

## CYCLICAL ELEMENTS OF OPERATORS WHICH ARE LEFT-INVERSES TO MULTIPLICATION BY AN INDEPENDENT VARIABLE

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**ABSTRACT.** We study properties of operators which are left-inverses to the operator of multiplication by an independent variable in the space  $\mathcal{H}(G)$  of functions that are analytic in an arbitrary domain  $G$ . This space is endowed with topology of compact convergence. A description of cyclic elements for such operators is obtained. The obtained statements generalize known results in this direction.

In the process of studying the structure of invariant subspaces of a fixed linear continuous operator  $A$  acting on a space of analytic functions, it is important to have a description of cyclical elements for the operator  $A$ , that is, to have necessary and sufficient conditions for completeness, in the given space, of a system of functions  $(A^n r(z))_{n=0}^\infty$ , where  $r(z)$  is a fixed function from this space. These problems were investigated for a differential operator acting on various spaces of analytic functions by A. F. Leontev and his students. For spaces of analytic functions, the Pomme operator plays an important role and it is close, in a certain sense, to a differential operator. The Pomme problems mentioned above were solved in works of Yu. A. Kazmin, N. I. Nagnibida and other mathematicians. In this paper, we study a description of cyclic elements in spaces of analytic functions for the operator which is a left inverse to the operator of multiplication by an independent variable.

Let  $G$  be an arbitrary domain in the complex plane. By  $\mathcal{H}(G)$  we denote the space of all functions analytic in  $G$  and endowed with the topology of compact convergence [1]; by  $\mathcal{L}(\mathcal{H}(G))$  we denote the set of all linear continuous operators that act on  $\mathcal{H}(G)$ . For a fixed function  $\varphi \in \mathcal{H}(G)$ , the operator  $U_\varphi$  of multiplication by the function  $\varphi(z)$  acts linearly and continuously on  $\mathcal{H}(G)$  by  $(U_\varphi f)(z) = \varphi(z)f(z)$ . If  $0 \in G$ , the formula  $(\Delta f)(z) = \frac{f(z)-f(0)}{z}$  at  $z \neq 0$  and  $(\Delta f)(0) = f'(0)$  defines the Pomme operator  $\Delta$ , which belongs to the class  $\mathcal{L}(\mathcal{H}(G))$ . In this case the formula

$$(1) \quad A = \Delta + U_\varphi \delta_0,$$

where  $\varphi \in \mathcal{H}(G)$  and  $\delta_0(f) = f(0)$ , defines a general form of a linear continuous operator from the class  $\mathcal{L}(\mathcal{H}(G))$  which are left-inverses to the operator  $U_z$  of multiplication by the independent variable.

Let us study conditions for completeness of a systems of functions  $(A^n r(z))_{n=0}^\infty$  in  $\mathcal{H}(G)$ , where  $A$  is a fixed linear continuous operator which is a left-inverse to the operator  $U_z$ , and  $r(z)$  is a function from space  $\mathcal{H}(G)$ .

At the beginning, let us give auxiliary statements.

**Lemma 1.** *Let  $G$  be an arbitrary domain in  $\mathbb{C}$ , which contains the origin, and  $\varphi(z)$  be a fixed function from  $\mathcal{H}(G)$ . An operator  $T \in \mathcal{L}(\mathcal{H}(G))$  commutes with the operator*

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$A = \Delta + U_\varphi \delta_0$  if and only if it has the form

$$(2) \quad (Tf)(z) = L \left[ \frac{zf(z)\psi(\zeta) - \zeta f(\zeta)\psi(z)}{z - \zeta} \right],$$

where  $L$  is an arbitrary linear continuous functional on space  $\mathcal{H}(G)$  and  $\psi(z) = 1 - z\varphi(z)$ .

*Proof.* Necessity. Let a linear continuous operator  $T \in \mathcal{L}(\mathcal{H}(G))$  with the characteristic function  $t(\lambda, z) = T \left[ \frac{1}{\lambda - z} \right]$  [1] commute with  $A$ , that is, we have

$$(3) \quad TA = AT.$$

Substituting the functions  $\frac{1}{\lambda - z}$  in (3) we see that, on the set  $\mathcal{F}$  [1], the following identity holds:

$$\frac{1}{\lambda} (t(\lambda, z) + \varphi_1(z)) = \frac{t(\lambda, z) - t(\lambda, 0)}{z} + \varphi(z)t(\lambda, 0),$$

where  $\varphi_1(z) = (T\varphi)(z)$ ,  $\varphi_1 \in \mathcal{H}(G)$ . Solving the last equation with respect to  $t(\lambda, z)$  we have that on the set  $\mathcal{F}$ ,

$$(4) \quad t(\lambda, z) = \frac{z\varphi_1(z)}{\lambda - z} + \frac{(1 - z\varphi(z))\lambda t(\lambda, 0)}{\lambda - z}.$$

Since the function  $t(\lambda, z)$  is locally analytic on  $\mathbb{C}G \times G$ , the function  $l(\lambda) = t(\lambda, 0)$  is analytic on  $\mathbb{C}G$ . Therefore there exists a linear continuous functional  $L$  on space  $\mathcal{H}(G)$  for which the function  $t(\lambda, 0)$  is characteristic, that is,  $t(\lambda, 0) = L \left[ \frac{1}{\lambda - \zeta} \right]$  [1].

Consider an operator  $T_1$  defined by

$$(T_1f)(z) = z\varphi_1(z)f(z) + (1 - z\varphi(z))L \left[ \frac{zf(z) - \zeta f(\zeta)}{z - \zeta} \right].$$

It is clear that  $T_1 \in \mathcal{L}(\mathcal{H}(G))$  and the characteristic function of the operator  $T_1$  coincides with the function  $t(\lambda, z)$  defined by the formula (4). Therefore,  $T = T_1$ . To find the function  $\varphi_1(z)$ , we apply the operator  $T$  to  $\varphi(z)$ . Then

$$\varphi_1(z)(1 - z\varphi(z)) = (1 - z\varphi(z))L \left[ \frac{z\varphi(z) - \zeta\varphi(\zeta)}{z - \zeta} \right].$$

Since  $1 - z\varphi(z) \not\equiv 0$  in  $G$ , the last equalities imply that, in  $z \in G$ ,

$$\varphi_1(z) = L \left[ \frac{z\varphi(z) - \zeta\varphi(\zeta)}{z - \zeta} \right].$$

Thus,

$$(Tf)(z) = zL \left[ \frac{z\varphi(z) - \zeta\varphi(\zeta)}{z - \zeta} \right] f(z) + (1 - z\varphi(z))L \left[ \frac{zf(z) - \zeta f(\zeta)}{z - \zeta} \right].$$

Denoting  $\psi(z) = 1 - z\varphi(z)$  we get that the operator  $T$  can also be represented as

$$(Tf)(z) = L \left[ \frac{zf(z)\psi(\zeta) - \zeta f(\zeta)\psi(z)}{z - \zeta} \right].$$

Necessity of conditions in Lemma 1 is proved.

Their sufficiency is established by a direct calculation.  $\square$

Let conditions of Lemma 1 be satisfied. Since for an arbitrary nonnegative integer  $n$ , the operator  $A^n$  commutes with the operator  $A$ , by Lemma 1, the operator  $A^n$  can be represented as

$$(5) \quad (A^n f)(z) = L_n \left[ \frac{zf(z)\psi(\zeta) - \zeta f(\zeta)\psi(z)}{z - \zeta} \right],$$

where  $(L_n)_{n=0}^\infty$  is a sequence of linear continuous functionals on space  $\mathcal{H}(G)$ .

By immediate evaluations we see that

$$\begin{aligned} L_0(f) &= f(0); \\ L_1(f) &= f'(0) + \varphi(0)f(0); \\ L_2(f) &= \frac{1}{2}f''(0) + \varphi(0)f'(0) + (\varphi'(0) + [\varphi(0)]_2)f(0). \end{aligned}$$

In the general case, the following statement holds.

**Lemma 2.** *For every nonnegative integer  $n$ , the functional  $L_n$  defined by (5), can be represented in the following form:*

$$(6) \quad L_n(f) = \frac{1}{n!}f^{(n)}(0) + \sum_{k=0}^{n-1} c_{k,n}f^{(k)}(0),$$

where  $c_{k,n}$ ,  $k = \overline{0, n-1}$ , are some complex numbers.

*Proof.* We prove the lemma by induction on  $n$ . As it was remarked before, formula (6) holds if  $n = 0, 1, 2$ . Assume that formula (6) is valid for some positive integer  $n$  and prove that it holds for  $n + 1$ .

Let  $t_{n+1}(\lambda, z)$  be the characteristic function of the operator  $A^{n+1}$ , that is,  $t_{n+1}(\lambda, z) = A^{n+1} \left[ \frac{1}{\lambda-z} \right]$ . Since  $t_1(\lambda, z) = A \left[ \frac{1}{\lambda-z} \right] = \frac{1}{\lambda} \frac{1}{\lambda-z} + \varphi(z) \frac{1}{\lambda}$  on the set  $\mathcal{F} [1]$ , we have

$$t_{n+1}(\lambda, z) = A^n[t_1(\lambda, z)] = L_n \left[ \frac{z \left( \frac{1}{\lambda} \frac{1}{\lambda-z} + \varphi(z) \frac{1}{\lambda} \right) \psi(\zeta) - \zeta \left( \frac{1}{\lambda} \frac{1}{\lambda-\zeta} + \varphi(\zeta) \frac{1}{\lambda} \right) \psi(z)}{z - \zeta} \right].$$

But the characteristic function  $l_{n+1}(\lambda) = L_{n+1} \left[ \frac{1}{\lambda-\zeta} \right]$  of the functional  $L_{n+1}$  coincides with  $t_{n+1}(\lambda, 0)$ . Therefore,

$$l_{n+1}(\lambda) = L_n \left[ \frac{1}{\lambda} \frac{1}{\lambda-\zeta} + \varphi(\zeta) \frac{1}{\lambda} \right] = \frac{1}{\lambda} L_n \left[ \frac{1}{\lambda-\zeta} \right] + \frac{1}{\lambda} L_n [\varphi(\zeta)].$$

According to the induction assumption,

$$L_n \left[ \frac{1}{\lambda-\zeta} \right] = \frac{1}{n!} \left( \frac{1}{\lambda-\zeta} \right)^{(n)} \Big|_{\zeta=0} + \sum_{k=0}^{n-1} c_{k,n} \left( \frac{1}{\lambda-\zeta} \right)^{(k)} \Big|_{\zeta=0} = \frac{1}{\lambda^{n+1}} + \sum_{k=0}^{n-1} c_{k,n} \frac{k!}{\lambda^{k+1}}.$$

Therefore,

$$l_{n+1}(\lambda) = \frac{1}{\lambda^{n+2}} + \sum_{k=0}^{n-1} c_{k,n} \frac{k!}{\lambda^{k+2}} + \frac{1}{\lambda} L_n(\varphi).$$

Consider the functional  $\Lambda$  which linearly and continuously acts on the space  $\mathcal{H}(G)$  by

$$\Lambda(f) = \frac{1}{(n+1)!}f^{(n+1)}(0) + \sum_{k=0}^{n-1} \frac{c_{k,n}}{(k+1)}f^{k+1}(0) + L_n(\varphi)f(0).$$

Since the characteristic function of the functional  $\Lambda$  coincides with  $l_{n+1}(\lambda)$ , we have  $L_{n+1} = \Lambda$ , and the functional  $L_{n+1}$  has the form

$$L_{n+1}(f) = \frac{1}{(n+1)!}f^{(n+1)}(0) + \sum_{k=0}^n c_{k,n+1}f^{(k)}(0),$$

where  $c_{0,n+1} = L_n(\varphi)$  and  $c_{k,n+1} = \frac{c_{k-1,n}}{k}$ ,  $k = \overline{1, n}$ .

Thus, Lemma 2 is proved. □

*Remark.* From the proof of Lemma 1 it follows that the numbers  $c_{k,n}$ ,  $k = \overline{0, n-1}$ , by means of which the functional  $L_n$  is defined in formula (6) are uniquely expressed in terms of the numbers  $\varphi(0), \varphi'(0), \dots, \varphi^{(n-1)}(0)$ .

**Lemma 3.** *Let  $G$  be an arbitrary domain of the complex plane, which contains the point  $0$ ,  $A = \Delta + \varphi(z)\delta_0$ , where  $\varphi(z)$  is a function from  $\mathcal{H}(G)$ , and  $r(z)$  be a fixed function from  $\mathcal{H}(G)$ .*

*A system of functions  $(A^n r(z))_{n=0}^\infty$  is complete in  $\mathcal{H}(G)$  if and only if for an arbitrary nonzero functional  $L \in \mathcal{H}'(G)$  the function*

$$(7) \quad R(\zeta) = L_z \left[ \frac{zr(z)\psi(\zeta) - \zeta r(\zeta)\psi(z)}{z - \zeta} \right],$$

where  $\psi(z) = 1 - z\varphi(z)$ , is not identically equal to zero in  $G$ .

*Proof.* Formula (5) and Lemma 1 imply that

$$A^n r(z) = L_n \left[ \frac{zr(z)\psi(\zeta) - \zeta r(\zeta)\psi(z)}{z - \zeta} \right]$$

for each nonnegative integer  $n$ , where  $(L_n)_{n=0}^\infty$  is a sequence of linear continuous functionals on the space  $\mathcal{H}(G)$ , which are defined by formulas (6). Therefore, according to the Banach criterion of completeness, a condition for completeness of the system  $(A^n r(z))_{n=0}^\infty$  in the space  $\mathcal{H}(G)$  is equivalent to the property that, for each linear continuous functional  $L \in \mathcal{H}'(G)$ , it follows from the identities

$$(8) \quad L L_n \left[ \frac{zr(z)\psi(\zeta) - \zeta r(\zeta)\psi(z)}{z - \zeta} \right] = 0$$

that  $L = 0$  for  $n = 0, 1, \dots$ . Since the functionals  $L_z$  and  $L_n$  act on different variables, they commute [2].

The function  $R(\zeta)$  defined by the formula (7) is analytic in  $G$ . Therefore, in view of identities (6), we get that relations (8), for  $n = 0, 1, \dots$ , are equivalent to  $R^{(n)}(0) = 0$  for  $n = 0, 1, \dots$ , which means that  $R(\zeta) \equiv 0$  in  $G$ . Thus the system  $(A^n r(z))_{n=0}^\infty$  is complete in  $\mathcal{H}(G)$  if and only if the equalities  $R(\zeta) \equiv 0$  in  $G$  imply that  $L = 0$ .

Lemma 3 is proved.  $\square$

Lemmas 1–3 imply the following result.

**Theorem.** *Let  $G$  be an arbitrary domain in the complex plane containing the origin,  $A = \Delta + \varphi(z)\delta_0$ , where  $\varphi(z)$  is a function from  $\mathcal{H}(G)$ , and let  $r(z)$  be a fixed function from  $\mathcal{H}(G)$ . The system of functions  $(A^n r(z))_{n=0}^\infty$  is complete in  $\mathcal{H}(G)$  if and only if the function  $r(z)$  is not a zero of any nontrivial operator  $T \in \mathcal{L}(\mathcal{H}(G))$  that commutes with the operator  $A$ .*

In the case where the space  $\mathcal{H}(G)$  coincides with some of the spaces  $A_R$ ,  $0 < R \leq \infty$ , sufficient conditions for completeness of systems of the functions  $(\Delta^n r(z))_{n=0}^\infty$  in  $A_R$  were obtained by M. G. Haplanov in [3], and necessary and sufficient conditions by N. I. Nag-nibida [4]. For an arbitrary simply connected domain  $G$ , criteria for completeness of the system  $(\Delta^n r(z))_{n=0}^\infty$  in  $\mathcal{H}(G)$  where obtained by Yu. A. Kazmin in [5], and, for an arbitrary domain  $G$ , by N. E. Linchuk in [6].

The previous theorem implies the following.

**Corollary.** *Let the domain  $G$  be simply connected, the functions  $\varphi(z)$  and  $r(z)$  satisfy the conditions of the theorem, and the function  $\psi(z) = 1 - z\varphi(z)$  do not become zero on  $G$ . The system of functions  $(A^n r(z))_{n=0}^\infty$  is complete in  $\mathcal{H}(G)$  if and only if the function  $r(z)$  can be represented in the form  $r(z) = \psi(z)g(z)$ , where  $g(z)$  is some rational function from the space  $\mathcal{H}(G)$ .*

*Proof.* To prove the corollary, according to the theorem, it is necessary to check that, with the assumptions made, the function  $f(z)$  is a zero of some nontrivial operator  $T \in \mathcal{L}(\mathcal{H}(G))$  which acts by the rule

$$(Tf)(z) = L \left[ \frac{zf(z)\psi(\zeta) - \zeta f(\zeta)\psi(z)}{z - \zeta} \right],$$

where  $L \in \mathcal{H}'(G)$ ,  $L \neq 0$ , if and only if

$$(9) \quad f(z) = \psi(z)g(z), \quad z \in G,$$

where  $g(z)$  is some rational function from the space  $\mathcal{H}(G)$ .

The equation  $(Tf)(z) = 0$  with  $z \in G$  is equivalent to

$$(10) \quad L \left( \frac{1}{\psi(\zeta)} \left[ \frac{z \frac{f(z)}{\psi(z)} - \zeta \frac{f(\zeta)}{\psi(\zeta)}}{z - \zeta} \right] \right) = 0, \quad \forall z \in G.$$

Denote by  $L_1$  a linear continuous functional on the space  $\mathcal{H}(G)$  defined by  $L_1(h) = L \left( \frac{h}{\psi} \right)$ . Then identity (10) means that the function  $\frac{f(z)}{\psi(z)}$  is a zero of the operator  $T_1 \in \mathcal{L}(\mathcal{H}(G))$  that acts by the rule

$$(T_1h)(z) = L_1 \left[ \frac{zh(z) - \zeta h(\zeta)}{z - \zeta} \right].$$

Therefore according to statements from [6], relation (10) is equivalent to  $\frac{f(z)}{\psi(z)} = g(z)$ , where  $g(z)$  is some rational function from  $\mathcal{H}(G)$ , which means that  $f(z)$  is represented in the form (9).  $\square$

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