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A GENERALIZED STOCHASTIC DERIVATIVE ON THE KONDRATIEV-TYPE SPACE OF REGULAR GENERALIZED FUNCTIONS OF GAMMA WHITE NOISE

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This paper is dedicated to the blessed memory of my first teacher Professor Yu. L. Daletsky.

Abstract. We introduce and study a generalized stochastic derivative on the Kondratiev-type space of regular generalized functions of Gamma white noise. Properties of this derivative are quite analogous to the properties of the stochastic derivative in the Gaussian analysis. As an example we calculate the generalized stochastic derivative of the solution of some stochastic equation with Wick-type nonlinearity.

0. Introduction

Let $\mathcal{S}'$ be the Schwartz distributions space, μ be the Gaussian measure on $\mathcal{S}'$. As is well known, every square integrable function $f \in L^2(\mathcal{S}',\mu)$ can be presented in the form

$$f = \sum_{n=0}^{\infty} \langle H_n, f^{(n)} \rangle, \tag{0.1}$$

where $\{\langle H_n, f^{(n)}\rangle\}_{n=0}^{\infty}$ are the generalized Hermite polynomials, $f^{(n)} \in \mathcal{H}^{\widehat{\otimes} n}$, $\mathcal{H}$ (in the simplest case) is the complexification of $L^2(\mathbb{R})$, $\widehat{\otimes}$ denotes a symmetric tensor product. A stochastic derivative $\mathcal{D} : L^2(\mathcal{S}',\mu) \to \mathcal{L}(\mathcal{H}, L^2(\mathcal{S}',\mu))$ can be defined on its domain $\{f \in L^2(\mathcal{S}',\mu) : \sum_{n=1}^{\infty} n!n|f^{(n)}|^2_{\mathcal{H}^{\otimes n}} < \infty\}$ by the formula

$$(\mathcal{D}f)(g^{(1)}) := \sum_{n=1}^{\infty} n\langle H_{n-1}, \langle f^{(n)}, g^{(1)}\rangle\rangle \quad \forall g^{(1)} \in \mathcal{H}, \tag{0.2}$$

where $\langle f^{(n)}, g^{(1)}\rangle \in \mathcal{H}^{\widehat{\otimes} n-1}$ is defined by

$$\langle\langle f^{(n)}, g^{(1)}\rangle, h^{(n-1)}\rangle = \langle f^{(n)}, h^{(n-1)}\widehat{\otimes} g^{(1)}\rangle \quad \forall h^{(n-1)} \in \mathcal{H}^{\widehat{\otimes} n-1}$$

(here $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathcal{H}^{\widehat{\otimes} n}$).

In the paper [1] Fred E. Benth extended the derivative $\mathcal{D}$ on the Kondratiev generalized functions space $(\mathcal{S})^{-1}$ (elements of $(\mathcal{S})^{-1}$ can be presented in the similar to (0.1) form, but the kernels $\{f^{(n)}\}_{n=0}^{\infty}$ are singular). This generalization is useful for different applications. For example, as opposed to $L^2(\mathcal{S}',\mu)$, in the space $(\mathcal{S})^{-1}$ one can introduce the Wick product $\lozenge$ by setting for the Hermite polynomials $\langle H_n, f^{(n)}\rangle \lozenge \langle H_m, g^{(m)}\rangle := \langle H_{n+m}, f^{(n)}\widehat{\otimes} g^{(m)}\rangle$, and $\mathcal{D}$ is a differentiation with respect to $\lozenge$, for all $F, J \in (\mathcal{S})^{-1}$ $\mathcal{D}(F\lozenge J) = (\mathcal{D}F)\lozenge J + F\lozenge(\mathcal{D}J)$. Using this result (and another properties of $\mathcal{D}$) one can study properties of solutions of stochastic equations with Wick type nonlinearity.

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Another possible applications are connected with the fact that the stochastic derivative is the adjoint operator to the extended (Skorokhod) stochastic integral.

If instead of the Gaussian measure we consider the (introduced in [12]) Gamma measure μ on $\mathcal{S}'$ (see the definition in Section 1) then the analog of representation (0.1) for each $f \in L^2(\mathcal{S}',\mu)$ (with the generalized Laguerre polynomials instead of the Hermite ones) holds true; but the kernels $\{f^{(n)}\}_{n=0}^{\infty}$ belong to special Hilbert spaces $\mathcal{H}_{ext}^{(n)}$ (see [11] and Section 1), these spaces have the structure more complicated than that of $\mathcal{H}^{\widehat{\otimes}n}$. Such a situation is natural because the Gamma measure does not have the so-called Chaotic Representation Property, see, e.g., [8] for a more detailed explanation. But if we construct a stochastic derivative on $L^2(\mathcal{S}',\mu)$ or on wider Kondratiev-type spaces by analogy with (0.2) (on $L^2(\mathcal{S}',\mu)$ such a derivative is the adjoint operator to an extended stochastic integral) then the mentioned property of the Gamma measure leads to complications with a study of this derivative. Nevertheless, in the paper [7] the author generalized the results of [1] to the Kondratiev-type space $(\mathcal{S}')'$ of *nonregular* generalized functions of Gamma white noise. But $(\mathcal{S}')'$ is too wide a space and the kernels from the natural orthogonal decompositions of elements of $(\mathcal{S}')'$ belong to the distributions spaces without "good" description, this is inconvenient for applications. Moreover, the "specifics" of the Gamma measure are such that the generalized stochastic derivatives on $(\mathcal{S}')'$ and on $L^2(\mathcal{S}',\mu)$ are different (see [7] for more details).

The main aim of this paper is to transfer the results of [7] to the Kondratiev-type space $(L^2)^{-1}$ of *regular* generalized functions of Gamma white noise. This space is smaller than $(\mathcal{S}')'$ and there is no the mentioned problem with the natural orthogonal decompositions of elements of $(L^2)^{-1}$. Simultaneously it turned out that *all* main results of [7] can be transferred to $(L^2)^{-1}$. Moreover, the generalized stochastic derivative on $(L^2)^{-1}$ is a direct generalization of the corresponding derivative on $L^2(\mathcal{S}',\mu)$. Finally, we note that properties of our derivative are quite analogous to the properties of the stochastic derivative in the Gaussian analysis.

The paper is organized in the following manner. In the first section we recall some elements of the Gamma analysis, including the stochastic integration and the Wick calculus. The second section is devoted to the generalized stochastic derivative on $(L^2)^{-1}$.

1. Preliminaries

Let σ be a non-atomic positive regular σ-finite measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ satisfying some additional condition, see Remark 1.1 for details (here and below the symbol $\mathcal{B}$ denotes the Borel σ-algebra). We denote $\mathcal{H} := L^2(\mathbb{R},\sigma)$ (the space of square integrable with respect to σ functions on $\mathbb{R}$). Let $\mathcal{S}$ be the Schwartz test functions space on $\operatorname{supp}\sigma$ (if, e.g., σ is the Lebesgue measure then $\mathcal{S}$ is the usual Schwartz space of rapidly decreasing infinitely differentiable functions; a more detailed description of $\mathcal{S}$ is given in Remark 1.1). As is well known, there exist Hilbert spaces $\mathcal{H}_p \equiv \mathcal{H}_p(\mathbb{R}) \subset \mathcal{H}$, $p \in \mathbb{N}$, such that there is the nuclear chain

$$\mathcal{S}' = \operatorname*{ind\,lim}_{p'\in\mathbb{N}} \mathcal{H}_{-p'} \supset \mathcal{H}_{-p} \supset \mathcal{H} \equiv \mathcal{H}_0 \supset \mathcal{H}_p \supset \operatorname*{pr\,lim}_{p'\in\mathbb{N}} \mathcal{H}_{p'} = \mathcal{S},$$

where the spaces $\mathcal{H}_{-p}$ ($p \in \mathbb{N}$), $\mathcal{S}'$ are dual to $\mathcal{H}_p$, $\mathcal{S}$ with respect to the zero space $\mathcal{H}$. Note that one can select spaces $\mathcal{H}_p$ ($p \in \mathbb{N}$) such that for each $p > p'$, we will have $|\cdot|_p \geq |\cdot|_{p'}$ (here $|\cdot|_p$ denotes the norm in $\mathcal{H}_p$, $p \in \mathbb{Z}$, in particular, $|\cdot|_0 = |\cdot|_{\mathcal{H}}$). We preserve the notation $|\cdot|_p$ for norms in the tensor powers and complexifications of $\mathcal{H}_p$, $p \in \mathbb{Z}$.

Remark 1.1. Let us describe the construction of the spaces $\mathcal{H}_p$, $p \in \mathbb{N}$, in details, following [11]. Let $\big(e_j\big)_{j=0}^{\infty}$ be the system of Hermite functions on $\mathbb{R}$. For each $p \geq 1$ we denote by $\widetilde{\mathcal{H}}_p \equiv \widetilde{\mathcal{H}}_p(\mathbb{R})$ the Hilbert space that is constructed by the orthogonal basis

$\big(e_j(2j+2)^{-p}\big)_{j=0}^{\infty}$, and assume that the measure σ is such that for some $\varepsilon \geq 0$ the space $\widetilde{\mathcal{H}}_{1+\varepsilon}$ is continuously embedded into $\mathcal{H} = L^2(\mathbb{R}, \sigma)$. Further, let $O_p : \widetilde{\mathcal{H}}_p \to \mathcal{H}$ be the embedding operator. Without loss of generality one can suppose that for defined above ε the operator $O_{1+\varepsilon}$ is of Hilbert-Schmidt type (for example, if σ is the Lebesgue measure then one can put $\varepsilon = 0$). Now we can put $\mathcal{H}_p := \widetilde{\mathcal{H}}_{p+\varepsilon}|_{KER\ O_{p+\varepsilon}}$ (the Hilbert factor space), $\mathcal{S} :=$ pr $\lim_{p\in\mathbb{N}} \mathcal{H}_p$.

We use the index $\mathbb{C}$ to denote complexifications of corresponding spaces. By $\langle\cdot,\cdot\rangle$ denote the (real) dual pairing between elements of $\mathcal{S}'_{\mathbb{C}}$ and $\mathcal{S}_{\mathbb{C}}$ (and also $\mathcal{H}_{-p,\mathbb{C}}$ and $\mathcal{H}_{p,\mathbb{C}}$), this pairing is generated by the scalar product in $\mathcal{H}$. This notation will be preserved for pairings in tensor powers of spaces. Let $\mathcal{F}$ be the σ-algebra on $\mathcal{S}'$ generated by cylinder sets.

Definition 1.1. The measure μ on the measurable space $(\mathcal{S}', \mathcal{F})$ with the Laplace transform

$$l_\mu(\lambda) = \int_{\mathcal{S}'} e^{\langle x,\lambda\rangle}\mu(dx) = \exp\{-\langle 1, \log(1-\lambda)\rangle\}, \quad 1 > \lambda \in \mathcal{S},$$

is called the *Gamma measure*.

The correctness of this definition was proved in [12].

Remark 1.2. Note that l_μ can be continued to a function holomorphic at zero. A more detailed description of μ is given in [12].

Remark 1.3. The term "Gamma measure" is connected with the fact that μ is the measure of the so-called *Gamma white noise*, see [12, 9, 8] for a more detailed information.

By $(L^2) \equiv L^2(\mathcal{S}', \mu)$ we denote the space of complex-valued functions on $\mathcal{S}'$ square integrable with respect to μ, and construct orthogonal in (L^2) polynomials. Let $\alpha : \mathcal{S}_{\mathbb{C}} \to \mathcal{S}_{\mathbb{C}}$ be the function that is defined on some neighborhood of $0 \in \mathcal{S}_{\mathbb{C}}$ by the formula $\alpha(\lambda) := \frac{\lambda}{\lambda-1}$. We define the so-called *Wick exponential* (a generating function of the orthogonal polynomials)

$$(1.1) \qquad :\exp(x;\lambda) :\stackrel{def}{=} \frac{\exp\{\langle x, \alpha(\lambda)\rangle\}}{l_\mu(\alpha(\lambda))} = \exp\big\{\langle x, \frac{\lambda}{\lambda-1}\rangle - \langle 1, \log(1-\lambda)\rangle\big\},$$

where $\lambda \in \mathcal{U}_0 \subset \mathcal{S}_{\mathbb{C}}$, $x \in \mathcal{S}'$, $\mathcal{U}_0$ is some neighborhood of $0 \in \mathcal{S}_{\mathbb{C}}$.

Remark 1.4. Note that (1.1) is the infinite-dimensional analog of the generating functions of the one-dimensional Laguerre polynomials. These polynomials are orthogonal "with respect to the one-dimensional Gamma measure", see, e.g., [16].

It is clear that $:\exp(x;\cdot):$ is a function on $\mathcal{S}_{\mathbb{C}}$ holomorphic at zero for each $x \in \mathcal{S}'$. So, using the Cauchy inequalities (see, e.g., [4]) and the kernel theorem (see, e.g., [3]) one can obtain the representation

$$:\exp(x;\lambda) := \sum_{n=0}^{\infty} \frac{1}{n!}\langle L_n(x), \lambda^{\otimes n}\rangle, \quad L_n(x) \in \mathcal{S}'^{\widehat{\otimes} n}_{\mathbb{C}}, \quad \lambda \in \mathcal{S}_{\mathbb{C}},$$

here and below $\widehat{\otimes}$ denotes a symmetric tensor product, $\lambda^{\otimes 0} = 1$ even for $\lambda \equiv 0$. (Note that actually for $x \in \mathcal{S}'$, $L_n(x) \in \mathcal{S}'^{\widehat{\otimes} n}$.)

Definition 1.2. The polynomials $\{\langle L_n(x), f^{(n)}\rangle,\ f^{(n)} \in \mathcal{S}_{\mathbb{C}}^{\widehat{\otimes} n},\ n \in \mathbb{Z}_+\}$ are called *generalized Laguerre polynomials*.

In order to formulate a statement on orthogonality of the generalized Laguerre polynomials we need the following.

Definition 1.3. We define the scalar product $\langle\cdot,\cdot\rangle_{ext}$ on $\mathcal{S}_{\mathbb{C}}^{\widehat{\otimes}n}$ by the formula

$$\langle f^{(n)}, g^{(n)}\rangle_{ext} := \sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n}} \frac{n!}{l_1^{s_1}\dots l_k^{s_k}s_1!\dots s_k!}$$

$$\times\int_{\mathbb{R}^{s_1+\dots+s_k}} f^{(n)}(\underbrace{\tau_1,\dots,\tau_1}_{l_1},\dots,\underbrace{\tau_{s_1},\dots,\tau_{s_1}}_{l_1},\dots,\underbrace{\tau_{s_1+\dots+s_k},\dots,\tau_{s_1+\dots+s_k}}_{l_k}) \tag{1.2}$$

$$\times g^{(n)}(\underbrace{\tau_1,\dots,\tau_1}_{l_1},\dots,\underbrace{\tau_{s_1},\dots,\tau_{s_1}}_{l_1},\dots,\underbrace{\tau_{s_1+\dots+s_k},\dots,\tau_{s_1+\dots+s_k}}_{l_k})$$

$$\times\sigma(d\tau_1)\dots\sigma(d\tau_{s_1+\dots+s_k}).$$

By $|\cdot|_{ext}$ denote the corresponding norm, i.e., $|f^{(n)}|_{ext}^2 = \langle f^{(n)}, \overline{f^{(n)}}\rangle_{ext}$.

Example. It follows from (1.2) that for $n=1$, $\langle f^{(1)}, g^{(1)}\rangle_{ext} = \langle f^{(1)}, g^{(1)}\rangle$. Further, for $n=2$,

$$\langle f^{(2)}, g^{(2)}\rangle_{ext} = \langle f^{(2)}, g^{(2)}\rangle + \int_{\mathbb{R}} f^{(2)}(\tau,\tau)g^{(2)}(\tau,\tau)\sigma(d\tau);$$

for a general $n\in\mathbb{N}$, $\langle f^{(n)}, g^{(n)}\rangle_{ext} = \langle f^{(n)}, g^{(n)}\rangle + \cdots$.

Theorem 1.1. [12]. *The generalized Laguerre polynomials are orthogonal in (L^2) in the sense that*

$$\int_{\mathcal{S}'}\langle L_n(x), f^{(n)}\rangle\langle L_m(x), g^{(m)}\rangle\mu(dx) = \delta_{mn}n!\langle f^{(n)}, g^{(n)}\rangle_{ext}.$$

By $\mathcal{H}_{ext}^{(n)}$ ($n\in\mathbb{N}$) denote the closure of $\mathcal{S}_{\mathbb{C}}^{\widehat{\otimes}n}$ with respect to the norm $|\cdot|_{ext}$ (see Definition 1.3), $\mathcal{H}_{ext}^{(0)} := \mathbb{C}$. For $f^{(n)}\in\mathcal{H}_{ext}^{(n)}$ we define $(L^2)\ni\langle L_n, f^{(n)}\rangle := \lim_{k\to\infty}\langle L_n, f_k^{(n)}\rangle$ in (L^2), where $\mathcal{S}_{\mathbb{C}}^{\widehat{\otimes}n}\ni f_k^{(n)}\to f^{(n)}$ (as $k\to\infty$) in $\mathcal{H}_{ext}^{(n)}$ (the correctness of this definition can be proved by analogy with the classical Gaussian case, see also [9, 7]). It follows from results of [11] that the generalized Laguerre polynomials with kernels $f^{(n)}\in\mathcal{H}_{ext}^{(n)}$ form an orthogonal basis in (L^2).

Now let us introduce the Kondratiev-type spaces of regular test and generalized functions. First we consider the set $\mathcal{P} := \{f = \sum_{n=0}^{N_f}\langle L_n, f^{(n)}\rangle,\ f^{(n)}\in\mathcal{H}_{ext}^{(n)},\ N_f\in\mathbb{Z}_+\}\subset(L^2)$ of polynomials and $\forall q\in\mathbb{N}$ introduce on this set the scalar product $(\cdot,\cdot)_q$ by setting, for $f=\sum_{n=0}^{N_f}\langle L_n, f^{(n)}\rangle$, $g=\sum_{n=0}^{N_g}\langle L_n, g^{(n)}\rangle$,

$$(f,g)_q := \sum_{n=0}^{\min(N_f,N_g)}(n!)^2 2^{qn}\langle f^{(n)}, g^{(n)}\rangle_{ext}.$$

Let $\|\cdot\|_q$ be the corresponding norm: $\|f\|_q = \sqrt{(f,\overline{f})_q} = \sqrt{\sum_{n=0}^{N_f}(n!)^2 2^{qn}|f^{(n)}|_{ext}^2}$.

Definition 1.4. We define the Kondratiev-type spaces of ("regular") test functions $(L^2)_q^1$ ($q\in\mathbb{N}$) as the closures of $\mathcal{P}$ with respect to the norms $\|\cdot\|_q$; $(L^2)^1 := \mathrm{pr}\lim_{q\in\mathbb{N}}(L^2)_q^1$.

It is not difficult to see that $f\in(L^2)_q^1$ if and only if f can be presented in the form

$$f=\sum_{n=0}^{\infty}\langle L_n, f^{(n)}\rangle,\quad f^{(n)}\in\mathcal{H}_{ext}^{(n)} \tag{1.3}$$

with

$$\|f\|_q^2 = \sum_{n=0}^{\infty}(n!)^2 2^{qn}|f^{(n)}|_{ext}^2<\infty,$$

therefore the generalized Laguerre polynomials play a role of an orthogonal basis in $(L^2)_q^1$.

It was proved in [8] that $\forall q \in \mathbb{N}$ $(L^2)^1_q \hookrightarrow (L^2)$ and this embedding is dense. Therefore one can consider the chain

$$(L^2)^{-1} = \operatorname*{ind\,lim}_{\widetilde{q}\in\mathbb{N}} (L^2)^{-1}_{-\widetilde{q}} \supset (L^2)^{-1}_{-q} \supset (L^2) \supset (L^2)^1_q \supset (L^2)^1,$$

where the spaces $(L^2)^{-1}_{-q}$, $(L^2)^{-1}$ are dual correspondingly to $(L^2)^1_q$, $(L^2)^1$ with respect to (L^2).

Definition 1.5. The spaces $(L^2)^{-1}_{-q}$, $(L^2)^{-1}$ are called the Kondratiev-type spaces of regular generalized functions (cf. [5]).

It is easy to see that $F \in (L^2)^{-1}_{-q}$ if and only if F can be presented as the formal series

$$F = \sum_{m=0}^{\infty} \langle L_m, F^{(m)} \rangle, \quad F^{(m)} \in \mathcal{H}^{(m)}_{ext}, \tag{1.4}$$

with

$$\|F\|^2_{-q} := \sum_{m=0}^{\infty} 2^{-qm} |F^{(m)}|^2_{ext} < \infty.$$

Moreover, the generalized Laguerre polynomials play a role of an orthogonal basis in $(L^2)^{-1}_{-q}$ in the sense that for $F, J \in (L^2)^{-1}_{-q}$ we have $(F,J)_{-q} = \sum_{m=0}^{\infty} 2^{-qm} \langle F^{(m)}, J^{(m)} \rangle_{ext}$ (here $(\cdot,\cdot)_{-q}$ denotes the (real) scalar product in $(L^2)^{-1}_{-q}$, $\|F\|_{-q} = \sqrt{(F,\overline{F})_{-q}}$; the kernels $F^{(m)}, J^{(m)} \in \mathcal{H}^{(m)}_{ext}$ are from decompositions (1.4) for F, J correspondingly).

By $\langle\langle \cdot,\cdot \rangle\rangle$ denote the dual pairing between elements of $(L^2)^{-1}_{-q}$ and $(L^2)^1_q$ (correspondingly $(L^2)^{-1}$ and $(L^2)^1$), this pairing is generated by the scalar product in (L^2). If $F \in (L^2)^{-1}_{-q}$ and $f \in (L^2)^1_q$ then we have

$$\langle\langle F, f \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F^{(n)}, f^{(n)} \rangle_{ext},$$

where $F^{(n)}, f^{(n)} \in \mathcal{H}^{(n)}_{ext}$ are the kernels from decompositions (1.4), (1.3) for F and f, respectively.

Now let us recall the construction of the extended stochastic integral on the space of regular generalized functions (see [8] for a more detailed presentation). By analogy with the classical Gaussian analysis one can consider the compensated Gamma process $G_s = \langle L_1, 1_{[0,s)} \rangle \in (L^2)$ $(s \in \mathbb{R}_+)$ on the probability space $(\mathcal{S}', \mathcal{F}, \mu)$ (from this point of view μ is the measure of the Gamma white noise G'_s, *formally* $G'_s = \langle L_1, \delta_s \rangle$, where δ_s is the "concentrated at s" delta-function, see more details below). Let $F_\cdot \in (L^2)^{-1}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$, $q \in \mathbb{N}$. Then (see (1.4))

$$F_\cdot = \sum_{m=0}^{\infty} \langle L_m, F^{(m)}_\cdot \rangle, \quad F^{(m)}_\cdot \in \mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}. \tag{1.5}$$

Lemma 1.1. [9]. *For given $F^{(m)}_\cdot \in \mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$ and $t \in [0,+\infty]$ we construct an element $\widehat{F}^{(m)}_{[0,t)} \in \mathcal{H}^{(m+1)}_{ext}$ in the following way. Let us consider a sequence $\{f^{(m)}_{i,\cdot} \in \mathcal{S}^{\widehat{\otimes}m}_{\mathbb{C}} \otimes \mathcal{S}_{\mathbb{C}}\}_{i=1}^{\infty}$ such that $F^{(m)}_\cdot = \lim_{i\to\infty} f^{(m)}_{i,\cdot}$ in $\mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$ and put*

$$\widetilde{f}^{(m)}_{[0,t),i}(\tau_1,\dots,\tau_m,\tau) := \begin{cases} f^{(m)}_{i,\tau}(\tau_1,\dots,\tau_m) 1_{[0,t)}(\tau), & \text{if } \tau \neq \tau_1,\dots,\tau \neq \tau_m, \\ 0, & \text{in other cases}, \end{cases} ,$$

$\widehat{f}^{(m)}_{[0,t),i} := P\widetilde{f}^{(m)}_{[0,t),i}$, where $1_{[0,t)}(\tau)$ denotes the indicator of $\{\tau \in [0,t)\}$, P is the symmetrization operator. Then $\widehat{F}^{(m)}_{[0,t)} := \lim_{i\to\infty} \widehat{f}^{(m)}_{[0,t),i}$ in $\mathcal{H}^{(m+1)}_{ext}$. This limit does not

depend on the sequence $\{f_{i,\cdot}^{(m)}\}_{i=1}^{\infty}$ *and the estimate*

$$|\widehat{F}_{[0,t)}^{(m)}|_{ext} \leq |F_{\cdot}^{(m)}|_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}_{\mathbb{C}}}$$

holds.

Definition 1.6. Let $F \in (L^2)_{-q}^{-1} \otimes \mathcal{H}_{\mathbb{C}}$, $q \in \mathbb{N}$. For each $t \in [0, +\infty]$ we define the extended stochastic integral $\int_0^t F_s \widehat{d}G_s \in (L^2)_{-q}^{-1}$ by setting

$$\int_0^t F_s \widehat{d}G_s := \sum_{m=0}^{\infty} \langle L_{m+1}, \widehat{F}_{[0,t)}^{(m)} \rangle, \tag{1.6}$$

where the kernels $\widehat{F}_{[0,t)}^{(m)}$ are constructed in Lemma 1.1 starting with the kernels $F_{\cdot}^{(m)}$ from decomposition (1.5) for F. If $F \in (L^2)^{-1} \otimes \mathcal{H}_{\mathbb{C}}$ then $\int_0^t F_s \widehat{d}G_s \in (L^2)^{-1}$.

The correctness of this definition was proved in [8].

Note that $\int_0^t F_s \widehat{d}G_s$ is a direct generalization of the defined in [9] extended stochastic integral on $(L^2) \otimes \mathcal{H}_{\mathbb{C}}$, and therefore $\int_0^t F_s \widehat{d}G_s$ is a direct generalization of the Itô stochastic integral (see [8, 9, 7] for more details).

Finally, let us recall elements of the Wick calculus on $(L^2)^{-1}$ (a more detailed presentation is given in [8]).

Definition 1.7. For $F \in (L^2)^{-1}$ we define the integral S-transform $(SF)(\lambda)$, λ belongs to some neighborhood of zero in $\mathcal{S}_{\mathbb{C}}$, by setting (see (1.1))

$$(SF)(\lambda) := \langle\langle F, :\exp(\cdot;\lambda): \rangle\rangle.$$

The correctness of this definition was proved in [8].

Note that a simple calculation gives

$$(SF)(\lambda) = \sum_{n=0}^{\infty} \langle F^{(n)}, \lambda^{\otimes n} \rangle_{ext}, \tag{1.7}$$

where $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ $(n \in \mathbb{Z}_+)$ are the kernels from decomposition (1.4) for F. In particular, $(SF)(0) = F^{(0)}$, $S1 = 1$.

Definition 1.8. We define a set B (a characterization set of $(L^2)^{-1}$ in terms of the S-transform) by setting $B := S\big((L^2)^{-1}\big) \equiv \big\{K \in Hol_0 | \exists F \in (L^2)^{-1} : K = SF\big\} \subset Hol_0$ (see [8]), where Hol_0 (see, e.g., [13]) is the set of germs of holomorphic at zero functions on $\mathcal{S}_{\mathbb{C}}$.

Remark 1.5. In this paper we do not need a topology on B. Nevertheless we note that Hol_0 was introduced in [13] as a topological space, therefore it seems to be natural to introduce on B the induced from Hol_0 topology; in this case S will be a topological isomorphism between $(L^2)^{-1}$ and B.

Proposition 1.1. [8]. *The set B is an algebra with respect to the usual (pointwise) multiplication of functions. Moreover, if $K \in B$ and $h : \mathbb{C} \to \mathbb{C}$ is a holomorphic at $K(0)$ function, then $\widetilde{K}(\cdot) := h(K(\cdot)) \in B$. In particular, for each entire $h : \mathbb{C} \to \mathbb{C}$ and $K \in B$ we have $h(K) \in B$.*

Definition 1.9. For $F, J \in (L^2)^{-1}$ and a holomorphic at $(SF)(0)$ function $h : \mathbb{C} \to \mathbb{C}$ we define the *Wick product* $F \Diamond J \in (L^2)^{-1}$ and the *Wick version of h* $h^{\Diamond}(F) \in (L^2)^{-1}$ by setting

$$F \Diamond J := S^{-1}(SF \cdot SJ), \quad h^{\Diamond}(F) := S^{-1} h(SF).$$

The correctness of this definition from Proposition 1.1 follows.

Remark 1.6. It is easy to see that if $F \in (L^2)^{-1}$ and h from Definition 1.9 is presented in the form $h(u) = \sum_{n=0}^{\infty} h_n(u - (SF)(0))^n$ then $h^{\Diamond}(F) = \sum_{n=0}^{\infty} h_n(F - (SF)(0))^{\Diamond n}$, where for $n \in \mathbb{N}$ $F^{\Diamond n} := \underbrace{F\Diamond \dots \Diamond F}_{n\ times}$, $F^{\Diamond 0} := 1$.

In order to make calculations with Wick products and with Wick versions of holomorphic functions we need coordinate forms of these objects.

Lemma 1.2. [8]. *Let $n, m \in \mathbb{N}$, $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $J^{(m)} \in \mathcal{H}_{ext}^{(m)}$. We construct the element $F^{(n)} \diamond J^{(m)} \in \mathcal{H}_{ext}^{(n+m)}$ by the following way. Let $\mathcal{S}_{\mathbb{C}}^{\widehat{\otimes} n} \ni F_k^{(n)} \to F^{(n)}$ as $k \to \infty$ in $\mathcal{H}_{ext}^{(n)}$, $\mathcal{S}_{\mathbb{C}}^{\widehat{\otimes} m} \ni J_k^{(m)} \to J^{(m)}$ as $k \to \infty$ in $\mathcal{H}_{ext}^{(m)}$. We put*

$$(\widetilde{F^{(n)} J^{(m)}})_k(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m}) := \begin{cases} F_k^{(n)}(t_1, \dots, t_n) J_k^{(m)}(t_{n+1}, \dots, t_{n+m}), & if \ {}_{\forall j \in \{n+1,\dots,n+m\}}^{\forall i \in \{1,\dots,n\},} \ t_i \neq t_j, \\ 0, & in\ other\ cases \end{cases},$$

$(\widehat{F^{(n)} J^{(m)}})_k := P(\widetilde{F^{(n)} J^{(m)}})_k$, where P is the symmetrization operator. Then $F^{(n)} \diamond J^{(m)} := \lim_{k\to\infty} (\widehat{F^{(n)} J^{(m)}})_k$ in $\mathcal{H}_{ext}^{(n+m)}$, this limit does not depend on the sequences $(F_k^{(n)})_{k=0}^{\infty}$, $(J_k^{(m)})_{k=0}^{\infty}$. For $n = 0$ we put $F^{(0)} \diamond J^{(m)} := F^{(0)} J^{(m)} \in \mathcal{H}_{ext}^{(m)}$ ($F^{(0)} \in \mathbb{C}$), by analogy $F^{(n)} \diamond J^{(0)} := F^{(n)} J^{(0)} \in \mathcal{H}_{ext}^{(n)}$. For $n, m \in \mathbb{Z}_+$

$$|F^{(n)} \diamond J^{(m)}|_{ext} \leq |F^{(n)}|_{ext} |J^{(m)}|_{ext}.$$

Remark 1.7. Note that nonstrictly speaking $\widehat{F^{(n)} J^{(m)}}$ is the symmetrization of the function

$$\widetilde{F^{(n)} J^{(m)}}(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m}) := \begin{cases} F^{(n)}(t_1, \dots, t_n) J^{(m)}(t_{n+1}, \dots, t_{n+m}), & if \ {}_{\forall j \in \{n+1,\dots,n+m\}}^{\forall i \in \{1,\dots,n\},} \ t_i \neq t_j, \\ 0, \ in\ other\ cases \end{cases}$$

with respect to $n + m$ variables.

Remark 1.8. Let us consider the riggings

$$\mathcal{S}'^{(n)}_{\mathbb{C}} \supset \mathcal{H}_{ext}^{(n)} \supset \mathcal{S}_{\mathbb{C}}^{\widehat{\otimes} n}, \quad \mathcal{S}'^{\widehat{\otimes} n}_{\mathbb{C}} \supset \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n} \supset \mathcal{S}_{\mathbb{C}}^{\widehat{\otimes} n}, \quad n \in \mathbb{N},$$

and let $U_n : \mathcal{S}'^{(n)}_{\mathbb{C}} \to \mathcal{S}'^{\widehat{\otimes} n}_{\mathbb{C}}$ be the natural isomorphism between $\mathcal{S}'^{(n)}_{\mathbb{C}}$ and $\mathcal{S}'^{\widehat{\otimes} n}_{\mathbb{C}}$, i.e., $\forall F_{ext}^{(n)} \in \mathcal{S}'^{(n)}_{\mathbb{C}}$, $\forall f^{(n)} \in \mathcal{S}_{\mathbb{C}}^{\widehat{\otimes} n}$ $\langle F_{ext}^{(n)}, f^{(n)} \rangle_{ext} = \langle U_n F_{ext}^{(n)}, f^{(n)} \rangle$ (here by $\langle \cdot, \cdot \rangle_{ext}$ denote the dual pairing between elements of $\mathcal{S}'^{(n)}_{\mathbb{C}}$ and $\mathcal{S}_{\mathbb{C}}^{\widehat{\otimes} n}$, this pairing is generated by the scalar product in $\mathcal{H}_{ext}^{(n)}$). One can prove (see [8]) that for $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $J^{(m)} \in \mathcal{H}_{ext}^{(m)}$ $U_{n+m}^{-1}(U_n F^{(n)} \widehat{\otimes} U_m J^{(m)})$ can be continued to a linear continuous functional on $\mathcal{H}_{ext}^{(n+m)}$ that coincides with $F^{(n)} \diamond J^{(m)}$, i.e., $F^{(n)} \diamond J^{(m)} = U_{n+m}^{-1}(U_n F^{(n)} \widehat{\otimes} U_m J^{(m)})$.

It follows from results of [8] that the "multiplication" $\diamond$ is associative, commutative, and distributive (over the field $\mathbb{C}$).

Proposition 1.2. *Let $F, J \in (L^2)^{-1}$ and a function $h : \mathbb{C} \to \mathbb{C}$ be holomorphic at $(SF)(0)$. Then $F\Diamond J$ and $h^{\Diamond}(F)$ can be presented in the form*

$$F\Diamond J = \sum_{k=0}^{\infty} \langle L_k, \sum_{n=0}^{k} F^{(n)} \diamond J^{(k-n)} \rangle,$$

$$h^{\Diamond}(F) = h_0 + \sum_{k=1}^{\infty} \langle L_k, \sum_{n=1}^{k} h_n \sum_{m_1,\dots,m_n \in \mathbb{N}: m_1 + \dots + m_n = k} F^{(m_1)} \diamond \dots \diamond F^{(m_n)} \rangle,$$

where the coefficients $h_n \in \mathbb{C}$ *(*$n \in \mathbb{Z}_+$*) are from the decomposition* $h(u) = \sum_{n=0}^{\infty} h_n(u - F^{(0)})^n$*;* $F^{(n)}, J^{(n)} \in \mathcal{H}_{ext}^{(n)}$ *are the kernels from decompositions* (1.4) *for* F, J*. In particular,*

$$\langle L_n, F^{(n)}\rangle \Diamond \langle L_m, J^{(m)}\rangle = \langle L_{n+m}, F^{(n)} \diamond J^{(m)}\rangle.$$

Proof. It was proved in [8] that for $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $J^{(m)} \in \mathcal{H}_{ext}^{(m)}$, $\lambda \in \mathcal{S}_{\mathbb{C}}$

$$\langle F^{(n)}, \lambda^{\otimes n}\rangle_{ext}\langle J^{(m)}, \lambda^{\otimes m}\rangle_{ext} = \langle F^{(n)} \diamond J^{(m)}, \lambda^{\otimes(n+m)}\rangle_{ext}. \tag{1.8}$$

Using this fact and (1.7) we obtain

$$\begin{aligned}
F\Diamond J &= S^{-1}(SF \cdot SJ) = S^{-1}\Big(\sum_{n,m=0}^{\infty} \langle F^{(n)}, \lambda^{\otimes n}\rangle_{ext}\langle J^{(m)}, \lambda^{\otimes m}\rangle_{ext}\Big) \\
&= S^{-1}\Big(\sum_{n,m=0}^{\infty} \langle F^{(n)} \diamond J^{(m)}, \lambda^{\otimes(n+m)}\rangle_{ext}\Big) \\
&= S^{-1}\Big(\sum_{k=0}^{\infty}\langle \sum_{n=0}^{k} F^{(n)} \diamond J^{(k-n)}, \lambda^{\otimes k}\rangle_{ext}\Big) = \sum_{k=0}^{\infty}\langle L_k, \sum_{n=0}^{k} F^{(n)} \diamond J^{(k-n)}\rangle; \\
h^{\Diamond}(F) &= S^{-1}h(SF) = S^{-1}h\Big(\sum_{m=0}^{\infty}\langle F^{(m)}, \lambda^{\otimes m}\rangle_{ext}\Big) \\
&= S^{-1}\Big(h_0 + \sum_{n=1}^{\infty} h_n\Big(\sum_{m=1}^{\infty}\langle F^{(m)}, \lambda^{\otimes m}\rangle_{ext}\Big)^n\Big) \\
&= h_0 + S^{-1}\Big(\sum_{n=1}^{\infty} h_n \sum_{m_1,\dots,m_n=1}^{\infty} \langle F^{(m_1)}, \lambda^{\otimes m_1}\rangle_{ext} \dots \langle F^{(m_n)}, \lambda^{\otimes m_n}\rangle_{ext}\Big) \\
&= h_0 + S^{-1}\Big(\sum_{n=1}^{\infty} h_n \sum_{m_1,\dots,m_n=1}^{\infty} \langle F^{(m_1)} \diamond \dots \diamond F^{(m_n)}, \lambda^{\otimes(m_1+\dots+m_n)}\rangle_{ext}\Big) \\
&= h_0 + S^{-1}\Big(\sum_{k=1}^{\infty}\langle \sum_{n=1}^{k} h_n \sum_{m_1,\dots,m_n \in \mathbb{N}: m_1+\dots+m_n=k} F^{(m_1)} \diamond \dots \diamond F^{(m_n)}, \lambda^{\otimes k}\rangle_{ext}\Big) \\
&= h_0 + \sum_{k=1}^{\infty}\langle L_k, \sum_{n=1}^{k} h_n \sum_{m_1,\dots,m_n \in \mathbb{N}: m_1+\dots+m_n=k} F^{(m_1)} \diamond \dots \diamond F^{(m_n)}\rangle,
\end{aligned}$$

where $F^{(n)}, J^{(n)} \in \mathcal{H}_{ext}^{(n)}$ are the kernels from decompositions (1.4) for F, J correspondingly. □

Remark 1.9. In the classical Gaussian analysis the (Gaussian) Wick exponential coincides with $\exp^{\diamond}(\langle H_1, \lambda\rangle)$ (here $H_1(x) = x$ is the kernel of the generalized Hermite polynomial, see, e.g., [13] for more details). But now

$$\exp^{\diamond}(\langle L_1, \lambda\rangle) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle L_1, \lambda\rangle^{\Diamond n} = \sum_{n=0}^{\infty} \frac{1}{n!}\langle L_n, \lambda^{\diamond n}\rangle \neq: \exp(\cdot; \lambda) := \sum_{n=0}^{\infty} \frac{1}{n!}\langle L_n, \lambda^{\otimes n}\rangle$$

and therefore the inherited from the Gaussian analysis term "Wick exponential" for $: \exp(\cdot; \lambda) :$ is, strictly speaking, inaccurate. Similar situations are typical for the Gamma analysis.

There is a simple interconnection between the Wick calculus and the stochastic integration. In order to explain this interconnection let us consider the Kondratiev-type space of *nonregular* generalized functions $(\mathcal{S}')'$, this space can be constructed by analogy with $(L^2)^{-1}$, but with using of singular kernels in decompositions of type (1.4) (see

[9, 7, 8] for details). One can consider the Gamma white noise at $s \in \mathbb{R}_+$ $G'_s = \langle L_1, \delta_s \rangle$ as an element of $(\mathcal{S}')'$, and construct for $t \in [0, +\infty]$ and $F \in (L^2)^{-1} \otimes \mathcal{H}_{\mathbb{C}}$ the Pettis-type integral $\int_0^t F_s \Diamond G'_s \sigma(ds) \in (\mathcal{S}')'$, i.e., this integral is a unique element of $(\mathcal{S}')'$ such that

$$\langle\langle \int_0^t F_s \Diamond G'_s \sigma(ds), f \rangle\rangle = \int_0^t \langle\langle F_s \Diamond G'_s, f \rangle\rangle \sigma(ds) \quad \forall f \in (\mathcal{S})$$

(here $(\mathcal{S})$ is the corresponding to $(\mathcal{S}')'$ test functions space, see [9, 7, 8] for details; $F_s \in (L^2)^{-1} \subset (\mathcal{S}')'$ is a representative from the corresponding equivalence class).

Theorem 1.2. [8]. *For all $t \in [0, +\infty]$ and $F \in (L^2)^{-1} \otimes \mathcal{H}_{\mathbb{C}}$ $\int_0^t F_s \Diamond G'_s \sigma(ds)$ can be extended to a linear continuous functional on $(L^2)^1$ that coincides with $\int_0^t F_s \widehat{d} G_s$, i.e.,*

$$\int_0^t F_s \Diamond G'_s \sigma(ds) = \int_0^t F_s \widehat{d} G_s \in (L^2)^{-1}. \tag{1.9}$$

By another words, one can "forget" now about the spaces $(\mathcal{S})$ and $(\mathcal{S}')'$, and operate with $\Diamond$ in the left hand side of (1.9) as with the usual Wick product.

2. Generalized stochastic derivatives

We begin from some "technical preparation". For $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ and $f^{(m)} \in \mathcal{H}_{ext}^{(m)}$ $(n > m)$ we define a "pairing" $\langle F^{(n)}, f^{(m)} \rangle_{ext} \in \mathcal{H}_{ext}^{(n-m)}$ by the formula

$$\langle \langle F^{(n)}, f^{(m)} \rangle_{ext}, g^{(n-m)} \rangle_{ext} = \langle F^{(n)}, g^{(n-m)} \diamond f^{(m)} \rangle_{ext} \quad \forall g^{(n-m)} \in \mathcal{H}_{ext}^{(n-m)}$$

(see Lemma 1.2). Since

$$|\langle F^{(n)}, g^{(n-m)} \diamond f^{(m)} \rangle_{ext}| \le |F^{(n)}|_{ext} |g^{(n-m)} \diamond f^{(m)}|_{ext} \le |F^{(n)}|_{ext} |g^{(n-m)}|_{ext} |f^{(m)}|_{ext},$$

this definition is correct and

$$|\langle F^{(n)}, f^{(m)} \rangle_{ext}|_{ext} \le |F^{(n)}|_{ext} |f^{(m)}|_{ext}.$$

In order to define an extended stochastic derivative on $(L^2)^{-1}$ we need the following statement.

Proposition 2.1. [7]. *Let $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$. Then there exists a unique $F^{(n)}(\cdot) \in \mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ such that*

$$\int_{\mathbb{R}} F^{(n)}(\tau) f^{(1)}(\tau) \sigma(d\tau) = \langle F^{(n)}, f^{(1)} \rangle_{ext} \in \mathcal{H}_{ext}^{(n-1)} \quad \forall f^{(1)} \in \mathcal{H}_{\mathbb{C}} \tag{2.1}$$

and

$$|F^{(n)}(\cdot)|_{\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}} \le |F^{(n)}|_{ext}.$$

Definition 2.1. Let $F \in (L^2)^{-1}$. We define the generalized stochastic derivative $\partial_{\cdot} F \in (L^2)^{-1} \otimes \mathcal{H}_{\mathbb{C}}$ by setting

$$\partial_{\cdot} F := \sum_{n=1}^{\infty} n \langle L_{n-1}, F^{(n)}(\cdot) \rangle, \tag{2.2}$$

where the kernels $F^{(n)}(\cdot) \in \mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ are constructed in Proposition 2.1 starting from the kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ from decomposition (1.4) for F.

Let us prove the correctness of this definition. Since $F \in (L^2)^{-1}$, there exists $q \in \mathbb{N}$ such that $F \in (L^2)^{-1}_{-(q-1)}$ and therefore $\|F\|_{-(q-1)} < \infty$. We can estimate as follows:

$$\begin{aligned}\|\partial_\cdot F\|^2_{(L^2)^{-1}_{-q}\otimes\mathcal{H}_{\mathbb{C}}} &= \sum_{n=1}^{\infty} 2^{-q(n-1)} n^2 |F^{(n)}(\cdot)|^2_{\mathcal{H}^{(n-1)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}} \\ &\le 2^q \sum_{n=1}^{\infty} [n^2 2^{-n}] 2^{-(q-1)n} |F^{(n)}|^2_{ext} \le 9\cdot 2^{q-3}\|F\|^2_{-(q-1)} < \infty\end{aligned}$$

(we used the equality $\max_{n\in\mathbb{N}}[n^2 2^{-n}] = 9/8$). Hence $\partial_\cdot$ is a well-defined linear *continuous* operator acting from $(L^2)^{-1}$ to $(L^2)^{-1}\otimes\mathcal{H}_{\mathbb{C}}$.

Remark 2.1. Let us consider a generalized differential operator $\langle f^{(m)}, :D:^{\otimes m}\rangle_{ext}$ (with a constant coefficient $f^{(m)} \in \mathcal{H}^{(m)}_{ext}$) that is defined on monomials $\langle L_n, F^{(n)}\rangle$ ($F^{(n)} \in \mathcal{H}^{(n)}_{ext}$) by the formula

$$\langle f^{(m)}, :D:^{\otimes m}\rangle_{ext}\langle L_n, F^{(n)}\rangle := 1_{\{n\ge m\}}\frac{n!}{(n-m)!}\langle L_{n-m}, \langle F^{(n)}, f^{(m)}\rangle_{ext}\rangle,$$

where $1_{\{n\ge m\}}$ denotes the indicator of $\{n \ge m\}$. This operator can be continued by linearity and continuity on $(L^2)^1_q$ ($q \in \mathbb{N}$) and can be used in order to study differential equations on $(L^2)^1_q$ by analogy with [10]; another applications are connected with the stochastic integration (see, e.g., [8]). Note that *formally* $\partial_\cdot = \langle\delta_\cdot, :D:\rangle_{ext}$, where δ is the delta-function. The operator of this type in the Gaussian analysis is called the Hida derivative (see, e.g., [6]).

Remark 2.2. Note that Definition 2.1 is a *direct generalization* of the definition of the stochastic derivative on (L^2), see [7]. But as it was explained in [7], this definition *can not be "transferred" to the space of nonregular generalized functions* $(\mathcal{S}')'$, therefore in a sense $(L^2)^{-1}$ is a "more natural" space for study of the stochastic derivative than the space $(\mathcal{S}')'$.

In [8] the operator $\partial_\cdot$ was introduced by formula (2.2) on the space $(L^2)^1_q$ ($q \in \mathbb{N}$). Then it was established that $\partial_\cdot$ is connected with the extended stochastic integral by the formula

$$\langle\!\langle \int_0^t F_s \widehat{d}G_s, f\rangle\!\rangle = \int_0^t \langle\!\langle F_s, \partial_s f\rangle\!\rangle \sigma(ds), \quad \forall t \in [0, +\infty], \tag{2.3}$$

where $f \in (L^2)^1_q$, $F \in (L^2)^{-1}_{-q}\otimes\mathcal{H}_{\mathbb{C}}$, $q \in \mathbb{N}$. For the operator $\partial_\cdot$ on $(L^2)^{-1}$ we have the similar property.

Theorem 2.1. *Let* $F \in (L^2)^{-1}$, $f \in (L^2)^1\otimes\mathcal{H}_{\mathbb{C}}$. *Then*

$$\langle\!\langle F, \int_0^t f_s \widehat{d}G_s\rangle\!\rangle = \int_0^t \langle\!\langle \partial_s F, f_s\rangle\!\rangle \sigma(ds), \quad \forall t \in [0, +\infty]. \tag{2.4}$$

Proof. First we note that $\forall q \in \mathbb{N}$

$$\begin{aligned}\|\int_0^t f_s \widehat{d}G_s\|^2_q &= \sum_{n=0}^{\infty} ((n+1)!)^2 2^{q(n+1)} |\widehat{f}^{(n)}_{[0,t)}|^2_{ext} \\ &\le 2^q \sum_{n=0}^{\infty} [(n+1)^2 2^{-n}](n!)^2 2^{(q+1)n} |f_\cdot^{(n)}|^2_{\mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}} \\ &\le 9\cdot 2^{q-2}\|f\|^2_{(L^2)^1_{(q+1)}\otimes\mathcal{H}_{\mathbb{C}}} < \infty\end{aligned}$$

(see (1.6)), here $f_\cdot^{(n)} \in \mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}$ are the kernels from decomposition (1.5) for f, $\widehat{f}^{(n)}_{[0,t)} \in \mathcal{H}^{(n+1)}_{ext}$ are constructed in Lemma 1.1 starting from $f_\cdot^{(n)}$; therefore $\int_0^t f_s \widehat{d}G_s \in (L^2)^1$ and the left hand side of (2.4) is well-defined. Further, using decompositions (1.4) for F and (1.6) for $\int_0^t f_s \widehat{d}G_s$ we obtain

$$\langle\langle F, \int_0^t f_s \widehat{d}G_s \rangle\rangle = \langle\langle \sum_{m=0}^{\infty} \langle L_m, F^{(m)} \rangle, \sum_{n=0}^{\infty} \langle L_{n+1}, \widehat{f}^{(n)}_{[0,t)} \rangle \rangle\rangle$$
$$= \sum_{n=0}^{\infty} (n+1)! \langle F^{(n+1)}, \widehat{f}^{(n)}_{[0,t)} \rangle_{ext}.$$

On the other hand, using decompositions (2.2) for $\partial_\cdot F$ and (1.5) for $f_\cdot$ we have

$$\langle\langle \partial_\cdot F, f_\cdot \rangle\rangle = \langle\langle \sum_{n=0}^{\infty} (n+1) \langle L_n, F^{(n+1)}(\cdot) \rangle, \sum_{m=0}^{\infty} \langle L_m, f_\cdot^{(m)} \rangle \rangle\rangle$$
$$= \sum_{n=0}^{\infty} (n+1)! \langle F^{(n+1)}(\cdot), f_\cdot^{(n)} \rangle_{ext}.$$

Therefore in order to finish the proof it is sufficient to show that

$$\langle F^{(n+1)}, \widehat{f}^{(n)}_{[0,t)} \rangle_{ext} = \int_0^t \langle F^{(n+1)}(s), f_s^{(n)} \rangle_{ext} \sigma(ds),$$

but this equality was proved in [8]. □

Remark 2.3. Formula (2.4) (in the same way as (2.3)) can be rewritten in the form

$$\langle\langle F, \int_0^t f_s \widehat{d}G_s \rangle\rangle = \int_0^t \langle\langle F, \partial_s^\dagger f_s \rangle\rangle \sigma(ds) \equiv \langle\langle F, \int_0^t \partial_s^\dagger f_s \sigma(ds) \rangle\rangle,$$

where $\partial_\cdot^\dagger$ is the adjoint to $\partial_\cdot$ with respect to the scalar product in (L^2) operator. Therefore

$$\int_0^t f_s \widehat{d}G_s = \int_0^t \partial_s^\dagger f_s \sigma(ds).$$

The analogous result is well-known in the Gaussian analysis, see, e.g., [6].

For further presentation it will be convenient to introduce another "stochastic differential operator" $\mathcal{D}$, in a sense $\mathcal{D}$ is equivalent to $\partial_\cdot$ (see Proposition 2.2 below).

Definition 2.2. We define a generalized stochastic derivative $\mathcal{D} : (L^2)^{-1} \to \mathcal{L}(\mathcal{H}_{\mathbb{C}}, (L^2)^{-1})$ (here and below $\mathcal{L}$ denotes a set of linear continuous operators) by the formula

$$(\mathcal{D}F)(\circ) := \sum_{n=1}^{\infty} n \langle L_{n-1}, \langle F^{(n)}, \circ \rangle_{ext} \rangle,$$

where $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ ($n \in \mathbb{N}$) are the kernels from decomposition (1.4) for $F \in (L^2)^{-1}$. Hence, for $f = \sum_{n=0}^{\infty} \langle L_n, f^{(n)} \rangle \in (L^2)^1$ (see (1.3)) and $g^{(1)} \in \mathcal{H}_{\mathbb{C}} = \mathcal{H}_{ext}^{(1)}$ we have

$$\langle\langle (\mathcal{D}F)(g^{(1)}), f \rangle\rangle = \sum_{n=0}^{\infty} (n+1)! \langle F^{(n+1)}, f^{(n)} \diamond g^{(1)} \rangle_{ext}.$$

Let us prove the correctness of this definition. Since $\exists q \in \mathbb{N}$ such that $F \in (L^2)^{-1}_{-(q-1)}$, we can estimate as follows:

$$\begin{aligned}
|\langle\!\langle (\mathcal{D}F)(g^{(1)}), f\rangle\!\rangle| &\leq \sum_{n=0}^{\infty}(n+1)!|\langle F^{(n+1)}, f^{(n)} \diamond g^{(1)}\rangle_{ext}| \\
&\leq \sum_{n=0}^{\infty}(n+1)!|F^{(n+1)}|_{ext}|f^{(n)}|_{ext}|g^{(1)}|_{ext} \\
&= |g^{(1)}|_{ext}\sum_{n=0}^{\infty} n!2^{qn/2}|f^{(n)}|_{ext}[(n+1)2^{-n/2}]2^{-(q-1)n/2}|F^{(n+1)}|_{ext} \\
&\leq |g^{(1)}|_{ext}\sqrt{\sum_{n=0}^{\infty}(n!)^2 2^{qn}|f^{(n)}|^2_{ext}}\sqrt{\sum_{n=0}^{\infty}[(n+1)^2 2^{-n}]2^{-(q-1)n}|F^{(n+1)}|^2_{ext}} \\
&\leq |g^{(1)}|_{ext}\|f\|_q \cdot \frac{3}{2}\cdot 2^{\frac{q-1}{2}}\sqrt{\sum_{n=0}^{\infty}2^{-(q-1)(n+1)}|F^{(n+1)}|^2_{ext}} \\
&\leq 3\cdot 2^{\frac{q-3}{2}}|g^{(1)}|_{ext}\|f\|_q\|F\|_{-(q-1)} < \infty
\end{aligned}$$

(because $\max_{n\in\mathbb{Z}_+}[(n+1)^2 2^{-n}] = 9/4$). Therefore in fact $(\mathcal{D}F)(\circ) \in \mathcal{L}(\mathcal{H}_{\mathbb{C}}, (L^2)^{-1})$ and, moreover, $\forall g^{(1)} \in \mathcal{H}_{\mathbb{C}}$ $(\mathcal{D}\circ)(g^{(1)})$ is a linear *continuous* operator acting from $(L^2)^{-1}$ to $(L^2)^{-1}$.

Remark 2.4. Note that $(\mathcal{D}F)(\circ) = \langle \circ, : D :\rangle_{ext}F$ (see Remark 2.1).

Proposition 2.2. *For all $F \in (L^2)^{-1}$, $g^{(1)} \in \mathcal{H}_{\mathbb{C}}$*

$$\int_{\mathbb{R}} \partial_s F \cdot g^{(1)}(s)\sigma(ds) = (\mathcal{D}F)(g^{(1)}) \in (L^2)^{-1}. \tag{2.5}$$

Proof. Using (2.1), $\forall f = \sum_{n=0}^{\infty}\langle L_n, f^{(n)}\rangle \in (L^2)^1$ (see (1.3)) we obtain

$$\begin{aligned}
&\langle\!\langle \int_{\mathbb{R}} \partial_s F \cdot g^{(1)}(s)\sigma(ds), f\rangle\!\rangle \\
&= \langle\!\langle \int_{\mathbb{R}}\sum_{m=0}^{\infty}(m+1)\langle L_m, F^{(m+1)}(s)\rangle g^{(1)}(s)\sigma(ds), \sum_{n=0}^{\infty}\langle L_n, f^{(n)}\rangle\rangle\!\rangle \\
&= \int_{\mathbb{R}}\sum_{n=0}^{\infty}(n+1)!\langle F^{(n+1)}(s), f^{(n)}\rangle_{ext} g^{(1)}(s)\sigma(ds) \\
&= \sum_{n=0}^{\infty}(n+1)!\langle \int_{\mathbb{R}} F^{(n+1)}(s)g^{(1)}(s)\sigma(ds), f^{(n)}\rangle_{ext} \\
&= \sum_{n=0}^{\infty}(n+1)!\langle\langle F^{(n+1)}, g^{(1)}\rangle_{ext}, f^{(n)}\rangle_{ext} = \sum_{n=0}^{\infty}(n+1)!\langle F^{(n+1)}, f^{(n)}\diamond g^{(1)}\rangle_{ext} \\
&= \langle\!\langle (\mathcal{D}F)(g^{(1)}), f\rangle\!\rangle.
\end{aligned}$$

The proposition is proved. □

Theorem 2.1 can be reformulated "in terms of $\mathcal{D}$" as follows:

Theorem 2.2. *For all $F \in (L^2)^{-1}$, $f \in (L^2)^1$ and $g^{(1)} \in \mathcal{H}_{\mathbb{C}}$*

$$\langle\!\langle F, \int_0^{\infty} f \cdot g^{(1)}(s)\widehat{d}G_s\rangle\!\rangle = \langle\!\langle F, f\Diamond\langle L_1, g^{(1)}\rangle\rangle\!\rangle = \langle\!\langle (\mathcal{D}F)(g^{(1)}), f\rangle\!\rangle.$$

Proof. First we note that the equality $\int_0^\infty f \cdot g^{(1)}(s)\widehat{d}G_s = f\Diamond\langle L_1, g^{(1)}\rangle$ follows directly from Lemma 3.1 in [9]. Further, using (2.4) and (2.5) we obtain

$$\langle\!\langle F, \int_0^\infty f \cdot g^{(1)}(s)\widehat{d}G_s\rangle\!\rangle = \int_0^\infty \langle\!\langle \partial_s F, f \cdot g^{(1)}(s)\rangle\!\rangle \sigma(ds)$$
$$= \langle\!\langle \int_0^\infty \partial_s F \cdot g^{(1)}(s)\sigma(ds), f\rangle\!\rangle = \langle\!\langle (\mathcal{D}F)(g^{(1)}), f\rangle\!\rangle.$$

The theorem is proved. □

Let us study another properties of the operator $\mathcal{D}$.

Theorem 2.3. *For any $F \in (L^2)^{-1}$ the kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ from decomposition* (1.4) *satisfy the equalities*

$$\langle F^{(n)}, g_1^{(1)} \diamond \cdots \diamond g_n^{(1)}\rangle_{ext} = \frac{1}{n!}\mathbf{E}(\mathcal{D}^n F)(g_1^{(1)} \diamond \cdots \diamond g_n^{(1)}) \quad \forall g_1^{(1)}, \dots, g_n^{(1)} \in \mathcal{H}_{\mathbb{C}}, \tag{2.6}$$

where $\mathbf{E}$ *denotes the expectation.*

Proof. First we note that for $F \in (L^2)^{-1}$ and $g_1^{(1)} \in \mathcal{H}_{\mathbb{C}}$

$$(\mathcal{D}F)(g_1^{(1)}) = \sum_{m=0}^\infty (m+1)\langle L_m, \langle F^{(m+1)}, g_1^{(1)}\rangle_{ext}\rangle \in (L^2)^{-1},$$

where $F^{(m)} \in \mathcal{H}_{ext}^{(m)}$, $m \in \mathbb{N}$ are the kernels from decomposition (1.4) for F. Applying $\mathcal{D}$ to $(\mathcal{D}F)(g_1^{(1)})$ we obtain

$$(\mathcal{D}(\mathcal{D}F)(g_1^{(1)}))(g_2^{(1)}) = \sum_{m=0}^\infty (m+1)(m+2)\langle L_m, \langle\langle F^{(m+2)}, g_1^{(1)}\rangle_{ext}, g_2^{(1)}\rangle_{ext}\rangle \in (L^2)^{-1} \tag{2.7}$$

(here $g_2^{(1)} \in \mathcal{H}_{\mathbb{C}}$).

Lemma 2.1. *Let $n, m, k \in \mathbb{Z}_+$, $n \geq k + m$. For $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $f^{(m)} \in \mathcal{H}_{ext}^{(m)}$, $g^{(k)} \in \mathcal{H}_{ext}^{(k)}$*

$$\langle\langle F^{(n)}, f^{(m)}\rangle_{ext}, g^{(k)}\rangle_{ext} = \langle F^{(n)}, g^{(k)} \diamond f^{(m)}\rangle_{ext} \in \mathcal{H}_{ext}^{(n-k-m)}.$$

Proof. Taking into account the associativity of $\diamond$, $\forall \varphi^{(n-k-m)} \in \mathcal{H}_{ext}^{(n-k-m)}$ we obtain

$$\langle\langle\langle F^{(n)}, f^{(m)}\rangle_{ext}, g^{(k)}\rangle_{ext}, \varphi^{(n-k-m)}\rangle_{ext} = \langle\langle F^{(n)}, f^{(m)}\rangle_{ext}, \varphi^{(n-k-m)} \diamond g^{(k)}\rangle_{ext}$$
$$= \langle F^{(n)}, \varphi^{(n-k-m)} \diamond g^{(k)} \diamond f^{(m)}\rangle_{ext} = \langle\langle F^{(n)}, g^{(k)} \diamond f^{(m)}\rangle_{ext}, \varphi^{(n-k-m)}\rangle_{ext}.$$

The lemma is proved. □

Considering the result of Lemma 2.1 (and the commutativity of $\diamond$), we can rewrite (2.7) in the form

$$(\mathcal{D}^2 F)(g_1^{(1)} \diamond g_2^{(1)}) := (\mathcal{D}(\mathcal{D}F)(g_1^{(1)}))(g_2^{(1)})$$
$$= \sum_{m=0}^\infty (m+1)(m+2)\langle L_m, \langle F^{(m+2)}, g_1^{(1)} \diamond g_2^{(1)}\rangle_{ext}\rangle.$$

Now it is easy to show by induction that

$$(\mathcal{D}^n F)(g_1^{(1)} \diamond \cdots \diamond g_n^{(1)}) = \sum_{m=0}^\infty \frac{(m+n)!}{m!}\langle L_m, \langle F^{(m+n)}, g_1^{(1)} \diamond \cdots \diamond g_n^{(1)}\rangle_{ext}\rangle. \tag{2.8}$$

Therefore,

$$\begin{aligned}\mathbf{E}(\mathcal{D}^nF)(g_1^{(1)}\diamond\dots\diamond g_n^{(1)})&\equiv\langle\!\langle(\mathcal{D}^nF)(g_1^{(1)}\diamond\dots\diamond g_n^{(1)}),1\rangle\!\rangle\\&=\langle\!\langle\sum_{m=0}^{\infty}\frac{(m+n)!}{m!}\langle L_m,\langle F^{(m+n)},g_1^{(1)}\diamond\dots\diamond g_n^{(1)}\rangle_{ext}\rangle,1\rangle\!\rangle\\&=n!\langle F^{(n)},g_1^{(1)}\diamond\dots\diamond g_n^{(1)}\rangle_{ext}.\end{aligned}$$

The theorem is proved. □

Remark 2.5. In the classical Gaussian (and Poissonian) analysis one can write the corresponding analog of formula (2.6) in the form $F^{(n)}=\frac{1}{n!}\mathbf{E}(\mathcal{D}^nF)$. Now this form is incorrect because elements $g_1^{(1)}\diamond\dots\diamond g_n^{(1)}\in\mathcal{H}_{ext}^{(n)}$ do not form a total set in $\mathcal{H}_{ext}^{(n)}$. Nevertheless, it is possible to generalize the operator $\mathcal{D}^n$ such that formula (2.6) will be rewritten in the "classical" form. Namely, considering (2.8) we can *define* for arbitrary $f^{(n)}\in\mathcal{H}_{ext}^{(n)}$

$$(\mathcal{D}^nF)(f^{(n)}):=\sum_{m=0}^{\infty}\frac{(m+n)!}{m!}\langle L_m,\langle F^{(m+n)},f^{(n)}\rangle_{ext}\rangle.$$

The operator $(\mathcal{D}^n\circ)(f^{(n)})$ is well-defined as a linear continuous one in $(L^2)^{-1}$. In fact, for each $F\in(L^2)^{-1}$ there exists $q\in\mathbb{N}$ such that $F\in(L^2)^{-1}_{-(q-2)}$, and we can estimate as follows:

$$\begin{aligned}\|(\mathcal{D}^nF)(f^{(n)})\|_{-q}^2&=\|\sum_{m=0}^{\infty}\frac{(m+n)!}{m!}\langle L_m,\langle F^{(m+n)},f^{(n)}\rangle_{ext}\rangle\|_{-q}^2\\&\le\sum_{m=0}^{\infty}2^{-qm}(\frac{(m+n)!}{m!})^2|F^{(m+n)}|_{ext}^2|f^{(n)}|_{ext}^2\\&\le(n!)^2|f^{(n)}|_{ext}^2\sum_{m=0}^{\infty}2^{-qm}2^{2(m+n)}|F^{(m+n)}|_{ext}^2\\&=2^{qn}(n!)^2|f^{(n)}|_{ext}^2\sum_{m=0}^{\infty}2^{-(q-2)(m+n)}|F^{(m+n)}|_{ext}^2\\&\le2^{qn}(n!)^2|f^{(n)}|_{ext}^2\|F\|_{-(q-2)}^2<\infty\end{aligned}$$

(we used the estimate $\frac{(m+n)!}{m!}=n!C_{m+n}^m\le n!2^{m+n}$). Now $\forall F\in(L^2)^{-1}$, $\forall f^{(n)}\in\mathcal{H}_{ext}^{(n)}$

$$\mathbf{E}(\mathcal{D}^nF)(f^{(n)})=\langle\!\langle\sum_{m=0}^{\infty}\frac{(m+n)!}{m!}\langle L_m,\langle F^{(m+n)},f^{(n)}\rangle_{ext}\rangle,1\rangle\!\rangle=n!\langle F^{(n)},f^{(n)}\rangle_{ext},$$

this equality can be formally rewritten in the form $F^{(n)}=\frac{1}{n!}\mathbf{E}(\mathcal{D}^nF)$.

Finally we note that $(\mathcal{D}^nF)(f^{(n)})=\langle f^{(n)},:D:^{\otimes n}\rangle_{ext}F$ (see Remark 2.1). Therefore we proved that the generalized differential operator $\langle f^{(n)},:D:^{\otimes n}\rangle_{ext}$ can be continued to a linear continuous operator on $(L^2)^{-1}$.

As is well known, in the classical Gaussian and Poissonian analysis the analog of the operator $\mathcal{D}$ can be constructed as a pre-image of the directional derivative under the S-transform (see [7] for more details). In the Gamma analysis the situation is slightly more complicated, nevertheless the similar result holds true. Let us explain this explicitly.

Definition 2.3. Let $g\in\mathcal{H}_{\mathbb{C}}$. We define the "directional derivative" $D_g^\diamond:B\to B$ (see Definition 1.8) by setting for $(SF)(\cdot)=\sum_{n=0}^{\infty}\langle F^{(n)},\cdot^{\otimes n}\rangle_{ext}\in B$

$$(D_g^\diamond SF)(\cdot):=\sum_{n=1}^{\infty}n\langle F^{(n)},\cdot^{\otimes(n-1)}\diamond g\rangle_{ext}\equiv\sum_{n=0}^{\infty}(n+1)\langle\langle F^{(n+1)},g\rangle_{ext},\cdot^{\otimes n}\rangle_{ext}\in B.$$

Let us prove the correctness of this definition. Let $SF \in B$. It means that $S^{-1}(SF) = F = \sum_{n=0}^{\infty}\langle L_n, F^{(n)}\rangle \in (L^2)^{-1}$ (here $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$ are the kernels from decomposition (1.7) for SF) and therefore there exists $q \in \mathbb{N}$ such that $\|F\|^2_{-(q-1)} = \sum_{n=0}^{\infty} 2^{-(q-1)n}|F^{(n)}|^2_{ext} < \infty$. Let us consider (formally!)

$$S^{-1}(D_g^{\diamond}SF) = \sum_{n=0}^{\infty}\langle L_n, (n+1)\langle F^{(n+1)}, g\rangle_{ext}\rangle. \tag{2.9}$$

Since

$$\begin{aligned}\|S^{-1}(D_g^{\diamond}SF)\|^2_{-q} &= \sum_{n=0}^{\infty} 2^{-qn}(n+1)^2|\langle F^{(n+1)}, g\rangle_{ext}|^2_{ext} \\ &\le \sum_{n=0}^{\infty}[2^{-n}(n+1)^2]2^{-(q-1)n}|F^{(n+1)}|^2_{ext}|g|^2_{ext} \\ &\le \frac{9}{4}|g|^2_{ext}2^{q-1}\sum_{n=0}^{\infty}2^{-(q-1)(n+1)}|F^{(n+1)}|^2_{ext} \\ &\le 9\cdot 2^{q-3}|g|^2_{ext}\|F\|^2_{-(q-1)} < \infty\end{aligned}$$

(here as above we used the fact that $\max_{n\in\mathbb{Z}_+}[2^{-n}(n+1)^2] = 9/4$), series (2.9) defines a unique element $K \in (L^2)^{-1}$ such that $SK = D_g^{\diamond}SF \in B$.

Theorem 2.4. *The generalized stochastic derivative $\mathcal{D}$ of a regular generalized function $F \in (L^2)^{-1}$ is a pre-image of the "directional derivative" $D_\circ^{\diamond}$ of SF under the S-transform, i.e.,*

$$(\mathcal{D}F)(g) = S^{-1}(D_g^{\diamond}SF).$$

Proof. Using Definition 2.2 and (2.9) we obtain

$$\begin{aligned}(\mathcal{D}F)(g) &= \sum_{n=1}^{\infty} n\langle L_{n-1}, \langle F^{(n)}, g\rangle_{ext}\rangle \\ &= \sum_{n=0}^{\infty}(n+1)\langle L_n, \langle F^{(n+1)}, g\rangle_{ext}\rangle = S^{-1}(D_g^{\diamond}SF).\end{aligned}$$

The theorem is proved. □

Remark 2.6. It was shown in [7] that the natural generalized stochastic derivative on the space $(\mathcal{S}')'$ of *nonregular* generalized functions is the operator $\widetilde{\mathcal{D}} : (\mathcal{S}')' \to \mathcal{L}(\mathcal{S}_{\mathbb{C}}, (\mathcal{S}')')$ that can be defined by the formula $(\widetilde{\mathcal{D}}F)(g) = S^{-1}(D_g SF)$, where D_g is the directional derivative in the direction $g \in \mathcal{S}_{\mathbb{C}}$. But the operator $\widetilde{\mathcal{D}}$ can not be identified with $\partial.$, and for a *regular* generalized function $F \in (L^2)^{-1}$ and $g \in \mathcal{S}_{\mathbb{C}}$ $(\widetilde{\mathcal{D}}F)(g)$ can be a *nonregular* generalized function. Such situation is inconvenient for applications but natural for the Gamma analysis. The reader can find a more detailed discussion in [7], here we note only that the operator $\mathcal{D}$ can not be continued on $(\mathcal{S}')'$.

It was established in [1] that the generalized stochastic derivative in the Gaussian analysis is a differentiation with respect to the Wick product. The generalized stochastic derivatives in the Poissonian analysis and in the Gamma analysis on $(\mathcal{S}')'$ have the same property, this fact was proved in [7]. Let us prove now the similar property of $\mathcal{D}$.

Theorem 2.5. *The generalized stochastic derivative $\mathcal{D}$ is a differentiation with respect to the Wick product, i.e., $\forall F, J \in (L^2)^{-1}$ we have*

$$\mathcal{D}(F\lozenge J) = (\mathcal{D}F)\lozenge J + F\lozenge(\mathcal{D}J). \tag{2.10}$$

Proof. First we note that by Theorem 2.4 $\forall g\in\mathcal{H}_{\mathbb{C}}$

$$\begin{aligned}(\mathcal{D}(F\lozenge J))(g)&=S^{-1}(D_g^{\diamond}(S(F\lozenge J)))=S^{-1}(D_g^{\diamond}(SF\cdot SJ)),\\(\mathcal{D}F)(g)\lozenge J&=S^{-1}(S(\mathcal{D}F)(g)\cdot SJ)=S^{-1}(D_g^{\diamond}(SF)\cdot SJ),\\F\lozenge(\mathcal{D}J)(g)&=S^{-1}(SF\cdot S(\mathcal{D}J)(g))=S^{-1}(SF\cdot D_g^{\diamond}(SJ)),\end{aligned}$$

therefore it is sufficient to prove that

$$D_g^{\diamond}(SF\cdot SJ)=D_g^{\diamond}(SF)\cdot SJ+SF\cdot D_g^{\diamond}(SJ). \tag{2.11}$$

Let $F^{(n)},J^{(n)}\in\mathcal{H}_{ext}^{(n)}$ be the kernels from decompositions (1.4) for F,J respectively. Then (see (1.7), (1.8) and Definition 2.3)

$$(SF)(\lambda)=\sum_{n=0}^{\infty}\langle F^{(n)},\lambda^{\otimes n}\rangle_{ext},\ (SJ)(\lambda)=\sum_{m=0}^{\infty}\langle J^{(m)},\lambda^{\otimes m}\rangle_{ext},$$

$$(SF)(\lambda)\cdot(SJ)(\lambda)=\sum_{n,m=0}^{\infty}\langle F^{(n)}\diamond J^{(m)},\lambda^{\otimes(n+m)}\rangle_{ext},$$

$$D_g^{\diamond}((SF)(\lambda)\cdot(SJ)(\lambda))=\sum_{n,m=0}^{\infty}(n+m)\langle F^{(n)}\diamond J^{(m)},\lambda^{\otimes(n+m-1)}\diamond g\rangle_{ext},$$

$$D_g^{\diamond}(SF)(\lambda)=\sum_{n=0}^{\infty}n\langle F^{(n)},\lambda^{\otimes(n-1)}\diamond g\rangle_{ext},$$

$$D_g^{\diamond}(SJ)(\lambda)=\sum_{m=0}^{\infty}m\langle J^{(m)},\lambda^{\otimes(m-1)}\diamond g\rangle_{ext},$$

$$D_g^{\diamond}(SF)(\lambda)\cdot(SJ)(\lambda)=\sum_{n,m=0}^{\infty}n\langle F^{(n)},\lambda^{\otimes(n-1)}\diamond g\rangle_{ext}\langle J^{(m)},\lambda^{\otimes m}\rangle_{ext},$$

$$(SF)(\lambda)\cdot D_g^{\diamond}(SJ)(\lambda)=\sum_{n,m=0}^{\infty}m\langle F^{(n)},\lambda^{\otimes n}\rangle_{ext}\langle J^{(m)},\lambda^{\otimes(m-1)}\diamond g\rangle_{ext}.$$

Therefore in order to obtain (2.11) it is sufficient to prove that $\forall n,m\in\mathbb{Z}_+$

$$\begin{aligned}&(n+m)\langle F^{(n)}\diamond J^{(m)},\lambda^{\otimes(n+m-1)}\diamond g\rangle_{ext}\\&\quad=n\langle F^{(n)},\lambda^{\otimes(n-1)}\diamond g\rangle_{ext}\langle J^{(m)},\lambda^{\otimes m}\rangle_{ext}\\&\quad+m\langle F^{(n)},\lambda^{\otimes n}\rangle_{ext}\langle J^{(m)},\lambda^{\otimes(m-1)}\diamond g\rangle_{ext}.\end{aligned} \tag{2.12}$$

It is obvious that for $n=0$ (or $m=0$) (2.12) holds true. Let us consider the case $n,m\in\mathbb{N}$. Note that we can operate with $F^{(n)},J^{(m)},F^{(n)}\diamond J^{(m)}$ etc. as with functions. In fact, since all operations will be "under the integral symbol", one can select representatives from the corresponding equivalence classes, the result does not depend on the selected representatives. Taking into account the symmetry of $\lambda^{\otimes(n+m-1)}\diamond g$ (this symmetry gives us the possibility to rearrange sequences of equal arguments, if these sequences have equal lengths; for example $(\lambda^{\otimes(n+m-1)}\diamond g)(\underbrace{t_1,\dots,t_1}_{l},\dots,\underbrace{t_k,\dots,t_k}_{l})=(\lambda^{\otimes(n+m-1)}\diamond g)(\underbrace{t_k,\dots,t_k}_{l},\dots,\underbrace{t_1,\dots,t_1}_{l})$), by the full analogy with the proof of Lemma 3.1 in [8] we obtain (we use the notation of Lemma 1.2)

$$
\begin{aligned}
&(n+m)\langle F^{(n)} \diamond J^{(m)}, \lambda^{\otimes(n+m-1)} \diamond g\rangle_{ext} \\
&\quad = (n+m) \sum_{\substack{k,l_j,s_j\in\mathbb{N}: j=1,\dots,k, l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n+m}} \frac{(n+m)!}{l_1^{s_1}\dots l_k^{s_k} s_1!\dots s_k!} \\
&\quad \times \int_{\mathbb{R}^{s_1+\dots+s_k}} (F^{(n)} \diamond J^{(m)})(\underbrace{t_1,\dots,t_1}_{l_1},\dots,\underbrace{t_{s_1},\dots,t_{s_1}}_{l_1},\dots,\underbrace{t_{s_1+\dots+s_k},\dots,t_{s_1+\dots+s_k}}_{l_k}) \\
&\quad \times (\lambda^{\otimes(n+m-1)} \diamond g)(\underbrace{t_1,\dots,t_1}_{l_1},\dots,\underbrace{t_{s_1},\dots,t_{s_1}}_{l_1},\dots,\underbrace{t_{s_1+\dots+s_k},\dots,t_{s_1+\dots+s_k}}_{l_k}) \\
&\quad \times \sigma(dt_1)\dots\sigma(dt_{s_1+\dots+s_k}) \\
&\quad = (n+m) \sum_{\substack{k',k'',l'_1,\dots,l'_{k'},s'_1,\dots,s'_{k'},l''_1,\dots,l''_{k''},s''_1,\dots,s''_{k''}\in\mathbb{N}:\\ l'_1>\dots>l'_{k'},\ l''_1>\dots>l''_{k''},\\ l'_1s'_1+\dots+l'_{k'}s'_{k'}=n,\ l''_1s''_1+\dots+l''_{k''}s''_{k''}=m}} \frac{n!}{{l'_1}^{s'_1}\dots {l'_{k'}}^{s'_{k'}} s'_1!\dots s'_{k'}!} \\
&\quad \times \frac{m!}{{l''_1}^{s''_1}\dots {l''_{k''}}^{s''_{k''}} s''_1!\dots s''_{k''}!} \\
&\quad \times \int_{\mathbb{R}^{s'_1+\dots+s'_{k'}+s''_1+\dots+s''_{k''}}} F^{(n)}(\underbrace{t_1,\dots,t_1}_{l'_1},\dots,\underbrace{t_{s'_1+\dots+s'_{k'}},\dots,t_{s'_1+\dots+s'_{k'}}}_{l'_{k'}}) \\
&\quad \times J^{(m)}(\underbrace{t_{n+1},\dots,t_{n+1}}_{l''_1},\dots,\underbrace{t_{n+s''_1+\dots+s''_{k''}},\dots,t_{n+s''_1+\dots+s''_{k''}}}_{l''_{k''}}) \\
&\quad \times (\lambda^{\otimes(n+m-1)} \diamond g)(\underbrace{t_1,\dots,t_1}_{l'_1},\dots,\underbrace{t_{s'_1+\dots+s'_{k'}},\dots,t_{s'_1+\dots+s'_{k'}}}_{l'_{k'}}, \\
&\quad \underbrace{t_{n+1},\dots,t_{n+1}}_{l''_1},\dots,\underbrace{t_{n+s''_1+\dots+s''_{k''}},\dots,t_{n+s''_1+\dots+s''_{k''}}}_{l''_{k''}}) \\
&\quad \times \sigma(dt_1)\dots\sigma(dt_{s'_1+\dots+s'_{k'}})\sigma(dt_{n+1})\dots\sigma(dt_{n+s''_1+\dots+s''_{k''}}) \\
&\quad = \sum_{\substack{k',k'',l'_1,\dots,l'_{k'},s'_1,\dots,s'_{k'},l''_1,\dots,l''_{k''},s''_1,\dots,s''_{k''}\in\mathbb{N}:\\ l'_1>\dots>l'_{k'},\ l''_1>\dots>l''_{k''},\\ l'_1s'_1+\dots+l'_{k'}s'_{k'}=n,\ l''_1s''_1+\dots+l''_{k''}s''_{k''}=m}} \frac{n!}{{l'_1}^{s'_1}\dots {l'_{k'}}^{s'_{k'}} s'_1!\dots s'_{k'}!} \\
&\quad \times \frac{m!}{{l''_1}^{s''_1}\dots {l''_{k''}}^{s''_{k''}} s''_1!\dots s''_{k''}!} \\
&\quad \times \int_{\mathbb{R}^{s'_1+\dots+s'_{k'}+s''_1+\dots+s''_{k''}}} F^{(n)}(\underbrace{t_1,\dots,t_1}_{l'_1},\dots,\underbrace{t_{s'_1+\dots+s'_{k'}},\dots,t_{s'_1+\dots+s'_{k'}}}_{l'_{k'}}) \\
&\quad \times J^{(m)}(\underbrace{t_{n+1},\dots,t_{n+1}}_{l''_1},\dots,\underbrace{t_{n+s''_1+\dots+s''_{k''}},\dots,t_{n+s''_1+\dots+s''_{k''}}}_{l''_{k''}}) \\
&\quad \times [(\lambda^{\otimes\widetilde{(n+m-1)}} g)(\underbrace{t_1,\dots,t_1}_{l'_1},\dots,\underbrace{t_{s'_1+\dots+s'_{k'}},\dots,t_{s'_1+\dots+s'_{k'}}}_{l'_{k'}},
\end{aligned}
$$

$$\underbrace{t_{n+1},\dots,t_{n+1}}_{l''_1},\dots,\underbrace{t_{n+s''_1+\dots+s''_{k''}},\dots,t_{n+s''_1+\dots+s''_{k''}}}_{l''_{k''}}) + \cdots]$$
$$\times\,\sigma(dt_1)\dots\sigma(dt_{s'_1+\dots+s'_{k'}})\sigma(dt_{n+1})\dots\sigma(dt_{n+s''_1+\dots+s''_{k''}}),$$

where every next term from $(n+m)$ summands in the sum $[\dots]$ is obtained from the previous term by the "shift of arguments":

$$(\cdot_1,\cdot_2,\dots,\cdot_{n+m-1},\cdot_{n+m})\to(\cdot_{n+m},\cdot_1,\dots,\cdot_{n+m-2},\cdot_{n+m-1})$$

etc.

Let $A(k',k'',l'_1,\dots,l'_{k'},s'_1,\dots,s'_{k'},l''_1,\dots,l''_{k''},s''_1,\dots,s''_{k''},t.)$ denote the sum of first m terms in $[\dots]$ and $B(k',k'',l'_1,\dots,l'_{k'},s'_1,\dots,s'_{k'},l''_1,\dots,l''_{k''},s''_1,\dots,s''_{k''},t.)$ denote the sum of last n terms in $[\dots]$. Taking into consideration the symmetry of $(\lambda^{\otimes\widehat{(n+m-1)}}g)$ with respect to first $n+m-1$ arguments and a nonatomicity of the measure σ one can conclude that

$$\sum_{\substack{k',k'',l'_1,\dots,l'_{k'},s'_1,\dots,s'_{k'},l''_1,\dots,l''_{k''},s''_1,\dots,s''_{k''}\in\mathbb{N}:\\ l'_1>\dots>l'_{k'},\ l''_1>\dots>l''_{k''},\\ l'_1s'_1+\dots+l'_{k'}s'_{k'}=n,\ l''_1s''_1+\dots+l''_{k''}s''_{k''}=m}}\frac{n!m!}{{l'_1}^{s'_1}\dots {l'_{k'}}^{s'_{k'}}s'_1!\dots s'_{k'}!{l''_1}^{s''_1}\dots {l''_{k''}}^{s''_{k''}}s''_1!\dots s''_{k''}!}$$
$$\times\int_{\mathbb{R}^{s'_1+\dots+s'_{k'}+s''_1+\dots+s''_{k''}}}F^{(n)}(\underbrace{t_1,\dots,t_1}_{l'_1},\dots,\underbrace{t_{s'_1+\dots+s'_{k'}},\dots,t_{s'_1+\dots+s'_{k'}}}_{l'_{k'}})$$
$$\times J^{(m)}(\underbrace{t_{n+1},\dots,t_{n+1}}_{l''_1},\dots,\underbrace{t_{n+s''_1+\dots+s''_{k''}},\dots,t_{n+s''_1+\dots+s''_{k''}}}_{l''_{k''}})$$
$$\times A(k',k'',l'_1,\dots,l'_{k'},s'_1,\dots,s'_{k'},l''_1,\dots,l''_{k''},s''_1,\dots,s''_{k''},t.)$$
$$\times\,\sigma(dt_1)\dots\sigma(dt_{s'_1+\dots+s'_{k'}})\sigma(dt_{n+1})\dots\sigma(dt_{n+s''_1+\dots+s''_{k''}})$$
$$= m\langle F^{(n)},\lambda^{\otimes n}\rangle_{ext}\langle J^{(m)},\lambda^{\otimes(m-1)}\diamond g\rangle_{ext}$$

and

$$\sum_{\substack{k',k'',l'_1,\dots,l'_{k'},s'_1,\dots,s'_{k'},l''_1,\dots,l''_{k''},s''_1,\dots,s''_{k''}\in\mathbb{N}:\\ l'_1>\dots>l'_{k'},\ l''_1>\dots>l''_{k''},\\ l'_1s'_1+\dots+l'_{k'}s'_{k'}=n,\ l''_1s''_1+\dots+l''_{k''}s''_{k''}=m}}\frac{n!m!}{{l'_1}^{s'_1}\dots {l'_{k'}}^{s'_{k'}}s'_1!\dots s'_{k'}!{l''_1}^{s''_1}\dots {l''_{k''}}^{s''_{k''}}s''_1!\dots s''_{k''}!}$$
$$\times\int_{\mathbb{R}^{s'_1+\dots+s'_{k'}+s''_1+\dots+s''_{k''}}}F^{(n)}(\underbrace{t_1,\dots,t_1}_{l'_1},\dots,\underbrace{t_{s'_1+\dots+s'_{k'}},\dots,t_{s'_1+\dots+s'_{k'}}}_{l'_{k'}})$$
$$\times J^{(m)}(\underbrace{t_{n+1},\dots,t_{n+1}}_{l''_1},\dots,\underbrace{t_{n+s''_1+\dots+s''_{k''}},\dots,t_{n+s''_1+\dots+s''_{k''}}}_{l''_{k''}})$$
$$\times B(k',k'',l'_1,\dots,l'_{k'},s'_1,\dots,s'_{k'},l''_1,\dots,l''_{k''},s''_1,\dots,s''_{k''},t.)$$
$$\times\,\sigma(dt_1)\dots\sigma(dt_{s'_1+\dots+s'_{k'}})\sigma(dt_{n+1})\dots\sigma(dt_{n+s''_1+\dots+s''_{k''}})$$
$$= n\langle F^{(n)},\lambda^{\otimes(n-1)}\diamond g\rangle_{ext}\langle J^{(m)},\lambda^{\otimes m}\rangle_{ext}.$$

Thus, (2.12) is proved. □

Corollary. *Let $n \in \mathbb{N}$, $F \in (L^2)^{-1}$, and a function $h : \mathbb{C} \to \mathbb{C}$ be a holomorphic one at $(SF)(0)$. Then we have*

$$\mathcal{D}(F^{\lozenge n}) = nF^{\lozenge (n-1)} \lozenge (\mathcal{D}F), \qquad \mathcal{D}h^{\lozenge}(F) = {h'}^{\lozenge}(F) \lozenge (\mathcal{D}F), \tag{2.13}$$

where h' denotes the usual derivative of h.

Proof. The first formula in (2.13) can be obtained by induction from (2.10), the second formula is a consequence of the first one. □

The forthcoming statement is convenient for some applications (see an example below).

Theorem 2.6. *Let $F \in (L^2)^{-1} \otimes \mathcal{H}_{\mathbb{C}}$. Then $\forall t \in [0, +\infty]$*

$$\Big(\mathcal{D} \int_0^t F_s \widehat{d} G_s\Big)(\circ) = \int_0^t (\mathcal{D}F_s)(\circ)\, \widehat{d} G_s + \int_0^t F_s \circ (s) \sigma(ds). \tag{2.14}$$

Proof. First we note that (see (1.6) and Definition 2.2)

$$\int_0^t F_s \widehat{d} G_s = \sum_{n=0}^{\infty} \langle L_{n+1}, \widehat{F}^{(n)}_{[0,t)} \rangle, \quad \Big(\mathcal{D} \int_0^t F_s \widehat{d} G_s\Big)(\circ) = \sum_{n=0}^{\infty} (n+1) \langle L_n, \langle \widehat{F}^{(n)}_{[0,t)}, \circ \rangle_{ext} \rangle,$$

where the kernels $\widehat{F}^{(n)}_{[0,t)} \in \mathcal{H}^{(n+1)}_{ext}$, $n \in \mathbb{Z}_+$ are constructed in Lemma 1.1 starting from the kernels $F^{(n)}_{\cdot} \in \mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$ from decomposition (1.5) for F. On the other hand,

$$(\mathcal{D}F_{\cdot})(\circ) = \sum_{n=1}^{\infty} n \langle L_{n-1}, \langle F^{(n)}_{\cdot}, \circ \rangle_{ext} \rangle,$$
$$\int_0^t (\mathcal{D}F_s)(\circ) \widehat{d} G_s = \sum_{n=1}^{\infty} n \langle L_n, \widehat{\langle F^{(n)}_{\cdot}, \circ \rangle}_{ext\,[0,t)} \rangle,$$
$$\int_0^t F_s \circ (s) \sigma(ds) = \sum_{n=0}^{\infty} \langle L_n, \int_0^t F^{(n)}_s \circ (s) \sigma(ds) \rangle.$$

Therefore in order to prove the theorem it is sufficient to show that $\forall n \in \mathbb{Z}_+$

$$(n+1) \langle \widehat{F}^{(n)}_{[0,t)}, \circ \rangle_{ext} = n \widehat{\langle F^{(n)}_{\cdot}, \circ \rangle}_{ext\,[0,t)} + \int_0^t F^{(n)}_s \circ (s) \sigma(ds).$$

In order to prove this equality it is sufficient to verify that $\forall g^{(1)} \in \mathcal{H}_{\mathbb{C}}$, $\forall f^{(n)} \in \mathcal{H}^{(n)}_{ext}$

$$\begin{aligned} &(n+1) \langle \langle \widehat{F}^{(n)}_{[0,t)}, g^{(1)} \rangle_{ext}, f^{(n)} \rangle_{ext} \\ &= n \langle \widehat{\langle F^{(n)}_{\cdot}, g^{(1)} \rangle}_{ext\,[0,t)}, f^{(n)} \rangle_{ext} + \langle \int_0^t F^{(n)}_s g^{(1)}(s) \sigma(ds), f^{(n)} \rangle_{ext}. \end{aligned} \tag{2.15}$$

As in the proof of Theorem 2.5 above, we will operate with representatives from the corresponding equivalence classes. Using the formula

$$\langle \widehat{F}^{(n)}_{[0,t)}, f^{(n+1)} \rangle_{ext} = \int_0^t \langle F^{(n)}_s, f^{(n+1)}(s) \rangle_{ext} \sigma(ds)$$

(this fact was proved in [8], formula (2.7); $f^{(n+1)}(\cdot) \in \mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$ is obtained from $f^{(n+1)} \in \mathcal{H}^{(n+1)}_{ext}$ as in Proposition 2.1) and taking into account the nonatomicity of σ we obtain (using the notation of Lemma 1.2)

$$
\begin{aligned}
&(n+1)\langle\langle \widehat{F}^{(n)}_{[0,t)}, g^{(1)}\rangle_{ext}, f^{(n)}\rangle_{ext} = (n+1)\langle \widehat{F}^{(n)}_{[0,t)}, f^{(n)}\diamond g^{(1)}\rangle_{ext}\\
&= (n+1)\int_0^t \langle F^{(n)}_s, (f^{(n)}\diamond g^{(1)})(s)\rangle_{ext}\sigma(ds)\\
&= \int_0^t \langle F^{(n)}_s, f^{(n)}(\widetilde{\cdot_1,\dots,\cdot_n})g^{(1)}(s) + f^{(n)}(\widetilde{\cdot_2,\dots,s})g^{(1)}(\cdot_1)+\cdots\\
&+ f^{(n)}(\widetilde{s,\dots,\cdot_{n-1}})g^{(1)}(\cdot_n)\rangle_{ext}\sigma(ds) = \langle \int_0^t F^{(n)}_s g^{(1)}(s)\sigma(ds), f^{(n)}\rangle_{ext}\\
&+\int_0^t \langle F^{(n)}_s, f^{(n)}(\widetilde{\cdot_2,\dots,s})g^{(1)}(\cdot_1)+\cdots+f^{(n)}(\widetilde{s,\dots,\cdot_{n-1}})g^{(1)}(\cdot_n)\rangle_{ext}\sigma(ds).
\end{aligned}
\tag{2.16}
$$

On the other hand, taking into account the fact that $f^{(n)}$ is a symmetric function, by analogy with (2.16) we obtain

$$
\begin{aligned}
&n\langle\widehat{\langle F^{(n)}_{\cdot}, g^{(1)}\rangle_{ext}}_{[0,t)}, f^{(n)}\rangle_{ext} = n\int_0^t \langle\langle F^{(n)}_s, g^{(1)}\rangle_{ext}, f^{(n)}(s)\rangle_{ext}\sigma(ds)\\
&= n\int_0^t \langle F^{(n)}_s, f^{(n)}(s)\diamond g^{(1)}\rangle_{ext}\sigma(ds) = \int_0^t \langle F^{(n)}_s, f^{(n)}(\widetilde{\cdot_1,\dots,\cdot_{n-1}},s)g^{(1)}(\cdot_n)\\
&+ f^{(n)}(\widetilde{\cdot_2,\dots,\cdot_n},s)g^{(1)}(\cdot_1)+\cdots+f^{(n)}(\widetilde{\cdot_n,\dots,\cdot_{n-2}},s)g^{(1)}(\cdot_n)\rangle_{ext}\sigma(ds)\\
&= \int_0^t \langle F^{(n)}_s, f^{(n)}(\widetilde{\cdot_2,\dots,\cdot_n},s)g^{(1)}(\cdot_1) + f^{(n)}(\widetilde{\cdot_3,\dots,s},\cdot_1)g^{(1)}(\cdot_2)+\cdots\\
&+ f^{(n)}(\cdot_n, \widetilde{s,\dots,\cdot_{n-2}})g^{(1)}(\cdot_n) + f^{(n)}(s,\widetilde{\cdot_1,\dots,\cdot_{n-1}})g^{(1)})(\cdot_n)\rangle_{ext}\sigma(ds).
\end{aligned}
\tag{2.17}
$$

Substituting (2.17) in (2.16) we obtain (2.15). □

By analogy with [1, 7] as an application of our results we will calculate the generalized stochastic derivative of the solution of the stochastic equation

$$
(L^2)^{-1} \ni F_t = F_0 + \int_0^t h^{\Diamond}(F_s)\widehat{d}G_s, \tag{2.18}
$$

where $h : \mathbb{C} \to \mathbb{C}$ is some entire function, $F_0 \in \mathbb{C}$. Under certain conditions on h a unique solution of (2.18) $F_t \in (L^2)^{-1}$ exists. Applying $\mathcal{D}$ to (2.18) and taking into account (2.14) and (2.13), $\forall g^{(1)} \in \mathcal{H}_{\mathbb{C}}$ we obtain

$$
\begin{aligned}
(\mathcal{D}F_t)(g^{(1)}) &= \Big(\mathcal{D}\int_0^t h^{\Diamond}(F_s)\widehat{d}G_s\Big)(g^{(1)})\\
&= \int_0^t h'^{\Diamond}(F_s)\Diamond(\mathcal{D}F_s)(g^{(1)})\widehat{d}G_s + \int_0^t h^{\Diamond}(F_s)g^{(1)}(s)\sigma(ds).
\end{aligned}
\tag{2.19}
$$

Let $\phi^{g^{(1)}}_s(\lambda) := S((\mathcal{D}F_s)(g^{(1)}))(\lambda)$. Applying the S-transform to (2.19) and taking into account (1.9) we obtain

$$
\phi^{g^{(1)}}_t(\lambda) = \int_0^t h'((SF_s)(\lambda))\phi^{g^{(1)}}_s(\lambda)\lambda(s)\sigma(ds) + \int_0^t h((SF_s)(\lambda))g^{(1)}(s)\sigma(ds).
$$

The solution of this equation is

$$
\phi^{g^{(1)}}_t(\lambda) = \int_0^t h((SF_s)(\lambda))g^{(1)}(s)\cdot\exp\Big\{\int_s^t h'((SF_u)(\lambda))\lambda(u)\sigma(du)\Big\}\sigma(ds).
$$

By the inverse S-transform we obtain

$$(\mathcal{D}F_t)(g^{(1)}) = \int_0^t h^{\Diamond}(F_s)g^{(1)}(s)\Diamond \exp^{\Diamond}\left\{\int_s^t h'^{\Diamond}(F_u)\widehat{d}G_u\right\}\sigma(ds) \in (L^2)^{-1}.$$

Remark 2.7. The results of this paper hold true (up to the obvious isomorphism) if we consider the Pascal measure or the Meixner measure instead of the Gamma measure (see [2, 14, 15] for an information about these measures). Moreover, similar results can be obtained in the "nonstationary" case that is based on the so-called generalized Meixner measure ([17]). We'll present these results in details in another paper.

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