Methods of Functional Analysis and Topology
Vol. 13 (2007), no. 1, pp. 62–66

CLASSIFICATION OF NONCOMPACT SURFACES WITH BOUNDARY

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Abstract. We give a topological classification of noncompact surfaces with any number of boundary components.

0. Introduction

A well-known classification of compact surfaces belongs to H. R. Brahana [1]. The first attempt to classify noncompact surfaces was made by B. V. Kerékjártó in 1923 [2]. Later this problem was considered by I. Richards [3]. There, the case of surfaces without boundaries was considered. A theorem on a classification of noncompact surfaces with a finite number of boundary components was considered in A. O. Prishlyak, K. I. Mischenko [4]. A generalization of this theorem to the case of any number of boundary components is the main result of this paper.

We consider, as our basic subject of the study, triangulable connected surfaces with a finite base of topology. To work with noncompact surfaces, we give a few new definitions and invariants, a boundary component (or an end) of a noncompact surface and an ideal boundary (or the set of ends) of the surface. Then, defining four "orientability classes" of surfaces and genus, we describe characteristic properties of ideal boundaries of noncompact surfaces with any number of boundary components. Thus, a task is to classify surfaces without boundary.

1. Basic definitions

The fundamental classification theorem for compact surfaces with border is the following: two compact triangulable surfaces with border are homeomorphic if and only if they both have the same number of boundary curves, the same Euler characteristic and are either both orientable or nonorientable.

Let $P_1 \supset P_2 \supset \dots$ be a nested sequence of connected unbounded regions in S such that the following holds:

a) the boundary of P_n in S is compact for all n;

b) for any bounded subset A of S, $P_n \cap A = \emptyset$ for n sufficiently large.

We say that two sequences $P_1 \supset P_2 \supset P_3 \supset \dots$ and $Q_1 \supset Q_2 \supset Q_3 \supset \dots$ are equivalent if for any n there is a corresponding integer N such that $P_n \subset Q_N$ and vice versa.

Definition 1. The equivalence class of sequences containing $p = P_1 \supset P_2 \supset P_3 \supset \dots$ is called an end and denoted by p^*.

Definition 2. The ideal boundary (the set of ends) $B(S)$ of a surface S is a topological space having the ends of S as elements and endowed with the following topology: for any

2000 *Mathematics Subject Classification.* 57N05.
Key words and phrases. Classification, noncompact, surface, 2-manifold

set U in S whose boundary in S is compact, we define U^* to be the set of all ends p^*, represented by some $p = P_1 \supset P_2 \supset P_3 \supset \dots$, such that $P_n \subset U$ for a sufficiently large n; we take the set of all such $U*$ to be a basis for the topology of $B(S)$.

Another definition of the set of ends can be given as follows.

Definition 3. Let S be a noncompact surface. Then there exists a compactification S^* of S (its existence was proved in [5]) such that

1) S^* is a locally connected set;
2) $B(S) = S^* - S$ does not disconnect S^*;
3) $B(S)$ is a totally disconnected set.

Thus determined $B(S)$ is the ideal boundary of S.

Definition 4. Let p^*, represented by $p = P_1 \supset P_2 \supset P_3 \supset \dots$, be an end of S. We say that p^* is planar and\or orientable if the sets P_n are planar and\or orientable for all sufficiently large n.

Following Definition 4, we will consider the set of ends to be a nested triple of the sets $B(S) \supset B'(S) \supset B''(S)$, where $B(S)$ is the hole ideal boundary, $B'(S)$ is the part which is not planar, and $B''(S)$ is the part which is not orientable. It follows directly from the definitions that $B'(S)$ and $B''(S)$ are closed subsets of $B(S)$.

Definition 5. A surface S with boundary is of infinite genus and\or infinitely nonorientable if there is no a bounded subset A of S such that $S - A$ is of genus zero and\or orientable.

Clearly, an infinitely nonorientable surface is also of infinite genus.

So, we distinguished two "orientability classes" of noncompact surfaces. There are orientable and infinitely nonorientable surfaces. A surface which belongs to neither of these categories is said to be of odd or even nonorientability type according to whether every sufficiently large compact subsurface contains, respectively, an odd or an even number of "cross cups" (i.e., has half integral or integral reduced genus.)

Consequently, we defined four "orientability classes" of noncompact surfaces.

Kerékjártó [2] has obtained the following result.

Theorem 1. *Let S_1 and S_2 be two separable surfaces of the same genus and orientability class. Then S_1 and S_2 are homeomorphic if and only if their ideal boundaries, considered as triple of spaces, are topologically equivalent.*

In addition, Richards [3] extended and proofed Kerékjártó's results as follows.

Theorem 2. *1. Let X be a totally disconnected metrical space, U and V its open subsets $U \supset V$. Then there exists a noncompact surface S such that there is a homeomorphism from $B(S)$ onto X that maps $B'(S)$ onto U and $B''(S)$ onto V.*

2. The set of ends of a separable surface is totally disconnected, separable, and compact.

3. Let (X, Y, Z) be any triple of compact, separable, totally disconnected spaces with $Z \subset Y \subset X$. Then there exists a surface S whose ideal boundary $(B(S), B'(S), B''(S))$ is topologically equivalent to the triple (X, Y, Z).

2. Boundary ends

A disk punctured at the boundary is homeomorphic to a disk with an open interval cut outfrom the boundary. This was proved in [4].

Considering noncompact surfaces with a finite number of boundary components, we see that components of the boundary can be compact (circles) and noncompact (intervals). In the case of circles, obviously they can be gathered to a point or glued up by a disk.

This case is trivial. When we deal with intervals, our purpose is to show that these components can also be represented as circles which are punctured at a finite number of points.

Let p^* be an equivalence class of S which lies on the boundary. Then there are exactly two ends which are joined to p^* and, for these ends, p^* is a boundary point [4].

So, the components of the boundary and the boundary ends can be subdivided into groups such that one group is a cyclic sequence made up of ends and components of the boundary in which two any neighboring elements make a component of the boundary and its corresponding end. Such a sequence is called a boundary cycle.

Gathering boundary cycles on points or gluing them up by disks, we will get a noncompact surface without boundary. Thus a classification of noncompact surfaces with a finite number of boundary components becomes a classification of noncompact surfaces without boundary.

3. Surfaces with an infinite number of boundary components

Farther we will classify surfaces which can have an infinite number of boundary components. To do this, at first we classify the ends which lie on the boundary.

The set of boundary ends will be denoted by C. To classify them, we will consider pairs (B, C), (B', C') and (B'', C'') in lines with the given before classification of noncompact surfaces with a finite number of boundary component. Let us define an equivalence relation on the set C.

Definition 6. Several ends belong to the same class of equivalence if they can be connected in pairs by sequences of adjacent ends.

Definition 7. Two ends are adjacent if there exists a boundary component for which they are the ends of it.

An equivalence Relation on the sets of C' and C'' is defined similarly.

We note that, in the case of an infinite number of boundary components, boundary cycles can be infinite.

the equivalence relation $\sim$ defined above gives rise to the set $D := C/\sim$. Consequently, D is a set of boundary cycles or, which is the same, a set of circles with punctured points. Then we have the following lemma.

Lemma 1. *The union of boundary ends can be represented as circles with embedded subsets of the Cantor.*

Construction. Let S be a noncompact surface with boundary. It is possible to arrange the boundary components α_k in circles with the pricked points out. If we throw away the boundary of the considered surface, we obtain a noncompact surface without boundary. To classify such surfaces, we can apply Richard's theorem. Interiors of surfaces are homeomorphic and there exist a sequences of compact surfaces F_k such that every next contains the previous one, $\forall k \geq 1 : F_k \subset F_{k+1}$.

The compact connected bordered surface is topologically determined by its orientability, genus and the number of its boundary curves. Then, linking each of the compact surfaces F_k with the components α_k, we will get a complete classification of noncompact surfaces with boundary.

To do this, to every subsurface F_k, we glue strips which continuously connect F_k with the corresponding component α_k (see Fig.1).

As far as punctured points are on the circle, this circle can be represented as a limit of a sequence of closed segments. So there exist a sequence $\beta_k^i \to \alpha_k$, $i \to \infty$, for the boundary component α_k. Then there is a sequence of finite unions of $\beta_k^i : \bigcup\limits_{k,i} \beta_k^i, i \geq 1$. Fixing the numbers of these segments from the list, we make gluing as follows: we

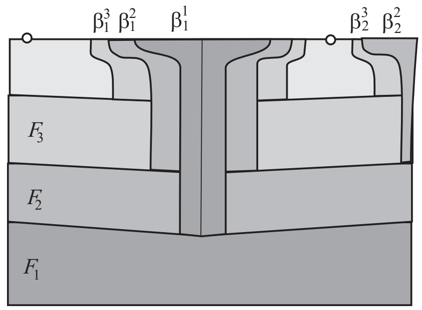

Figure 1

continuously connect the first segment with the first (the least) subsurface F_1. We require that crossing near every surface of F_k be transversal in one point. To get a surface with boundary, we extend this path to a closed neighborhood. To avoid an ambiguity in the subsequent gluing, the first time a point on a surface is picked arbitrarily. Each following time, a point on a surface is picked depending on the sets D and D'. All paths on the surface S_1 are constructed arbitrarily with a following condition satisfied: if a path goes out from the boundary of surface F_k, then it crosses the boundary of every surface F_n ($n > k$) transversely in one point only. The paths on the surface S_2 are constructed such that the first path is built arbitrarily, the other paths are chosen as follows: the curvilinear quadrangles are formed by parts of the boundary of the surface F_k and parts of this and the previous paths that were boundaries of the regions homeomorphic to the corresponding regions of the surface S_1. These quadrangles must have the same number of ends, caps or cross caps. It is easy to do that in a way that a neighborhood of each following path that connects the point pricked out from the circle with a subsurface of the corresponding index, would contain the previous one. At every step of gluing of the strips, we have the finite number of segments.

Lemma 2. *Lets for certain k there be two paths γ_1 and γ_2 that connect the surface F_k with the corresponding component of the boundary. Then the surfaces obtained by gluing the strips with these paths are homeomorphic.*

Proof. Since, by the construction, every path crosses the boundary transversely, we may assume that the paths γ_1 and γ_2 coincide in some neighborhood of the boundary component. Consequently, there is a number $n \in \mathbf{N}$ such that the paths γ_1 and γ_2 do not coincide on the set $M = F_{k+n} \setminus F_n$. This means that there exists a homeomorphism of this set onto itself, $\varphi : M \to M$, which maps γ_1 to $\gamma_2 : \varphi(\gamma_1) = \gamma_2$. So, Lemma 2 is proved. □

Consequently, every closed segment between two arbitrary points pricked out on a circle is continuously connected with the corresponding subsurface F_k. Then, passing to the limit, we obtain an initial noncompact surface with a boundary as a limit of compact surfaces with boundary.

Lemma 3. *The so obtained sequence of surfaces with boundary and the glued strips approaches the noncompact surface S.*

Proof. Denote by $\widetilde{F_k}$ the obtained sequence of surfaces with boundary with the glued strips. Obviously, $\forall k \geq 1 : F_k \subset \widetilde{F_k}$. Then $\lim\limits_{k\to\infty} F_k \subset \lim\limits_{k\to\infty} \widetilde{F_k}$, and $\lim\limits_{k\to\infty} F_k = \operatorname{Int} S \subset \lim\limits_{k\to\infty} \widetilde{F_k}$. By the construction, $\widetilde{F_k}$ contains each boundary component. Consequently, the union of boundary components with the interior of the surface S is the surface S. Lemma 3 is proved. □

Theorem 3. *Two any noncompact separable surfaces with boundary, S_1 and S_2, are homeomorphic if and only if they have the same genus, orientability class and there exists a homeomorphism which maps $B(S_1)$ on $B(S_2)$, $C(S_1)$ on $C(S_2), B'(S_1)$ on $B'(S_2)$, $C'(S_1)$ on $C'(S_2)$, $B''(S_1)$ on $B''(S_2)$, $C''(S_1)$ on $C''(S_2)$ and $D(S_1)$ on $D(S_2)$.*

Proof. The necessity is obviously, because if noncompact separable surfaces with a boundary are homeomorphic, then they have the same genus, orientability class and there exists a homeomorphism which maps all indicated sets of one surface on the corresponding sets of other surface.

Sufficiency. Lets there be two noncompact separable surfaces with boundary, S_1 and S_2, which have the same genus, orientability class and there exist a homeomorphism that maps $B(S_1)$ on $B(S_2)$, and there exist a homeomorphism which maps $B(S_1)$ on $B(S_2)$, $C(S_1)$ on $C(S_2), B'(S_1)$ on $B'(S_2)$, $C'(S_1)$ on $C'(S_2)$, $B''(S_1)$ on $B''(S_2)$, $C''(S_1)$ on $C''(S_2)$ and $D(S_1)$ on $D(S_2)$. We apply the construction described above. Then the surfaces $\widetilde{F_k^1}$ and F_k^1 are homeomorphic by Lemma 2. The same can be stated about $\widetilde{F_k^2}$ and F_k^2. Then $\widetilde{F_k^1}, \to S_1, \widetilde{F_k^2}, \to S_2, k \to \infty$ and, for all $k \geq 1$, $F_k \subset F_{k+1}$ by Lemma 3. Consequently, there exist homeomorphisms $f_k : \widetilde{F_k^1} \to \widetilde{F_k^2}$, and the sequence of homeomorphisms $\{f_k, k \geq 1\}$ defines a homeomorphism $f : S_1 \to S_2$, where $f = \lim_{k\to\infty} f_k$. So, the theorem is proved. □

Conclusion. Thus, we obtained a complete classification of noncompact surfaces with an arbitrary number of boundary components. It may be used for a study of functions with isolated critical points on surfaces, harmonic functions and forms.

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Received 16/11/2005