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$*$-WILDNESS OF SOME CLASSES OF C^*-ALGEBRAS

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Dedicated to the memory of Yurii Daletskii.

Abstract. We consider the complexity of the representation theory of free products of C^*-algebras. Necessary and sufficient conditions for the free product of finite-dimensional C^*-algebras to be $*$-wild is presented. As a corollary we get criteria for $*$-wildness of free products of finite groups. It is proved that the free product of a non-commutative nuclear C^*-algebra and the algebra of continuous functions on the one-dimensional sphere is $*$-wild. This result is applied to estimate the complexity of the representation theory of certain C^*-algebras generated by isometries and partial isometries.

Introduction

The problem of estimating the complexity of the representation theory of a certain C^*-algebra ($*$-algebra) is one of important questions of representations theory. In order to compare C^*-algebras according to the complexity of their categories of representations S. A. Kruglyak and Yu. S. Samoilenko proposed the partial order $\succ$, see [6, 7]. Informally, if $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, then $\mathcal{A} \succ \mathcal{B}$ if the problem of classification of $*$-representations of $\mathcal{A}$ contains, as a subproblem, a classification of $*$-representations of $\mathcal{B}$, see Preliminaries below for precise definitions. A C^*-algebra $\mathcal{A}$ is called $*$-wild if $\mathcal{A} \succ C^*(\mathcal{F}_2)$, where $\mathcal{F}_2$ is free group with two generators. In fact, if a C^*-algebra is $*$-wild, then the problem of unitary classification of its irreducible representations contains, as a subproblem, the problem of classification of irreducible representations of any finitely-generated C^*-algebra, see [6].

The notion of $*$-wildness is closely related to the notion of a free product, see [13] for details on the free products of algebras like C^*-algebras and von Neumann algebras. Indeed, the model example of a $*$-wild C^*-algebra, the algebra $C^*(\mathcal{F}_2)$, is the C^*-free product of two copies of $C(\mathbf{T})$, the algebra of continuous functions on the circle. A natural problem that arises is to establish when free products of C^*-algebras are $*$-wild.

In this paper we give a complete answer for free products of finite-dimensional C^*-algebras.

The next step is to consider free products of non-commutative and commutative algebras. In this paper we show that the free product of a non-commutative nuclear algebra having non-trivial projection and $C(\mathbf{T})$ is $*$-wild. In particular, this fact implies $*$-wildness of the algebras $M_n(\mathbb{C}) * C(\mathbf{T})$, $n \geq 2$. This result is then applied to prove the $*$-wildness of C^*-algebras generated by partial isometries with linearly dependent range projections and C^*-algebras generated by families of q-commuting isometries (pairs of q-commuting isometries were also studied in [4]).

The paper is organized as follows. In preliminaries we modify the definition of majorization given in [6, 7], to make it less restrictive but preserving the most important properties of majorization. Then we give a definition of $*$-wild C^*-algebra and list basic

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properties of $*$-wild C^*-algebras. Finally, we recall the definitions and some properties of free products of C^*-algebras.

In Section 1, we study free products of finite-dimensional C^*-algebras. Namely, we give criteria of $*$-wildness for such free products. As a corollary we obtain necessary and sufficient conditions for $*$-wildness of free products of finite groups.

In Section 2.1, it is proved that the C^*-free product of a nuclear C^*-algebra with a projection and $C(\mathbf{T})$ is $*$-wild. In Section 2.2, we apply this result to prove $*$-wildness of certain C^*-algebras generated by isometries and partial isometries.

0. Preliminaries

0.1. **Enveloping C^*-algebras.** In this subsection we recall the definitions of the enveloping C^*-algebra and the group C^*-algebra, which can be found for example in [3], [5].

Let $\mathcal{A}$ be a $*$-algebra. The C^*-algebra $\widetilde{\mathcal{A}}$ with a $*$-homomorphism $\varphi : \mathcal{A} \mapsto \widetilde{\mathcal{A}}$ is called an enveloping C^*-algebra of the algebra $\mathcal{A}$ if for every representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ there exists a unique representation $\widetilde{\pi} : \widetilde{\mathcal{A}} \to \mathcal{B}(\mathcal{H})$ of $\widetilde{\mathcal{A}}$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{A} \arrow[d, "\varphi"'] \arrow[dr, "\pi"] & \\ \widetilde{\mathcal{A}} \arrow[r, "\widetilde{\pi}"'] & \mathcal{B}(\mathcal{H}) \end{tikzcd}$$

A group C^*-algebra of a group G is an enveloping C^*-algebra of the group ring $\mathbb{C}[G]$. More precisely, the group C^*-algebra, $C^*(G)$, is the completion of $\mathbb{C}[G]$ with respect to the C^*-norm

$$\|a\| = \sup\{\pi(a) : \pi \in \operatorname{Rep}(\mathbb{C}[G])\}.$$

0.2. **$*$-Wild C^*-algebras and their properties.** Recall that the category of representations of a certain C^*-algebra $\mathcal{A}$, denoted by $\operatorname{Rep}\mathcal{A}$, has representations of $\mathcal{A}$ as objects and intertwining operators as morphisms.

Definition 1. We say that a C^*-algebra $\mathcal{A}$ is majorized by a C^*-algebra $\mathcal{B}$, $\mathcal{A} \prec \mathcal{B}$, if there exists a homomorphism

$$\varphi \colon \mathcal{B} \to \mathcal{C} \otimes \mathcal{A},$$

where $\mathcal{C}$ is a nuclear C^*-algebra, and an irreducible representation

$$\widetilde{\pi} \colon \mathcal{C} \to B(\mathcal{H})$$

such that the functor

$$\mathcal{F}_{\mathcal{C},\varphi,\widetilde{\pi}} \colon \operatorname{Rep}\mathcal{A} \to \operatorname{Rep}\mathcal{B}$$

defined by

$$\mathcal{F}_{\mathcal{C},\varphi,\widetilde{\pi}}(\pi) = (\widetilde{\pi} \otimes \pi) \circ \varphi, \quad \pi \in \operatorname{Ob}(\operatorname{Rep}\mathcal{A}),$$
$$\mathcal{F}_{\mathcal{C},\varphi,\widetilde{\pi}}(A) = 1 \otimes A, \quad A \in \operatorname{Mor}(\pi_1, \pi_2)$$

is full.

Remark 1. It can be shown, see [11], that if the functor $\mathcal{F}_{\mathcal{C},\varphi,\widetilde{\pi}}$ is full then it is automatically faithful.

Remark 2. Originally in [6, 7], the authors considered the matrix algebras $M_n(\mathbb{C})$ with the identical representation instead of an arbitrary nuclear C^*-algebra $\mathcal{C}$ in the definition of majorization. However that definition was quite restrictive. Namely, if $\mathcal{A}$ has not finite-dimensional irreducible representations, then $\mathcal{A} \not\succ C^*(\mathcal{F}_2)$ according to the original definition. In Section 2 we present an example of such a majorization in our sense.

Let us show that the majorization is a partial order.

Proposition 1. *If $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{B} \prec \mathcal{C}$, then $\mathcal{A} \prec \mathcal{C}$.*

Proof. Let $\mathcal{C}_i$, $i = 1, 2$, be nuclear C^*-algebras and

$$\widetilde{\pi}_i \colon \mathcal{C}_i \to B(\mathcal{H}_i), \quad i = 1, 2$$

be their irreducible representations. By Definition 1, there are homomorphisms

$$\varphi_1 \colon \mathcal{B} \to \mathcal{C}_1 \otimes \mathcal{A}, \quad \varphi_2 \colon \mathcal{C} \to \mathcal{C}_2 \otimes \mathcal{B}$$

such that the induced functors $\mathcal{F}_{\mathcal{C},\varphi_i,\widetilde{\pi}}$ are full. Consider the homomorphism

$$\varphi := (\mathrm{id}_{\mathcal{C}_2} \otimes \varphi_1) \circ \varphi_2 \colon \mathcal{C} \to \mathcal{C}_2 \otimes \mathcal{C}_1 \otimes \mathcal{A}$$

and the irreducible representation

$$\widetilde{\pi} := \widetilde{\pi}_2 \otimes \widetilde{\pi}_1 \colon \mathcal{C}_2 \otimes \mathcal{C}_1 \to B(\mathcal{H}_2 \otimes \mathcal{H}_1).$$

Now we prove that the functor

$$\mathcal{F}_{\mathcal{C},\varphi,\widetilde{\pi}} \colon \operatorname{Rep} \mathcal{A} \to \operatorname{Rep} \mathcal{C}$$

is full. Denote $\mathcal{F}_{\mathcal{C},\varphi,\widetilde{\pi}}(\pi) = (\widetilde{\pi} \otimes \pi) \circ \varphi$ by $\widehat{\pi}$. Then

$$\widehat{\pi} = (\widetilde{\pi}_2 \otimes \mathcal{F}_{\mathcal{C}_1,\varphi_1,\widetilde{\pi}_1}(\pi)) \circ \varphi_2 = \mathcal{F}_{\mathcal{C}_2,\varphi_2,\widetilde{\pi}_2}(\mathcal{F}_{\mathcal{C}_1,\varphi_1,\widetilde{\pi}_1}(\pi)).$$

Evidently, for any $\Lambda \in \operatorname{Mor}(\pi_1, \pi_2)$, $\pi_i \in \operatorname{Ob}(\operatorname{Rep} \mathcal{A})$, one has

$$\mathcal{F}_{\mathcal{C},\varphi,\widetilde{\pi}}(\Lambda) = \mathbf{1} \otimes \mathbf{1} \otimes \Lambda = \mathcal{F}_{\mathcal{C}_2,\varphi_2,\widetilde{\pi}_2}(\mathcal{F}_{\mathcal{C}_1,\varphi_1,\widetilde{\pi}_1}(\Lambda)).$$

Hence, $\mathcal{F}_{\mathcal{C},\varphi,\widetilde{\pi}}$ is full as a composition of functors. $\square$

To give a definition of a $*$-wild C^*-algebra one has to fix some C^*-algebra with a rather complicated category of representations. The group C^*-algebra of the free group in two generators, $C^*(\mathcal{F}_2)$, is a suitable candidate for this purpose, see [6].

Definition 2. A C^*-algebra $\mathcal{A}$ is $*$-wild if

$$C^*(\mathcal{F}_2) \prec \mathcal{A}.$$

By Proposition 1, a C^*-algebra majorizing a $*$-wild C^*-algebra is $*$-wild.
The following property of $*$-wild algebras is obvious.

Proposition 2. *An extension of a $*$-wild C^*-algebra is $*$-wild.*

Proof. Let $\mathcal{A}$ be a quotient of a $*$-wild C^*-algebra $\mathcal{B}$ and the majorization $\mathcal{B} \succ C^*(\mathcal{F}_2)$ be given by a nuclear C^*-algebra $\mathcal{C}$ via a homomorphism

$$\varphi \colon \mathcal{B} \to \mathcal{C} \otimes C^*(\mathcal{F}_2)$$

and an irreducible representation $\widetilde{\pi} \colon \mathcal{C} \to B(\mathcal{H})$. Denote by $\psi \colon \mathcal{A} \to \mathcal{B}$ the canonical homomorphism. Then the majorization $\mathcal{A} \succ C^*(\mathcal{F}_2)$ is defined by the triple $(\mathcal{C}, \varphi \circ \psi, \widetilde{\pi} \circ \psi)$. $\square$

In the next Proposition we show that $*$-wild C^*-algebras are not nuclear, see also [4].

Proposition 3. *Let $\mathcal{A}$ be a $*$-wild C^*-algebra. Then $\mathcal{A}$ is not nuclear.*

Proof. It is sufficient to show that $\mathcal{A}$ has a representation generating a non-hyperfinite factor, since any factor-representation of a nuclear C^*-algebra is hyperfinite. Since $C^*(\mathcal{F}_2) \prec \mathcal{A}$, one has the homomorphism

$$\varphi\colon \mathcal{A} \to \mathcal{C} \otimes C^*(\mathcal{F}_2),$$

where $\mathcal{C}$ is a nuclear C^*-algebra, and the irreducible representation

$$\widetilde{\pi}\colon \mathcal{C} \to B(\mathcal{H})$$

is as in Definition 1.

Consider the representation π of $C^*(\mathcal{F}_2)$ generating a non-hyperfinite factor. Put $(\widetilde{\pi} \otimes \pi) \circ \varphi := \pi_1$ and note that

$$(\widetilde{\pi} \otimes \pi)(\mathcal{C} \otimes C^*(\mathcal{F}_2))'' = B(\mathcal{H}) \otimes \pi(C^*(\mathcal{F}_2))''$$

is also a non-hyperfinite factor. Since the functor

$$F\colon \ \operatorname{Rep} C^*(\mathcal{F}_2)) \to \operatorname{Rep}\mathcal{A}$$

defined in Definition 1 is full, we have

$$\pi_1(\mathcal{A})' = (\widetilde{\pi} \otimes \pi)(\mathcal{C} \otimes C^*(\mathcal{F}_2))'.$$

Hence,

$$\pi_1(\mathcal{A})'' = (\widetilde{\pi} \otimes \pi)(\mathcal{C} \otimes C^*(\mathcal{F}_2))''$$

is a non-hyperfinite factor. $\square$

Similarly to the definition of a $*$-wild C^*-algebra, one can define the notion of a $*$-wild group, see [11].

Definition 3. A group G is called $*$-wild iff its group C^*-algebra is $*$-wild.

Note that there exist non-amenable but not $*$-wild groups, see [11].

0.3. **C^*-free products.** Here we recall the definition of the free product of a family of C^*-algebras, see [13] for details.

Definition 4. If $(\mathcal{A}_i)_{i\in I}$ is a family of unital C^*-algebras, then their free product is a unique unital C^*-algebra $\mathcal{A} = *_{i\in I}\mathcal{A}_i$ and a unital $*$-homomorphisms $\psi_i : \mathcal{A}_i \to \mathcal{A}$ such that, given any unital C^*-algebra $\mathcal{B}$ and a unital $*$-homomorphisms $\Phi_i : \mathcal{A}_i \to \mathcal{B}$, there exists a unique unital $*$-homomorphism $\Phi = *_{i\in I}\Phi_i : \mathcal{A} \to \mathcal{B}$ making the following diagram commutative,

$$\begin{tikzcd}
\mathcal{A}_i \arrow[d, "\psi_i"'] \arrow[dr, "\Phi_i"] & \\
\mathcal{A} \arrow[r, "\Phi"'] & \mathcal{B}
\end{tikzcd}$$

The definition of the free product of groups (algebras) is absolutely analogous. Moreover, it is easy to see that the free product of the family of group C^*-algebras is isomorphic to the group C^*-algebra of the free product of the corresponding family of groups. Indeed, let $(G_i)_{i\in I}$ be a family of groups. Then a unitary representation of the free product $*_{i\in I}G_i$ is determined uniquely by its restrictions to the groups G_i. Since unitary representations of a group are in one-to-one correspondence with $*$-representations of its group C^*-algebra, by the universal property of the C^*-free product we get the required isomorphism,

$$C^*(*_{i\in I}G_i) \simeq *_{i\in I}C^*(G_i).$$

1. The free products of finite-dimensional C^*-algebras

In this section we show that the free product of finite-dimensional C^*-algebras is either of type **1** or is $*$-wild. In fact, we give a criterion for $*$-wildness of such free products. Throughout this section we use the old definition of majorization, i.e., $\mathcal{A} \succ \mathcal{B}$ if there exists $n \in \mathbb{N}$ and

$$\varphi\colon \mathcal{A} \to M_n(\mathcal{B}) \simeq M_n(\mathbb{C}) \otimes \mathcal{B}$$

such that the corresponding functor $\mathcal{F}_\varphi\colon \operatorname{Rep}\mathcal{B} \to \operatorname{Rep}\mathcal{A}$ is full. Here by $\mathcal{F}_\varphi$ we mean $\mathcal{F}_{M_n(\mathbb{C}),\varphi,\mathrm{id}_n}$ and

$$\mathrm{id}_n\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

denotes the identical representation.

Since any finite-dimensional C^*-algebra is $*$-isomorphic to the direct sum of matrix algebras over $\mathbb{C}$, see for example [10], our idea is to consider "elementary" free products of the form $*_{i\in I} M_{n_i}(\mathbb{C})$ for all possible values of $n_i \in \mathbb{N}$.

In the following, for $A \in M_k(\mathbb{C})$ and $B \in M_n(\mathbb{C})$, by $A \oplus B$ we mean the $n+k \times n+k$ block-diagonal matrix of the form

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

If $A = (a_{ij})_{i,j=1}^k \in M_k(\mathbb{C})$ and the element b belongs to some $*$-algebra $\mathcal{B}$, we denote by $A \otimes b$ the element of $M_k(\mathcal{B})$ of the form $(b\ a_{ij})_{i,j=1}^k$.

1. Firstly we prove the $*$-wildness of the C^*-free product of two matrix-algebras with $k > 1$ and $n > 1$.

Proposition 4. *The C^*-algebra $M_k(\mathbb{C}) * M_n(\mathbb{C})$ is $*$-wild for every $k > 1$, $n > 1$.*

Proof. Let $(e_{ij}^{(k)})_{i,j=1}^k$ and $(e_{ij}^{(n)})_{i,j=1}^n$ be the matrix units of the algebras $M_k(\mathbb{C})$ and $M_n(\mathbb{C})$, respectively. Let u, v be the generators and e be the unit of $\mathcal{F}_2$. Set $L := LCM(n,k)$. We shall construct a $*$-homomorphism $\psi : M_k(\mathbb{C}) * M_n(\mathbb{C}) \to M_L(C^*(\mathcal{F}_2))$ by the following rule:

$$\psi(e_{12}^{k}) = e_{12}^{(k)} \otimes v \oplus \bigoplus_{1}^{\frac{1}{k}L-1} e_{12}^{(k)} \otimes e,$$

$$\psi(e_{ii+1}^{(k+1)}) = \bigoplus_{1}^{\frac{1}{k}L} e_{ii+1}^{(k)} \otimes e, \qquad \text{for } 2 \le i \le k;$$

$$\psi(e_{12}^{(n)}) = T(e_{12}^{(n)} \otimes u \oplus \bigoplus_{1}^{\frac{1}{n}L-1} e_{12}^{(n)} \otimes e)T^*,$$

$$\psi(e_{ii+1}^{(n)}) = T(\bigoplus_{1}^{\frac{1}{n}L} e_{ii+1}^{(n)} \otimes e)T^*, \qquad \text{for } 2 \le i \le n.$$

Here

$$T = \begin{pmatrix} 1 & -\tan x \\ \tan x & 1 \end{pmatrix} \otimes e \oplus I_{L-2} \otimes e, \quad x \in \big(0, \frac{\pi}{2}\big),$$

I_k denotes the $k \times k$ identity matrix. In the right-hand sides of the above formulas, by $e_{ij}^{(k)}$, $(e_{ij}^{(n)})$ we mean concrete $k \times k$ ($n \times n$ resp.) matrices.

Consider a representation π of $\mathcal{F}_2$ and any operator $A = \big(a_{rs}\big)_{r,s=1}^L$ commuting with the operators $(\mathrm{id}_L \otimes \pi)(\psi(e_{ij}^{(k)}))$, $i,j = 1,\dots,k$, and $(\mathrm{id}_L \otimes \pi)(\psi(e_{ij}^{(n)}))$, $i,j = 1,\dots,n$.

It is just a routine to verify that

$$[A, (\mathrm{id}_L \otimes \pi)(\psi(e_{ij}^{(k)}))] = 0, \quad i, j = 1, \dots, k,$$

and

$$[A, (\mathrm{id}_L \otimes \pi)(\psi(e_{ij}^{n}))] = 0, \quad i, j = 1 \dots, n,$$

implies $A = \mathrm{diag}(c, c, \dots, c)$ with $[c, \pi(u)] = [c, \pi(v)] = 0$, proving the fullness of the functor F_ψ. $\square$

2. In the following proposition we study the free products of $\mathbb{C}^k$, $k \geq 2$.

Proposition 5. *The C^*-algebra $\mathbb{C}^n * \mathbb{C}^k$ is $*$-wild iff $n \geq 2$, $k \geq 3$, $n, k \in \mathbb{N}$. The C^*-algebra $\mathbb{C}^2 * \mathbb{C}^2 * \mathbb{C}^2$ is $*$-wild.*

Proof. Since for any family of groups $\{G_i,\ i \in I\}$ one has $C^*(*_{i\in I} G_i) = *_{i\in I} C^*(G_i)$, we have the following isomorphisms:

$$\mathbb{C}^n * \mathbb{C}^k \simeq C^*(\mathbb{Z}_n * \mathbb{Z}_k),$$
$$\mathbb{C}^2 * \mathbb{C}^2 * \mathbb{C}^2 \simeq C^*(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2).$$

As it was proved in [11], $C^*(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2)$ is $*$-wild and $C^*(\mathbb{Z}_n * \mathbb{Z}_k)$ is $*$-wild if and only if $n \geq 2$, $k \geq 3$. This yields the statement of the proposition. $\square$

3. It remains to consider the product $\mathbb{C}^k * M_n(\mathbb{C})$.

Proposition 6. *The C^*-algebra $\mathbb{C}^n * M_k(\mathbb{C})$ is $*$-wild iff $n \geq 2$, $k \geq 2$, $n, k \in \mathbb{N}$.*

Proof. Let $(e_{ij})_{i,j=1}^k$ be matrix units of the algebra $M_k(\mathbb{C})$ and let z be a generator of $C^*(\mathbb{Z}_n) \simeq \mathbb{C}^n$. It is known, see [11], that the group

$$\mathbb{Z}_2 * \mathbb{Z}_3 =< u, v | v^2 = u^3 = e >$$

is $*$-wild. To prove the statement of the proposition we are going to construct a $*$-homomorphism

$$\varphi : \mathbb{C}^n * M_k(\mathbb{C}) \to M_l(C^*(\mathbb{Z}_2 * \mathbb{Z}_3)), \quad \text{for sufficient large } \ l \in \mathbb{N}$$

such that the corresponding functor F_φ is full.

a) Let $n = k = 2$. Define

$$\varphi : \mathbb{C}^2 * M_2(\mathbb{C}) \to M_4(C^*(\mathbb{Z}_2 * \mathbb{Z}_3))$$

by the following rule:

$$\varphi(z) = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & 0 & -e \end{pmatrix}, \quad \varphi(e_{12}) = T \begin{pmatrix} 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{pmatrix} T^*,$$

$$\text{here } T = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e & -e\tan x \\ 0 & 0 & e\tan x & e \end{pmatrix}, \quad x \in \big(0, \pi/2\big).$$

One can directly check that the functor F_φ generated by φ is full. Thus, $\mathbb{C}^2 * M_2(\mathbb{C})$ is $*$-wild.

b) Let $n = 2$, $k \geq 3$. In this case we construct

$$\varphi : \mathbb{C}^2 * M_k(\mathbb{C}) \to M_k(C^*(\mathbb{Z}_2 * \mathbb{Z}_3))$$

as follows:

$$\varphi(z) = \begin{pmatrix} 0 & u^{-1} & 0 \\ u & 0 & 0 \\ 0 & 0 & v \end{pmatrix} \oplus (I_{k-3} \otimes e), \quad \varphi(e_{ij}) = e_{ij} \otimes e, \quad i,j = 1,\dots,k.$$

To prove that F_φ is full we fix an arbitrary representation π of $\mathbb{Z}_2 * \mathbb{Z}_3$. Let $A = \left(a_{rs}\right)_{r,s=1}^2$ commute with $(\mathrm{id}_k \otimes \pi)(\varphi(e_{ij}))$, $i,j = 1,\dots,k$, and $(\mathrm{id}_k \otimes \pi)(\varphi(z))$. Then

$$[A, (\mathrm{id}_k \otimes \pi)(\varphi(e_{ij}))] = 0, \quad i,j = 1,\dots,k,$$

implies $A = \mathrm{diag}(a_{11},\dots,a_{11})$, and

$$[A, (\mathrm{id}_k \otimes \pi)(\varphi(z))] = 0$$

gives $[a_{11}, \pi(u)] = [a_{11}, \pi(v)] = 0$, proving the fullness of the functor F_φ.

c) Let $n \geq 3$, $k \geq 2$. Define a $*$-homomorphism

$$\varphi : \mathbb{C}^n * M_k(\mathbb{C}) \to M_k(C^*(\mathbb{Z}_n * \mathbb{Z}_n))$$

by the following rule:

$$\varphi(z) = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \oplus (I_{k-2} \otimes e), \quad \varphi(e_{ij}) = e_{ij} \otimes e, \quad i,j = 1,\dots,k.$$

One can directly check, using similar arguments as in item **b)**, that the functor F_φ generated by φ is full. Since $C^*(\mathbb{Z}_n * \mathbb{Z}_n)$ is $*$-wild, see [11], $\mathbb{C}^n * M_k(\mathbb{C})$ is also $*$-wild for $n \geq 3$, $k \geq 2$. □

Proposition 7. *Let $\mathcal{A}$ be a $*$-wild C^*-algebra and $\mathcal{B}$ be a finite-dimensional C^*-algebra. Then $\mathcal{A} * \mathcal{B}$ is $*$-wild.*

Proof. Evidently, it is sufficient to consider only the case $\mathcal{B} = M_n(\mathbb{C})$, $n \in \mathbb{N}$. Construct a homomorphism

$$\phi \colon \mathcal{A} * \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$$

defined by $\phi(a) = 1 \otimes a$, $a \in \mathcal{A}$, $\phi(b) = b \otimes 1$, $b \in \mathcal{B}$.

Let the majorization $\mathcal{A} \succ C^*(\mathcal{F}_2)$ be given by the homomorphism

$$\psi \colon \mathcal{A} \to \mathcal{C} \otimes C^*(\mathcal{F}_2), \quad \widehat{\pi} \colon \mathcal{C} \to B(\mathcal{K}),$$

where $\mathcal{C}$ is a nuclear C^*-algebra and $\widehat{\pi}$ is its irreducible representation. Then the majorization $\mathcal{A} * \mathcal{B}$ is given by the homomorphism

$$(\mathrm{id}_{\mathcal{B}} \otimes \psi) \circ \phi \colon \mathcal{A} * \mathcal{B} \to \mathcal{B} \otimes \mathcal{C} \otimes C^*(\mathcal{F}_2)$$

and the irreducible representation $\mathrm{id}_{\widetilde{B}} \otimes \widehat{\pi} \colon \mathcal{B} \otimes \mathcal{C} \to B(\mathbb{C}^n \otimes \mathcal{K})$. □

Remark 3. The statement of the proposition above remains true if we suppose $\mathcal{B}$ to be a nuclear C^*-algebra.

Now we combine the results of the previous propositions to get a criterion for $*$-wildness of the C^*-free products of finite-dimensional C^*-algebras.

Theorem 1. *The C^*-free product $*_{i \in I} \mathcal{A}_i$ of a family of finite-dimensional C^*-algebras $(\mathcal{A}_i)_{i \in I}$ is $*$-wild if and only if one of the following conditions holds:*

1. *there exist i_1, $i_2 \in I$ such that $\mathcal{A}_{i_1}$ has $M_k(\mathbb{C})$ and $\mathcal{A}_{i_2}$ has $M_n(\mathbb{C})$ as a direct summand with $k > 1$, $n > 1$;*
2. *there exist i_1, $i_2 \in I$ such that $\mathcal{A}_{i_1}$ has $\mathbb{C}^n$ and $\mathcal{A}_{i_2}$ has $\mathbb{C}^k$ as a direct summand with $n \geq 2$, $k \geq 3$;*
3. *there exist i_1, i_2, $i_3 \in I$ such that $\mathcal{A}_{i_1}$ has $\mathbb{C}^2$, $\mathcal{A}_{i_2}$ has $\mathbb{C}^2$, and $\mathcal{A}_{i_3}$ has $\mathbb{C}^2$ as a direct summand;*

4. *there exist* i_1, $i_2 \in I$ *such that* $\mathcal{A}_{i_1}$ *has* $\mathbb{C}^n$ *and* $\mathcal{A}_{i_2}$ *has* $M_k(\mathbb{C})$ *as a direct summand with* $n \geq 2$, $k \geq 2$.

Proof. In fact, the $*$-wildness of the algebras specified in items of the statement follows from Propositions 2,4,5,6,7. The C^*-algebras products which are left aside in the list of concerned C^*-free products are the following:

1. $\mathbb{C}^2 * \mathbb{C}^2$,
2. the C^*-free product of any number of copies of $\mathbb{C}$ and one C^*-algebra $M_n(\mathbb{C})$; evidently, such a product is isomorphic to $M_n(\mathbb{C})$.

The $*$-representations of the C^*-algebra $\mathbb{C}^2 * \mathbb{C}^2$ are described in [11] and this C^*-algebra is of type **1**. $\square$

Corollary 1. $G_1 * G_2$, *where* $G_i \neq \langle e \rangle$, $i = 1, 2$, *are finite groups, is* $*$-*wild unless* $G_i = \mathbb{Z}_2$, $i = 1, 2$.

2. $*$-Wildness of the free product of a nuclear non-commutative C^*-algebra and $C(T)$

2.1. Recall that the C^*-algebra generated by a single unitary element is isomorphic to the algebra $C(\mathbf{T})$ of the continuous functions on the one-dimensional torus $\mathbf{T}$.

Theorem 2. *Let* $\mathcal{A}$ *be non-commutative nuclear* C^**-algebra having a non-trivial projection, say* p. *Then the* C^**-algebraic free product* $\mathcal{A} * C(\mathbf{T})$ *is* $*$*-wild.*

Proof. In the following we denote by u the standard generator of $C(\mathbf{T})$, and denote by u_1, u_2 the standard free generators of $C^*(\mathcal{F}_2)$. Then the needed majorization is given by the homomorphism

$$\varphi\colon \mathcal{A} * C(\mathbf{T}) \to \mathcal{A} \otimes C^*(\mathcal{F}_2)$$

defined by

$$\varphi(x) = x \otimes 1, \quad x \in \mathcal{A}, \qquad \varphi(u) = p \otimes u_1 + (1-p) \otimes u_2,$$

and the irreducible representation $\widetilde{\pi}\colon \mathcal{A} \to B(\mathcal{H})$, where $\dim \mathcal{H} \geq 2$ such that $\widetilde{\pi}(p)$ is non-trivial.

We prove that the functor

$$\mathcal{F}_{\mathcal{A},\varphi,\widetilde{\pi}}\colon \mathrm{Rep} C^*(\mathcal{F}_2) \to \mathrm{Rep} \mathcal{A} * C(\mathbf{T})$$

is full.

Put $\mathcal{F}_{\mathcal{A},\varphi,\widetilde{\pi}}(\pi) := \pi_1$, $\widetilde{\pi}(p) := P$, and $\pi(u_i) = U_i$, $i = 1, 2$. Then

$$\pi_1(u) = P \otimes U_1 P + (1-P) \otimes U_2.$$

Let $\Lambda \in \pi_1(\mathcal{A} * C(\mathbf{T}))'$, then it follows from the irreducibility of $\widetilde{\pi}$ and the inclusion $\mathbf{1} \otimes \widetilde{\pi}(\mathcal{A}) \subset \pi_1(\mathcal{A} * C(\mathbf{T}))$ that $\Lambda = \mathbf{1} \otimes \Lambda_1$. We now show that $\Lambda_1 \in \pi(C^*(\mathcal{F}_2))'$. In fact, from

$$\pi_1(up) = P \otimes U_1, \quad \pi_1(u(1-p)) = (1-P) \otimes U_2,$$

one has

$$P \otimes \Lambda_1 U_1^{\varepsilon} = P \otimes U_1^{\varepsilon} \Lambda_1 \otimes P, \quad (1-P) \otimes \Lambda_1 U_2^{\varepsilon} = (1-P) \otimes U_2^{\varepsilon} \Lambda_1,$$

hence $\Lambda_1 U_i^{\varepsilon} = U_i^{\varepsilon} \Lambda_1$, $\varepsilon \in \{1, *\}$, $i = 1, 2$, and $\Lambda_1 \in \pi(C^*(\mathcal{F}_2))'$. $\square$

Remark 4.

1. In fact, the claim of the proposition remains true if we suppose that $\mathcal{A}$ has an irreducible representation $\widehat{\pi}$ such that $\widehat{\pi}(\mathcal{A})$ is nuclear and contains the non-trivial projection p. Indeed, in this case one can replace

$$\varphi\colon \mathcal{A} * C(\mathbf{T}) \to \mathcal{A} \otimes C^*(\mathcal{F}_2)$$

with

$$\varphi\colon \mathcal{A} * C(\mathbf{T}) \to \widehat{\pi}(\mathcal{A}) \otimes C^*(\mathcal{F}_2)$$

defined as in the proof of Theorem 2 and put $\widetilde{\pi}$ to be the identical representation of $\widehat{\pi}(\mathcal{A})$.

2. It follows form the results of [11, 8] that the C^*-algebra $C(\mathbf{T})$ can be replaced with $C([a,b])$ or $C_0(\mathbb{R})$.

2.2. In the following propositions we present some applications of the above result.

Proposition 8. *Let B be a C^*-algebra generated by partial isometries $s_1, \dots, s_d$ satisfying the relation*

$$\sum_{i=1}^{d} \alpha_i s_i s_i^* = 1.$$

Suppose that there exist irreducible projections P_i, $i = 1, \dots, d$, acting on a finite-dimensional Hilbert space $\mathcal{H}$, $\dim \mathcal{H} \geq 2$, satisfying the relation

$$\sum_{i=1}^{d} \alpha_i P_i = 1. \tag{1}$$

Then B is $*$-wild.

Proof. Consider the C^*-algebra $\mathcal{P}$ generated by the projections P_i, $i = 1, \dots, d$. Construct the homomorphism

$$\varphi\colon B \to \mathcal{P} * C(\mathbf{T})$$

by the formulas

$$\varphi(s_i) = uP_i, \quad i = 1, \dots, d.$$

Evidently, φ is surjective, since $\varphi(s_i^* s_i) = p_i$ and $\varphi(\sum_i \alpha_i s_i) = u$. By Theorem 2 the C^*-algebra $\mathcal{P} * C^*(\mathbf{T})$ is $*$-wild, therefore Proposition 2 implies that B has the same property. □

Consider the C^*-algebra $\mathcal{A}_n^q$ generated by the isometries s_i, $i = 1, \dots, n$, satisfying the q-commutation relations of the form

$$s_i s_j = q s_j s_i, \quad \mid q \mid = 1, \quad i > j.$$

Below we show that $\mathcal{A}_n^q$ is $*$-wild for any q specified above.

Remark 5.

1. It is interesting to note that if we consider the C^*-algebra $\widetilde{B}_n^q$ generated by isometries s_i satisfying relations of the form

$$s_j^* s_i = q s_i s_j^*, \quad \mid q \mid = 1, \quad i > j,$$

then relations $s_i s_j = q s_j s_i$, $i > j$, hold automatically, see [12], and $\mathcal{B}_q$ is nuclear, so it is not $*$-wild.

2. If $\mid q \mid \neq 1$, then there are no isometries satisfying the relation $s_i s_j = q s_j s_i$. Indeed, let s_i, s_j act on a Hilbert space $\mathcal{H}$. Then for any $x \in \mathcal{H}$ one has

$$\|x\| = \|s_i s_j x\| = \mid q \mid^2 \|s_j s_i x\| = \mid q \mid \|x\|.$$

Below we denote by $\mathcal{O}_2$ the Cuntz algebra with two generators, see [2]. Namely

$$\mathcal{O}_2 = C^*(t_1,\ t_2 \mid t_i^* t_i = 1,\ i = 1,2,\ t_1 t_1^* + t_2 t_2^* = 1).$$

Proposition 9. *The C^*-algebra $\mathcal{A}_n^q$ is $*$-wild for any q, $\mid q \mid = 1$.*

Proof. We show that $\mathcal{O}_2 * C([0,1]) \prec \mathcal{A}_n^q$.

Let us construct the homomorphism
$$\varphi\colon \mathcal{A}_n^q \to \mathcal{B}^{\otimes n-1} \otimes \big(\mathcal{O}_2 * C([0,1])\big),$$
where
$$\mathcal{B} = C^*(s,u \mid s^*s = 1,\ u^*u = uu^* = 1,\ us = qsu).$$
Evidently $\mathcal{B}$ is nuclear, since it is the crossed product $\mathcal{B} = \mathcal{T}(C(\mathbf{T})) \rtimes \mathbb{Z}$, where $\mathcal{T}(C(\mathbf{T}))$ is the Toeplitz C^*-algebra. Let c, $0 < c < 1$, denote the generator of $C([0,1])$ and let t_1, t_2 be generators of $\mathcal{O}_2$. Put
$$a_1 := t_1 c, \quad a_2 := t_2(1-c).$$
It is easy to verify that the following relations are satisfied
$$a_1^* a_1 + a_2^* a_2 = 1, \quad a_2^* a_1 = 0. \tag{2}$$
Then φ is defined by following formulas:
$$\varphi(s_i) = \bigotimes_{j<i} u \otimes s \otimes \bigotimes_{i<j\le n} 1, \quad i = 1,\dots,n-1\,,$$
$$\varphi(s_n) = \bigotimes_{1\le j\le n-2} u \otimes (u \otimes a_1 + su \otimes a_2).$$
Further we fix the irreducible representation $\widetilde{\pi}$ of $\mathcal{B}$ acting on $\mathcal{K} = l_2(\mathbb{N})$,
$$\widetilde{\pi}(s) = S, \quad \widetilde{\pi}(u) = D(q), \tag{3}$$
where
$$Se_n = e_{n+1}, \quad D(q)e_n = q^{n-1}e_n, \quad n \in \mathbb{N}.$$
Then the majorization $\mathcal{A}_n^q$ is given by functor the $\mathcal{F}_{\mathcal{B}^{\otimes n-1},\varphi,\widetilde{\pi}}$.

The proof of the fullness of $\mathcal{F}_{\mathcal{B}^{\otimes n-1},\varphi,\widetilde{\pi}}$ is essentially the same as in Theorem 2. For $\pi \in$ ObRep$\mathcal{O}_2 * C([0,1])$, put $\mathcal{F}_{\mathcal{B}^{\otimes n-1},\varphi,\widetilde{\pi}}(\pi) = \pi_1$ and denote $\pi(a_i)$ by A_i, $\pi(t_i) = T_i$, $\pi(c) = C$.

Then the equalities $C^2 = A_1^* A_1$ and $T_i = A_i C^{-1}$, $i = 1,2$, imply that
$$\{A_i, A_i^*,\ i = 1,2\}' = \{T_i, T_i^*, C,\ i = 1,2\}' = \pi(\mathcal{O}_2 \star C([0,1]))'.$$
So one has to show that any $\Lambda \in \pi_1(\mathcal{A}_n^q)'$ has the form $\Lambda = \mathbf{1}^{\otimes n-1} \otimes \Lambda_1$ with $\Lambda_1 \in \{A_i, A_i^*,\ i = 1,2\}'$. $\square$

The following corollary is immediate.

Corollary 2. *The C^*-algebra $\mathcal{A}_n^q$ is not nuclear.*

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