $W_p^n[0,1]$ if either $\alpha > \max(n-1-1/p; 1)$ or $\alpha = \max(n-1-1/p; 1)$ and $\beta \le 0$.

Example 2.3. Consider an operator

$$
F_{\alpha\beta\gamma}: \quad f \to \int_0^x \left[1 + (x-t)^\alpha \ln^\beta \ln^\gamma \frac{a}{x-t}\right] f(t)\,dt, \quad \gamma > 0, \quad a > 1 \tag{10}
$$

on $W_p^n[0,1]$. By Theorem 2.1, for the operator $F_{\alpha\beta\gamma}$ to be similar to the operator J in $W_p^n[0,1]$ it is sufficient that $\alpha > \max(n-1-1/p\,; 1)$.

We next assert the following.

Lemma 2.4. *Let the convolution operator K be bounded on $W_p^s[0,1]$. Then the kernel $k(\cdot)$ belongs to the space $W_p^{s-1}[0,1]$.*

Proof. Differentiating the equality $K\mathbf{1} = \int_0^x k(x-t)\,dt = \int_0^x k(t)\,dt \in W_p^s[0,1]$, we obtain $k(x) \in W_p^{s-1}[0,1]$. □

Example 2.5. Let $p \in [1, +\infty]$ and either $\alpha > s - \frac{1}{p}$ or $\alpha \in \mathbb{N}$. Then the operator J^α (3) is bounded in $W_p^s[0,1]$.

Combining Lemma 2.4 for $s = n \in \mathbb{Z}_+$ with Theorem 2.1, we get the following criterion.

Theorem 2.6. *Let K be an operator of the form (1) with a kernel $k(x-t)$, $k(0) = 1$, and $n \ge 3$. Then for any $p \in [1, +\infty]$ the operator K is similar to the operator J of integration in the spaces $W_p^n[0,1]$ if and only if $k(\cdot) \in W_p^{n-1}[0,1]$.*

Theorem 2.6 immediately yields.

Corollary 2.7. *Let $n \geq 3$. Then for any $p \in [1, +\infty]$ the operator*

$$(J + J^{\alpha}): \quad f \to \int_0^x \left[1 + \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\right] f(t)\,dt \tag{11}$$

is similar to the operator J in $W_p^n[0,1]$ if and only if $\alpha > n - \frac{1}{p}$.

The following result is presented for the sake of completeness. The proof can be found in [2].

Proposition 2.8. *Suppose that $n \in \{1, 2\}$ and the kernel $k(\cdot) \in W_p^1[0,1]$ satisfies the following conditions:*

(i) $k(0) = 0$;
(ii) $k'(x) \geq 0$ *for a. e.* $x \in [0,1]$;
(iii) $k'(\cdot)$ *is not bounded in a neighborhood of zero and does not increase on* $[0,1]$.

Then the operator

$$(J + K): \quad f \to \int_0^x [1 + k(x-t)] f(t)\,dt \tag{12}$$

is not similar to the operator J in the spaces $W_p^n[0,1]$ for any $p \in [1, +\infty]$, $n \in \{1, 2\}$.

Combining Proposition 2.8 and Theorem 2.1, we obtain

Example 2.9. Let $n \in \{1, 2\}$. Then for any $p \in [1, +\infty]$ the operator $J + J^{\alpha}$ is similar to the operator J in $W_p^n[0,1]$ if and only if $\alpha \geq 2$.

Example 2.10. For the operator

$$F_{\alpha}: \quad f \to \int_0^x \left[1 + (x-t)\ln^{\alpha}\frac{a}{x-t}\right] f(t)\,dt, \quad a > 1, \quad \alpha \in \mathbb{R}, \tag{13}$$

to be similar to the operator J of integration in $W_p^n[0,1]$ ($n \in \{1, 2\}$) it is necessary and sufficient that $\alpha \leq 0$.

Example 2.11. Consider an operator

$$F_{\alpha\beta}: \quad f \to \int_0^x \left[1 + (x-t)\ln^{\alpha}\ln^{\beta}\frac{a}{x-t}\right] f(t)\,dt, \quad a > 1, \quad \alpha \in \mathbb{R}, \quad \beta > 0, \tag{14}$$

in $W_p^n[0,1]$ ($n \in \{1, 2\}$). For the operator $F_{\alpha\beta}$ to be similar to the operator J of integration in $W_p^n[0,1]$ it is necessary and sufficient that $\alpha \leq 0$.

Remark 2.12. For the space $L_p[0,1]$ both Theorem 2.1 and Proposition 2.8 have been obtained by M. M. Malamud in [13]. Note in conclusion, that the assumptions on an operator K to be similar to the integration operator in the spaces $W_n^j[0,1]$ ($n \in \{1, 2\}$) do not require an additional smoothness comparatively with the case of the space $L_p[0,1]$ (the case $n = 0$).

3. Sufficient conditions of similarity of K and J^{α} in $W_p^n[0,1]$

In this section we consider sufficient conditions of similarity.

We next need the following lemmas.

Lemma 3.1. *Let $k(\cdot)$, $k_m(\cdot) \in W_p^{n-1}[0,1] \cap W_1^2[0,1]$ and generate in $W_p^n[0,1]$ operators K and K_m by (1), and $k(0) = k_m(0) = 1$. Let also $\|K_m - K\|_{W_p^{n-1}[0,1]} \to 0$ as $m \to \infty$.*

Then there exists a sequence V_m of transformation operators ($V_m^{-1} K_m V_m = J$) such that $\lim_{m\to\infty} \|V_m - V\|_{W_p^n[0,1]} = 0$, where V is a transformation operator for K, i. e., $V^{-1} K V = J$.

Proof. For the space $W_p^0[0,1] := L_p[0,1]$ the proof has been given in [14].

Note that under the conditions of lemma the operators K_m and J are similar in $L_p[0,1]$. Hence we have a sequence of transformation operators $\tilde{V}_m$ ($J = \tilde{V}_m^{-1} K_m \tilde{V}_m$) such that $\lim_{m\to\infty} \|\tilde{V}_m - \tilde{V}\|_{L_p[0,1]} = 0$.

The similarity of the operators K_m and J in Sobolev space $W_p^n[0,1]$ follows from the Theorem 2.1.

Moreover the transformation operator has the form

$$V_m: \quad f \to \begin{pmatrix} J^n \tilde{V}_m J^{-n} & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} f_0 \\ c_{n-1}x^{n-1} \\ \dots \\ c_1 x \\ c_0 \end{pmatrix}$$

where I is an identity operator and $f = f_0 + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \dots + c_1 x + c_0$, $f_0 \in W_{p,0}^n[0,1]$.

Hence $\|V_m - V\|_{W_p^n[0,1]} = \|J^n(\tilde{V}_m - \tilde{V})J^{-n}\|_{W_{p,0}^n[0,1]} = \|\tilde{V}_m - \tilde{V}\|_{L_p[0,1]} \to 0$ as $m \to \infty$. □

We next need the following lemma, due to M. M. Malamud [14].

Lemma 3.2 ([14]). *Let $f(x) \in W_p^n[0,1]$, and $\alpha_m = p_m/q_m$ be a sequence of rational numbers such that $\alpha_m \to \alpha > 0$ as $m \to \infty$. Then for each m the equation*

(15)
$$\sum_{l=2}^{p_m} C_{p_m}^l \int_0^x \int_0^{s_1} \cdots \int_0^{s_{l-2}} u(x-s_1)u(s_1-s_2)\cdots u(s_{l-1})\, ds_{l-1}\cdots ds_1 + p_m u_m(x)$$
$$= \sum_{j=2}^{q_m} C_{q_m}^j \int_0^x \int_0^{s_1} \cdots \int_0^{s_{j-2}} f(x-s_1)f(s_1-s_2)\cdots f(s_{j-1})\, ds_{j-1}\cdots ds_1 + q_m f(x)$$

has a unique solution $u_m(x) \in W_p^n[0,1]$, while

$$\lim_{m,n\to\infty} \|u_n - u_m\|_{W_p^n[0,1]} = 0. \tag{16}$$

We next assert:

Theorem 3.3. *Suppose $k(\cdot) \in W_p^{n+m-2}[0,1]$, $k(0) = k'(0) = \dots = k^{(m-2)}(0) = 0$, and $k^{(m-1)}(0) = 1$; then there exists a function $h(t) \in W_p^{n-1}[0,1]$ such that the operator*

$$H: \quad f \to \int_0^x h(x-t)f(t)\, dt \tag{17}$$

is bounded on $W_p^n[0,1]$ and satisfies the condition $H^m = K$, where $h(0) = 1$.

Proof. Let us prove that there exists a function $h(t) \in W_p^{n-1}[0,1]$ such that

$$\int_0^x \int_0^{s_1} \cdots \int_0^{s_{m-2}} h(x-s_1)h(s_1-s_2)\cdots h(s_{m-1})\, ds_{m-1}\cdots ds_1 = k(x). \tag{18}$$

Differentiating (18) m times, we get

$$\sum_{l=2}^{m} C_l^m \int_0^x \int_0^{s_1} \cdots \int_0^{s_{l-2}} u(x-s_1)u(s_1-s_2)\cdots u(s_{l-1})\, ds_{l-1}\cdots ds_1 + (m-1)u(x) = k^{(m)}(x). \tag{19}$$

Since $k^{(m)}(x) \in W_p^{n-2}[0,1]$, then by Lemma 3.2 there exists a unique solution $u(t)$ of the equation (19) such that $u(t) \in W_p^{n-2}[0,1]$. Hence

$$h(x) = \int_0^x u(t)\,dt + 1. \tag{20}$$

is a solution of (18) and $h(t) \in W_p^{n-1}[0,1]$. This proves the theorem. □

Now we are ready to prove the following result on similarity between K and J^m.

Theorem 3.4. *Let K be an operator of the form (1) with the kernel $k(\cdot) \in W_p^{n+m-2}[0,1] \cap W_1^2[0,1]$ and $k(0) = \cdots = k^{(m-2)}(0) = 0$, $k^{(m-1)}(0) = 1$. Then the operator K is similar to the operator J^m in the spaces $W_p^n[0,1]$ for $p \in [1,+\infty]$.*

Proof. By Theorem 3.3 there exists an operator H of the form (17) such that $H^m = K$, where $h(t) \in W_p^{n-1}[0,1]$. It follows from (20) that $h(0) = 1$. Therefore by Theorem 2.1, the operators H and J are similar ($LHL^{-1} = J$). Raising this equation to the mth power, we obtain the required result. □

Theorem 3.5. *Let $k(\cdot) \in L_p^{\alpha+n-2}[0,1] \cap L_{p,0}^{\alpha}[0,1] \cap L_1^{\alpha+1}[0,1]$, either $\alpha > n - \frac{1}{p}$ or $\alpha \in \mathbb{N}$. Then the operator*

$$J^\alpha + K: \quad f \to \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt + \int_0^x k(x-t) f(t)\,dt \tag{21}$$

is similar to the operator J^α in Sobolev spaces $W_p^n[0,1]$, $p \in [1,+\infty]$.

Proof. The proof is similar to that of Theorem 2 in [14].

1. Let $\alpha = r/q$ be rational.

Then $(J^\alpha + K)^q = J^r + \sum_{l=1}^q C_q^l J^{(q-l)r/q} K^l = J^r + F$. It easy to see that the kernel $f(x-t)$ of operator F satisfies the hypothesis of Theorem 3.4 for $m = r$. Consider one summand $F_l = J^{(q-l)r/q} K^l$. The kernel $f_l(x-t)$ of the operator F_l has the form

$$f_l(x) = \frac{x^{r-rl/q-1}}{\Gamma(r - rl/q)} * k_l(x) \tag{22}$$

where $k_l(x-t)$ is the kernel of the operator K^l. By hypothesis, $k(x) \in L_p^{\alpha+n-2}[0,1] \cap L_{p,0}^{\alpha}[0,1]$. It follows from (4) that $k(x) = \frac{x^{r/q-1}}{\Gamma(r/q)} * \phi(x)$, where $\phi(x) \in W_p^{n-2}[0,1]$. Hence, using the commutativity of convolution, we get

$$k_l(x) = [k(x)]^l = \left(\frac{x^{r/q-1}}{\Gamma(r/q)} * \phi(x)\right) * \cdots * \left(\frac{x^{r/q-1}}{\Gamma(r/q)} * \phi(x)\right) = \frac{x^{lr/q-1}}{\Gamma(lr/q)} * \phi_l(x), \tag{23}$$

where $\phi_l(x) = [\phi]^l = \underbrace{\phi * \phi * \cdots * \phi}_{l}$. Combining (22) and (23), we obtain

$$f_l(x) = \frac{x^{r-rl/q-1}}{\Gamma(r-rl/q)} * \frac{x^{lr/q-1}}{\Gamma(lr/q)} * \phi_l(x) = \frac{x^{r-1}}{\Gamma(r)} * \phi_l(x) = J^r \phi_l(x). \tag{24}$$

Thus, it follows from (22) that

$$f(x) = \sum_{l=1}^q C_q^l J^r \phi_l(x) = \sum_{l=1}^q \frac{1}{\Gamma(r)} C_q^l \int_0^x (x-t)^{r-1} \phi_l(t)\,dt. \tag{25}$$

Since $\phi(x) \in W_p^{n-2}[0,1]$, then $\phi_l(x) \in W_p^{n-2}[0,1]$. It follows from (25) that $f(x) \in W_p^{r+n-2}[0,1] \cap W_{p,0}^r[0,1]$. Hence by Theorem 3.4, the operator $J^r + F$ is similar to the operator J^r in $W_p^n[0,1]$, i. e., $V(J^r + F)V^{-1} = J^r$. Consider the operator $G = V(J^{r/q} + K)V^{-1}$. It is clear that $G^q = J^r$ and G is the Hilbert-Schmidt operator commuting with the operator of integration J. Hence, G is a convolution operator. It is follows from [20, Theorem 3] that $G = \varepsilon J^{r/q}$ ($\varepsilon^q = 1$). It is clear that $\varepsilon = 1$. In fact, the transformation operator V has the form $Vf = \exp(cx)\left[f(x) + \int_0^x N_m(x,t)f(t)\,dt\right]$. By definition, put $\beta(x) = x^{r/q-1} + \Gamma(r/q)k(x)$. Using the equation $G = \varepsilon J^{r/q}$, we get

$$\varepsilon\left[(x-t)^{r/q-1}e^{ct} + (x-t)^{r/q-1} * (e^{cx}N(x,t))\right] = e^{cx}\beta(x-t) + e^{cx}N(x,t) * \beta(x-t). \tag{26}$$

Since by hypothesis $k(x) \in L_p^{\alpha+n-2}[0,1] \cap L_{p,0}^{\alpha}[0,1]$, it follows from (4) that $k(x) = x^{r/q}\beta_1(x)$, where $\beta_1(x) = \int_0^1 (1-u)^{\alpha-1}k^{(\alpha)}(ux)\,du$. Substituting $s = x-(x-t)u$ in (26), we get

$$\begin{aligned}\varepsilon\Big[(x-t)^{r/q-1}e^{ct} &+ (x-t)^{r/q}\int_0^1 u^{r/q-1}e^{c(x-(x-t)u)}N(x-(x-t)u,t)\,du\Big]\\ &= e^{cx}(x-t)^{r/q-1} + e^{cx}(x-t)^{r/q-1}\beta_1(x-t)\\ &+ (x-t)^{r/q}\int_0^1 e^{cx}N(x,x-(x-t)u)\Big[(1-u)^{r/q-1} + (1-u)^{r/q-1}\beta_2(x-t)u\Big]\,du,\end{aligned}$$

where $\beta_2(x) = x\beta_1(x)$. Dividing the last equation by $(x-t)^{r/q-1}$ and setting $x = t$, we get $\varepsilon = 1$.

2. Let α be irrational.

By hypothesis of the theorem, $k(x) = \frac{x^{r/q-1}}{\Gamma(r/q)} * \phi(x)$, where $\phi(x) \in W_p^{n-2}[0,1]$. Then there exists a sequence of rational numbers $\alpha_m = r_m/q_m$ such that $\alpha_m \to \alpha$ as $m \to \infty$. Consider the sequence of kernels $p_m(x) = \frac{x^{\alpha_m-1}}{\Gamma(\alpha_m)} * \phi(x)$. It is easy to see that $p_m(x) \in L_p^{\alpha_m+n-2}[0,1] \cap L_{p,0}^{\alpha_m}[0,1]$. Then it follows from the previous step that the operators $J^{\alpha_m} + P_m$ of the form

$$J^{\alpha_m} + P_m:\quad f \to \frac{1}{\Gamma(\alpha_m)}\int_0^x (x-t)^{\alpha_m-1}f(t)\,dt + \int_0^x p_m(x-t)f(t)\,dt$$

are similar to the operators J^{α_m}. We have $(J^{\alpha_m} + P_m)^{q_m} = J^{r_m} + G_m$, where G_m are the operators of the form (1) with the kernels $g_m(x-t)$, where

$$g_m(x) = \sum_{l=1}^{q_m} C_{q_m}^l J^{\alpha_m(q_m-l)}\phi_l(x). \tag{27}$$

Consider the equation

$$\begin{aligned}\sum_{l=2}^{r_m} C_{r_m}^l \int_0^x\int_0^{s_1}\cdots\int_0^{s_{l-2}} u(x-s_1)u(s_1-s_2)\cdots u(s_{l-1})\,ds_{l-1}\cdots ds_1&\\ +r_m u_m(x) = g_m(x).&\end{aligned} \tag{28}$$

It follows from Lemma 3.2 that the sequence $u_m(x)$ is fundamental in $W_p^{n-2}[0,1]$. Let H_m ($H_m f = \int_0^x h_m(x-t)f(t)dt$) be the root of degree r_m of the operator $J^{r_m} + G_m$ such that $h_m(0) = 1$. Then, as is known [20, 13], $h_m(x) = \int_0^x u_m(t)dt + 1$, where $u_m(x)$ is a solution of (28). Hence the sequence $h_m(x)$ is fundamental in $W_p^{n-1}[0,1]$. By Lemma 3.1, there exists a sequence V_m of transformation operators ($V_m H_m V_m^{-1} = J$) that converge to some (invertible) operator V in the uniform norm in $W_p^n[0,1]$. It remains to note that V_m is a transformation operator for the operator $J^{\alpha_m} + K$, and pass to the limit in the equation $V_m(J^{\alpha_m} + K) = J^{\alpha_m}V_m$ as $m \to \infty$. □

Example 3.6. Let $p \in [1,+\infty]$ and either $\alpha > n - \frac{1}{p}$ or $\alpha \in \mathbb{N}$. For an operator

$$(J^\alpha + J^\beta):\quad f \to \int_0^x \left[\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(x-t)^{\beta-1}}{\Gamma(\beta)}\right] f(t)\,dt \tag{29}$$

to be similar to the operator J^α in $W_p^n[0,1]$ it is sufficient that $\beta > \max\{\alpha + n - 1 - \frac{1}{p}, \alpha + 1\}$.

4. Necessary conditions of similarity of K and J^α in $W_p^n[0,1]$

We consider necessary conditions of similarity.

Theorem 4.1. *Let $k^{(l)}(x) \ge 0$, $x \in [0,1]$, $l \in \{0, \dots, n-1\}$ and $k(x)$ not increase on some segment $[0, \varepsilon]$, and $k(x)x^{\alpha-1} \to \infty$ as $x \to 0$. Then for any $n \in \mathbb{Z}_+$ and $p \in 1, +\infty]$ the operator*

$$(J^\alpha + K): \quad f \to \int_0^x \left[\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(x-t)^{2\alpha-1}}{\Gamma(2\alpha)} k(x-t) \right] f(t)\, dt, \quad \alpha > 0, \tag{30}$$

is not similar to the operator J^α in the space $W_p^n[0,1]$.

Proof. The proof is similar to that of Theorem 9 in [17] and Theorem 2 in [14]. If the operator Q with the kernel $q(x)$ acts in the space $W_{p,0}^n[0,1]$, then

$$\begin{aligned}\|Qf\|_{W_p^n[0,1]}^p &= \left\| \int_0^x q(x-t) f(t)\, dt \right\|_{W_p^n[0,1]}^p = \sum_{k=0}^{n-1} \left| \left(\int_0^x q(x-t) f(t)\, dt \right)^{(k)} (0) \right|^p \\ &+ \left\| \left(\int_0^x q(x-t) f(t)\, dt \right)^{(n)} \right\|_{L_p[0,1]}^p = \sum_{k=0}^{n-2} \left| \sum_{l=0}^{k} q^{(l)}(0) f^{(k-l)}(0) \right|^p \\ &+ \int_0^1 \left| \sum_{k=0}^{n-1} q^{(k)}(x) f^{(n-k-1)}(0) + \int_0^x q(x-t) f^{(n)}(t)\, dt \right|^p dx.\end{aligned}$$

It is clear that the norm of the operator Q is reached for the functions f such that $f^{(k)}(0) > 0$ ($k \in \{0, \dots, n-1\}$) and $f^{(n)}(x) > 0$.

Let Q_1 and Q_2 be the operators of the form (1). It follows from the inequalities

$$0 \le q_1^{(j)}(x-t) \le q_2^{(j)}(x-t), \quad j \in \{0, \dots, n-1\},$$

that $\|Q_1\|_{W_p^n[0,1]} \le \|Q_2\|_{W_p^n[0,1]}$.

It is clear that $\|(J^\alpha + K)^m\| \ge m \|J^{\alpha(m-1)} K\|$. It is easy to see that the kernel of $J^{\alpha(m-1)} K$ has the form

$$k_1(x-t) = \int_0^{x-t} \frac{[(x-t) - u]^{\alpha(m-1)-1}}{\Gamma(\alpha(m-1))} \frac{u^{2\alpha-1}}{\Gamma(2\alpha)} k(u)\, du.$$

Setting $u = (x-t)s$, we obtain

$$k_1(x-t) = \frac{(x-t)^{\alpha m + \alpha - 1}}{\Gamma(\alpha(m-1))\Gamma(2\alpha)} \int_0^1 k(s(x-t))(1-s)^{\alpha(m-1)-1} s^{2\alpha-1} ds.$$

Since $k(x)$ is monotonically nonincreasing on $[0,1]$, we have $k(s(x-t)) \ge k(s)$, $t \in [0,1]$. Therefore

$$\begin{aligned}k_1(x-t) &\ge \frac{(x-t)^{\alpha m + \alpha - 1}}{\Gamma(\alpha(m-1))\Gamma(2\alpha)} \int_0^1 k(s)(1-s)^{\alpha(m-1)-1} s^{2\alpha-1} ds \\ &\ge \frac{(x-t)^{\alpha m + \alpha - 1}}{\Gamma(\alpha(m-1))\Gamma(2\alpha)} \int_0^{1/m} k(s)(1-s)^{\alpha(m-1)-1} s^{2\alpha-1} ds \\ &\ge \frac{(x-t)^{\alpha m + \alpha - 1}}{\Gamma(\alpha(m-1))\Gamma(2\alpha)} k\left(\frac{1}{m}\right) \int_0^{1/m} (1-s)^{\alpha(m-1)-1} s^{2\alpha-1} ds \\ &\ge \frac{(x-t)^{\alpha m + \alpha - 1}}{\Gamma(\alpha m + \alpha)} \cdot \frac{\Gamma(\alpha m + \alpha)}{\Gamma(\alpha(m-1)) m^{2\alpha}} \cdot \frac{1}{\Gamma(2\alpha+1)} k\left(\frac{1}{m}\right) \left(1 - \frac{1}{m}\right)^{\alpha(m-1)-1}.\end{aligned}$$

Finally,

$$\frac{\|(J^\alpha+K)^m\|}{\|J^{\alpha m}\|} \geq c\cdot m\cdot k\left(\frac{1}{m}\right)\cdot\left(1-\frac{1}{m}\right)^{\alpha(m-1)-1}\cdot\frac{\|J^{\alpha m+\alpha}\|}{\|J^{\alpha m}\|}\cdot\frac{\Gamma(\alpha m+\alpha)}{\Gamma(\alpha(m-1))m^{2\alpha}}$$
$$= c\cdot k\left(\frac{1}{m}\right)\cdot\left(1-\frac{1}{m}\right)^{\alpha m-\alpha-1}\cdot\frac{\Gamma(\alpha m+\alpha)(\alpha m)^{\alpha m}}{\Gamma(\alpha m-\alpha)m^{2\alpha-1}(\alpha m+\alpha)^{\alpha m+\alpha}}$$
$$\times\frac{(\alpha m+\alpha)^{\alpha m+\alpha}\|J^{\alpha m+\alpha}\|}{(\alpha m)^{\alpha m}\|J^{\alpha m}\|}.$$

By assumption, $k\left(\frac{1}{m}\right)\left(\frac{1}{m}\right)^{\alpha-1}\to\infty$ as $m\to\infty$.

It follows from the asymptotic of gamma-function $\Gamma(x)=\sqrt{2\pi x}x^xe^{-x}[1+O(1)]$ as $x\to\infty$ that

$$\left(1-\frac{1}{m}\right)^{\alpha(m-1)-1}k\left(\frac{1}{m}\right)\frac{\Gamma(\alpha m+\alpha)(\alpha m)^{\alpha m}}{\Gamma(\alpha(m-1))m^{2\alpha-1}(\alpha m+\alpha)^{\alpha m+\alpha}}\to\infty\quad\text{as}\quad m\to\infty.$$

Since the order of growth of the resolvent of J^α equals $1/\alpha$, and the type of growth is the length of the interval of integration, by the theorem on the connection between the growth of an entire function and the rate of decrease of the coefficients of its power series expansion, we obtain

$$\overline{\lim_{m\to+\infty}}\frac{(\alpha(m+1))^{\alpha(m+1)}\|J^{\alpha(m+1)}\|}{(\alpha m)^{\alpha m}\|J^{\alpha m}\|}\geq\overline{\lim_{m\to+\infty}}(\alpha m)^\alpha\sqrt[m]{\|J^{\alpha m}\|}=(e)^\alpha.$$

Hence $\frac{\|(J^\alpha+K)^m\|}{\|J^{\alpha m}\|}\to+\infty$. This contradiction concludes the proof.

□

Example 4.2. Suppose that $p\in[1,+\infty]$ and either $\alpha>n-\frac{1}{p}$ or $\alpha\in\mathbb{N}$. If $\beta<\min\{2\alpha,\alpha+1\}$ then the operator $J^\alpha+J^\beta$ of the form (29) is not similar to the operator J^α in $W_p^n[0,1]$.

Theorem 4.3. *Let $k^{(l)}(x)\geq 0$, $x\in[0,1]$, $l\in\{0,\dots,n-1\}$ and $k(x)$ not increase on some segment $[0,\varepsilon]$, for some $\varepsilon>0$.*

Then the operator

$$(31)\qquad (J^\alpha+K):\quad f\to\int_0^x\left[\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(x-t)^\alpha}{\Gamma(\alpha)}k(x-t)\right]f(t)\,dt,\quad \alpha\geq 1$$

is not similar to the operator J^α in $W_p^n[0,1]$ for any $p\in[1,+\infty]$, $n\in\mathbb{Z}_+$.

The proof is similar to that of Theorem 4.1.

Combining Example 3.6 and Example 4.2, we obtain the following statement.

Corollary 4.4. *Suppose either $\alpha>n-1/p$ or $\alpha\in\mathbb{N}$. Then the operator $J^\alpha+J^\beta$ is similar to the operator J^α if and only if $\beta\geq\alpha+1$ in the space $W_p^n[0,1]$ for any $p\in[1,+\infty]$, $n\in\{1,2\}$.*

Remark 4.5. I. Domanov and M. Malamud [1] have described the lattices $\operatorname{Lat}J_n^\alpha$ and $\operatorname{Hyplat}J_n^\alpha$ of invariant and hyperinvariant subspaces of the operator J_n^α defined on $W_p^n[0,1]$ and investigated the operator algebras $\operatorname{Alg}J_n^\alpha$, commutant $\{J_n^\alpha\}'$ and double commutant $\{J_n^\alpha\}''$.

Note that similar operators have equivalent geometrical structure. So as a corollary of similarity of the operators K and J^α we obtain a description of the lattices $\operatorname{Lat}K$ and $\operatorname{Hyplat}K$ of invariant and hyperinvariant subspaces of the operator K in $W_p^n[0,1]$, $(1\leq p<+\infty)$. We also can investigate the operator algebras $\operatorname{Alg}K$, commutant $\{K\}'$ and double commutant $\{K\}''$ of the operator K being similar to J^α in $W_p^n[0,1]$, $(1\leq p<+\infty)$.

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