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ON ENVELOPING C^*-ALGEBRA OF ONE AFFINE TEMPERLEY-LIEB ALGEBRA

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Abstract. C^*-algebras generated by orthogonal projections satisfying relations of Temperley-Lieb type are constructed.

1. Introduction

In the present paper we construct an enveloping C^*-algebra of $*$-algebra A_τ, generated by orthogonal projections satisfying relations of Temperley-Lieb type. Temperley-Lieb algebras appeared in the context of ice-type models, in the analysis of subfactors of II_1 factors and in the theory of knots (see, e.g. [7, 4]). And were later studied in depth, see [3, 11] for example.

Let a number $\tau \in [0,1]$ be fixed. Then the algebra A_τ is a $*$-algebra over $\mathbb{C}$ generated by three orthogonal projections p_1, p_2, p_3 ($p_i^* = p_i^2 = p_i$) with relations of Temperley-Lieb type between any two of them

$$p_i p_j p_i = \tau p_i, \quad p_j p_i p_j = \tau p_j, \quad i, j \in \{1,2,3\}, \quad i \neq j. \tag{1}$$

This $*$-algebra is sometimes called an affine Temperley-Lieb algebra $TL(\widetilde{A}_3)$.

By a representation of $*$- or C^*-algebra we mean a $*$-homomorphism into the algebra $B(H)$ of bounded operators on a Hilbert space.

We recall the definition of an enveloping C^*-algebra.

Definition 1. Let $\mathbf{A}$ be a $*$-algebra, having at least one representation. Then a pair $(\mathcal{A}, \rho)$ of a C^*-algebra $\mathcal{A}$ and a homomorphism $\rho : \mathbf{A} \longrightarrow \mathcal{A}$ is called an enveloping pair for $\mathbf{A}$ if every irreducible representation π:$\mathbf{A} \longrightarrow B(H)$ factors uniquely through $\mathcal{A}$, i.e., there is precisely one irreducible representation π_1 of the algebra $\mathcal{A}$ satisfying $\pi_1 \circ \rho = \pi$. The algebra $\mathcal{A}$ is called an enveloping C^*-algebra for $\mathbf{A}$.

If A is $*$-algebra we denote by $C^*(A)$ it's enveloping C^*-algebra (if it exists).

Below we will need an analogue of Stone-Weierstrass theorem for C^*-algebras. To formulate the theorem we need the following.

Definition 2. Let $B \subseteq A$ be two $*$-algebras. An algebra B is called massive in A if two conditions hold:

(1) for every irreducible representation π of $*$-algebra A the restriction $\pi|B$ is irreducible;
(2) if π and π' are non-equivalent irreducible representations, then $\pi|B$ and $\pi'|B$ are again non-equivalent.

Statement 1. *For every $GCR - C^*$-algebra A, and massive sub-C^*-algebra $B \subseteq A$, we have $A = B$.*

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We will use the well-known fact that C^*-algebras having only finite-dimensional irreducible representations are GCR-algebras (see [1]) so for such C^*-algebras the Stone-Weierstrass theorem holds.

Note that A_τ has only finite dimensional irreducible $*$-representations. The proof can be found in [10]. The next theorem gives a description of all irreducible $*$-representations of A_τ (see [6] for the proof).

Theorem 1. *Let π be a nontrivial irreducible $*$-representation of the algebra A_τ in a finite dimensional Hilbert space H. Then in H there exists an orthonormal basis such that, in this basis,*

$$\pi(p_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\pi(p_2) = \begin{pmatrix} \tau & \sqrt{\tau-\tau^2} & 0 \\ \sqrt{\tau-\tau^2} & 1-\tau & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2}$$

$$\pi(p_3) = \begin{pmatrix} \tau & \lambda & \mu \\ \overline{\lambda} & \frac{|\lambda|^2}{\tau} & \frac{\overline{\lambda}\mu}{\tau} \\ \mu & \frac{\mu\lambda}{\tau} & \frac{\mu^2}{\tau} \end{pmatrix},$$

where the parameter $\lambda \in \mathbb{C}$ is such that

$$|\tau^2 + \lambda\sqrt{\tau-\tau^2}| = \tau^{3/2}, \tag{3}$$

$$|\lambda| \le \sqrt{\tau-\tau^2}, \tag{4}$$

and $\mu = \sqrt{\tau-\tau^2-|\lambda|^2}$.

Remark 1. Representations corresponding to different admissible values of the parameter λ are not equivalent.

Remark 2. We consider all finitely-generated $*$-algebras to have a unit. So, for every fixed $\tau \in [0,1]$, the algebra A_τ has one additional one-dimensional representation, $p_i \mapsto 0$, $e \mapsto 1$, where e is the unit in A_τ.

For a fixed τ, the algebra A_τ can have a lot of $*$-representations. They are parameterized by λ. From (3) we obtain that λ belongs to a circle O_1 with center $\left(-\tau\sqrt{\frac{\tau}{1-\tau}}, 0\right)$ and radius $\tau\sqrt{\frac{1}{1-\tau}}$. From (4) we get that λ belongs to O_2 with center $(0,0)$ and radius $\sqrt{\tau-\tau^2}$. It is easy to show that the following holds.

(1) If $\tau = 0$ then there exist three one-dimensional representations; one projection is equal to 1 and other two are 0.
(2) If $\tau \in (0, 1/4)$ then the circle O_1 is entirely in the disk O_2. So λ belongs to the whole circle O_1 and dimensions of nontrivial irreducible $*$-representations are equal to 3.
(3) If $\tau = 1/4$ we get that the circle O_1 is still in the disk O_2 but they have one common point B. So, λ belongs to the whole circle O_1 and dimensions of nontrivial irreducible $*$-representations are equal to 3 when $\lambda \neq B$ and to 2 otherwise.
(4) If $\tau \in (1/4, 1)$ then λ belongs to the arc $\smile CD$ of the circle O_1 which is inside the disk O_2. And dimensions of nontrivial irreducible $*$-representations are equal to 3 when $\lambda \neq C, D$ and to 2 otherwise.
(5) If $\tau = 1$ we have that $\lambda = 0$, $\mu = 0$ and the only nontrivial irreducible $*$-representation is $\pi(p_1) = \pi(p_2) = \pi(p_3) = 1$.

Representations can be parameterized by values of the argument of λ. For every $\tau \in [0,1]$ choose φ_τ such that the arguments of the corresponding λ's lie in the interval $[-\varphi_\tau,\ \varphi_\tau]$. For $\tau \in [0,1/4]$ we have that $\varphi_\tau = \pi$, for $\tau \in (1/4,1)$ we have $0 < \varphi_\tau < \pi$ and for $\tau = 1$, $\varphi_\tau = 0$.

2. Enveloping C^*-algebras

In this section we describe some C^*-algebras related to a family of the $*$-algebras A_τ.

Let S^1 denote the unit circle in the complex plane. If $f(\cdot)$ is a matrix-valued function, we denote by $f_{ij}(\cdot)$ the corresponding matrix element.

Theorem 2. *The C^*-algebra $C^*(A_\tau)$ can be realized as follows:*

(1) *if $\tau = 0$ then*
$$C^*(A_\tau) \simeq \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C};$$
(2) *if $0 < \tau < 1/4$ then*
$$C^*(A_\tau) \simeq \left\{ f \in C(S^1 \to M_3(\mathbb{C})) \right\} \oplus \mathbb{C};$$
(3) *if $\tau = 1/4$ then*
$$C^*(A_\tau) \simeq \left\{ f \in C(S^1 \to M_3(\mathbb{C})) | f(-1) \in M_2(\mathbb{C}) \oplus \mathbb{C} \right\};$$
(4) *let $1/4 < \tau < 1$ and $I_\tau = \left\{ e^{i\varphi} | -\varphi_\tau \le \varphi \le \varphi_\tau \right\}$, where φ_τ is specified above, then*
$$C^*(A_\tau) \simeq \big\{ f \in C(I_\tau \to M_3(\mathbb{C}))\,|\, f(e^{-i\varphi_\tau}), f(e^{i\varphi_\tau}) \in M_2(\mathbb{C}) \oplus \mathbb{C}, \\ f_{33}(e^{-i\varphi_\tau}) = f_{33}(e^{i\varphi_\tau}) \big\};$$
(5) *if $\tau = 1$ then*
$$C^*(A_\tau) \simeq \mathbb{C} \oplus \mathbb{C}.$$

Proof. We give proof only for the case where $0 < \tau < 1/4$. For other cases, the proof is similar. Since $\pi(p_i)$ in formulas (2) can be considered as functions of the argument of λ, we can introduce the functions $P_i(e^{i\varphi}) := (\pi(p_i))(\lambda),\ i = 1,2,3$, where $\lambda = |\lambda| e^{i\varphi}$. To obtain the assertion of the theorem we will make the next two steps.

(1) Let $\widehat{A} \subseteq C(S^1 \longrightarrow M_3(\mathbb{C})) \oplus \mathbb{C}$ be the C^*-algebra generated by $(P_i(e^{i\varphi}), 0)$ and the unit, $(E^{i\varphi}, 1)$, where $E(e^{i\varphi}) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It is easy to check that $\widehat{A}$ is an enveloping C^*-algebra of A_τ. Indeed, we have a homomorphism of A_τ into $\widehat{A}$, which satisfies the universal property, so $\widehat{A}$ is an enveloping algebra by definition.

(2) We will show that $\widehat{A}$ coincides with
$$\overline{A} = \left\{ f \in C(S^1 \to M_3(\mathbb{C})) \right\}$$
using Statement 1.

Let us check that $\widehat{A} \subseteq \overline{A}$. Indeed, it is easy to check that $P_i(e^{i\varphi})$ are continuous matrix-functions, so we have $P_i(e^{i\varphi}) \in \overline{A}$.

We need to check that $\widehat{A}$ is a massive sub-C^*-algebra in $\overline{A}$. To check that $\widehat{A}$ satisfies the definition of a massive subalgebra, we use the description of all irreducible representations of $\overline{A}$. Let π_1 and π_2 be two irreducible 3-dimensional non-equivalent representations of $\overline{A}$. Then there exist $e^{i\varphi_1},\ e^{i\varphi_2} \in S^1, e^{i\varphi_1} \neq e^{i\varphi_2}$ such that the representations $\pi_i,\ i = 1,2$, are unitarily equivalent to the representations
$$\widetilde{\pi}_i : \overline{A} \to M_3(\mathbb{C}), \quad f \mapsto f(e^{i\varphi}), \quad i = 1,2.$$

To check that the restrictions of $\widetilde{\pi}_i$ to $\widehat{A}$ are again irreducible and non-equivalent, note that their restrictions to A_τ are irreducible and non-equivalent by Theorem 1.

So, by Statement 1 we have $\widehat{A} = \overline{A}$.

To prove the theorem for other values of τ, use the description of irreducible representations for the corresponding $\overline{A}$ and $\widehat{A}$. □

Remark 3. The previous theorem implies that there exist five isomorphism classes of the algebras $C^*(A_\tau)$. Isomorphisms of algebras $C^*(A_{\tau_1}),\ C^*(A_{\tau_2}),\ 1/4 < \tau_1,\ \tau_2 < 1$, can be constructed by using the natural homeomorphisms of the intervals I_{τ_1} and I_{τ_2}.

Now we consider the so-called com-algebra related to the family A_τ. Namely, let

$$A_{\text{com}} = \mathbb{C}\big\langle P_1,\ P_2,\ P_3,\ T \mid P_i^2 = P_i = P_i^*,\ P_iT = TP_i,\ T^* = T,\\ P_iP_jP_i = TP_i,\ i \in \{1,2,3\}\big\rangle.$$

Our aim is to describe $C^*(A_{\text{com}})$, which in some sense is a direct integral of the algebras $C^*(A_\tau)$. Indeed, by Schur's lemma, in every irreducible representation, $T = \tau I$ and it is known from [10] that $\tau \in [0,1]$, so the algebra A_{com} "contains" all representations of the algebras A_τ.

To describe $C^*(A_{\text{com}})$ we consider the space $D_{\text{com}} \subset I \times S^1$, where $I = [0,1]$. The space D_{com} consists of points $(\tau,\ e^{i\varphi})$, where $-\varphi_\tau \le \varphi \le \varphi_\tau$.

Theorem 3. *The algebra $C^*(A_{\text{com}})$ can be realized as an algebra of pairs of matrix-functions,*

$$\begin{gathered} C^*(A_{\text{com}}) \simeq \Big\{(f,g)| f \in C(D_{\text{com}} \to M_3(\mathbb{C})),\ g \in C(I \to \mathbb{C}),\\ f(0,e^{i\varphi}) = f(0,1) \in \mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C},\ \forall\varphi \in (-\pi,\pi],\ f(1/4,-1) \in M_2(\mathbb{C})\oplus\mathbb{C},\\ f(\tau,e^{i\varphi_\tau}),\ f(\tau,e^{-i\varphi_\tau}) \in M_2(\mathbb{C})\oplus\mathbb{C},\\ f_{33}(\tau,e^{i\varphi_\tau}) = f_{33}(\tau,e^{-i\varphi_\tau}) = g(\tau),\ \tau \in [1/4,1),\\ f(1,-1) \in \mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C},\ f_{22}(1,-1) = f_{33}(1,-1) = g(1)\Big\}. \end{gathered}$$

(*Operations in the algebra are pointwise and the norm is the usual,* $\|(f,g)\| = \max(\|f\|,\|g\|)$.)

Proof. We introduce the matrix-functions $P_i(\tau,e^{i\varphi})$ given by formulas (2) defined on the space $\big\{(\tau,e^{i\varphi)} \in D_{\text{com}}|\ \tau \ne 0\big\}$. Use (3) and (4) to define $P_i(\tau,e^{i\varphi})$ on the whole space D_{com} by continuity. To end the proof note that the elements $(P_i(\tau,e^{i\varphi}),0),\ i = 1,2,3$, and the unit element $\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},1\right)$ generate $C^*(A_{\text{com}})$ and use Statement 1 to establish the required isomorphism. □

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