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ON $*$-REPRESENTATIONS OF ALGEBRAS GIVEN BY GRAPHS

S. V. IVANOV

Abstract. *-algebras given by trees generated by projections with Temperley-Lieb type relations are considered in this work. Formulas of representations are built by algorithms for *-algebras, associated with the Dynkin diagrams, and we get estimates for the parameters at which non-trivial *-representations of the *-algebras exist.

1. Introduction

The problem of describing $*$-representations of $*$-algebras generated by projections which is linked with some additional relations was considered in [1], [2], [3], [7]. Relations of the kind

$$p_ip_jp_i = \tau p_i, \quad p_jp_ip_j = \tau p_j, \tag{1}$$

where τ is some number, first arose in works of physicists Temperley and Lieb (see [4]) on statistical physics in the case of studies of two-dimensional model of ice and Potts model. The algebra of Temperley-Lieb,

$$TL_n(\tau) = C\langle 1, p_1, \dots, p_{n-1} \,|\, p_ip_{i\pm 1}p_i = \tau p_i;\, p_ip_j = p_jp_i,\, i \neq j \pm 1\rangle,$$

later arose in case of being an index of a sub-factor in a Neiman factor of $\prod_1$-type (see [5]), and in the theory of invariants of knots (see [6]).

The $*$-algebras $A_{\Gamma,\tau,\perp}$, defined by finite nonoriented graphs Γ without multiple edges and loops, with numbers τ placed at the edges, were considered in the works [3], [8]. To the nodes of the graph there correspond generating projections. If between the nodes of the graph there is an edge, marked with a number τ, then for the corresponding pair of generators, relation (1) holds, and if there is no an edge, then $p_ip_j = p_jp_i = 0$. Results about the dimension of such algebras were obtained in [3], there is given a description of $*$-representations in the case where the graph Γ is a tree, or a cycle with glued trees. In work [8], there was grounded an algorithm allowing to write formulas for representations of the algebra $A_{\Gamma,\tau,\perp}$, if Γ is a tree.

In this work, we consider $*$-algebras $A_{\Gamma,\tau,\perp}$, where Γ is a tree with numbers placed at its edges. Necessary definitions and facts about previously studied $*$-algebras are given in Section 2. Section 3 is devoted to algorithms considered in work [8]. In Section 4, formulas for $*$-representations of the $*$-algebras associated with Dynkin diagrams, which are trees, are given, as well as estimates for parameters for which non-trivial $*$-representations of the corresponding $*$-algebras exists.

2. The $*$-algebra $A_{\Gamma,\tau,\perp}$ and its representations

Thoughout the paper, Γ is some finite nonoriented graph whithout multiple edges and loops with the number of nodes denoted by $V\Gamma$ ($|V\Gamma| = n$) and the number of edges $E\Gamma$.

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In the sequel, $\tau : E\Gamma \to (0;1)$ assigns numbers to its edges. An indexing of the nodes of Γ in a random way with numbers from 1 to n and will denoted by $\tau(i,j) = \tau_{ij} = \tau_{ji}$.

Definition 1. $A_{\Gamma,\tau,\perp}$ is a $*$-algebra with 1 over the field C, generated by projections p_i, $i \in V\Gamma$ ($p_i^2 = p_i^* = p_i\ \forall i$), with the relations

$$\begin{cases} p_i p_j p_i = \tau_{ij} p_i, p_j p_i p_j = \tau_{ij} p_j, & \text{if } (i,j) \in E\Gamma, \\ \qquad p_i p_j = p_j p_i = 0, & \text{if } (i,j) \overline{\in} E\Gamma. \end{cases}$$

Such algebras were considered in [3], where their growth is determined and, in particular, the dimension depending on the type of the graph, and in the case where the graph is a tree, on placing τ, such that the algebra $A_{\Gamma,\tau,\perp}$ has non-trivial $*$-representations.

Lemma 1. *If the graph Γ does not contain cycles, then the $*$-algebra $A_{\Gamma,\tau,\perp}$ is finite dimensional.*

Lemma 2. *If the graph Γ is a tree, then the algebra $A_{\Gamma,\tau,\perp}$ has no infinite dimensional irreducible $*$-representations in the Hilbert space. If $A_{\Gamma,\tau,\perp}$ has non-trivial irreducible $*$-representations, then the rank of all generating projections in them is equal* 1.

Consider a self-adjoint matrix $A(\Gamma,\tau) = \|A_{ij}\|_{i,j=1}^{n}$, where $A_{ii} = 1\ \forall i$, $A_{ij} = 0$ if $(i,j)\overline{\in} E\Gamma$, and $A_{ij} = \sqrt{\tau_{ij}}$ otherwise.

Theorem 1. *Let the graph Γ be a tree. Non-trivial $*$-representations of the algebra $A_{\Gamma,\tau,\perp}$ exist if and only if the matrix $A(\Gamma,\tau)$ is non-negative definite. Irreducible non-trivial *-representation of $A_{\Gamma,\tau,\perp}$ are unique up to unitary equivalence, and its dimension is equal to the rank of the matrix $A(\Gamma,\tau)$.*

3. Algorithm for constructing a representation of the $*$-algebra $A_{\Gamma,\tau,\perp}$

Let Γ be some finite nonoriented tree without multiple edges and loops. The number of nodes in the set $\mid V\Gamma \mid= n$, and that of edges is by one less then $\mid V\Gamma \mid$, that is, $\mid E\Gamma \mid= n-1$. Arbitrarily index the nodes of the graph with the numbers $i = \overline{1,n}$ and place numbers on the edges, $\tau : E\Gamma \to (0;1)$. Thus every edge of the graph with nodes i and j gets the index $\tau_{ij} = \tau_{ji}$. Consider the following algorithm.

The algorithm of marking the tree

$\underline{1}$. $s := 1$, $G^{(1)} := \Gamma$.

$\underline{2}$. Choose an arbitrary node i in the graph $G^{(1)}$ with valence 1 in the graph $G^{(s)}$ and not marked.

$\underline{3}$. $s_i := s$, $a_i := 1$.

$\underline{4}$. If there exist nodes of the graph $G^{(1)}$, contiguous with node i and not marked, index all such nodes with $i_1, i_2, \ldots, i_l$. Then $a_i := a_i - \sum\limits_{j=1}^{l} \frac{\tau_{i,i_j}}{a_{i_j}}$. If $a_i < 0$ or if $a_i = 0$ and in graph $G^{(1)}$ there are not marked nodes, then the algorithm is interrupted.

$\underline{5}$. Consider a node k of the graph $G^{(1)}$, contiguous with node i in the graph $G^{(s)}$ and not marked. If such a node does not exist, then go to the item $\underline{10}$.

$\underline{6}$. If the valence of k in the graph $G^{(s)}$ is more than 2, then go to the item $\underline{7}$, else go to item $\underline{3}$.

$\underline{7}$. If the set of nodes of the graph $G^{(s)}$ with valence 1 is empty, then go to item $\underline{8}$, else go to item $\underline{2}$.

$\underline{8}$. $s := s+1$. Build graph $G^{(s)}$. The nodes of the graph $G^{(s)}$ are:

1) the nodes of the graph $G^{(s-1)}$ with valence 2 and unmarked;

2) the nodes of the graph $G^{(s-1)}$ with valence more than 2 and unmarked.

The graph $G^{(s)}$ is a subgraph of the graph $G^{(s-1)}$ generated by the indicated set of nodes.

9. If $G^{(s)} = P_1$, then sole node i gets marks $s_i = s$, $a_i = 1 - \sum_{j=1}^{l} \frac{\tau_{i,i_j}}{a_{i_j}}$, where $i_1, i_2, \dots, i_l$ are all the nodes of graph $G^{(1)}$ that are contiguous with the node i and not marked and, go to item 10, else go to item 2.

10. If $a_i \geq 0$, then marking of the tree is completed. Otherwise the algorithm is interrupted.

We shall say that marking of a tree is executed correctly, if it is completed. We shall notice that marking of a tree Γ will be executed correctly, if during the work of the algorithm, the marks will be $a_i > 0$, and the last mark $a_j \geq 0$. Let us denote by Ω_Γ the set of placing τ, for which there exists correct markings of the tree Γ.

Suppose that, for Γ, a correct marking is executed, that is, $\tau \in \Omega_\Gamma$. Then the marking conducted by the algorithm allows to build a non-trivial $*$-representation π of the $*$-algebra $A_{\Gamma,\tau,\perp}$, thus, if the last mark obtained in the algorithm is $a_i \neq 0$ then the dimension of the representation is equal to n, and if $a_i = 0$, then the dimension is equal to $n - 1$.

Let us denote $P_i = \pi(p_i)$, $i = \overline{1,n}$. The following algorithm describes a construction of this $*$-representation. The first part places the main diagonal elements in matrices of operators of orthogonal projection. The first element in the first line is considered as a current position at the start of the algorithm and all nodes supposed not used. The second part of the algorithm completes the other elements of matrices of operators.

An algorithm for constructing a the $*$-representation

I part:

1. $s := 1$.

2. Choose an arbitrary used node i in the graph $G^{(1)}$ which has the valence 1 in $G^{(s)}$.

3. If $a_i \neq 0$ then place a_i in the current position in the matrix of the operator P_i and pass to the following diagonal element.

4. If the nodes, which are contiguous with the node i in the graph $G^{(1)}$ and used, exist, then we let $i_1, i_2, \dots, i_l$ for all such nodes. Consider that nodes on $m = \overline{1,l}$ and in matrix of operator P_i stage a number $\frac{\tau_{i,i_m}}{a_{i_m}}$ in that position in which number a_{i_m} is placed in matrix of operator P_{i_m}. Work with P_i is closed after completion of this considering. Node i is considered as used.

5. If the set of not used nodes in the graph $G^{(1)}$ is empty then go to item 8.

6. Consider a node j of the graph $G^{(1)}$ contiguous with the node i in the graph $G^{(s)}$ and not used. If $s_j > s_i$ then go to 7 else $i := j$ and go to item 3.

7. If set of not used nodes with valence 1 in the graph $G^{(s)}$ is empty then $s := s + 1$. Go to item 2.

8. The work of the algorithm is finished.

II part:

To complete the construction of the operators consider them on $i = \overline{1,n}$ and do the following.

1. If on the diagonal of the matrix P_i there is placed only 1 then all other elements make equal to zero and P_i is built.

2. If on the diagonal of matrix P_i in the positions $i_1, i_2, \dots, i_l$ there are the numbers $s_{i_1}, s_{i_2}, \dots, s_{i_l}$, consider all possible combinations from $\{i_1, i_2, \dots, i_l\}$ for two and for each $\{i_k, i_m\}$ matrix elements of P_i with the indexes i_k, i_m and i_m, i_k, and put them equal to $\sqrt{s_{i_k} s_{i_m}}$. Other elements of the matrix P_i are set to zero. P_i is built.

By the construction, the matrix P_i satisfies $P_i^2 = P_i = P_i^*$ and, consequently, make an operator of orthogonal projection.

Thus Σ_Γ is a set of placing τ for which there exists non-trivial $*$-representations of the $*$-algebra $A_{\Gamma,\tau,\perp}$. Then taking into account the foresaid we get that $\Omega_\Gamma \subseteq \Sigma_\Gamma$.

Theorem 2. *If* Γ *is a tree then, for the* $*$*-algebra* $A_{\Gamma,\tau,\perp}$*, the identity* $\Omega_\Gamma = \Sigma_\Gamma$ *holds.*

The proof of this theorem can be found in [8]. From the theorem we get the following.

Corollary 1. *If for the tree* Γ*, a correct marking has been carried out, then for the algebra* $A_{\Gamma,\tau,\perp}$*, the dimension of the representation is either equal to* n*, if the last mark* $a_j > 0$*, or is equal to* $n-1$*, if* $a_j = 0$*. Other dimensions are impossible.*

4. Formulas for $*$-representations of $*$-algebras associated with Dynkin's diagrams and estimations of the parameters

Let us consider Dynkin diagrams. Numbering the nodes of these trees for every tree Γ will conduct placing numbers on edges $\tau : E\Gamma \to (0;1)$ (fig. 1).

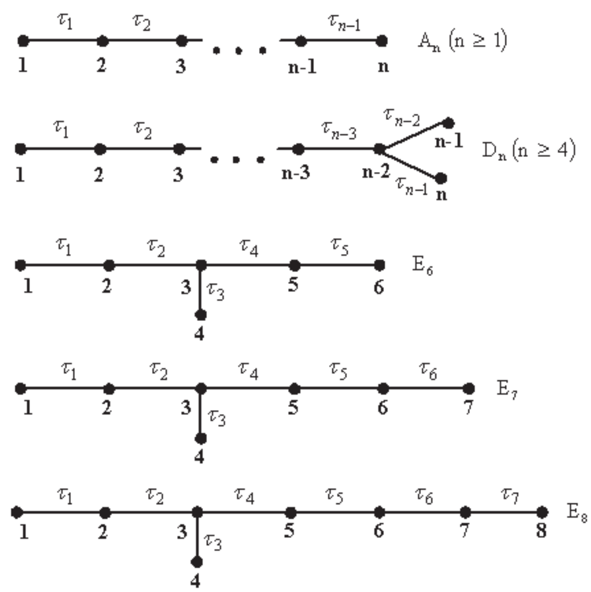

Figure 1

Suppose that for these numbers τ_i, the representations of the algebras $A_{A_n,\tau,\perp}$, $A_{D_n,\tau,\perp}$, $A_{E_6,\tau,\perp}$, $A_{E_7,\tau,\perp}$, $A_{E_8,\tau,\perp}$ exist. So that $\tau_i \in \Sigma_\Gamma$ for the corresponding tree.

Denote by $P(p_{11}, p_{22}, \ldots, p_{nn})$ the matrix of the orthogonal projection P with numbers on the main diagonal being $p_{11}, p_{22}, \ldots, p_{nn}$, and the other entries of the matrix, p_{ij}, when $i \neq j$, are determined as follows:

$$p_{ij} = p_{ji} = \begin{cases} \sqrt{p_{ii} \cdot p_{jj}} & \text{if } p_{ii} \neq 0 \text{ and } p_{jj} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We will write down formulas for representations of these algebras obtained from the algorithms. For the sake of brevity, assume that the last marks obtained from the algorithms are different from zero. That is, the dimensions of the presentations are equal to the orders of the proper graphs. Notice that if the last mark is equal to zero, it is enough to delete the last zeros in the corresponding formulas.

1. The algebra $A_{A_n,\tau,\perp}$. After an implementation of marking, every node will get the following marks:

$$\begin{array}{lll} a_1=1, & a_2=1-\tau_1, & a_3=1-\frac{\tau_2}{1-\tau_1}=1-\frac{\tau_2}{a_2}, \quad \dots, \\ a_i=1-\frac{\tau_{i-1}}{a_{i-1}}, & \dots, & a_n=1-\frac{\tau_{n-1}}{a_{n-1}}. \end{array}$$

Then

$$\begin{array}{ll} P_1=(1,0,0,0,\dots,0), & P_2=(\tau_1,a_2,0,0,\dots,0), \\ P_3=(0,\frac{\tau_2}{a_2},a_3,0,\dots,0), & \dots, \\ P_i=(0,0,\dots,0,\frac{\tau_{i-1}}{a_{i-1}},a_i,0,\dots,0), & \dots, \\ P_n=(0,0,0,\dots,0,\frac{\tau_{n-1}}{a_{n-1}},a_n). & \end{array}$$

2. The algebra $A_{D_n,\tau,\perp}$. Executing the marking we will get:

$$\begin{array}{lll lll} a_1 &=& 1, & a_2 &=& 1-\tau_1, \\ a_3 &=& 1-\frac{\tau_2}{1-\tau_1}=1-\frac{\tau_2}{a_2},\dots, & a_i &=& 1-\frac{\tau_{i-1}}{a_{i-1}},\dots, \\ a_{n-3} &=& 1-\frac{\tau_{n-4}}{a_{n-4}}, & a_{n-2} &=& 1-\frac{\tau_{n-3}}{a_{n-3}}-\tau_{n-2}-\tau_{n-1}, \\ a_{n-1} &=& 1, & a_n &=& 1. \end{array}$$

Then

$$\begin{array}{lll lll} P_1 &=& (1,0,0,0,\dots,0), & P_2 &=& (\tau_1,a_2,0,0,\dots,0), \\ P_3 &=& (0,\frac{\tau_2}{a_2},a_3,0,\dots,0), & \dots, && \\ P_i &=& (0,0,\dots,0,\frac{\tau_{i-1}}{a_{i-1}},a_i,0,\dots,0), & \dots, && \\ P_{n-2} &=& (0,0,0,\dots,0,\frac{\tau_{n-3}}{a_{n-3}},\tau_{n-2},\tau_{n-1},a_{n-2}), & P_{n-1} &=& (0,0,0,\dots,0,1,0,0), \\ P_n &=& (0,0,0,\dots,0,1,0). &&& \end{array}$$

3. The algebra $A_{E_6,\tau,\perp}$. After implementation of marking we will get:

$$\begin{array}{l} a_1=a_4=a_6=1,\ a_2=1-\tau_1,\ a_5=1-\tau_5, \\ a_3=1-\frac{\tau_2}{1-\tau_1}-\frac{\tau_4}{1-\tau_5}-\tau_3=1-\frac{\tau_2}{a_2}-\frac{\tau_4}{a_5}-\frac{\tau_3}{a_4}. \end{array}$$

Then

$$\begin{array}{lll} P_1=(1,0,0,0,0,0), & P_2=(\tau_1,a_2,0,0,0,0), & P_3=(0,\frac{\tau_2}{a_2},\tau_3,0,\frac{\tau_4}{a_5},a_3), \\ P_4=(0,0,1,0,0,0), & P_5=(0,0,0,\tau_5,a_5,0), & P_6=(0,0,0,1,0,0). \end{array}$$

4. The algebra $A_{E_7,\tau,\perp}$. Conducting the marking we will get:

$$\begin{array}{l} a_1=a_4=a_7=1,\ a_2=1-\tau_1,\ a_6=1-\tau_6, \\ a_5=1-\frac{\tau_5}{a_6},\ a_3=1-\frac{\tau_2}{a_2}-\frac{\tau_4}{a_5}-\frac{\tau_3}{a_4}. \end{array}$$

Then

$$\begin{array}{lll} P_1=(1,0,0,0,0,0,0), & P_2=(\tau_1,a_2,0,0,0,0,0), & P_3=(0,\frac{\tau_2}{a_2},\tau_3,0,0,\frac{\tau_4}{a_5},a_3), \\ P_4=(0,0,1,0,0,0,0), & P_5=(0,0,0,0,\frac{\tau_5}{a_6},a_5,0), & P_6=(0,0,0,\tau_6,a_6,0,0), \\ P_7=(0,0,0,1,0,0,0). && \end{array}$$

5. The algebra $A_{E_8,\tau,\perp}$. The results of the marking are

$$\begin{array}{l} a_1=a_4=a_8=1,\ a_2=1-\tau_1,\ a_7=1-\tau_7, \\ a_6=1-\frac{\tau_6}{a_7},\ a_5=1-\frac{\tau_5}{a_6},\ a_3=1-\frac{\tau_2}{a_2}-\frac{\tau_4}{a_5}-\frac{\tau_3}{a_4}. \end{array}$$

Then

$$\begin{array}{ll} P_1=(1,0,0,0,0,0,0,0), & P_2=(\tau_1,a_2,0,0,0,0,0,0), \\ P_3=(0,\frac{\tau_2}{a_2},\tau_3,0,0,0,\frac{\tau_4}{a_5},a_3), & P_4=(0,0,1,0,0,0,0,0), \\ P_5=(0,0,0,0,0,\frac{\tau_5}{a_6},a_5,0), & P_6=(0,0,0,0,\frac{\tau_6}{a_7},a_6,0,0), \\ P_7=(0,0,0,\tau_7,a_7,0,0,0), & P_8=(0,0,0,1,0,0,0,0). \end{array}$$

Consider the extended Dynkin diagrams which are trees (all, except for $\widetilde{A}_n$). We will number the nodes of the graphs and execute placing of the numbers on edges in the same arbitrary way (fig. 2).

Supposing that for these numbers τ, the representations of the proper algebras exist and using the algorithms we get formulas for the representations.

1. The algebra $A_{\widetilde{D}_n,\tau,\perp}$. Executing the marking we will get:

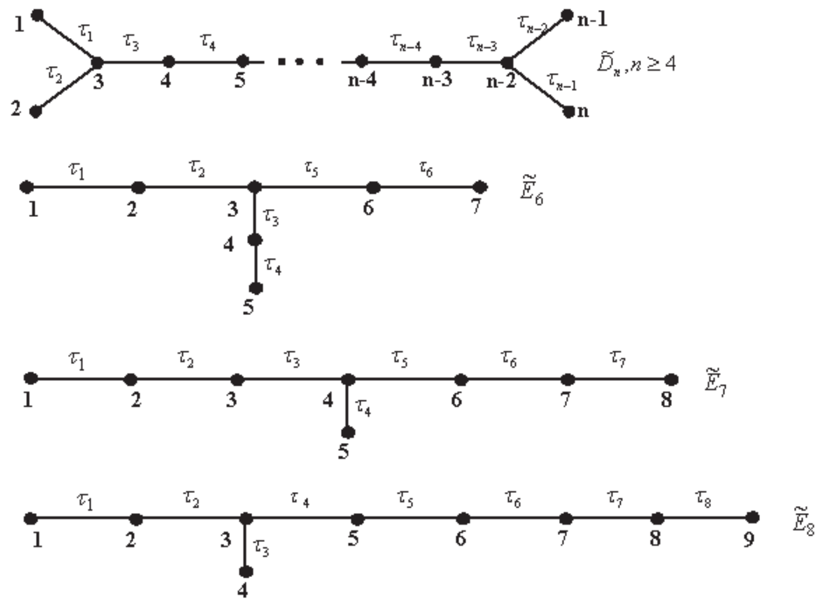

FIGURE 2

$$
\begin{array}{lll}
a_1 = a_2 = 1, & a_3 = 1 - \tau_1 - \tau_2 = 1 - \frac{\tau_1}{a_1} - \frac{\tau_2}{a_2}, & a_4 = 1 - \frac{\tau_3}{a_3}, \\
\dots, & a_i = 1 - \frac{\tau_{i-1}}{a_{i-1}}, & \dots, \\
a_{n-3} = 1 - \frac{\tau_{n-4}}{a_{n-4}}, & a_{n-2} = 1 - \frac{\tau_{n-3}}{a_{n-3}} - \tau_{n-2} - \tau_{n-1}, & a_{n-1} = a_n = 1.
\end{array}
$$

Then

$$
\begin{array}{c}
P_1 = (1,0,0,0,\dots,0), \qquad P_2 = (0,1,0,0,\dots,0), \\
P_3 = (\tau_1,\tau_2,a_3,0,\dots,0), \quad P_4 = (0,0,\frac{\tau_3}{a_3},a_4,\dots,0), \\
\dots, \quad P_i = (0,0,\dots,0,\frac{\tau_{i-1}}{a_{i-1}},a_i,0,\dots,0), \quad \dots, \\
P_{n-2} = (0,0,0,\dots,0,\frac{\tau_{n-3}}{a_{n-3}},\tau_{n-2},\tau_{n-1},a_{n-2}), \\
P_{n-1} = (0,0,0,\dots,0,1,0,0), \quad P_n = (0,0,0,\dots,0,1,0).
\end{array}
$$

2. The algebra $A_{\widetilde{E}_6,\tau,\perp}$. After implementation of the marking we will get:

$$
\begin{array}{l}
a_1 = a_5 = a_7 = 1,\ a_2 = 1 - \tau_1,\ a_4 = 1 - \tau_4, \\
a_6 = 1 - \tau_6,\ a_3 = 1 - \frac{\tau_2}{a_2} - \frac{\tau_3}{a_4} - \frac{\tau_5}{a_6}.
\end{array}
$$

Then

$$
\begin{array}{ll}
P_1 = (1,0,0,0,0,0,0), & P_2 = (\tau_1,a_2,0,0,0,0,0), \\
P_3 = (0,\frac{\tau_2}{a_2},\frac{\tau_3}{a_4},0,\frac{\tau_5}{a_6},0,a_3), & P_4 = (0,0,a_4,\tau_4,0,0,0), \\
P_5 = (0,0,0,1,0,0,0), & P_6 = (0,0,0,0,a_6,\tau_6,0), \\
P_7 = (0,0,0,0,0,1,0). &
\end{array}
$$

3. The algebra $A_{\widetilde{E}_7,\tau,\perp}$. Conducting the marking we will get:

$$
\begin{array}{l}
a_1 = a_5 = a_8 = 1,\ a_2 = 1 - \tau_1,\ a_3 = 1 - \frac{\tau_2}{a_2}, \\
a_7 = 1 - \tau_7,\ a_6 = 1 - \frac{\tau_6}{a_7},\ a_4 = 1 - \frac{\tau_3}{a_3} - \frac{\tau_5}{a_6} - \tau_4.
\end{array}
$$

Then

$$
\begin{array}{ll}
P_1 = (1,0,0,0,0,0,0,0), & P_2 = (\tau_1,a_2,0,0,0,0,0,0), \\
P_3 = (0,\frac{\tau_2}{a_2},a_3,0,0,0,0,0), & P_4 = (0,0,\frac{\tau_3}{a_3},\tau_4,\frac{\tau_5}{a_6},0,0,a_4), \\
P_5 = (0,0,0,1,0,0,0,0), & P_6 = (0,0,0,0,a_6,\frac{\tau_6}{a_7},0,0), \\
P_7 = (0,0,0,0,0,a_7,\tau_7,0), & P_8 = (0,0,0,0,0,0,1,0).
\end{array}
$$

4. The algebra $A_{\widetilde{E}_8,\tau,\perp}$. Results of the marking:

$$a_1 = a_4 = a_9 = 1,\ a_2 = 1 - \tau_1,\ a_8 = 1 - \tau_8,$$
$$a_7 = 1 - \tfrac{\tau_7}{a_8},\ a_6 = 1 - \tfrac{\tau_6}{a_7},\ a_5 = 1 - \tfrac{\tau_5}{a_6},\ a_3 = 1 - \tfrac{\tau_2}{a_2} - \tfrac{\tau_4}{a_5} - \tau_3.$$

Then

$$P_1 = (1,0,0,0,0,0,0,0,0), \qquad P_2 = (\tau_1, a_2, 0,0,0,0,0,0,0),$$
$$P_3 = (0, \tfrac{\tau_2}{a_2}, \tau_3, \tfrac{\tau_4}{a_5}, 0,0,0,0, a_3), \qquad P_4 = (0,0,1,0,0,0,0,0,0),$$
$$P_5 = (0,0,0,a_5, \tfrac{\tau_5}{a_6}, 0,0,0,0), \qquad P_6 = (0,0,0,0,a_6, \tfrac{\tau_6}{a_7}, 0,0,0),$$
$$P_7 = (0,0,0,0,0,a_7, \tfrac{\tau_7}{a_8}, 0,0), \qquad P_8 = (0,0,0,0,0,0,a_8,\tau_8,0),$$
$$P_9 = (0,0,0,0,0,0,0,1,0).$$

Conducting an estimation of parameters τ_i is possible by algorithms at which the non-trivial representations of algebras exist. Consider an example of the algebras $A_{E_6,\tau,\perp}$, $A_{E_7,\tau,\perp}$, $A_{E_8,\tau,\perp}$, $A_{\widetilde{E}_6,\tau,\perp}$, $A_{\widetilde{E}_7,\tau,\perp}$, $A_{\widetilde{E}_8,\tau,\perp}$ with the two parameters τ_1 and τ_2 (fig. 3).

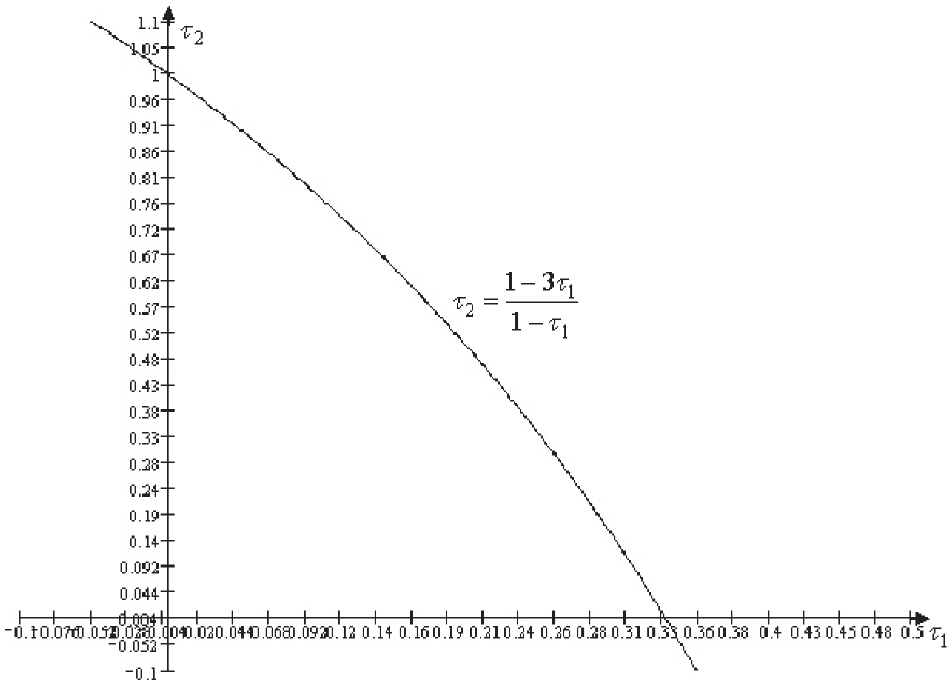

FIGURE 3

1. The algebra $A_{E_6,\tau,\perp}$. After implementation of the marking we will get:

$$a_1 = a_4 = a_6 = 1,\ a_2 = 1 - \tau_1,\ a_5 = 1 - \tau_1,$$
$$a_3 = 1 - \tfrac{\tau_1}{a_2} - \tfrac{\tau_1}{a_5} - \tfrac{\tau_2}{a_4} = 1 - \tfrac{\tau_1}{1-\tau_1} - \tfrac{\tau_1}{1-\tau_1} - \tau_2 = 1 - \tfrac{2\tau_1}{1-\tau_1} - \tau_2.$$

Then we will get the following limitations on parameters:

$$\begin{cases} 1 - \tau_1 > 0 \\ 1 - \frac{2\tau_1}{1-\tau_1} - \tau_2 \ge 0 \end{cases} \quad \text{or} \quad \begin{cases} \tau_1 < 1 \\ \tau_2 \le \frac{1-3\tau_1}{1-\tau_1}. \end{cases}$$

2. The algebra $A_{E_7,\tau,\perp}$. Conducting the marking we will get:

$$a_1 = a_4 = a_7 = 1,\ a_2 = 1 - \tau_1,\ a_6 = 1 - \tau_1,$$
$$a_5 = 1 - \tfrac{\tau_1}{a_6} = \tfrac{1-2\tau_1}{1-\tau_1},\ a_3 = 1 - \tfrac{\tau_1}{a_2} - \tfrac{\tau_1}{a_5} - \tfrac{\tau_2}{a_4} = \tfrac{1-5\tau_1+6\tau_1^2-\tau_1^3}{(1-\tau_1)(1-2\tau_1)} - \tau_2.$$

Then we will get the following limitations on parameters:

$$\begin{cases} 1 - \tau_1 > 0 \\ \frac{1-2\tau_1}{1-\tau_1} > 0 \\ \frac{1-5\tau_1+6\tau_1^2-\tau_1^3}{(1-\tau_1)(1-2\tau_1)} - \tau_2 \ge 0 \end{cases} \quad \text{or} \quad \begin{cases} \tau_1 < \frac{1}{2} \\ \tau_2 \le \frac{1-5\tau_1+6\tau_1^2-\tau_1^3}{(1-\tau_1)(1-2\tau_1)}. \end{cases}$$

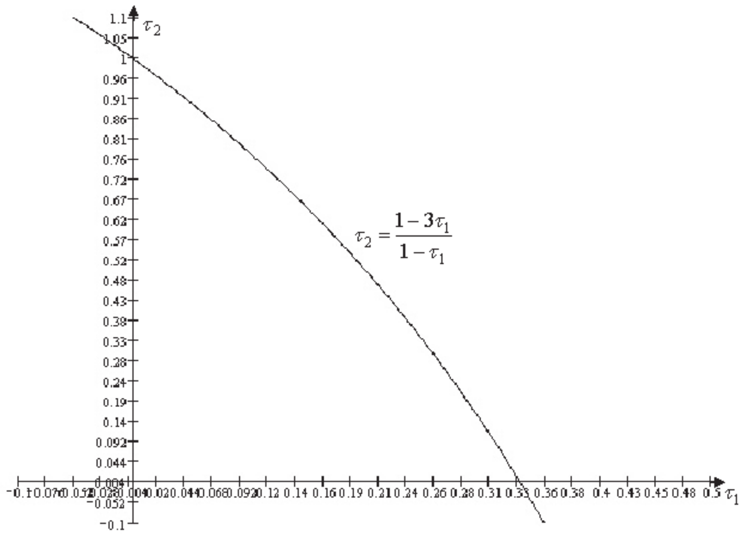

FIGURE 4

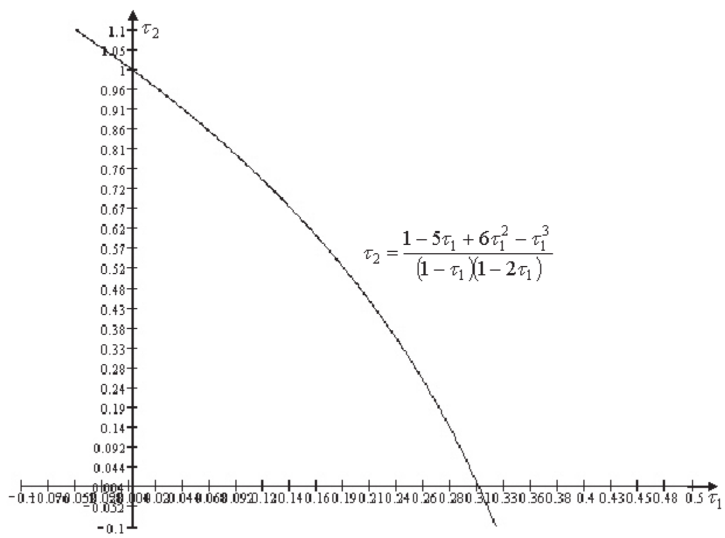

FIGURE 5

3. The algebra $A_{E_8,\tau,\perp}$. The results of the marking are

$$a_1 = a_4 = a_8 = 1,\ a_2 = 1-\tau_1,\ a_7 = 1-\tau_1,$$
$$a_6 = 1 - \frac{\tau_1}{1-\tau_1} = \frac{1-2\tau_1}{1-\tau_1},\ a_5 = 1 - \frac{\tau_1}{a_6} = \frac{1-3\tau_1+\tau_1^2}{1-2\tau_1},$$
$$a_3 = 1 - \frac{\tau_1}{a_2} - \frac{\tau_1}{a_5} - \frac{\tau_2}{a_4} = \frac{1-6\tau_1+10\tau_1^2-4\tau_1^3}{(1-\tau_1)(1-3\tau_1+\tau_1^2)} - \tau_2.$$

Then we will get the following limitations on parameters:

$$\begin{cases} 1-\tau_1 > 0 \\ \frac{1-2\tau_1}{1-\tau_1} > 0 \\ \frac{1-3\tau_1+\tau_1^2}{1-2\tau_1} > 0 \\ \frac{1-6\tau_1+10\tau_1^2-4\tau_1^3}{(1-\tau_1)(1-3\tau_1+\tau_1^2)} - \tau_2 \geq 0 \end{cases} \quad \text{or} \quad \begin{cases} \tau_1 \in (0; \frac{3-\sqrt{5}}{2}) \\ \tau_2 \leq \frac{1-6\tau_1+10\tau_1^2-4\tau_1^3}{(1-\tau_1)(1-3\tau_1+\tau_1^2)}. \end{cases}$$

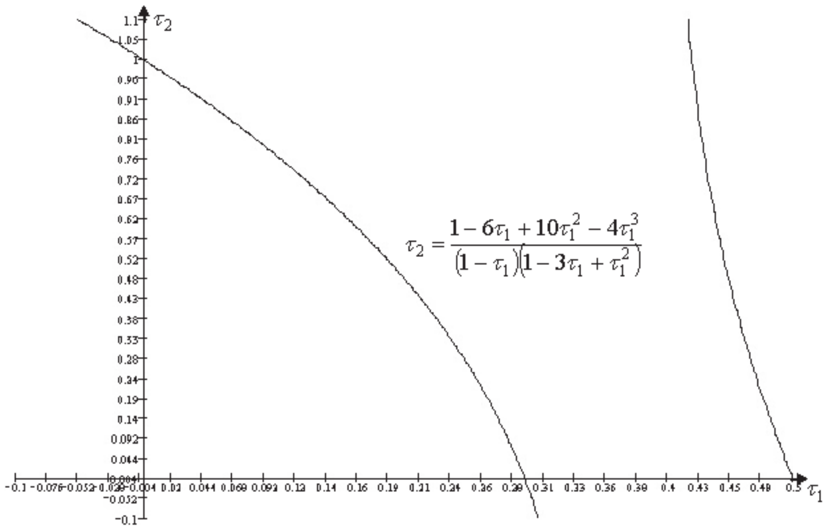

Figure 6

4. The algebra $A_{\widetilde{E}_6,\tau,\perp}$. After implementation of the marking we will get:

$$a_1 = a_5 = a_7 = 1,\ a_2 = 1-\tau_1,\ a_4 = 1-\tau_2,$$
$$a_6 = 1-\tau_1,\ a_3 = 1 - \tfrac{\tau_2}{a_2} - \tfrac{\tau_2}{a_4} - \tfrac{\tau_1}{a_6} = \tfrac{1-3\tau_1}{1-\tau_1} - \tfrac{\tau_2}{1-\tau_2}.$$

Then we will get the following limitations on the parameters:

$$\begin{cases} 1-\tau_1 > 0 \\ 1-\tau_2 > 0 \\ \frac{1-3\tau_1}{1-\tau_1} - \frac{\tau_2}{1-\tau_2} \geq 0 \end{cases} \quad \text{or} \quad \begin{cases} \tau_1 < 1 \\ \tau_2 < 1 \\ \frac{\tau_2}{1-\tau_2} \leq \frac{1-3\tau_1}{1-\tau_1}. \end{cases}$$

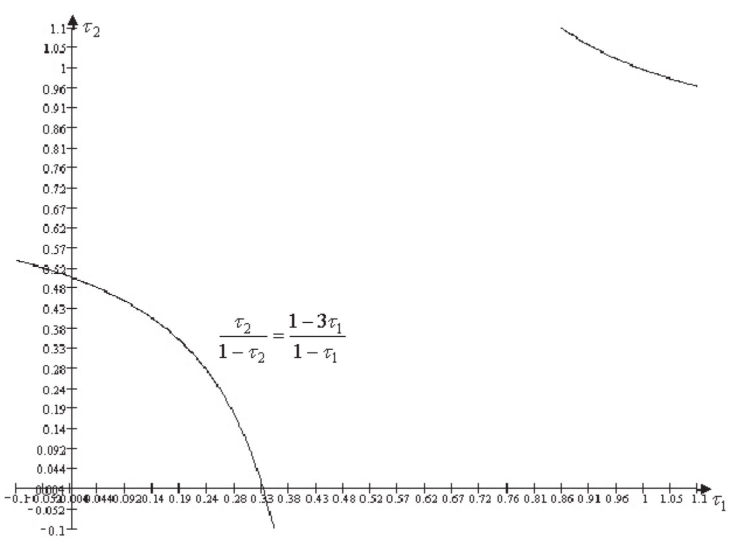

Figure 7

5. The algebra $A_{\widetilde{E}_7,\tau,\perp}$. Conducting the marking we will get:

$$a_1 = a_5 = a_8 = 1,\ a_2 = 1 - \tau_1,\ a_3 = 1 - \tfrac{\tau_1}{a_2} = \tfrac{1-2\tau_1}{1-\tau_1},$$
$$a_7 = 1 - \tau_1,\ a_6 = 1 - \tfrac{\tau_1}{a_7} = \tfrac{1-2\tau_1}{1-\tau_1},$$
$$a_4 = 1 - \tfrac{\tau_1}{a_3} - \tfrac{\tau_1}{a_6} - \tau_2 = \tfrac{1-4\tau_1+2\tau_1^2}{1-2\tau_1} - \tau_2.$$

Then we will get the following limitations on the parameters:

$$\begin{cases} 1-\tau_1 > 0 \\ \frac{1-2\tau_1}{1-\tau_1} > 0 \\ \frac{1-4\tau_1+2\tau_1^2}{1-2\tau_1} - \tau_2 \geq 0 \end{cases} \quad \text{or} \quad \begin{cases} \tau_1 < \frac{1}{2} \\ \tau_2 \leq \frac{1-4\tau_1+2\tau_1^2}{1-2\tau_1}. \end{cases}$$

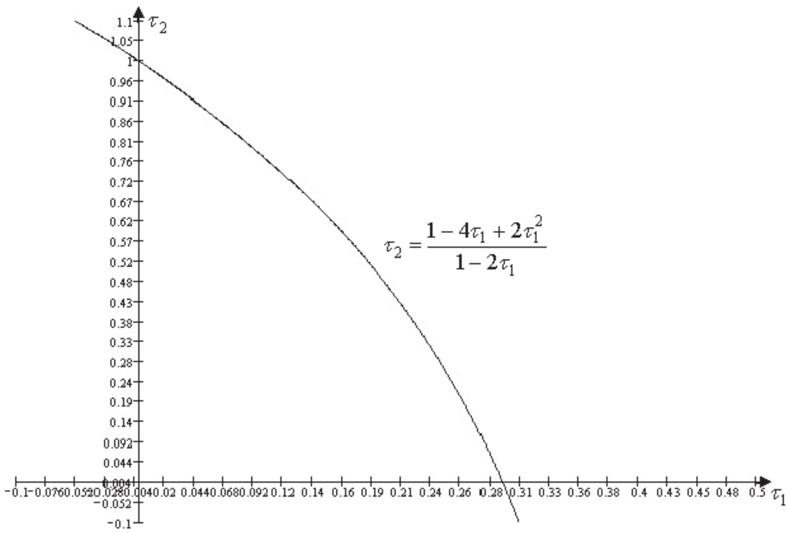

FIGURE 8

6. The algebra $A_{\widetilde{E}_8,\tau,\perp}$. The results of the marking are

$$a_1 = a_4 = a_9 = 1,\ a_2 = 1 - \tau_1,\ a_8 = 1 - \tau_1,$$
$$a_7 = 1 - \tfrac{\tau_1}{a_8} = \tfrac{1-2\tau_1}{1-\tau_1},\ a_6 = 1 - \tfrac{\tau_1}{a_7} = \tfrac{1-3\tau_1+\tau_1^2}{1-2\tau_1},\ a_5 = 1 - \tfrac{\tau_1}{a_6} = \tfrac{1-4\tau_1+3\tau_1^2}{1-3\tau_1+\tau_1^2},$$
$$a_3 = 1 - \tfrac{\tau_1}{a_2} - \tfrac{\tau_1}{a_5} - \tau_2 = \tfrac{1-7\tau_1+15\tau_1^2-10\tau_1^3+\tau_1^4}{(1-\tau_1)(1-4\tau_1+3\tau_1^2)} - \tau_2.$$

Then we will get the following limitations on parameters:

$$\begin{cases} 1-\tau_1 > 0 \\ \frac{1-2\tau_1}{1-\tau_1} > 0 \\ \frac{1-3\tau_1+\tau_1^2}{1-2\tau_1} > 0 \\ \frac{1-4\tau_1+3\tau_1^2}{1-3\tau_1+\tau_1^2} > 0 \\ \frac{1-7\tau_1+15\tau_1^2-10\tau_1^3+\tau_1^4}{(1-\tau_1)(1-4\tau_1+3\tau_1^2)} - \tau_2 \geq 0 \end{cases} \quad \text{or} \quad \begin{cases} \tau_1 \in (0; \frac{1}{3}) \\ \tau_2 \leq \frac{1-7\tau_1+15\tau_1^2-10\tau_1^3+\tau_1^4}{(1-\tau_1)(1-4\tau_1+3\tau_1^2)}. \end{cases}$$

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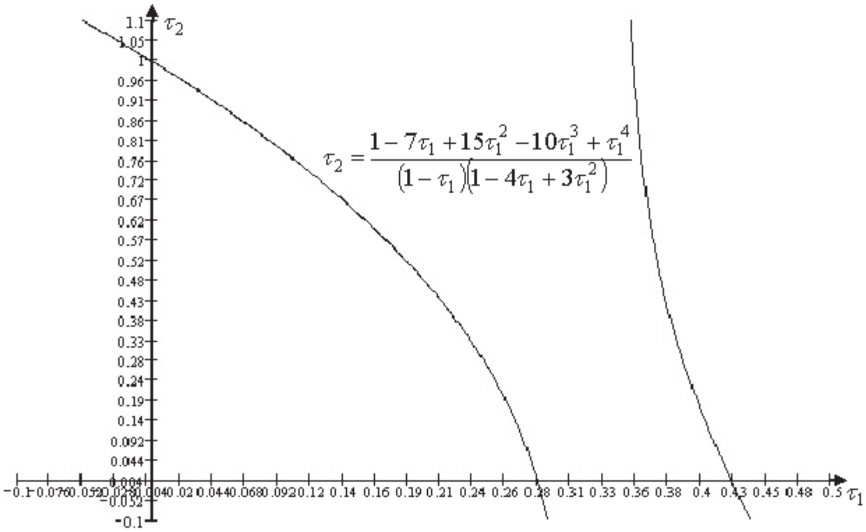

FIGURE 9

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V. I. VERNADSKY TAURIDA NATIONAL UNIVERSITY, 4, VERNADSKIY AVE., SIMFEROPOL, 95007, AUTONOMOUS REPUBLIC OF CRIMEA, UKRAINE

E-mail address: serg_h-g@mail.ru

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