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PERMUTATIONS IN TENSOR PRODUCTS OF FACTORS, AND L^2 COHOMOLOGY OF CONFIGURATION SPACES

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Dedicated to the memory of Yuri Daletskii.

Abstract. We prove that the natural action of permutations in a tensor product of type II factors is free, and compute the von Neumann trace of the projection onto the space of symmetric and antisymmetric elements respectively. We apply this result to computation of von Neumann dimensions of the spaces of square-integrable harmonic forms (L^2-Betti numbers) of N-point configurations in Riemannian manifolds with infinite discrete groups of isometries.

1. Introduction

It is difficult to overestimate the role of the theory of von Neumann algebras and their traces in different areas of mathematics and mathematical physics. One of the important applications is the definition of regularized dimensions of certain infinite-dimensional Hilbert modules, in particular of the spaces of harmonic forms over certain non-compact manifolds. Indeed, let X be a non-compact Riemannian manifold admitting an infinite discrete group G of isometries such that the quotient $M = X/G$ is compact. Then G acts by isometries on the spaces $L^2\Omega^m(X)$ of square-integrable m-forms over X. The projection P_m onto the space $\mathcal{K}^m(X)$ of square-integrable harmonic m-forms (the m-th L^2cohomology space) commutes with the action of G and thus belongs to the commutant $\mathcal{A}_m$ of this action, which is a von Neumann algebra. The corresponding von Neumann trace of P_m gives a regularized dimension of the space $\mathcal{K}^m(X)$ and is called the L^2-Betti number. L^2-Betti numbers were introduced in [4] and have been studied by many authors (see e.g. [20] and references given there).

Thus an important problem is construction of von Neumann algebras containing particular projections, and computation of the corresponding traces of these projections. In particular, let us consider the space $X^{(N)}$ of all N-point subsets (configurations) of X. Such spaces have been actively studied by geometers and topologists, see e.g. [11]. They play a significant role in the quantum field theory ([13], [14]), representation theory ([25], [15]) and statistical mechanics ([23], [12]), see also references given in these works. Clearly,

$$X^{(N)} = X \widetilde{\times \cdots \times} X/S_N\,, \tag{1}$$

where $X \widetilde{\times \cdots \times} X$ is the Cartesian product of N copies of X without coinciding components and S_N is the symmetric group. $X^{(N)}$ is a Riemannian manifold equipped with the Riemannian structure induced from X. The space $\mathbf{K}^{(p)}$ of square-integrable harmonic p-forms on $X^{(N)}$ can be described in terms of symmetric and antisymmetric tensor products of the spaces $\mathcal{K}^{(m)}(X)$, $m = 1, 2, \ldots, p$ (a "symmetrized" version of the Künneth formula). If X is as above, the construction of a von Neumann algebra containing the

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projection $\mathbf{P}_p$ onto $\mathbf{K}^{(p)}$ and computation of the trace of $\mathbf{P}_p$ involves the study of the algebras $\{\mathcal{A}_m^{\otimes n}, P_s\}''$ and $\{\mathcal{A}_m^{\otimes n}, P_a\}''$, $n = 2, \ldots, p$, where P_s and P_a are the projections onto the spaces of symmetric and antisymmetric elements respectively, $\otimes n$ denoting the n-th tensor power.

More generally, let $\mathcal{M}$ be a von Neumann algebra acting in a separable Hilbert space H. An interesting problem is to describe of the structure of the von Neumann algebra $\{\mathcal{M}^{\otimes n}, \alpha\}''$, where α is the natural action of the symmetric group S_n by permutations in $H^{\otimes n}$. It is clear that the answer is spatial dependent, i. e. it depends on the choice of the concrete $\mathcal{M}$-module H. For example, if $H = \mathbb{C}^m$ is the module over the I_m-factor $\mathcal{M} = M_m(\mathbb{C})$, then (for $n = 2$) $\mathcal{M} \otimes \mathcal{M}$ coincides with the space of all linear operators in $\mathbb{C}^m \otimes \mathbb{C}^m$. Therefore the permutation operator U belongs to $\mathcal{M} \otimes \mathcal{M}$ and

$$\{\mathcal{M} \otimes \mathcal{M}, U\}'' = \mathcal{M} \otimes \mathcal{M}. \tag{2}$$

However, if the same I_m factor $\mathcal{M}$ acts on its standard form $H = \mathbb{C}^m \otimes \mathbb{C}^m$ by operators $x(f \otimes g) = xf \otimes g,\ x \in \mathcal{M},\ f, g \in \mathbb{C}^m$, then U does not belong to $\mathcal{M} \otimes \mathcal{M}$. Thus U induces an outer action α of the symmetric group S_2 on the factor $\mathcal{M} \otimes \mathcal{M}$, and the von Neumann algebra $\{\mathcal{M} \otimes \mathcal{M}, U\}''$ is isomorphic to the cross-product $\mathcal{M} \otimes \mathcal{M} \times_\alpha S_2$.

In this paper, we consider the case where H is a separable module over a type II factor $\mathcal{M}$. In Section 2, we prove that the action α of the group S_n in $\mathcal{M}^{\otimes n}$ generated by the representation U is outer and free and thus there exists an isomorphism

$$\{\mathcal{M}^{\otimes n}, \{U_i\}_{i=1}^{n-1}\}'' \simeq \mathcal{M}^{\otimes n} \times_\alpha S_n, \tag{3}$$

where U_i is an operator in $H^{\otimes n}$ that permutes i-th and $i+1$-th components. We compute the von Neumann trace of the projections P_s and P_a onto the spaces of symmetric and antisymmetric elements of $H^{\otimes n}$ respectively. Moreover we show that the factors $\{\mathcal{M}^{\otimes n}, P_s\}''$, $\{\mathcal{M}^{\otimes n}, P_a\}''$ and $\{\mathcal{M}^{\otimes n}, \{U_i\}_{i=1}^{n-1}\}''$ are isomorphic. In Section 3 we use the finite dimensional approximation of $\mathcal{M}$ and prove that the obtained formulae are consistent with the well-known formulae of the dimensions of symmetric and antisymmetric tensor products of finite dimensional spaces. In Section 4, we apply these results to computation of the von Neumann dimensions of the spaces of square-integrable harmonic forms (L^2-cohomologies) of the spaces of N-point configurations in Riemannian manifolds with infinite discrete groups of isometries (L^2-Betti numbers of configuration spaces).

Let us remark that the results of the first two sections are valid for general type II modules. In the particular case where $\mathcal{M}$ is the commutant of a free action of an infinite discrete group, similar results were obtained (by different methods) in [9] ($n = 2$) and [2] (n arbitrary). In the latter work, the L^2-Betti numbers of infinite configuration spaces Γ_X equipped with Poisson measures were computed. Geometry and analysis on the infinite configuration spaces, which naturally appear in different problems of statistical mechanics and quantum physics, have been a very active topic of research in recent years, see references in [2]. Unlike $X^{(N)}$, Γ_X does not possess a natural manifold structure, and many objects on Γ_X are studied via the limit transition from $X^{(N)}$ as $N \to \infty$.

The preliminary version of this paper was published in [8]. Here, we give more detailed proofs and include important result on isomorphism of factors $\{\mathcal{M}^{\otimes n}, P_s\}''$, $\{\mathcal{M}^{\otimes n}, P_a\}''$ and $\mathcal{M} \otimes \mathcal{M} \times_\alpha S_n$ (Theorem 4) and on finite dimensional approximation (Theorem 6).

In what follows we denote by $\mathcal{L}(H)$ the algebra of all bounded operators in Hilbert space H. We refer to [7], [24] for general notions of the theory of von Neumann algebras.

2. Permutations in a tensor product of type II factors

Let $L^2(\mathcal{M})$ be the standard form of a finite factor $\mathcal{M}$, that is, the completion of M in the norm generated by the canonical trace on M. Denote by Ω the corresponding cyclic and separating vector for $\mathcal{M}$ (recall that separating means that $z\Omega = 0$, $z \in \mathcal{M}$ implies

$z = 0$). Let τ be a faithful normal trace on $\mathcal{M}$. Since $(\mathcal{M} \otimes \mathcal{M})' = \mathcal{M}' \otimes \mathcal{M}'$, $\mathcal{M} \otimes \mathcal{M}$ is a finite factor acting in Hilbert space $L^2(\mathcal{M}) \otimes L^2(\mathcal{M})$. Let U be the permutation operator in $L^2(\mathcal{M}) \otimes L^2(\mathcal{M})$. Denote by α_U the corresponding automorphism of the factor $\mathcal{M} \otimes \mathcal{M}$,

$$\alpha_U(x \otimes y) = U(x \otimes y)U^* = y \otimes x, \quad x, y \in \mathcal{M}. \tag{4}$$

This automorphism generates a natural action α of the group S_2 on $\mathcal{M} \otimes \mathcal{M}$. Recall that the action of an automorphism β on $\mathcal{M}$ is called free, if each element $x \in \mathcal{M}$ satisfying the equality $xy = \beta(y)x$ for all $y \in \mathcal{M}$ is zero. It is well known that an automorphism of a factor is free iff it is outer (that is, not generated by automorphisms of the form $x \mapsto uxu^*$, where u is unitary in M). If $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ is a free action of a discrete group G on $\mathcal{M}$ then the cross-product $\mathcal{M} \times_\alpha G$ is a factor (see e.g. [16], Proposition 1.4.4).

We have the following statement.

Proposition 1.

(i). *The Hilbert space $L^2(\mathcal{M}) \otimes L^2(\mathcal{M})$ is the standard form of the factor $\mathcal{M} \otimes \mathcal{M}$;*

(ii). *The action α of the group S_2 on $\mathcal{M} \otimes \mathcal{M}$ is free;*

(iii). *There exists a natural isomorphism of the finite factor $\mathcal{M} \otimes \mathcal{M} \times_\alpha S_2$ and the von Neumann algebra $\{\mathcal{M} \otimes \mathcal{M}, U\}''$.*

Proof. Note that the vector $\Omega_1 = \Omega \otimes \Omega$ is cyclic for both $\mathcal{M} \otimes \mathcal{M}$ and $(\mathcal{M} \otimes \mathcal{M})' = \mathcal{M}' \otimes \mathcal{M}'$. Hence Ω_1 is separating for $\mathcal{M} \otimes \mathcal{M}$.

Denote $\tau_1 = \tau \otimes \tau$. It is obvious that τ_1 is a trace on $\mathcal{M} \otimes \mathcal{M}$. Moreover the trace τ_1 is faithful on $\mathcal{M} \otimes \mathcal{M}$. Indeed, since Ω_1 is separating for $\mathcal{M} \otimes \mathcal{M}$, for $x = \sum_i x_i \otimes y_i \in \mathcal{M} \otimes \mathcal{M}$ we have

$$\begin{aligned} \tau_1(x^* x) &= \sum_i \tau(x_i^* x_k) \tau(y_i^* y_k) \\ &= \sum_i (x_i^* x_k \Omega, \Omega)_{L^2(\mathcal{M})} (y_i^* y_k \Omega, \Omega)_{L^2(\mathcal{M})} \\ &= (x^* x \Omega_1, \Omega_1)_{L^2(\mathcal{M}) \otimes L^2(\mathcal{M})} = \|x\Omega_1\|^2 \neq 0. \end{aligned} \tag{5}$$

The equality

$$\tau_1(x) = (x\Omega_1, \Omega_1)_{L^2(\mathcal{M}) \otimes L^2(\mathcal{M})} \tag{6}$$

for any $x \in \mathcal{M} \otimes \mathcal{M}$ implies that $L^2(\mathcal{M}) \otimes L^2(\mathcal{M})$ is the standard form of $\mathcal{M} \otimes \mathcal{M}$.

Let us show that α_U given by (4) is a nontrivial outer automorphism of $\mathcal{M} \otimes \mathcal{M}$, i. e. that the operator U does not belong to $(\mathcal{M} \otimes \mathcal{M}) \cup (\mathcal{M} \otimes \mathcal{M})'$. Suppose that $U \in \mathcal{M} \otimes \mathcal{M}$. Rewrite the equality $U\Omega \otimes \Omega = \Omega \otimes \Omega$ in the form $(U - 1)\Omega_1 = 0$. Moreover Ω_1 is a separating vector for $\mathcal{M} \otimes \mathcal{M}$, which implies that $U = 1$. Thus $U \notin \mathcal{M} \otimes \mathcal{M}$. It can be shown by similar arguments that $U \notin (\mathcal{M} \otimes \mathcal{M})'$.

Since $\mathcal{M} \otimes \mathcal{M}$ is a factor, it is known (see e.g. [16], Proposition 1.4.4) that

$$(\mathcal{M} \otimes \mathcal{M})' \cap (\mathcal{M} \otimes \mathcal{M} \times_\alpha S_2) = \mathbb{C}. \tag{7}$$

This implies in particular that the crossed product $\mathcal{M} \otimes \mathcal{M} \times_\alpha S_2$ is also a finite factor. We conclude from the equality $\alpha_U(x)\Omega_1 = Ux\Omega_1$, $x \in \mathcal{M} \otimes \mathcal{M}$, that the map

$$\mathcal{M} \otimes \mathcal{M} \times_\alpha S_2 \ni (x, \sigma) \mapsto (x, u) \in \{\mathcal{M} \otimes \mathcal{M}, U\}'' \tag{8}$$

is a surjective homomorphism of $\mathcal{M} \otimes \mathcal{M} \times_\alpha S_2$ onto $\{\mathcal{M} \otimes \mathcal{M}, U\}''$. Moreover, since finite factors do not contain two-sided ideals, any normal homomorphism between them is either identically zero or injective. Hence factors $\mathcal{M} \otimes \mathcal{M} \times_\alpha S_2$ and $\{\mathcal{M} \otimes \mathcal{M}, U\}''$ are isomorphic. □

Now we can consider the case where $\mathcal{M}$ is a type II factor. Let H be a separable $\mathcal{M}$-module. Denote by U the operator of permutation in $H \otimes H$ and let α_U be the corresponding (nontrivial) automorphism of $\mathcal{M} \otimes \mathcal{M}$.

Theorem 2. *The automorphism α_U defines an outer action of the group S_2 on the* II*-factor $\mathcal{M}\otimes\mathcal{M}$, and there exists an isomorphism of factors*

$$\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2 \simeq \{\mathcal{M}\otimes\mathcal{M}, U\}''. \tag{9}$$

Proof. II_1 **case**. Let $\mathcal{M}$ be a II_1-factor. Denote by K the standard form of $\mathcal{M}$. Using the theorem on the structure of normal isomorphisms of von Neumann algebras [24] we can conclude that H as $\mathcal{M}$-module is isomorphic to $\mathcal{M}$-module

$$H_d = p(K\otimes l_2) \tag{10}$$

for some $d\in[0,\infty]$, where $p\in\mathcal{M}'\otimes\mathcal{L}(l_2)$ is a projection with $\mathrm{Tr}\, p = d$. Here Tr denotes the natural trace in $\mathcal{M}'\otimes\mathcal{L}(l_2)$, with the normalization $\mathrm{Tr}(1_{\mathcal{M}}\otimes q) = 1$, where q is a projection of rank 1 in $\mathcal{L}(l_2)$. The action of $\mathcal{M}$ on H_d is given by

$$x(p(f\otimes\xi)) = p(xf\otimes\xi), \quad x\in\mathcal{M}, \quad f\in K, \quad \xi\in l_2. \tag{11}$$

Let us remark that the Hilbert spaces $K\otimes l_2\otimes K\otimes l_2$ and $K\otimes K\otimes l_2\otimes l_2$ are isomorphic. Thus there exists a projection $\tilde{p}$ such that $\mathcal{M}\otimes\mathcal{M}$-modules

$$H_d\otimes H_d = (p\otimes p)(K\otimes l_2\otimes K\otimes l_2) \tag{12}$$

and

$$\tilde{p}(K\otimes K\otimes l_2\otimes l_2) \tag{13}$$

are isomorphic, where the action of $\mathcal{M}\otimes\mathcal{M}$ on the latter space is defined by

$$(x\otimes y)(\tilde{p}(f\otimes g\otimes\xi\otimes\eta)) = \tilde{p}(xf\otimes yg\otimes\xi\otimes\eta), \tag{14}$$

$x, y\in\mathcal{M}, f, g\in K, \xi, \eta\in l_2$. The operator U of permutation in $H_d\otimes H_d$ is unitarily isomorphic to the operator $U_1\otimes U_2$ in $\tilde{p}(K\otimes K\otimes l_2\otimes l_2)$, where U_1 and U_2 are the operators of permutation in $K\otimes K$ and $l_2\otimes l_2$ respectively. It follows from Proposition 1 that $U_1\notin\mathcal{M}\otimes\mathcal{M}$. Hence the operator U does not belong to $\mathcal{M}\otimes\mathcal{M}$ and thus α_U is outer (and consequently, free) automorphism of $\mathcal{M}\otimes\mathcal{M}$. Repeating the arguments from the proof of the Proposition 1 we conclude that the factors $\mathcal{M}\otimes\mathcal{M}\times S_2$ and $\{\mathcal{M}\otimes\mathcal{M}, U\}''$ are isomorphic.

II_∞ **case**. Let $\mathcal{M}$ be a II_∞ factor. Fix an arbitrary finite projection $p\in\mathcal{M}$. Then there exists [24] a spatial isomorphism of $\mathcal{M}$ and the II_∞ factor $\mathcal{M}_p\otimes\mathcal{L}(l_2)$, where $\mathcal{M}_p = p\mathcal{M}p$ (the so-called "corner" of $\mathcal{M}$) is a II_1 factor. Denote $H_p = pH$. Then the II_∞ factor $\mathcal{M}\otimes\mathcal{M}$ is isomorphic to $\mathcal{M}_p\otimes\mathcal{M}_p\otimes\mathcal{L}(l_2\otimes l_2)$ and the permutation operator U in $H\otimes H$ is unitarily equivalent to the operator $U_1\otimes U_2$, where U_1 and U_2 are the operators of permutation in $H_p\otimes H_p$ and $l_2\otimes l_2$ respectively. Note that the operator U_2 belongs to $\mathcal{L}(l_2\otimes l_2)$.

It follows from the arguments presented above that the operator U_1 does not belong to $\mathcal{M}_p\otimes\mathcal{M}_p$. Thus the operator U does not belong to $\mathcal{M}\otimes\mathcal{M}$ and as above α is a free automorphism of $\mathcal{M}\otimes\mathcal{M}$. Therefore $\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2$ is a II_∞ factor. It follows from the arguments presented in the proof of Proposition 1 that there exists a normal homomorphism of $\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2$ onto $\{\mathcal{M}\otimes\mathcal{M}, U\}''$, which is in fact an isomorphism (since any normal homomorphism between factors is either identically zero or injective). Thus we have that $\{\mathcal{M}\otimes\mathcal{M}, U\}''$ is also a II_∞ factor isomorphic to $\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2$. □

The following result is the extension of the theorem above to the case of the symmetric group S_n acting in $\mathcal{M}^{\otimes n}$, $n\geq 2$.

Theorem 3. *Let $\mathcal{M}$ be a type* II *factor and H be a separable $\mathcal{M}$-module. Let U_{ij} $(i, j = 1, \ldots, n)$ be the operator in $H^{\otimes n}$ permuting i-th and j-th components. Then the family of operators $\{U_{ij}\}_{i,j=1}^n$ defines an outer action of the symmetric group S_n on the factor $\mathcal{M}^{\otimes n}$, and there exists an isomorphism*

$$\mathcal{M}^{\otimes n}\times_\alpha S_n \simeq \{\mathcal{M}^{\otimes n}, \{U_{ij}\}_{i,j=1}^n\}''. \tag{15}$$

Proof. It follows from Theorem 2 that the operator U_{ij} does not belong to the factor $\mathcal{M}^{\otimes n}$ and therefore determines the outer automorphism $\alpha_{U_{ij}}$ of $\mathcal{M}^{\otimes n}$. It is obvious that the action of the symmetric group S_n on the factor $\mathcal{M}^{\otimes n}$ generated by automorphisms $\alpha_{U_{ij}}$, $i,j=1,\dots,n$ is free. Therefore the factors $\mathcal{M}^{\otimes n}\times_\alpha S_n$ and $\{\mathcal{M}^{\otimes n},\{U_{ij}\}_{i,j=1,\dots,n}\}''$ are isomorphic (see the proof of Theorem 2). □

Let P_s and P_a be projections in $H^{\otimes n}$ onto symmetric tensor product $H^{\hat{\otimes} n}$ and anti-symmetric tensor product $H^{\wedge n}$ respectively,

$$P_s=\frac{1}{n!}\sum_{g\in S_n}U_g \tag{16}$$

and

$$P_a=\frac{1}{n!}\sum_{g\in S_n}(-1)^{\mathrm{sign}(g)}U_g. \tag{17}$$

It is obvious that P_s and P_a belong to $\mathcal{M}_n\times_\alpha S_n$.

Denote

$$\mathcal{M}_s=\{\mathcal{M}^{\otimes n},P_s\}'',\quad \mathcal{M}_a=\{\mathcal{M}^{\otimes n},P_a\}''. \tag{18}$$

Theorem 4. *Let $\mathcal{M}$ be a type* II *factor. Then*

$$\mathcal{M}_s=\mathcal{M}_a=\mathcal{M}^{\otimes n}\times_\alpha S_n. \tag{19}$$

Proof. II$_1$ **case**. Let $\mathcal{M}$ be a II$_1$ factor. The inclusions $\mathcal{M}^{\otimes n}\subset\mathcal{M}_s\subset\mathcal{M}^{\otimes n}\times_\alpha S_n$ of II$_1$ factors are obvious. Then by virtue of Proposition A.4.2 in [16] there exists a subgroup G of S_n such that

$$\mathcal{M}_s=\mathcal{M}^{\otimes n}\times_\alpha G. \tag{20}$$

It is known that the index $[\mathcal{M}_s:\mathcal{M}^{\otimes n}]$ of the inclusion $\mathcal{M}^{\otimes n}\subset\mathcal{M}_s$ is given by

$$[\mathcal{M}_s:\mathcal{M}^{\otimes n}]=|G|, \tag{21}$$

where $|G|$ is the number of elements in G. Moreover, the collection $\{U_g|g\in G\}$ is a basis for $\mathcal{M}_s/\mathcal{M}^{\otimes n}$, thus every element $x\in\mathcal{M}_s$ can be decomposed into the sum

$$x=\sum_{g\in G}E_{\mathcal{M}^{\otimes n}}(xU_g^*)U_g, \tag{22}$$

where

$$E_{\mathcal{M}^{\otimes n}}:\mathcal{M}_s\to\mathcal{M}^{\otimes n} \tag{23}$$

is the canonical conditional expectation arising in the basic construction of towers of factors (see [16, Proposition 4.3.3]). Then since $P_sU_g=P_s$ for each $g\in G$ we have

$$\begin{aligned}P_s&=\sum_{g\in G}E_{\mathcal{M}^{\otimes n}}(P_sU_g^*)U_g=\sum_{g\in G}E_{\mathcal{M}^{\otimes n}}(P_s)U_g\\&=E_{\mathcal{M}^{\otimes n}}(P_s)\sum_{g\in G}U_g=\frac{1}{|G|}\sum_{g\in G}U_g\end{aligned} \tag{24}$$

(note that the equality $E_{\mathcal{M}^{\otimes n}}(P_s)=[\mathcal{M}_s:\mathcal{M}^{\otimes n}]^{-1}\mathbf{1}_{H^{\otimes n}}=|G|^{-1}\mathbf{1}_{H^{\otimes n}}$ follows from the Proposition 3.1.2 in [16]). Comparing formulae (24) and (16) we obtain $G=S_n$.

The case of $\mathcal{M}_a$ can be treated in the similar way using the relation $P_aU_g=(-1)^{\mathrm{sign}(g)}P_a$.

II$_\infty$ **case**. Let $\mathcal{M}$ be a II$_\infty$ factor. The inclusion $\mathcal{M}_s\subset\mathcal{M}^{\otimes n}\times_\alpha S_n$ is obvious. For an inverse inclusion it suffices to show that the operators $P_{ij}=\frac{1}{2}(1+U_{ij})\ \ i,j=1,\dots,n$ belong to $\mathcal{M}_s$. Since $\mathcal{M}\simeq\mathcal{M}_p\otimes\ \mathcal{L}(l_2)$, the factor $\mathcal{M}$ contains an isometry V such that $(V^*)^m\to 0,\ m\to\infty$ strongly (for example $V=1\otimes W$ where W is unilateral shift in

l_2: $We_k = e_{k+1}$ for a standard basis $\{e_k\}_{k=1}^{\infty}$ in l_2). Since $(V^*)^m V^m = 1$ we have strong convergence

$$(1 \otimes 1 \otimes (V^*)^m \otimes \cdots \otimes (V^*)^m) P_s (1 \otimes 1 \otimes V^m \otimes \cdots \otimes V^m) \to \frac{2}{n!} P_{12}, \tag{25}$$

$m \to \infty$. Thus $P_{12} \in \mathcal{M}_s$. Similar arguments show that $P_{ij} \in \mathcal{M}_s$ for any $i, j = 1, \ldots, n$.
The case of $\mathcal{M}_a$ can be treated in a completely similar way. □

Remark 1. The proof of the theorem above is essentially different for II_1 and II_∞ Indeed, Proposition A.4.2 in [16] does not in general hold for II_∞ factors (cf. (20)). In the latter case, our considerations are based on the existence of the isometry $V \in M$.This is certainly wrong for II_1 factors, which do not contain any isometries.

In what follows we denote by $\mathrm{Tr}_{\mathcal{N}}$ the faithful normal finite resp. semifinite trace on a II_1 resp. II_∞ factor $\mathcal{N}$.

Corollary 5. *Let $\mathcal{M}$ be a type* II *factor. Then for any $A \in \mathcal{M}$ we have*

$$\mathrm{Tr}_{\mathcal{M}_s}(A^{\otimes n} P_s) = \mathrm{Tr}_{\mathcal{M}_a}(A^{\otimes n} P_a) = \frac{(\mathrm{Tr}_{\mathcal{M}} A)^n}{n!}. \tag{26}$$

Proof. According to Theorem 4 we will use the faithful normal finite (or semifinite) trace on the type II factor $\mathcal{M}^{\otimes n} \times_\alpha S_n$. It is obvious that $\alpha_g(A^{\otimes n}) = A^{\otimes n}$ for any $g \in S_n$. Therefore for any $g \in S_n$ we have

$$\mathrm{Tr}_{\mathcal{M}^{\otimes n} \times_\alpha S_n}(A^{\otimes n} U_g) = \delta_{e,g} \mathrm{Tr}_{\mathcal{M}^{\otimes n}}(A^{\otimes n}) \tag{27}$$

(here $\delta_{g,h}$ is the Kronecker symbol). Then (26) follows from (16) and (17). □

3. Finite dimensional approximation

We first recall well known formulae for the traces of projections P_s and P_a onto symmetric and antisymmetric n-th tensor powers $H^{\hat{\otimes} n}$ and $H^{\wedge n}$ of d-dimensional Hilbert space $H = \mathbb{C}^d$ respectively:

$$\begin{aligned} \mathrm{Tr} P_s &= \frac{d(d+1)\cdots(d+n-1)}{n!}, \\ \mathrm{Tr} P_a &= \frac{d(d-1)\cdots(d-n+1)}{n!}. \end{aligned} \tag{28}$$

Here by Tr we denote the usual trace in $\mathcal{L}(H) = M_d(\mathbb{C})$.

We shall discuss the relationship between formulae (26) and (28). The first question is whether they are consistent. In case when $\mathcal{M} = \mathcal{R}$ is a hyperfinite II_1 factor, positive answer can be given by a finite dimensional approximation. Indeed, $\mathcal{R}$ is a weak closure of the union of an increasing sequence

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_m \subset \cdots, \tag{29}$$

where $\mathcal{A}_m = \otimes_1^m M_N(\mathbb{C}) = M_{N^m}(\mathbb{C})$. Let us denote by $\mathrm{Tr}_{M_d(\mathbb{C})}$ and $\mathrm{Tr}_{\mathcal{R}}$ the traces on factors $M_d(\mathbb{C})$ and $\mathcal{R}$ normalized by the conditions $\mathrm{Tr}_{M_d(\mathbb{C})}(1) = \frac{1}{d}$ and $\mathrm{Tr}_{\mathcal{R}}(1) = 1$ respectively. Let $P_m \in \mathcal{A}_m$ be a sequence of projections weakly converging to some $P \in \mathcal{R}$. Then

$$\lim_{m \to \infty} \mathrm{Tr}_{\mathcal{A}_m}(P_m) = \mathrm{Tr}_{\mathcal{R}}(P). \tag{30}$$

It is clear that all permutation operators U_g, $g \in S_n$, in corresponding tensor products of finite-dimensional Hilbert spaces, belong to $\mathcal{A}_m^{\otimes n} = M_{N^{nm}}(\mathbb{C})$ (cf. (2)). Therefore the projections P_s^m and P_a^m defined by formulae (16) and (17) respectively belong to

$\mathcal{A}_m^{\otimes n}$. Since the projections $P_m^{\otimes n}$ and P_s^m (resp. P_a^m) commute, we can define projections $P_m^{\otimes n} P_s^m$ and $P_m^{\otimes n} P_a^m$. Then, according to (28),

$$\begin{aligned} \operatorname{Tr}_{\mathcal{A}_m}(P_m^{\otimes n} P_s^m) &= \frac{1}{n!}\frac{\operatorname{Tr}(P_m)(\operatorname{Tr}(P_m)+1)\cdots(\operatorname{Tr}(P_m)+n-1)}{N^{mn}}, \\ \operatorname{Tr}_{\mathcal{A}_m}(P_m^{\otimes n} P_a^m) &= \frac{1}{n!}\frac{\operatorname{Tr}(P_m)(\operatorname{Tr}(P_m)-1)\cdots(\operatorname{Tr}(P_m)-n+1)}{N^{mn}}. \end{aligned} \tag{31}$$

Theorem 6.

$$\operatorname{Tr}_{\mathcal{R}_s}(P^{\otimes n} P_s) = \lim_{m\to\infty} \operatorname{Tr}_{\mathcal{A}_m}(P_m^{\otimes n} P_s^m) \tag{32}$$

and

$$\operatorname{Tr}_{\mathcal{R}_m}(P^{\otimes n} P_a) = \lim_{m\to\infty} \operatorname{Tr}_{\mathcal{A}_m}(P_m^{\otimes n} P_a^m). \tag{33}$$

Proof. We have according to (31) and (26),

$$\begin{aligned} \lim_{m\to\infty} \operatorname{Tr}_{\mathcal{A}_m}(P_m^{n} P_s^m) &= \lim_{m\to\infty}\left(\frac{1}{n!}\frac{\operatorname{Tr}(P_m)(\operatorname{Tr}(P_m)+1)\cdots(\operatorname{Tr}(P_m)+n-1)}{N^{mn}}\right) \\ &= \frac{(\operatorname{Tr}_{\mathcal{R}}(P))^n}{n!} = \operatorname{Tr}_{\mathcal{R}}(P^{\otimes n} P_s) \end{aligned} \tag{34}$$

because $\frac{\operatorname{Tr}(P_m)}{N^m} = \operatorname{Tr}_{\mathcal{A}_m}(P_m) \to \operatorname{Tr}_{\mathcal{R}}(P)$, $m \to \infty$. Formula (33) can be obtained by similar arguments. □

Remark 2. Using the similar arguments one can prove the analogue of Theorem 6 for the II_∞ factor $\mathcal{R} \otimes L(l^2)$.

4. L^2 cohomology of N-point configuration spaces

In this section, we apply the results described above to computation of L^2-Betti numbers of the spaces of N-point configurations in the manifolds possessing infinite groups of isometries. We start with the discussion of the structure of square-integrable differential forms over configuration spaces.

Let X be a smooth connected Riemannian manifold. Consider the N-point configuration space

$$X^{(N)} := \{\{x_1, \ldots, x_N\} \subset X\}, \tag{35}$$

the set of all N-point subsets of X. Clearly,

$$X^{(N)} = X \widetilde{\times \cdots \times} X / S_N, \tag{36}$$

where $X \widetilde{\times \cdots \times} X$ is the Cartesian product of N copies of X without coinciding components. $X^{(N)}$ is a Riemannian manifold equipped with the Riemannian structure induced from X.

For a Riemannian manifold R, we denote by $L^2\Omega^p(R)$ the space of square-integrable (w.r.t. the Riemannian volume) p-forms on R, and let Δ_R^p to be the Hodge-de Rham Laplacian in $L^2\Omega^p(R)$. Consider the space

$$\mathcal{H}^p(R) := \operatorname{Ker} d_p / \overline{\operatorname{Im} d_{p-1}} \tag{37}$$

of (reduced) L^2-cohomologies of R [4]. Here

$$d_j : L^2\Omega^j(R) \to L^2\Omega^{j+1}(R), \tag{38}$$

$j = 0, 1, \ldots, d$, is the Hodge differential of R. It is known [10] that the spaces $\mathcal{H}^p(R)$ and $\mathcal{K}^p(R) = \operatorname{Ker}\Delta_R^p$ are isomorphic, and we will identify them.

Remark 3. The isomorphism of $\mathcal{H}^p(R)$ and $\mathrm{Ker}\Delta_R^p$ is due to the weak Hodge–de Rham decomposition

$$L^2\Omega^p(R) = \mathrm{Ker}\Delta_R^p \oplus \overline{\mathrm{Im}\, d_{p-1}} \oplus \overline{\mathrm{Im}\, d_p^*}, \tag{39}$$

which follows from the general operator theory in Hilbert spaces. This fact is in general simpler than the Hodge–de Rham decomposition of smooth forms.

For a Hilbert space $\mathcal{K}$, we use the notation

$$\mathcal{K}^{\overset{k}{\diamond} s} = \begin{cases} \mathcal{K}^{\widehat{\otimes} s}, & k \text{ is even}, \\ \mathcal{K}^{\wedge s}, & k \text{ is odd}. \end{cases} \tag{40}$$

Let $d = \dim X$. The following result is a symmetrized version of the Künneth formula.

Theorem 7. *For $p \le dN$ we have a natural unitary isomorphism*

$$\mathcal{H}^p(X^{(N)}) \simeq \bigoplus_{\substack{s_0, s_1, \ldots, s_d = 0,1,2,\ldots \\ \sum_k s_k = N,\ \sum_k k s_k = p}} \bigotimes_{m=0}^{d} (\mathcal{H}^m(X))^{\overset{m}{\diamond} s_m} \tag{41}$$

and

$$\mathcal{H}^p(X^{(N)}) = \emptyset, \quad p > dN. \tag{42}$$

Proof. $X^{(N)}$ is an dN-dimensional manifold, which implies that $\mathcal{H}^p(X^{(N)}) = \emptyset$ if $p > dN$.

Let $p \le dN$. Remark that the space $L^2\Omega^p(X^{(N)})$ is unitarily isomorphic to $L^2\Omega^p_{\mathrm{sym}}(X^N)$, the latter being the space of square-integrable p-forms on $X^N := \overbrace{X \times \cdots \times X}^{N}$ which are invariant w.r.t. the permutations of variables. It is easy to see that there exists a natural unitary isomorphism

$$L^2\Omega^p_{\mathrm{sym}}(X^N) \simeq \bigoplus_{s_0, s_1, \ldots, s_d = 0,1,2,\ldots} \bigotimes_{k=0}^{d} (L^2\Omega^k(X))^{\overset{k}{\diamond} s_k}, \tag{43}$$

where $\sum_{k=0}^d s_k = N$, $\sum_{k=0}^d k s_k = p$. It has been proved in [1] that the restriction of $\Delta^p_{X^N}$ onto $L^2\Omega^p_{\mathrm{sym}}(X^N)$ is essentially self adjoint on the space of smooth forms with compact support, and coincides on this space with $\Delta^p_{X^{(N)}}$. Thus we have

$$\mathcal{H}^p(X^{(N)}) = \mathrm{Ker}(\Delta^p_{X^N})_{\mathrm{sym}} = \mathrm{Ker}(\Delta^p_{X^N}) \cap L^2\Omega^p_{\mathrm{sym}}(X^N). \tag{44}$$

By the Künneth formula,

$$\mathrm{Ker}(\Delta^p_{X^N}) \simeq \bigoplus_{\substack{k_1, \ldots, k_N = 0,1,2,\ldots,d \\ k_1 + \cdots + k_N = p}} \mathcal{H}^{k_1}(X) \otimes \cdots \otimes \mathcal{H}^{k_N}(X). \tag{45}$$

This together with formula (43) imply the result. □

Corollary 8. *If all the spaces $\mathcal{H}^k(X)$ are finite dimensional, then all the spaces $\mathcal{H}^p(X^{(N)})$ are so. Their dimensions are given by the following formula:*

$$\dim \mathcal{H}^p(X^{(N)}) = \sum_{\substack{s_1, \ldots, s_d = 0,1,2,\ldots \\ \sum_k s_k = N,\ \sum_k k s_k = p}} \prod_{m=0}^{d} \beta_m^{(s_m)}, \quad p \le d\,N, \tag{46}$$

and

$$\dim \mathcal{H}^p(X^{(N)}) = 0, \quad p > d\,N, \tag{47}$$

where

$$\beta_k^{(s)} := \begin{cases} \begin{pmatrix} \beta_k \\ s \end{pmatrix}, & k = 1, 3, \dots\ , \\ \begin{pmatrix} \beta_k + s - 1 \\ s \end{pmatrix}, & k = 2, 4, \dots \end{cases} \tag{48}$$

$s \neq 0$, and $\beta_k^{(s)} = 1$ for $s = 0$. Here $\beta_k := \dim \mathcal{H}^k(X)$, $k = 0, 1, \dots, d$.

This case occurs for instance when X is compact or has finite number of ends, as in the following example.

Example 1. Let X be a manifold with a cylindrical end (that is, $X = M \cup (S \times \mathbb{R}^1_+)$ for some compact manifold M with boundary S). It is proved in [5] that $\mathcal{H}^k(X)$ is isomorphic to the image of the canonical map $H^k_0(X) \to H^k(X)$, where $H^k(X)$ resp. $H^k_0(X)$ is the space of the de Rham cohomologies resp. compactly supported de Rham cohomologies of X. By e.g. [6], the spaces $H^k(X)$ are finite-dimensional. Thus, all $\mathcal{H}^k(X)$ are finite-dimensional and, in general, non-trivial, and hence so are all spaces $\mathcal{H}^p(X^{(N)})$. For a bigger class of examples of manifolds X with finite-dimensional spaces $\mathcal{H}^k(X)$ see [21].

Example 2. In the framework of the previous example, let $\dim M = 2$. Then $\beta_0 = \beta_2 = 0$ and it is easy to see from (46) that

$$\dim \mathcal{H}^p(X^{(N)}) = \begin{cases} \begin{pmatrix} \beta_1 \\ N \end{pmatrix}, & p = N, \\ 0, & p \neq N. \end{cases} \tag{49}$$

In particular, $\dim \mathcal{H}^p(X^{(N)}) = 0$ for all p, if $N > \beta_1$.

An important example of a manifold X with infinite dimensional spaces $\mathcal{H}^p(X)$ is given by the universal covering of a compact Riemannian manifold (say M) with the infinite fundamental group $G = \pi_1(M)$. In this case, G acts by isometries on X and consequently on all spaces $L^2\Omega^p(X)$. The orthogonal projection

$$\mathcal{P}_p : L^2\Omega^p(X) \to \mathcal{H}^p(X), \quad p = 0, 1, \dots, \dim X, \tag{50}$$

commutes with the action of G and thus belongs to the commutant $\mathcal{A}_p$ of this action which is a semifinite von Neumann algebra. The corresponding von Neumann trace $b_p := \mathrm{Tr}_{\mathcal{A}} \mathcal{P}_p$ gives a regularized dimension of the space $\mathcal{H}^{(p)}(X)$ and is called the L^2-Betti number of X (or M). L^2-Betti numbers were introduced in [4] studied by many authors (see e.g. [20] and references given there). It is known [4] that (because of the elliptic regularity of Δ^p_X) $b_p < \infty$. It follows from the general theory of von Neumann algebras that $b_p = 0$ iff $\dim \mathcal{H}^p(X) < \infty$.

It is natural to ask whether the notion of L^2-Betti numbers can be extended to configuration spaces over infinite coverings. It particular, is formula (48) valid in this case (with β_k replaced by b_k)?

More generally, let X be a Riemannian manifold admitting an infinite discrete group G of isometries such that the quotient $M = X/G$ is a compact connected Riemannian manifold. In what follows, we use the results of the first section in order to construct a von Neumann algebra containing the projection

$$\mathbf{P}_p : L^2\Omega^p(X^{(N)}) \to \mathcal{H}^p(X^{(N)}), \tag{51}$$

and to compute its von Neumann trace.

From now on, we assume that G is an ICC group (that is, all non-trivial classes of conjugate elements are infinite) that is acting freely on the manifold X. Under this conditions we have that the von Neumann algebra $\mathcal{A}_p$ is a II_∞ factor. Indeed, let U be a fundamental domain for the action of G, then we can make an identification

$$L^2\Omega^p(X) \cong L^2(G) \otimes L^2\Omega^p(U) \cong L^2(G) \otimes L^2\Omega^p(M). \tag{52}$$

The action of G on $L^2\Omega^p(X)$ corresponds by (52) to the left regular representation of G on $L^2(G)$ extended by the identity on $L^2\Omega^p(M)$. Then the commutant of this action is given by $R(G) \otimes \mathcal{L}(L^2\Omega^p(M))$, where $R(G)$ is the von Neumann algebra generated by the right regular representation of G. Since G is ICC group then $R(G)$ is a II_1 factor. Therefore $\mathcal{A}_p$ is a II_∞ factor.

Let us define the operator

$$\mathcal{P}_p^{(n)} := \begin{cases} \mathcal{P}_p^{\otimes n} P_s, & \text{p is even,} \\ \mathcal{P}_p^{\otimes n} P_a, & \text{p is odd} \end{cases} \tag{53}$$

and the von Neumann algebra

$$\mathcal{A}_p^{(n)} := \begin{cases} \left\{\mathcal{A}_p^{\otimes n}, P_s\right\}'', & \text{p is even,} \\ \left\{\mathcal{A}_p^{\otimes n}, P_a\right\}'', & \text{p is odd} \end{cases} \tag{54}$$

generated by $\mathcal{A}_p^{\otimes n}$ and the projections P_s and P_a respectively. Thus,

$$\mathcal{P}_p^{(n)} : \left(L^2\Omega^p(X)\right)^{\otimes n} \to \left(\mathcal{H}^p(X)\right)^{\overset{p}{\Diamond} n}, \tag{55}$$

$n = 1, 2, \ldots,$ is the orthogonal projection. Obviously, $\mathcal{P}_p^{(n)} \in \mathcal{A}_p^{(n)}$. It follows from Theorem 4 that $\mathcal{A}_p^{(n)} = \mathcal{A}_p^{\otimes n} \times_\alpha S_n$. We will use the convention $\mathcal{A}_p^{(0)} = \mathbb{C}^1$.

Further, for $0 \le p \le dN$, we introduce the von Neumann algebra

$$\mathbf{A}^{(p)} = \bigoplus_{\substack{s_0,\ldots,s_d=0,1,2,\ldots \\ \sum_k s_k=N,\ \sum_k ks_k=p}} \bigotimes_{m=0}^{d} \mathcal{A}_m^{(s_m)}. \tag{56}$$

Since all algebras $\mathcal{A}_k^{(s_k)}$ are II_∞-factors, so is $\mathbf{A}^{(p)}$, with the trace given by the product of the traces in $\mathcal{A}_k^{(s_k)}$.

Theorem 9. *$\mathbf{P}_p \in \mathbf{A}^{(p)}$, and its trace is given by the formula*

$$\mathrm{Tr}_{\mathbf{A}^{(p)}} \mathbf{P}_p = \sum_{\substack{s_1,\ldots,s_{d-1}=0,1,2,\ldots \\ \sum_k s_k=N,\ \sum_k ks_k=p}} \frac{(b_1)^{s_1}}{s_1!} \cdots \frac{(b_d)^{s_{d-1}}}{s_{d-1}!}, \tag{57}$$

if $N \le p \le (d-1)N$, and

$$\mathrm{Tr}_{\mathbf{A}^{(p)}} \mathbf{P}_p = 0, \tag{58}$$

if $p < N$, or $p > (d-1)N$. Here b_k are the L^2-Betti numbers of X.

Proof. In the case where $p > dN$, the equality $\mathbf{P}_p = 0$ is obvious.

Let $0 \le p \le dN$. It follows from Theorem 7 that

$$\mathbf{P}_p = \sum_{\substack{s_0,\ldots,s_d=0,1,2,\ldots \\ \sum_k s_k=N,\ \sum_k ks_k=p}} \prod_{m=0}^{d} \mathcal{P}_m{}^{(s_m)} \tag{59}$$

with the convention $\mathcal{P}_k^{(0)} = id$. Clearly $\mathbf{P}_p \in \mathbf{A}^{(p)}$. Corollary 5 implies that

$$\mathrm{Tr}_{\mathbf{A}^{(p)}} \mathbf{P}_p = \sum_{\substack{s_0,\ldots,s_d=0,1,2,\ldots \\ \sum_k s_k=N,\ \sum_k ks_k=p}} \prod_{m=0}^{d} \frac{(b_m)^{s_m}}{s_m!}. \tag{60}$$

The manifold X is connected, which implies that the spaces $\mathcal{H}^0(X)$ and $\mathcal{H}^d(X)$ are one-dimensional, and their von Neumann dimensions $b_0 = b_d = 0$. This proves formula (57). Let us remark that, in the case where $p < N$ resp. $p > (d-1)\,N$, we have $s_0 > 0$ resp. $s_d > 0$ in every term of the latter sum, which implies (58). $\square$

We will use the notation $\mathbf{b}_p = \mathrm{Tr}_{\mathbf{A}^{(p)}}\mathbf{P}_p$ and call $\mathbf{b}_p$ the p-th L^2-Betti number of $X^{(N)}$.

Remark 4. It is easy to see that formula (57) can be rewritten in the form

$$\mathbf{b}_p = \frac{1}{N!} \sum_{\substack{k_1,\dots,k_N = 0,1,2,\dots,d \\ k_1+\cdots+k_N = p}} b_{k_1}\cdots b_{k_N}, \tag{61}$$

or, according to the Künneth formula (45),

$$\mathbf{b}_p = \frac{1}{N!}\mathrm{Tr}_{\mathcal{A}^{\otimes N}} P, \tag{62}$$

where P is the orthogonal projection $L^2\Omega^p(X^N) \to H^p(X^N)$.

Example 3. Let $X = \mathbb{H}^d$, the hyperbolic space of dimension d. It is known that the only non-zero L^2-Betti number of $\mathbb{H}^d$ is $b_{d/2}$ (provided d is even). Then

$$\mathbf{b}_p = \begin{cases} \frac{\left(b_{d/2}\right)^N}{N!}, & p = \frac{Nd}{2}, \\ 0, & p \neq \frac{Nd}{2}. \end{cases} \tag{63}$$

The precise value of $b_{d/2}$ depends on the choice of the corresponding group G of isometries of $\mathbb{H}^d$, see [3]. In the particular case $m = 2$, we obtain the formula

$$\mathbf{b}_p = \begin{cases} \frac{(b_1)^N}{N!}, & p = N, \\ 0, & p \neq N, \end{cases} \tag{64}$$

which is quite different from the case of 2-dimensional non-compact surfaces with finite-dimensional spaces of harmonic forms, cf. (49).

Remark 5. Let us consider the space of finite configurations

$$\Gamma_{X,0} = \bigsqcup_{N=0,1,2,\dots} X^{(N)}, \tag{65}$$

with the convention $X^{(0)} = \{\emptyset\}$. The natural measure on $\Gamma_{X,0}$ generated by Riemannian volume measures on $X^{(N)}$ is called the Lebesgue-Poisson measure. The space $\Gamma_{X,0}$ appear as a "dual object" in the harmonic analysis on the infinite configuration space Γ_X [19], [17], [18]. Formulae for the L^2-Betti numbers of $\Gamma_{X,0}$ can be easily obtained from (57), (58). It turns out that they coincide with the corresponding formulae for L^2-Betti numbers of Γ_X equipped with the Poisson measure [2] (actually, there exists a natural unitary isomorphism of the spaces of harmonic forms on $\Gamma_{X,0}$ and Γ_X, which are square-integrable w.r.t. the Lebesgue-Poisson and Poisson measure respectively).

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