Methods of Functional Analysis and Topology
Vol. 12 (2006), no. 4, pp. 389–396

ABOUT KRONROD-REEB GRAPH OF A FUNCTION ON A MANIFOLD

V. V. SHARKO

Dedicated to the memory of Yuri Daletskii.

Abstract. We study Kronrod-Reeb graphs of functions with isolated critical points on smooth manifolds. We prove that any finite graph, which satisfies the condition $\Im$ is a Kronrod-Reeb graph for some such function on some manifold. In this connection, monotone functions on graphs are investigated.

1. Introduction

Let M^n be a closed smooth manifold and denote by $C^\infty(M^n)$ the space of smooth functions on M^n with isolated critical points. A connected component of the level surface $f^{-1}(a)$ of a function f from $C^\infty(M^n)$ is often referred to as a layer. Considering all layers of the function f, we get a decomposition of the manifold M^n into the union of layers. The property of a point of the manifold to belong to a layer determines an equivalence relation and, by introducing the natural quotient topology in the set of layers, we obtain a quotient set, which we denote in sequel by $\Gamma_{K-R}(f)$. This quotient set $\Gamma_{K-R}(f)$ is homeomorphic to a finite graph. The set $\Gamma_{K-R}(f)$ is called the **Kronrod-Reeb graph of the function** f from the space $C^\infty(M^n)$.

The Kronrod-Reeb graph $\Gamma_{K-R}(f)$ of the function f on a manifold M^n admits an orientation, in order to show the direction in which the function f grows. It is a **special orientation** of the graph $\Gamma_{K-R}(f)$ (see Definition 3.1). We will be denote such orientation by **f-orientation** of the graph $\Gamma_{K-R}(f)$. The graph $\Gamma_{K-R}(f)$ with the f-orientation we will called in the sequel by the **f-oriented Kronrod-Reeb graph**.

It is obvious that the function f determines, in a canonical way, a partial order on vertexes of its f-oriented Kronrod-Reeb graph $\Gamma_{K-R}(f)$. A vertex $x \in \Gamma_{K-R}(f)$ precedes a vertex $y \in \Gamma_{K-R}(f)$ if values of f at the corresponding vertexes x and y of the layers N_x and N_y satisfy the inequality $f(N_x) < f(N_y)$. Such an order on the graph $\Gamma_{K-R}(f)$ we will be called the **f-order**. The graph $\Gamma_{K-R}(f)$ with the f-orientation and the f-order we will called a **f-pooriented Kronrod-Reeb graph**.

A graph $\Sigma = (V, E)$ (oriented (pooriented) graph $\Sigma = (V, E)$) for which there exist a manifold M^n and a smooth function $f \in C^\infty(M^n)$ such that the Kronrod-Reeb graph (f-oriented (f-pooriented) Kronrod-Reeb graph) $\Gamma_{K-R}(f)$ for this function f is isomorphic (orientation (orientation and partial order) preserve isomorphic) to the graph $\Sigma = (V, E)$ is called an K-R-graph (oriented (pooriented) K-R-graph).

It should be noted that not every finite graph having at least two vertexes of order one is a Kronrod-Reeb graph for a certain smooth function with finitely many critical points on a smooth n-manifold ($n \geq 2$).

2000 *Mathematics Subject Classification.* 05C10, 57R45.
Key words and phrases. Smooth function, manifold, graph, critical point.

Theorem 2.1. *Any finite graph $\Sigma = (V, E)$ is a K-R-graph of some function on some manifold M^n if and only if $\Sigma = (V, E)$ admits some f-orientation (see Definition 3.1).*

A proof of this theorem is based on the following result.

Theorem 3.1. *Let $\Sigma = (V, E)$ be a finite f-oriented graph. Then on the graph Σ there exists a monotone (increasing) function $g : \Sigma \to R$ such that its oriented Kronrod-Reeb graph is orientation preserving homeomorphic to an f-oriented graph $\Sigma = (V, E)$.*

Let $\Sigma = (V, E)$ be a finite graph. Denote by Ω the set of vertexes of order 1 in the graph $\Sigma = (V, E)$. The graph $\Sigma = (V, E)$ satisfies the condition $\Im$, if it is connected, for any dividing vertex v from Σ and for any connected component $\Sigma_v^i \subset \Sigma_v$ we have $\Sigma_v^i \bigcap \Omega \neq \varnothing$ and the set Ω consists of at least of two vertexes.

Theorem 5.2. *A finite graph $\Sigma = (V, E)$ is an K-R-graph of some function with finite critical points on some manifold M^n if and only if it satisfies the condition $\Im$.*

2. The Kronrod-Reeb graph of a function on a manifold

By a smooth n-manifold M^n (resp., manifold with boundary) we mean an n-dimensional smooth compact manifold without boundary (resp., with boundary ∂M^n). The word "smooth" always indicates that the appropriate object belongs to the class C^∞. By a critical point x of a function f defined on a manifold M^n we mean a point at which the partial derivatives of f vanish.

Let $C^\infty(M^n, \partial M^n)$ denote the space of smooth functions on a manifold M^n with boundary ∂M^n, with a finite number of critical points. Suppose that all critical points of this function lie in the interior of M^n. Assume also that the functions from the space $C^\infty(M^n, \partial M^n)$ take constant values on connected components of the boundary ∂M^n (the case where the boundary is absent is excluded). Let us consider an arbitrary connected component of the level surface $f^1(a)$ of a function f from $C^\infty(M^n, \partial M^n)$; such level surfaces are often referred to as layers. If a is a regular value of the function f, then the layer is a submanifold of dimension $n-1$ smoothly embedded in into M^n. In the case where a is a critical value, the layer is a closed set N. Since critical points of f on N are isolated, we have that N is a manifold with singularities of dimension $n-1$, where singularity set consists of critical points of f which lie on N. Considering all layers of the function f, we get a decomposition of the manifold M^n into the union of layers, i.e., there arises on M^n a foliation with singularities. The property of a point of the manifold to belong to a layer determines an equivalence relation and, by introducing the natural quotient topology in the set of layers, we obtain a quotient set, which we denote in sequel by $\Gamma_{K-R}(f)$.

Lemma 2.1. *Let $f : M^n \longrightarrow [a, b]$ be a function from $C^\infty(M^n, \partial M^n)$. Then the quotient set $\Gamma_{K-R}(f)$ is homeomorphic to a finite graph.*

A finite graph $\Sigma = (V, E)$ is understood as a finite one-dimensional simplicial complex. Here, V are the zero-dimensional simplices (vertexes) and E are the one-dimensional simplices (i.e., edges) [2].

Proof of Lemma 2.1. Let c be a regular value of the function f. Since the manifold M^n is compact, the set $f^{-1}(c)$ has a finite number of connected components. Since critical values of the function f are isolated, for any critical value $d \in (a, b)$ there exists $\varepsilon > 0$ such that the segment $[d-\varepsilon, d+\varepsilon]$ does not contain other critical values. Obviously, for any regular value $\overline{c}$ from the half-interval $[d-\varepsilon, d+\varepsilon)$ $((d-\varepsilon, d+\varepsilon])$, the number of connected components of the level surface $f^{-1}(\overline{c})$ is the same. Therefore, the image of the non-compact manifold $f^{-1}[d-\varepsilon, d+\varepsilon)$ $(f^{-1}(d-\varepsilon, b+\varepsilon])$ into the set $\Gamma_{K-R}(f)$ will be homeomorphic to a disconnected union of a finite number of half-intervals $[d-\varepsilon, d+\varepsilon)$

$((d-\varepsilon, d+\varepsilon])$. By virtue of continuity of the function f, there is a level surface $f^{-1}(d)$ corresponding to the set $\Gamma_{K-R}(f)$ with a finite number of points, which will be vertexes in the graph. Therefore, the image of the manifold $f^{-1}[d-\varepsilon, d+\varepsilon]$ into the set $\Gamma_{K-R}(f)$ will be a set homeomorphic to a finite subgraph.

It is clear that there exists $\varepsilon > 0$ such that the segment $[a, a+\varepsilon]$ ($[b-\varepsilon, b]$) does not contain other critical values. Using above arguments we can show that the set $f^{-1}(a)$ ($f^{-1}(b)$) consists of a finite number of points, which will be vertexes of order 1 in the graph, and the image of the manifold $f^{-1}[a, a+\varepsilon]$ ($f^{-1}[b-\varepsilon, b]$) in the set $\Gamma_{K-R}(f)$ will be a set homeomorphic to a finite subgraph. Consequently, the quotient set $\Gamma_{K-R}(f)$ is homeomorphic to a finite graph. □

Definition 2.1. The set $\Gamma_{K-R}(f)$ is called a Kronrod-Reeb graph for the function f from the space $C^\infty(M^n, \partial M^n)$.

Remark 2.1. For more details concerning the above definition, see [1]. To vertexes of a Kronrod-Reeb graph $\Gamma_{K-R}(f)$ there correspond connected components of those level surfaces that contain splitting critical points of the function f. The local extrema of the function f correspond to vertexes of order 1.

The Kronrod-Reeb graph $\Gamma_{K-R}(f)$ of the function f on a manifold M^n admits an orientation, i.e., one can put arrows on the edges in order to show the direction in which the function f grows. It is the **special orientation** of the graph $\Gamma_{K-R}(f)$ (see Definition 3.1). We will call such an orientation by an **f-orientation** of the graph $\Gamma_{K-R}(f)$.

It is obvious that a function f determines, in canonical manner, a partial order on the vertexes of its f-oriented Kronrod-Reeb graph $\Gamma_{K-R}(f)$. A vertex $x \in \Gamma_{K-R}(f)$ precedes a vertex $y \in \Gamma_{K-R}(f)$ if the values of f at the layers N_x and N_y corresponding to the vertexes x and y satisfy the inequality $f(N_x) < f(N_y)$. Such an order on the graph $\Gamma_{K-R}(f)$ we will be referred to as an **f-order**.

Definition 2.2. The Kronrod-Reeb graph $\Gamma_{K-R}(f)$ with the f-orientation (with the f-orientation and the f-partial order on vertexes) defined by the function f will be called an **f-oriented (f-pooriented)** Kronrod-Reeb graph for the function f from the space $C^\infty(M^n, \partial M^n)$.

Definition 2.3. The graph $\Sigma = (V, E)$ (oriented (pooriented) graph $\Sigma = (V, E)$) for which there exist a manifold M^n and a smooth function $f \in C^\infty(M^n)$ such that the Kronrod-Reeb graph (f-oriented (f-pooriented) Kronrod-Reeb graph) $\Gamma_{K-R}(f)$ for this function f is isomorphic (orientation (orientation and partial order) preserving isomorphic) to the graph $\Sigma = (V, E)$ is called a K-R-graph (oriented (pooriented) K-R-graph).

It should be noted that not every finite graph (with orientation (orientation and partial order)) having at least two vertexes of order one is a Kronrod-Reeb (f-oriented (f-pooriented) Kronrod-Reeb) graph for a certain smooth function with finitely many critical points on a smooth n-manifold ($n \geq 2$).

Fig. 1 shows a graph that is not a K-R-graph.

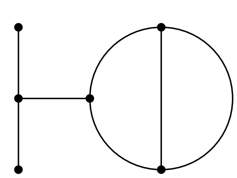

Figure 1. Not a K-R graph.

Theorem 2.1. *Any finite graph $\Sigma = (V, E)$ is a K-R-graph of some function on some manifold M^n if and only if $\Sigma = (V, E)$ admits some f-orientation (see Definition 3.1).*

For a proof of this theorem, we need to consider monotone functions on graphs.

3. Monotone functions on finite graphs

Let $\Sigma = (V, E)$ be a finite graph. Let us assume that the number of vertexes of order 1 in the graph $\Sigma = (V, E)$ is greater than 1. By an **f-orientation of the graph** $\Sigma = (V, E)$ we mean assigning arrows to the edges in such way that following conditions are satisfied:

a) there exist two vertexes of order 1 incident to edges with entering and outgoing arrows;
b) for every vertex of order $n \geq 2$ we can find two edges with entering and outgoing arrows incident to it;
c) the graph $\Sigma = (V, E)$ does not have oriented closed cycles.

By the definition, an oriented closed cycle of a graph $\Sigma = (V, E)$ is the set of oriented edges of Σ which forms a homeomorphic image of an oriented circle.

It is clear that the f-orientation of a graph $\Sigma = (V, E)$ determines, in a canonical way, a partial order P_o on the vertexes V of Σ. A vertex $x \in \Sigma = (V, E)$ precedes a vertex $y \in \Sigma = (V, E)$ if there is an f-oriented path beginning at the vertex x and ending at the vertex y.

The partial order P on the vertexes V of an f-oriented graph $\Sigma = (V, E)$ will be called **consistent with the f-orientation** of $\Sigma = (V, E)$, if for any two vertexes from $\Sigma = (V, E)$ that are in some relation relatively to the order P_o there exists the same relation between these vertexes relatively to the order P.

Definition 3.1. A finite graph $\Sigma = (V, E)$ with the f-orientation (with the f-orientation and some partial order on vertexes of Σ consistent with the f-orientation of Σ) will be called **f-oriented (f-pooriented)** graph.

Definition 3.2. Let $g : \Sigma \to R$ be a continuous function on the graph $\Sigma = (V, E)$. We say that the function g is monotone on the graph Σ, if

a) the restriction of the function g to edges E of the graph Σ is a strongly monotone function;
b) local extrema of the function g lie on vertexes of the order 1 .

Remark 3.1. For a monotone function g on the graph $\Sigma = (V, E)$, we can introduce the notion of the Kronrod-Reeb graph (oriented (pooriented) Kronrod-Reeb graph) for this function g, which we will call a K-R-graph (oriented (pooriented) K-R-graph) of g. Obviously, the K-R-graph of a monotone function g on the graph $\Sigma = (V, E)$ is homeomorphic to Σ.

Theorem 3.1. *Let $\Sigma = (V, E)$ be a finite f-oriented graph. Then on the graph $\Sigma = (V, E)$ there exists a monotone (increasing) function $g : \Sigma \to \mathbb{R}$ such that its oriented K-R-graph is homeomorphic (orientation preserving homeomorphic) to the f-oriented graph $\Sigma = (V, E)$.*

Proof. First we will construct some increasing function g on the f-oriented graph Σ. Our arguments will be of inductive character. Let v_i, $i = 1, 2, \ldots, s$, be an arbitrary indexing of vertexes in the graph Σ. Consider the first vertex v_1 from Σ and put $g(v_1) = 1$. Let v_2 be the second vertex. If there is not an f-oriented path w in Σ connecting vertexes v_1 and v_2, then put $g(v_2) = 1$. In the case when there is an f-oriented way w connecting these vertexes in Σ, we put $g(v_2) = 2$, $(g(v_2) = 0)$, if the vertex v_2 is the end (origin) of the path w. Suppose that we have values of the function g on the vertexes v_i, $i = 1, 2, \ldots, k - 1$.

Consider the vertex v_k from the graph Σ. There are four possibilities to connect the vertex v_k with vertexes v_i, $i = 1, 2, \ldots, k-1$, with an f-oriented path in the graph Σ,

a) the vertex v_k could not be connected in the graph Σ with an f-oriented path to the vertexes v_i, $i = 1, 2, \ldots, k-1$;
b) in the graph Σ there is an f-oriented path w with the origin at the vertex v_{i_0}, $1 \le i_0 \le k-1$, and the end in the vertex v_{i_1}, $1 \le i_1 \le k-1$, and the vertex v_k belongs to the path w;
c) in the graph Σ there exists only an f-oriented path w with the origin at the vertex v_{i_0}, $1 \le i_0 \le k-1$, and the end at the vertex v_k (there is not an f-oriented path w in Σ with the origin at the vertex v_k and the end at the vertex v_{i_0}, $1 \le i_0 \le k-1$;
d) in the graph Σ there exists only an f-oriented path w with the origin at the vertex v_k and with the end in vertex v_{i_0}, $1 \le i_0 \le k-1$, (there is not an f-oriented path w in Σ with the origin at the vertex v_{i_0}, $1 \le i_0 \le k-1$), and the end at the vertex v_k.

In the case a) we put $g(v_k) = 1$.

Consider the case b). Let w be an f-oriented path in the graph Σ which satisfies the following conditions:

1) the origin of the path w is the vertex v_{i_0}, $1 \le i_0 \le k-1$, on which the function g takes a maximal value $g(v_{i_0}) = a_{i_0}$ with the respect to the relation on other vertexes from v_i, $i = 1, 2, ..., k-1$, which are origins of f-oriented paths in Σ, going to the vertex v_k;
2) the end of the path w is the vertex v_{i_1}, $1 \le i_1 \le k-1$, on which the function g takes a minimal value $g(v_{i_0}) = b_{i_0}$ with the respect to the order relation on the vertexes v_i, $i = 1, 2, ..., k-1$, which are ends of oriented paths in Σ, going from the vertex v_k.

In this case we put

$$g(v_k) = a_{i_0} + 1/2(b_{i_0} - a_{i_0}) \ .$$

Consider the case c). Let c_k be a maximal value the function g takes on the set of the vertexes v_i, $i = 1, 2, \ldots, k-1$. Let us put

$$g(v_k) = c_k + 1 \ .$$

In the last case we shall act in similar manner. Let d_k be a minimal value the function g takes on the set of the vertexes v_i, $i = 1, 2, \ldots, k-1$. Set

$$g(v_k) = d_k - 1 \ .$$

This finishes the inductive step.

It is not difficult to show that the function g constructed as above on vertexes of Σ could be extended on edges to a strong increasing function on the graph Σ. Let e be an arbitrary f-oriented edge with the origin at a vertex v_i and the end at a vertex v_j. Suppose that $i < j$ $(i > j)$ then, in the process of constructing the function g, the vertex v_i (v_j) will appear first for the function g to be defined. By the method of the construction of the function g, the value of g on the vertex v_j (v_i) will be greater (less) than the value g on the vertex v_i (v_j). Therefore, the function g may be correctly extended from vertexes of the graph Σ on its edges.

The fact that the function g does not have local extrema in vertexes of the order greater than 1 follows from the condition that, in these vertexes, there exist both incoming and outgoing edges. Our construction will result from the fact that the function g is increasing on the f-oriented graph Σ. $\square$

Remark 3.2. The consideration of the case where the graph $\Sigma = (V, E)$ is (f-pooriented) reduces to the case of an f-oriented graph. The graph Σ has the partition P_o induced by the f-orientation which is compatible with the partition P for the pooriented graph Σ. If there exists a couple of vertexes v_1 and v_2 in the graph Σ, which are non-congruent in the partition P_o but $v_1 < v_2$ in the partition P, then we introduce a new oriented edge $\overline{e}$ from the vertex v_1 to the vertex v_2. As a result, we obtain a new oriented graph $\overline{\Sigma} \supseteq \Sigma$. Obviously, the graph $\overline{\Sigma}$ doe not have oriented cycles. On the graph $\overline{\Sigma}$, we construct an increasing function $\overline{g}$ by analogy with the previous case. The restriction of the function $\overline{g}$ to the subgraph Σ in the graph $\overline{\Sigma}$ gives the desired increasing function g.

4. The proof of Theorem 2.1

Necessity. It is obvious.

Sufficiency. Let $\Sigma = (V, E)$ be a finite f-oriented graph. First, we shall construct a closed surface M^2 and a smooth function $g : M^2 \longrightarrow \mathbb{R}$ such that its f-oriented Kronrod-Reeb graph $\Gamma_{K-R}(g)$ is isomorphic to the f-oriented graph $\Sigma = (V, E)$.

Consider a neighborhood U_i of the vertex $e_i \in V$ in the graph $\Sigma = (V, E)$ having the order $l_i \geq 3$. Take a two-dimension sphere S^2 and remove from it l non-intersecting open disks. As a result we obtain a smooth surface with boundary, N_i^2. Let the neighborhood U_i correspond to this surface N_i^2.

On Fig. 2, the order of vertexes e_i is $l_i = 5$.

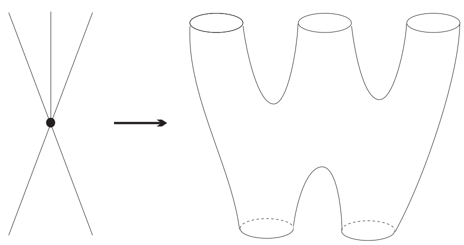

Figure 2. The neighborhood U_i corresponding to the surface N_i^2.

Using the function $\operatorname{Re} Z^{l_i - 1} + a_i$ ($a_i \in \mathbb{R}$) we construct on the surface N_i^2 a smooth function g_i, the Kronrod-Reeb graph $\Gamma_{K-R}(g_i)$ of which is isomorphic to U_i. Fig. 3 shows the function $g_i = \operatorname{Re} Z^4$.

If a neighborhood $\widetilde{U_j}$ of the vertex $e_j \in V$ has order two, then it correspondents to the Möbius band $\widetilde{N2}_j$ with an open disk removed. Fig. 4 shows this case.

Using the function $\operatorname{Re} Z^2 + b_j$ ($b_j \in \mathbb{R}$), we will construct on the Möbius band $\widetilde{N_j^2}$ a smooth function $\widetilde{g_j}$, the Kronrod-Reeb graph $\Gamma_{K-R}(\widetilde{g_j})$ of which is isomorphic to $\widetilde{U_j}$. The function $g_j = \operatorname{Re} Z^2$ on the Möbius band is shown in Fig. 5.

At last, if a neighborhood $\widehat{U_k}$ of the vertex $e_k \in V$ has order one, then it corresponds to the two-dimensional disk $\widehat{D_k^2}$. On this disk $\widehat{D_k^2}$, using orientation of the neighborhood $\widehat{U_k}$, we construct a smooth function $g_k = |z|^2 + c_k$ or $g_k = -|z|^2 + c_k$. Using the graph $\Sigma = (V, E)$, glue the surfaces N_i^2, $\widetilde{N2}_j$ and $\widehat{D_k^2}$ along their boundaries. As a result we obtain a closed surface M^2. By Theorem 3.1, on the graph $\Sigma = (V, E)$ there

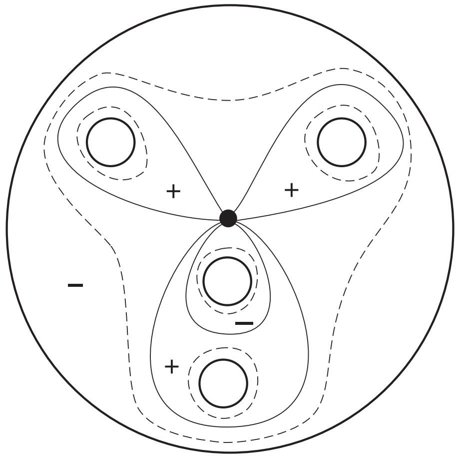

FIGURE 3. The level line of the function $g_i = \operatorname{Re} Z^4$ on the surface N_i^2.

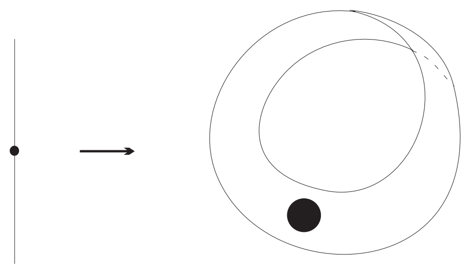

FIGURE 4. The neighborhood $\widetilde{U_j}$ corresponding to a Möbius band

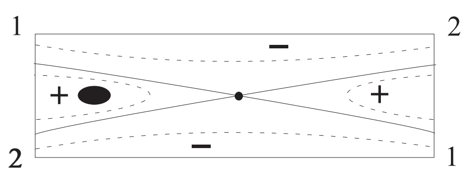

FIGURE 5. The function $\widetilde{g_j} = \operatorname{Re} Z^2$ on Möbius band

exists a monotone function f such that its f-oriented K-R-graph is orientation preserving homeomorphic to the graph Σ. Using the function f, we use the functions g_i, $\widetilde{g_j}$, $\widehat{g_k}$ to construct the increasing function $g : M^2 \longrightarrow \mathbb{R}$ such that its f-oriented K-R-graph is orientation preserving isomorphic to the f-oriented graph $\Sigma = (E, V)$.

Using the method of F. Takens [4] for tracing degenerate critical points from non-degenerate critical points we can construct an n-dimensional manifold M^n and a smooth function f on M^n such that its f-oriented K-R-graph is orientation preserving isomorphic to the f-oriented graph $\Sigma = (E, V)$.

Theorem 2.1 is proved. $\square$

Remark 4.1. There is a similar theorem in case where the graph $\Sigma = (E, V)$ is an f-pooriented graph. The proof of this theorem is analogous.

5. Necessary and sufficient condition for a finite graph to be a Kronrod-Reeb graph of a function on a manifold

Let $\Sigma = (V, E)$ be a finite graph. Let v be a vertex of graph $\Sigma = (V, E)$ and E_v denote the set of edges incident with the vertex v. The graph $\Sigma_v = (V \setminus v, E \setminus E_v)$ is obtained as a result of eliminating the vertex v and the incident to it edges E_v. The vertex v is called dividing, if the graph $\Sigma_v = (V \setminus v, E \setminus E_v)$ is a disconnected set. Denote by Ω the set vertexes of order 1 in the graph Σ.

Definition 5.1. A finite graph $\Sigma = (V, E)$ satisfies condition $\Im$ if

a) the graph Σ is a connected set;
b) for any dividing vertex v from Σ and any connected component $\Sigma_v^i \subset \Sigma_v$, we have $\Sigma_v^i \bigcap \Omega \neq \varnothing$;
c) the set Ω consists of at least of two vertexes.

In [3], the following theorem is proved.

Theorem 5.1. *Let $\Sigma = (V, E)$ be a finite graph which satisfies the condition $\Im$. Then on the graph $\Sigma = (V, E)$ there exists an increasing function.*

Corollary 5.1. *Any finite graph $\Sigma = (V, E)$ that satisfies the condition $\Im$ may be converted to an f-oriented (f-pooriented) graph.*

Theorem 5.2. *A finite graph $\Sigma = (V, E)$ is a K-R-graph of some function on some manifold M^n if and only if it satisfies condition $\Im$.*

Proof. Necessity. Let f be a smooth function on a connected manifold M^n, which belongs to $C^\infty(M^n, \partial M^n)$, and $\Gamma_{K-R}(f)$ be an f-oriented (f-pooriented) K-R-graph of f. Obviously, $\Gamma_{K-R}(f)$ satisfies condition a).

Consider condition b). Let $v \in V$ be a split vertex corresponding to the layer $N \in f^{-1}(x)$ for some critical value x. It is clear that set $M^n \backslash N$ consists of a finite a number of connected submanifolds $M_1^n, \dots, M_k^n$. The restriction of the function f to any submanifold M_i^n, $f|_{M_i^n}$, gives some smooth function f_i that has necessarily either a maximum or a minimum. Therefore, condition b) is fulfilled.

Validity of condition c) is obvious.

Sufficiency. Let $\Sigma = (V, E)$ be a finite graph $\Sigma = (V, E)$ that satisfies condition $\Im$. Then, by Corollary 5.1 and Theorem 2.1, it is a K-R-graph of some function on some manifold M^n. □

Remark 5.1. The question when a finite graph $\Sigma = (V, E)$ will be a K-R-graph of some function g on a fixed manifold M^n is more difficult and will be considered in the next paper.

References

1. A. S. Kronrod, *On functions of two variables*, Uspekhi Mat. Nauk **5** (1950), no. 1, 24-134.
2. O. Ore, *Theory of graphs*, Amer. Math. Soc. Colloq. Publ., Vol. 38, 1962.
3. V. Sharko, *Morse functions on non-compact surfaces*, Proc. Ukrainian Math. Congress–2001, Topology and Geometry, Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, 2003, pp. 91–102.
4. F. Takens, *The minimal number of critical points of a function on a compact manifold and the Lusternik–Schnirelman category*, Invent. Math. **6** (1968), 197–244.

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine

E-mail address: sharko@imath.kiev.ua; sharko@ukrpack.net

Received 07/09/2006