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ON QUADRUPLES OF LINEARLY CONNECTED PROJECTIONS AND TRANSITIVE SYSTEMS OF SUBSPACES

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Abstract. We study conditions under which the images of irreducible quadruples of linearly connected projections give rise to all transitive systems of subspaces in a finite dimensional Hilbert space.

Introduction

A number of recent papers are devoted to the study of families of projections $\{P_i\}_{i=1}^n$, in a complex separable Hilbert space $\mathcal{H}$, which satisfy the linear relation

$$\alpha_1 P_1 + \cdots + \alpha_n P_n = \gamma I, \tag{1}$$

where all α_i and γ are real non-negative numbers. In particular, the correspondence between such irreducible families and associated systems of n subspaces in $\mathcal{H}$, $S = (\mathcal{H}; \mathcal{H}_1, \dots, \mathcal{H}_n)$ where $\mathcal{H}_i = \operatorname{Im}(P_i)$, was noticed and studied in [3, 8].

The system of subspaces S is transitive (brick) if any operator in $\mathcal{H}$ which maps any $\mathcal{H}_i$ into itself is scalar. In this case, we also say that the family $\{P_i\}_{i=1}^n$ is transitive. In [3] it was shown that there exists a one-to-one correspondence between transitive quadruples of subspaces in a finite-dimensional Hilbert space and irreducible quadruples of projections, $P_1, \dots, P_4$, such that $P_1 + P_2 + P_3 + P_4 = \gamma I$ for some $\gamma \in \mathbb{R}$. For arbitrary n, in the finite-dimensional case the images of an irreducible family of projections $P_1, \dots, P_n$ satisfying (1) form a transitive n-tuple of subspaces (see [8]). In the infinite-dimensional case, the structure of transitive quadruples of subspaces is much more complicated (see, e.g., [2]). Also, it is still unknown if there exist infinite-dimensional transitive triples of subspaces.

In this paper we show directly that all irreducible families of projections that satisfy (1) are transitive in the case where $n \le 4$. The following question arises naturally: given a fixed $\chi_n = (\alpha_1, \dots, \alpha_n)$, $n \le 4$, will all transitive systems arise as images of the projections satisfying (1) with an appropriate γ? If $\chi_n = (1, \dots, 1)$ then the answer is positive (see [3]). The investigation in the case where $n < 4$ is trivial. For the case $n = 4$ we use the description of transitive systems in finite dimensional space given in [1] to show that given a fixed $\chi_4 = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ the irreducible families of projections satisfying (1) generate all transitive finite-dimensional quadruples of subspaces if and only if $\chi_4 = (1, 1, 1, 1)$.

1. Transitive systems of subspaces

Consider the category $\operatorname{Sys}(n)$, $n \in \mathbb{N}$. Each object in this category, $S \in \operatorname{Sys}_n$, is a system $S = (\mathcal{H}; \mathcal{H}_1, \dots, \mathcal{H}_n)$ of subspaces $\mathcal{H}_i$ in some Hilbert space $\mathcal{H}$. A morphism $A \in \operatorname{Mor}(S, \tilde{S})$ between two systems $S \in \operatorname{Sys}_n$ and $\tilde{S} \in \operatorname{Sys}_n$ is a linear bounded operator $A : \mathcal{H} \to \tilde{\mathcal{H}}$, such that

$$A(\mathcal{H}_i) \subset \tilde{\mathcal{H}}_i, \quad \text{for all } i = 1, \dots, n.$$

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Definition 1. *A system $S \in \mathrm{Sys}_n$ is transitive if the algebra of its endomorphisms is trivial, i.e.,* $\mathrm{Mor}(S,S) = \mathbb{C}I_{\mathcal{H}}$.

Definition 2. *Two systems $S \in \mathrm{Sys}_n$ and $\tilde{S} \in \mathrm{Sys}_n$ are isomorphic if there exists a bijective operator $A \in \mathrm{Mor}(S,\tilde{S})$ such that*

$$A(\mathcal{H}_i) = \tilde{\mathcal{H}}_i, \quad \text{for all } i = 1,\dots,n.$$

Definition 3. *Two systems $S \in \mathrm{Sys}_n$ and $\tilde{S} \in \mathrm{Sys}_n$ are called isomorphic up to a permutation if there exists a permutation $\sigma \in S_n$ such that the systems $S_\sigma = (\mathcal{H}, \mathcal{H}_{\sigma(1)}, \dots, \mathcal{H}_{\sigma(n)})$ and $\tilde{S}$ are isomorphic.*

Transitive systems are the simplest objects in the category Sys_n.

Theorem 1 (S. Brenner [1])**.**

(1) *For $n = 1$, there exist 2 non-isomorphic transitive systems,*

$$S_1^{(1)} = (\mathbb{C}; 0), \quad S_2^{(1)} = (\mathbb{C}; \mathbb{C}).$$

(2) *For $n = 2$, there exist 4 non-isomorphic transitive systems,*

$$S_1^{(2)} = (\mathbb{C}; 0, 0), \quad S_1^{(2)} = (\mathbb{C}; \mathbb{C}, 0), \quad S_3^{(2)} = (\mathbb{C}; 0, \mathbb{C}), \quad S_4^{(2)} = (\mathbb{C}; \mathbb{C}, \mathbb{C}).$$

(3) *For $n = 3$, there exist 9 non-isomorphic transitive systems, 8 one-dimensional,*

$$S_1^{(3)} = (\mathbb{C}; 0, 0, 0), \qquad S_2^{(3)} = (\mathbb{C}; \mathbb{C}, 0, 0), \qquad S_3^{(3)} = (\mathbb{C}; 0, \mathbb{C}, 0),$$
$$S_4^{(3)} = (\mathbb{C}; 0, 0, \mathbb{C}), \qquad S_5^{(3)} = (\mathbb{C}; \mathbb{C}, \mathbb{C}, 0), \qquad S_6^{(3)} = (\mathbb{C}; \mathbb{C}, 0, \mathbb{C}),$$
$$S_7^{(3)} = (\mathbb{C}; 0, \mathbb{C}, \mathbb{C}), \qquad S_8^{(3)} = (\mathbb{C}; \mathbb{C}, \mathbb{C}, \mathbb{C}),$$

and 1 two-dimensional,

$$S_9^3 = (\mathbb{C}^2; \mathbb{C}(0,1), \mathbb{C}(1,0), \mathbb{C}(1,1)).$$

For $n = 4$, the description depends in an essential way on an important integer valued invariant $\rho(S)$, called a *defect*.

Definition 4. *For a system $S \in \mathrm{Sys}_n$,*

$$\rho(S) = \sum_{i=1}^{n} \dim \mathcal{H}_i - 2 \dim \mathcal{H}.$$

It turned out that there exist a one-parameter continuous family of transitive systems with defect 0, and four countable series of transitive systems with defect $\rho(S) = \pm 2, \pm 1$, respectively.

Theorem 2 (S. Brenner [1])**.** *Let $B(u,\rho)$ denote the set of systems $S \in \mathrm{Sys}_4$ such that* $\dim(\mathcal{H}) = u$ *and $\rho(S) = \rho$. Then we have the following.*

(1) *For every $u > 2, u \in \mathbb{N}$, there exists a unique system $S \in B(u, \pm 1)$, up to isomorphism and permutation.*
(2) *For every $u = 2k+1, k \in \mathbb{N}$, there exists a unique system $S \in B(u, \pm 2)$, up to isomorphism and permutation. If the dimension of $\mathcal{H}$ is even, then there exist no systems with defect $\rho(S) = \pm 2$.*
(3) *Besides the trivial one-dimensional systems with defect $\rho(S) = 0$, there exists the one-parameter family $B(2,0)$. If $S_\lambda = (\mathbb{C}^2; \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4) \in B(2,0)$, then*

$$\mathcal{H}_1 = \mathbb{C}(1,0), \qquad \mathcal{H}_2 = \mathbb{C}(0,1),$$
$$\mathcal{H}_3 = \mathbb{C}(1,1), \qquad \mathcal{H}_4 = \mathbb{C}(1,\theta), \quad \theta \in \mathbb{C} \setminus \{0,1\}.$$

There exist no other transitive systems of four subspaces in a finite-dimensional Hilbert space.

2. Projections with linear relation and Coxeter functors

Let $\chi_n = (\alpha_1, \dots, \alpha_n)$ be a vector in $\mathbb{R}_+^n$ the components of which are ordered by values. Consider the finitely generated $*$-algebra

$$\mathcal{A}_{\chi_n} = \mathbb{C}\langle p_1, \dots, p_n, q | p_i = p_i^* = p_i^2,\ [q, p_i] = 0,\ \alpha_1 p_1 + \dots + \alpha_n p_n = q\rangle.$$

The generator q belongs to the center of the algebra, therefore, any irreducible $*$-representation of this algebra is given by an irreducible collections of projections $\{P_i\}_{i=1}^n$ that satisfy

$$\alpha_1 P_1 + \dots + \alpha_n P_n = \gamma I \tag{2}$$

for some γ.

Remark 1. If two vectors $\tilde{\chi}_n$ and χ_n are proportional, then the corresponding algebras $\mathcal{A}_{\chi_n}$ and $\mathcal{A}_{\tilde{\chi}_n}$ are $*$-isomorphic, so in what follows we will consider vectors χ_n from the projective space $\mathbb{PR}_+^n$.

Proposition 1.

1. *If $n < 3$, then for all vectors $\chi_n \in \mathbb{PR}_+^n$ all irreducible representations of the algebras $\mathcal{A}_{\chi_n}$ generate all transitive system of n subspaces.*
2. *If $n = 3$, then all irreducible representations of the algebras $\mathcal{A}_{\chi_3}$ generate all transitive systems of 3 subspaces iff, for the vector $\chi_3 = (\alpha_1, \alpha_2, \alpha_3)$, the following holds:*

$$\alpha_3 < \alpha_1 + \alpha_2.$$

Proof. The proof is trivial in the case where $n < 3$. Indeed, irreducible $*$-representations of the algebra $\mathcal{A}_{\chi_n}$ are one-dimensional and it is easy to see the statement.

For $n = 3$, there exist 8 one-dimensional $*$-representations of the algebra $\mathcal{A}_{\chi_3}$, but irreducible two-dimensional representations exists iff $\alpha_3 < \alpha_1 + \alpha_2$, hence this proves the statement. □

In the case of four subspaces the investigation is based on the structure of the set Σ_{χ_4}, which is the set of those $\gamma \in \mathbb{R}$ for which there are quadruples of projections that satisfy (2). Such set was completely described in paper [7] using the Coxeter functors technique, developed in [4]. Namely there are two functors Φ^+ and Φ^- which establish equivalence between the categories of $*$-representations of the algebras $\mathcal{A}_{\chi_n}$ with different values of the vector χ_n (see [4] for the details). Using these functors it was proved that all irreducible representations of the algebra $\mathcal{A}_{\chi_4}$ are finite dimensional and representations with defect $\rho(S_\pi) \neq 0$ could be obtained starting from one-dimensional representations. But there exists a hyperplane (corresponding to defect $\rho(S_\pi) = 0$) invariant with respect to the action of the Coxeter functors (it is defined by the condition $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 2\gamma$). In what follows we conduct an investigation of these two possibilities.

At first we notice that the Coxeter functors preserve the transitivity and the defect.

Proposition 2. *Coxeter functors Φ^+ and Φ^- preserve the defect value of the system in the following sense: if $\pi \in \operatorname{Rep} \mathcal{A}_{\chi_4}$ and $\pi_+ = \Phi^+(\pi)$ and $\pi_- = \Phi^-(\pi)$, then*

$$\rho(S_\pi) = \rho(S_{\pi_+}), \quad \rho(S_\pi) = \rho(S_{\pi_-}).$$

Proof. The proof is clear after extending the action of the Coxeter functors to the vectors of generalized dimension of representation $\pi : \mathcal{A}_{\chi_4} \to \mathcal{H}$, i.e., to the vectors

$$v_\pi = (\dim \mathcal{H}, \dim(\operatorname{Im}(\pi(p_1))), \dots, \dim(\operatorname{Im}(\pi(p_1)))).$$ □

In [3] it was proved that the functors Φ^+ and Φ^- map transitive families of representations of $\mathcal{A}_{(1,\ldots,1)}$ into transitive families. The proof can be easily modified for a more general situation of an arbitrary vector χ_n.

Proposition 3. *The functors Φ^+ and Φ^- map representations that generate transitive systems into representation that generate transitive systems. That is, if $\pi \in Rep\mathcal{A}_{\chi_n}$ and S_π is a transitive system, then the systems S_{π_+} and S_{π_-} are transitive, where $\pi_+ = \Phi^+(\pi)$, $\pi_- = \Phi^-(\pi)$.*

Corollary 1. *If for a pair (χ_4, γ) there exists an irreducible collection of projections P_1, P_2, P_3, P_4 in space $\mathcal{H}$ such that*

$$\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = \gamma I,$$

then there exist $\alpha \in \mathbb{R}$ and an irreducible collection of projections $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \tilde{P}_4$ in the space $\tilde{\mathcal{H}}$ such that

$$\tilde{P}_1 + \tilde{P}_2 + \tilde{P}_3 + \tilde{P}_4 = \alpha I,$$

and the systems $S = (\mathcal{H}; \mathrm{Im}(P_1), \mathrm{Im}(P_2), \mathrm{Im}(P_3), \mathrm{Im}(P_4))$ and $\tilde{S} = (\mathcal{H}; \mathrm{Im}(\tilde{P}_1), \mathrm{Im}(\tilde{P}_2), \mathrm{Im}(\tilde{P}_3), \mathrm{Im}(\tilde{P}_4))$, are isomorphic in Sys_4.

Proof. An arbitrary irreducible quadruple of projections such that $\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = \gamma I$ and $\gamma \neq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/2$ could be obtained by the functor Φ^+ starting from a one-dimensional quadruple, hence it is a transitive system. On the other hand irreducible collections of projections such that $\tilde{P}_1 + \tilde{P}_2 + \tilde{P}_3 + \tilde{P}_4 = \alpha I$ generate all transitive systems. Hence there exists α such that the statement holds. □

3. The case of nonzero defect

Proposition 4. *1. If the vector χ_4 satisfies*

$$\alpha_1 + \alpha_4 < \alpha_2 + \alpha_3,$$

then all irreducible $*$-representations of the algebra $\mathcal{A}_{\chi_4}$ generate all transitive systems of four subspaces with the defect value $\rho(S) = 1$.

2. If the vector χ_4 satisfies

$$\alpha_1 + \alpha_4 > \alpha_2 + \alpha_3,$$

then all irreducible $*$-representations of the algebra $\mathcal{A}_{\chi_4}$ generate all transitive systems of four subspaces with the defect value $\rho(S) = -1$.

Proof. Let χ_4 satisfy $\alpha_1 + \alpha_4 < \alpha_2 + \alpha_3$. Then (see [7]) the set Σ_{χ_4} includes the infinite series

$$\left\{ \frac{\alpha}{2} - \frac{\alpha_1}{2n} \mid n \in \mathbb{N} \right\}.$$

The corresponding infinite series of $*$-representations are representations of the dimensions $3, 4, \ldots$. Such series is generated by the action of the functor Φ^+ on one dimensional representation $P_1 = 0, P_2 = I, P_3 = I, P_4 = I$ with defect 1. Therefore using Proposition 2 and Theorem 3 we see that such series generate all transitive systems with defect value $\rho(S) = 1$.

The case $\alpha_1 + \alpha_4 < \alpha_2 + \alpha_3$ is similar. □

Corollary 2. *All irreducible $*$-representations of the algebra $\mathcal{A}_{\chi_4}$ generate all transitive systems with defect value $\rho(S) = \pm 1$ if and only if the vector χ_4 satisfies*

$$\alpha_1 + \alpha_4 = \alpha_2 + \alpha_3.$$

Proposition 5. *Irreducible $*$-representations of the algebra $\mathcal{A}_{\chi_4}$ generate all transitive systems with defect value $\rho(S) = \pm 2$ if and only if $\chi_4 = (1, 1, 1, 1)$ up to a multiplier.*

Proof. All irreducible representations with defect value ± 2 are obtained using the functor Φ^+ starting from one-dimensional $P_1 = I, P_2 = I, P_3 = I, P_4 = I$ and $P_1 = 0, P_2 = 0, P_3 = 0, P_4 = 0$. But due to the structure of the set Σ_{χ_4} [7] such series of representations are infinite if and only if $\chi_4 = (1,1,1,1)$ up to a multiplier. □

4. The case of zero defect

Transitive systems of four subspaces with defect value 0 are generated by the quadruples of projections that satisfy

$$\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = 2, \tag{3}$$

and with the equation

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 4.$$

Such irreducible quadruples exist in dimension not grater than 2. It is easy to describe one-dimensional quadruples and to see that for an arbitrary $\chi_4 = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ such quadruples do not generate all one-dimensional transitive systems with defect value 0.

To investigate two-dimensional case we use explicit formulas for the solutions of (3) (see [5])

$$P_1 = \frac{1}{2\alpha_1\lambda}\begin{pmatrix} (\lambda - u_1)(\lambda + u_2) & \sqrt{-(\lambda^2 - u_1^2)(\lambda^2 - u_2^2)} \\ \sqrt{-(\lambda^2 - u_1^2)(\lambda^2 - u_2^2)} & -(\lambda + u_1)(\lambda - u_2) \end{pmatrix},$$

$$P_2 = \frac{1}{2\alpha_2\lambda}\begin{pmatrix} -(\lambda - v_2)(\lambda + v_1) & e^{i\chi}\sqrt{-(\lambda^2 - v_2^2)(\lambda^2 - v_1^2)} \\ e^{-i\chi}\sqrt{-(\lambda^2 - v_2^2)(\lambda^2 - v_1^2)} & (\lambda + v_2)(\lambda - v_1) \end{pmatrix},$$

$$P_3 = \frac{1}{2\alpha_3\lambda}\begin{pmatrix} -(\lambda - v_2)(\lambda - v_1) & -e^{i\chi}\sqrt{-(\lambda^2 - v_2^2)(\lambda^2 - v_1^2)} \\ -e^{-i\chi}\sqrt{-(\lambda^2 - v_2^2)(\lambda^2 - v_1^2)} & (\lambda + v_2)(\lambda + v_1) \end{pmatrix},$$

$$P_4 = \frac{1}{2\alpha_4\lambda}\begin{pmatrix} (\lambda + u_2)(\lambda + u_1) & -\sqrt{-(\lambda^2 - u_1^2)(\lambda^2 - u_2^2)} \\ -\sqrt{-(\lambda^2 - u_1^2)(\lambda^2 - u_2^2)} & -(\lambda - u_2)(\lambda - u_1) \end{pmatrix},$$

$$(\alpha_4 - \alpha_1)/2 \le \lambda \le \min((\alpha_2 + \alpha_3)/2, (\alpha_1 + \alpha_4)/2)), \quad 0 \le \chi < 2\pi,$$

where $u_1 = \frac{1}{2}(\alpha_4 - \alpha_1)$, $u_2 = \frac{1}{2}(\alpha_4 + \alpha_1)$, $v_1 = \frac{1}{2}(\alpha_3 - \alpha_2)$, $v_2 = \frac{1}{2}(\alpha_3 + \alpha_2)$.

The following theorem holds.

Theorem 3. *All two-dimensional projections P_1, P_2, P_3, P_4 that satisfy* (3) *generate all transitive systems of four subspaces with defect value 0 if and only if*

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1 \tag{4}$$

up to a positive multiplier.

Proof. First we prove that the condition is sufficient. Let (4) hold, then the formulas for P_1, P_2, P_3, P_4 take the following form:

$$P_1 = \frac{1}{2}\begin{pmatrix} 1+\lambda & \sqrt{1-\lambda^2} \\ \sqrt{1-\lambda^2} & 1-\lambda \end{pmatrix}, \qquad P_2 = \frac{1}{2}\begin{pmatrix} 1-\lambda & e^{i\chi}\sqrt{1-\lambda^2} \\ e^{-i\chi}\sqrt{1-\lambda^2} & 1+\lambda \end{pmatrix},$$

$$P_3 = \frac{1}{2}\begin{pmatrix} 1-\lambda & -e^{i\chi}\sqrt{1-\lambda^2} \\ -e^{-i\chi}\sqrt{1-\lambda^2} & 1+\lambda \end{pmatrix}, \qquad P_4 = \frac{1}{2}\begin{pmatrix} 1+\lambda & -\sqrt{1-\lambda^2} \\ -\sqrt{1-\lambda^2} & 1-\lambda \end{pmatrix},$$

$$0 \le \lambda < 1, \quad \begin{cases} 0 < \chi < \pi, & \lambda = 0, \\ 0 \le \chi < 2\pi, & 0 < \lambda < 1. \end{cases}$$

Let $\Omega \subset \mathbb{C}$ be the set of complex numbers $z \in \mathbb{C}$ such that

$$|z| = \frac{1-\lambda}{1+\lambda}, \quad \arg z = -\chi.$$

The set Ω selects the set of all two-dimensional, unitary non equivalent quadruples of projections that satisfy (3). Topologically this set is homeomorphic to the sphere without three points.

Consider the following complex function (Zhukovski function):

$$\theta(z) = \frac{1}{4}\left(2 + z + \frac{1}{z}\right). \tag{5}$$

The following proposition proves sufficiency of the statement of the theorem.

Proposition 6. *The Zhukovski function $\theta(z)$ maps conformally the domain Ω into the domain $\mathbb{C}\backslash\{0,1\}$. The system of subspaces that corresponds to the parameter $z \in \Omega$ is isomorphic to transitive quadruples* (3) *with parameter $\theta = \theta(z)$.*

Proof. The domain Ω is the domain of univalence of the function $\theta(z)$. The function $\theta(z)$ maps every circle $|z| \in (0,1)$ in Ω to an ellipse with focuses at the points 0 and 1. And the arc $|z| = 1$ maps into the interval $(0,1)$.

The images of the projections P_1, P_2, P_3, P_4 are the following subspaces in $\mathbb{C}^2$:

$$\operatorname{Im}(P_1) = \mathbb{C}(1, \sqrt{|z|}), \quad \operatorname{Im}(P_4) = \mathbb{C}(1, -\sqrt{|z|}),$$
$$\operatorname{Im}(P_2) = \mathbb{C}(z, \sqrt{|z|}), \quad \operatorname{Im}(P_3) = \mathbb{C}(z, -\sqrt{|z|}).$$

A direct calculation shows that the matrix

$$M = \begin{pmatrix} 2e^{i\arg z} & -2e^{2i\arg z}\sqrt{|z|} \\ z+1 & (z+1)\sqrt{|z|}^{-1} \end{pmatrix} \in \mathcal{M}^2(\mathbb{C})$$

establishes an isomorphism between systems of subspaces generated by the images of projections and with transitive quadruples with parameter $\theta = \theta(z)$. ☐

Let us prove the necessary part of the statement. Let $\alpha_1 \neq \alpha_4$, and assume that the projections P_1, P_2, P_3, P_4 generate all transitive quadruples with defect value 0. Introduce the following notation:

$$A = \frac{1}{2}(\alpha_4 - \alpha_1), \quad B = \frac{1}{2}(\alpha_4 + \alpha_1), \quad C = \frac{1}{2}(\alpha_3 - \alpha_2), \quad D = \frac{1}{2}(\alpha_3 + \alpha_2),$$

and

$$K_1 = \sqrt{\frac{(\lambda + A)(B - \lambda)}{(\lambda - A)(B + \lambda)}}, \qquad K_2 = \sqrt{\frac{(\lambda - C)(D - \lambda)}{(\lambda + C)(D + \lambda)}},$$
$$K_3 = \frac{\lambda + C}{\lambda - C} K_2, \qquad K_4 = \frac{\lambda - A}{\lambda + A} K_1.$$

In terms of the latter values the images of the projections could be written as follows:

$$\operatorname{Im}(P_1) = \mathbb{C}(1, K_1), \qquad \operatorname{Im}(P_4) = \mathbb{C}(1, -K_4),$$
$$\operatorname{Im}(P_2) = \mathbb{C}(1, e^{-i\chi}K_2), \qquad \operatorname{Im}(P_3) = \mathbb{C}(1, -e^{-i\chi}K_3).$$

For a fixed λ and χ from the set of possible parameters the system of subspaces (2) is isomorphic to a transitive system with parameter θ that is defined as follows:

$$\theta = \frac{1}{(K_1 + K_4)(K_2 + K_3)}\left(K_1K_2 + K_3K_4 + K_1K_4e^{i\chi} + K_2K_3e^{-i\chi}\right).$$

The latter formula is equivalent to the following:

$$\theta = \frac{1}{4}\Bigg(2 - \frac{2AC}{\lambda^2} + \frac{K_1K_2^{-1}(\lambda - A)(\lambda - C)}{\lambda^2}e^{-i\chi}$$
$$+ \frac{(\lambda^2 - A^2)(\lambda^2 - C^2)}{\lambda^4}\frac{\lambda^2}{K_1K_2^{-1}(\lambda - A)(\lambda - C)}e^{-i\chi}\Bigg).$$

Let z be a complex number such that

$$|z| = \frac{K_1 K_2^{-1}(\lambda - A)(\lambda - C)}{\lambda^2}, \quad \arg z = -\chi,$$

and let M denote

$$M = \frac{(\lambda^2 - A^2)(\lambda^2 - C^2)}{\lambda^4},$$

then formula for θ takes the following form:

$$\theta = \frac{1}{4}\left(2 - \frac{2AC}{\lambda^2} + z + M\frac{1}{z}\right).$$

Let us show that this function is not surjective in $\mathbb{C} \setminus \{0, 1\}$. Indeed, for every fixed λ the set of the corresponding values $\theta \in \mathbb{C}$ is an ellipse in the complex plane symmetric with respect to the real axis. It is clear that when λ grows, then the axis of the ellipse also grows and the limit point for focuses are the points 0 and 1. Therefore there must exist λ such that one of the half-axis of the ellipse equals zero; in fact this means that $M = |z|^2$ and the latter is equivalent to

$$\frac{B - \lambda}{B + \lambda} = \frac{D + \lambda}{D - \lambda}.$$

But this means that $\lambda = 0$, which is possible if $\alpha_1 = \alpha_4$. This contradiction proves the theorem. □

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