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SPECTRAL MEASURE OF COMMUTATIVE JACOBI FIELD EQUIPPED WITH MULTIPLICATION STRUCTURE

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Abstract. The article investigates properties of the spectral measure of the Jacobi field constructed over an abstract Hilbert rigging $H_- \supset H \supset L \supset H_+$. Here L is a real commutative Banach algebra that is dense in H.

It is shown that with certain restrictions, the Fourier transform of the spectral measure can be found in a similar way as it was done for the case of the Poisson field with the zero Hilbert space $L^2(\Delta, d\nu)$. Here Δ is a Hausdorff compact space and ν is a probability measure defined on the Borel σ-algebra of subsets of Δ.

The article contains a formula for the Fourier transform of a spectral measure of the Jacobi field that is constructed over the above-mentioned abstract rigging.

1. Introduction

It is known that the Gaussian measure on the Schwartz space $S'(\mathbb{R})$ is the spectral measure of a commuting family of boson field of self-adjoint operators that act in the symmetric Fock space $\mathcal{F}(H)$ constructed with the use of the Hilbert space $H = L^2_{\mathrm{Re}}(\mathbb{R}, dt)$.

Similar results were obtained in [8] and [9] by Yoshifusa Ito and Izumi Kubo. They used results contained in the papers of T. Hida and N. Ikeda (see [5], [6]) where both the Gaussian and Poisson cases were considered. The results of these works gave a possibility to assume that the Poisson measure with intensity dt on the space $S'(\mathbb{R})$ is also the spectral measure of a family of self-adjoint operators perturbed by diagonal operators of some quite general form. The proof of this statement can be found in the paper of Yurij M. Berezansky [10].

In [10] the author claims that the same result can be obtained without a considerable change for the following more general construction.

Suppose, instead of the space $\mathbb{R}$ with the Lebesgue measure dt, one has a measurable space T with a σ-finite measure $d\nu(t)$. Construct the Fock space $\mathcal{F}(H)$ using $H = L^2_{\mathrm{Re}}(T, d\nu(t))$. In the space $\mathcal{F}(H)$ one defines a family of commuting self-adjoint operators which forms a Poisson field. The spectral measure of this field $d\rho(x)$ is a Borel measure on the negative Hilbert real space H_- of the rigging $H_- \supset L^2_{\mathrm{Re}}(T, d\nu(t)) \supset H_+$ with a Hilbert-Schmidt type embedding $H_+ \hookrightarrow L^2_{\mathrm{Re}}(T, d\nu(t))$. Then the Fourier transform of $d\rho(x)$ has the form

$$\int_{H_-} e^{i(x,\varphi)} d\rho(x) = \exp\left(\int_T \left(e^{i\varphi(t)} - 1 - i\varphi(t) \right) d\nu(t) \right), \quad \varphi \in H_+.$$

In this paper, a next step is carried out. The scalar complex multiplication is replaced with some abstract algebraic one. Origins of this idea can be found in [11, p. 24, remark 8.1]. Using constructions similar to the one from [10], a new Jacobi field ("*Generalized Poisson Field*") is constructed. The corresponding expression for the Fourier transform of the spectral measure of this field is calculated.

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2. Preliminaries

Consider a rigging of an infinite-dimensional real Hilbert space H with real Hilbert spaces H_+ and $H_- = H'_+ :\ \ H_- \supset H \supset H_+$ with a quasinuclear embedding $\mathcal{O} : H_+ \hookrightarrow H$ (i.e., the operator $\mathcal{O}$ is of Hilbert-Schmidt type).

Let $\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H)$ be the corresponding symmetric Fock space. Each element of $\mathcal{F}(H)$ can be associated with a sequence $f = (f_n)_{n=0}^{\infty}$, $f_n \in \mathcal{F}_n(H) = H_{\mathbb{C}}^{\widehat{\otimes} n}$, where $H_{\mathbb{C}}$ denotes the complexification of H and the sign $\widehat{\otimes}$ denotes the symmetric tensor product.

Denote by $\mathcal{F}_{\text{fin}}(H)$ the linear subset of finite vectors and by $\Omega = (1, 0, 0, \ldots) \in \mathcal{F}_{\text{fin}}(H)$ the so-called vacuum.

Consider a family $J = (J(\varphi))_{\varphi \in H_+}$ of operator-valued Jacobi matrices

$$J(\varphi) = \begin{pmatrix} b_0(\varphi) & a_0^*(\varphi) & 0 & 0 & 0 & \cdots \\ a_0(\varphi) & b_1(\varphi) & a_1^*(\varphi) & 0 & 0 & \cdots \\ 0 & a_1(\varphi) & b_2(\varphi) & a_2^*(\varphi) & 0 & \cdots \\ 0 & 0 & a_2(\varphi) & b_3(\varphi) & a_3^*(\varphi) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with the entries

$$\begin{aligned} a_n(\varphi) :&\quad \mathcal{F}_n(H) \longrightarrow \mathcal{F}_{n+1}(H), \\ a_n^*(\varphi) :&\quad \mathcal{F}_{n+1}(H) \longrightarrow \mathcal{F}_n(H), \\ b_n(\varphi) = b_n^*(\varphi) :&\quad \mathcal{F}_n(H) \longrightarrow \mathcal{F}_n(H). \end{aligned} \tag{1}$$

Every matrix $J(\varphi)$ generates a Hermitian operator $A(\varphi) : \mathcal{F}_{\text{fin}}(H) \longrightarrow \mathcal{F}_{\text{fin}}(H)$.

Definition 2.1. The family J is called a Jacobi field if the following conditions hold (see [11, p. 11], [10, p. 123]):

(a) The operators (1) are bounded and real (i.e., act from real subspaces of $\mathcal{F}_n(H)$ into real ones).

(b) The dependence of elements of $J(\varphi)$ on $\varphi \in H$ is linear and the operators

$$\begin{aligned} \forall f_n \in \mathcal{F}_n(H) &\qquad H \ni \varphi \mapsto a_n(\varphi) f_n \in \mathcal{F}_{n+1}(H), \\ \forall f_n \in \mathcal{F}_n(H) &\qquad H \ni \varphi \mapsto b_n(\varphi) f_n \in \mathcal{F}_n(H), \qquad n \in \mathbb{Z}_+ \\ \forall f_{n+1} \in \mathcal{F}_{n+1}(H) &\qquad H \ni \varphi \mapsto a_n^*(\varphi) f_{n+1} \in \mathcal{F}_n(H), \end{aligned} \tag{2}$$

are linear and bounded.

(c) The operators $A(\varphi)$, $\forall \varphi \in H_+$, are essentially self-adjoint and their closures $\tilde{A}(\varphi)$ commute strongly (i.e. their resolutions of identity commute).

(d) (regularity) Consider $\forall n \in \mathbb{N}$ a real operator $V_{n,n} : \mathcal{F}_n(H) \to \mathcal{F}_n(H)$ defined by the equality

$$V_{n,n}(\varphi_1 \hat{\otimes} \varphi_2 \hat{\otimes} \ldots \hat{\otimes} \varphi_n) = (J(\varphi_1) \ldots J(\varphi_n)\Omega)_n = a_{n-1}(\varphi_1) \ldots a_0(\varphi_n) 1. \tag{3}$$

We assume that this operator is continuous and, after being extended by continuity, is invertible in $\mathcal{F}_n(H)$; we also put $V_{0,0} = 1$.

(e) (smoothness) Properties **(a), (b), (d)** are preserved for restrictions of the operators (1), (2), (3) onto the space $\mathcal{F}_n(H_+)$, $\mathcal{F}_{n+1}(H_+), H_+$ and with values in $\mathcal{F}_{n+1}(H_+)$ and $\mathcal{F}_n(H_+)$, respectively.

Consider one example of such a field (see [11, p. 25]). Take $H = \ell_2$ and fix some orthonormal basis $(e_j)_{j=0}^{\infty}$ of this space. Choose some weight $\gamma = (\gamma_n)_{n=0}^{\infty}$ such that $\gamma_n \geqslant 1\ \forall n \geqslant 0$ and $\sum_{n=0}^{\infty} \frac{1}{\gamma_n} < \infty$. Denote

$$\ell_2(\gamma) = \{f \in \ell_2 : \sum_{n=0}^{\infty} |f_k|^2 \gamma_k < \infty\}$$

and take $H_+ = \ell_2(\gamma)$. The last inequality guarantees that the embedding $H_+ \hookrightarrow H$ is quasinuclear. At last, take $H_- = \ell_2(\gamma^{-1})$ where $\ell_2(\gamma^{-1}) = \{f = (f_n)_{n=0}^\infty : \sum_{n=0}^\infty |f_k|^2 \frac{1}{\gamma_k} < \infty\}$ and construct the corresponding symmetric Fock space $\mathcal{F}(H)$. The matrices $J(\varphi)$ are built in the following way: $\forall \varphi \in H_+, f_n \in \mathcal{F}_n(H)$

$$a_n(\varphi) f_n = \sqrt{n+1}\varphi \hat{\otimes} f_n,$$
$$b_n(\varphi) f_n = 0\,.$$

The family $J(\varphi), \varphi \in H_+$ satisfies the definition of a Jacobi field and is called the *Classical Free Field*.

Note that throughout this paper (particularly in the case of the Classical Free Field) all constructions of Jacobi Fields are based on Theorem 8.1 from [11, p. 24]. Consider this question in details.

In [11] after formula (5.5), one can find a description of *smoothness* condition. After Definition 5.1 in the same paper, a detailed description of *regularity* axiom is contained. This axiom uses the *smoothness* condition. After formula (8.19) and in the statement of Theorem 8.1, the author placed a set of necessary assumptions for building a Jacobi field.

Using the notations of [11], in our case $X = \mathbf{1}, H_1 = H_+$. So the assumption that $X : H_1 \longrightarrow H_1$ acts continuously and is invertible is fulfilled (see the paragraph after formula (8.21)).

The second assumption is that, for $\varphi \in H_1$, the operator $(b(\varphi)) \upharpoonright H_1$ acts continuously in H_1 and depends on $\varphi \in H_1$ continuously in the sense of the norm of operators in H_1. For the case of the *Classical Free Field*, this condition is obviously satisfied because $b(\varphi) = \mathbf{0}$. For our case, assumptions **(A)** and **(B)** from the next section correspond to this condition.

3. Generalized Poisson field

We represent each matrix $J(\varphi)$ in the form

$$J(\varphi) = J_+(\varphi) + B(\varphi) + J_-(\varphi), \quad \varphi \in H_+, \tag{4}$$

where $J_+(\varphi)$ ($B(\varphi)$, $J_-(\varphi)$) has nonzero entries on the lower (correspondingly main, upper) diagonals.

We define the action of $J_+(\varphi)$ as follows:

$$J_+(\varphi) f_n = a_n(\varphi) f_n = \sqrt{n+1}\varphi \widehat{\otimes} f_n \in \mathcal{F}_{n+1}(H), \quad f_n \in \mathcal{F}_n(H), \quad \varphi \in H_+.$$

The action of $J_-(\varphi)$ can be deduced from the previous formula for $J_+(\varphi)$,

$$J_-(\varphi) f_{n+1} = a_n^*(\varphi) f_{n+1} \in \mathcal{F}_n(H), \quad f_{n+1} \in \mathcal{F}_{n+1}(\varphi), \quad \varphi \in H_+.$$

Let us define the action of $b(\varphi) : H \longrightarrow H$, $\varphi \in H_+$. Suppose there is a real commutative Banach algebra between H_+ and H in the rigging

$$H_- \supset H \supset L \supset H_+. \tag{5}$$

We make the following assumptions:

(A) the algebraic multiplication $*$ can be extended by continuity from $L \times L$ onto $H_+ \times H$. This condition is equivalent to the following inequality:

$$\|a * b\|_H \leqslant C_1 \|a\|_L \cdot \|b\|_H, \quad \forall a \in H_+ \subset L, \quad b \in L;$$

(B) the algebraic multiplication $*$ generates a bounded operator in H_+. This condition is equivalent to the following inequality:

$$\|a * b\|_{H_+} \leqslant C_2 \|a\|_L \cdot \|b\|_{H_+}, \quad \forall a, b \in H_+ \subset L;$$

(C) the algebraic multiplication is Hermitian,

$$(a * b, c)_H = (b, a * c)_H, \quad \forall a, b, c \in L.$$

Under these assumptions we define the action of $b(\varphi)$ in the following way:

$$b(\varphi)f = \varphi * f, \quad \varphi \in H_+ \subset L, \quad \forall f \in H. \tag{6}$$

Assumptions **(A)**, **(C)** correspond to a condition on the operator $b(\varphi)$ of [11, Theorem 8.1]. Assumption **(B)** corresponds to the *smoothness* condition (axiom **(e)**), because

$$\|b(\varphi)\|_{\mathcal{L}(H_+)} \leqslant C_2\|\varphi\|_L \leqslant C_2\|\varphi\|_{H_+}$$

(see [11, paragraph after formula (8.19)]).

The operator $B(\varphi)$ is equal to the differential quantization of operator $b(\varphi)$. This means, that

$$B(\varphi)f_n = \{b(\varphi)\otimes \mathbf{1}\otimes \mathbf{1}\otimes \ldots + \mathbf{1}\otimes b(\varphi)\otimes \mathbf{1}\otimes \mathbf{1}\otimes \ldots + \cdots + \mathbf{1}\otimes \mathbf{1}\otimes \ldots \otimes \mathbf{1}\otimes b(\varphi)\}f_n.$$

We need the family $J = (J(\varphi))_{\varphi\in H_+}$ to be a commutative Jacobi field.

Assumptions **(a), (b)** are satisfied because of the way of construction of $J(\varphi)$.

Essential self-adjointness and commutativity **(c)** follow from the general result [11, p. 14]. Namely suppose that the family $J = (J(\varphi))_{\varphi\in H_+}$ consists of algebraically commuting matrices and for all $\varphi \in H_+ : \ \|\varphi\|_{H_+} = 1$ the series $\sum_{n=0}^{\infty} \|a_n(\varphi)\|^{-1}$ diverge. Then the corresponding operators $A(\varphi)$ are essentially self-adjoint and strongly commute. In our case, $\sum_{n=0}^{\infty} \frac{1}{\|a_n(\varphi)\|} \geqslant \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}\|\varphi\|_H} = \infty$.

Regularity condition holds in our case while $V_{n,n} = \sqrt{n!}\,\mathrm{Id}$.

Definition 3.1. The Jacobi field $J = (J(\varphi))_{\varphi\in H_+}$ that is built in the above specified way is called a **generalized Poisson field**.

It is possible to apply the projection spectral theorem (see [11, Theorem 5.1]) to the field J. Here we give only the final result. It is necessary for introduction of the term spectral measure.

Theorem 3.2. *For the Jacobi field J there exists a Borel probability measure ρ on the space H_- (the **spectral measure**) and a vector-valued function $H_- \ni \xi \mapsto P(\xi) \in (\mathcal{F}_{\mathrm{fin}}(H_+))'$ such that the following statements hold:*

1) *For every $\xi \in H_-$ the vector $P(\xi) = (P_n(\xi))_{n=0}^{\infty} \in (\mathcal{F}_{\mathrm{fin}}(H_+))'$ is a generalized joint eigenvector of J with eigenvalue ξ, i.e.,*

$$\langle P(\xi), J(\varphi)\psi\rangle = \langle \xi,\varphi\rangle\langle P(\xi),\psi\rangle, \quad \varphi \in H_+, \psi \in \mathcal{F}_{\mathrm{fin}}(H_+).$$

2) *After being extended by continuity to the whole space $\mathcal{F}(H)$the Fourier transform*

$$\mathcal{F}(H) \supset \mathcal{F}_{\mathrm{fin}}(H_+) \ni \Phi = (\Phi_n)_{n=0}^{\infty} \mapsto (I\Phi)(\xi)$$
$$= \langle \Phi, P(\xi)\rangle = \sum_{n=0}^{\infty}\langle \Phi_n, P_n(\xi)\rangle \in L^2(H_-, d\rho)$$

becomes a unitary operator between $\mathcal{F}(H)$ and $L^2(H_-, d\rho)$.

3) *I maps every operator $J(\varphi), \varphi \in H_+$ into the operator of multiplication by the function $H_- \ni \xi \mapsto \langle \xi,\varphi\rangle \in \mathbb{R}$ in the space $L^2(H_-, d\rho)$.*

The angle brackets $\langle\cdot,\cdot\rangle$ denote the pairing between the positive H_+ and the negative H_- spaces.

Now we can formulate the main result of this paper.

Theorem 3.3. *Let L be a real Banach algebra, H_+, H_- be real separable Hilbert spaces. Assume that L is dense in the real separable Hilbert space H. Let $H_+ \hookrightarrow H$ be a dense quasinuclear embedding. The Hilbert rigging has the following form: $H_- \supset H \supset L \supset H_+$. Additionally we make the following assumptions:*

1) $\|\cdot\|_H \leqslant \|\cdot\|_L \leqslant \|\cdot\|_{H+}$;
2) $\|a * b\|_H \leqslant C_1\|a\|_L \cdot \|b\|_H, \quad \forall a \in H_+, b \in L$;

3) $\|a*b\|_{H_+} \leqslant C_2\|a\|_L \cdot \|b\|_{H_+}, \quad \forall a,b \in H_+$;
4) *the multiplication is Hermitian,* $(a*b,c)_H = (b,a*c)_H, \quad \forall a,b,c \in L$;
5) $\exists\, \omega \in H$: *the set* $\{a*\omega | a \in L\}$ *is dense in* H.

Then on the space H_- *there exist a* σ*-algebra* $\mathfrak{U}$ *and a measure* σ*, that is spectral for the generalized Poisson field* $A = (A(\varphi))_{\varphi \in H_+}$*, for which the Fourier-transform has the form*

$$\int_{H_-} e^{i\langle x,\varphi\rangle} d\sigma(x) = \exp\left(\int_M (e^{i\varphi(t)} - 1 - i\varphi(t))\, d\nu(t)\right),$$

where M *is some Hausdorff compact,* ν *Borel measure.*

It is interesting to note that in comparison with the analogous result in the paper [10, p. 128], for the case of the generalized Poisson field, the support of the spectral measure is contained in the compact set M. Thus here we don't have a "pure" generalization of the classical Poisson field case (where the support is a subset of real axis).

The proof of the theorem will be conducted in several steps. The next two sections describe the way how we can pass from abstract Hilbert spaces to scalar-valued functional spaces without a considerable loss of generality.

4. Foundation of replacement of the base chain

Here we prove an abstract theorem that will substantiate the replacement of the Hilbert rigging with the one where, instead of the zero Hilbert space, one uses L^2 and the abstract algebraic multiplication maps into scalar one. The main result of this section is contained in the following theorem.

Theorem 4.1. *Let* $\mathfrak{L}$ *be a commutative complex Banach algebra with involution,* $\mathfrak{H}$ *be a complex separable Hilbert space and the following assumptions hold:*

1) $\mathfrak{L}$ *is dense in* $\mathfrak{H}$*;*
2) $\forall a,b \in \mathfrak{L}, \quad \|a*b\|_{\mathfrak{H}} \leqslant C\|a\|_{\mathfrak{L}} \cdot \|b\|_{\mathfrak{H}}$;
3) $\forall a,b,c \in \mathfrak{L}, \quad (a*b,c)_{\mathfrak{H}} = (b,a^* * c)_{\mathfrak{H}}$;
4) $\exists \omega \in \mathfrak{H}$ *such that the set* $\{a*\omega | a \in \mathfrak{L}\}$ *is dense in* $\mathfrak{H}$*,* $\|\omega\|_{\mathfrak{H}} = 1$.

Then there exists a Hausdorff compact set M *and a Borel measure* $\nu : \mathfrak{B}(M) \to [0;1]$ *such that*

a) there exists an isometric isomorphism $G : \mathfrak{H} \xrightarrow{\sim} L^2_{\mathbb{C}}(M, d\nu)$ *(the index* $\mathbb{C}$ *denotes that field of scalars of this space is* $\mathbb{C}$*);*

b) denote by $Y_a \in \mathcal{L}(\mathfrak{H})$ *the operator of multiplication by the element* $a \in \mathfrak{L}$ *(it exists due to assumption 2). Then its resolution of identity,* E_a*, is mapped, under* G*, into the operator of multiplication by a continuous function from* $C^{\mathbb{C}}(M)$*.*

Remark 4.2. Assumption 3) is equivalent to $Y_a^* = Y_{a^*}, \ \forall a \in \mathfrak{L}$. It's necessary to draw attention to the fact that the operators of multiplication have the same multiplication structure as the algebra, $Y_a Y_b = Y_{ab}\ \forall a,b \in \mathfrak{L}$. And as a consequence they form a commutative family of normal operators, $\forall a \in \mathfrak{L}$

$$Y_a Y_a^* = Y_a Y_{a^*} = Y_{aa^*} = Y_{a^*a} = Y_{a^*} Y_a = Y_a^* Y_a.$$

Proof. Let $\mathcal{A}$ be the algebra generated by the family of resolutions of identity $E_a(\cdot)$ of the operators Y_a (first build the linear span of all possible linear combinations of all possible products of resolutions of identity $E_a(\Delta), \forall \Delta \in \mathfrak{B}(\mathbb{C}), \forall a \in \mathfrak{L}$, then complete it in the norm $\|\cdot\|_{\mathcal{L}(\mathfrak{H})}$). It is well-known that the space $\mathcal{A}$ obtained in this way is a C^*-algebra with identity $\mathrm{Id} = E_a(\mathbb{C})$.

According to the Gelfand-Naimark theorem (see [7, p. 311]) there exists an algebraic isometric isomorphism $g : \mathcal{A} \longrightarrow C^{\mathbb{C}}(M)$, where M is a Hausdorff compact space of maximal ideals of the algebra $\mathcal{A}$.

In a similar way as it was done in [12, p. 208], consider the linear functional

$$\ell : C^{\mathbb{C}}(M) \longrightarrow \mathbb{C}, \quad \ell(\alpha) = \big(g^{-1}(\alpha)\omega, \omega\big)_{\mathfrak{H}}, \quad \alpha \in C^{\mathbb{C}}(M).$$

The functional ℓ is continuous due to the following inequality:

$$|\ell(\alpha)| = |(g^{-1}(\alpha)\omega, \omega)_{\mathfrak{H}}| \leqslant \|g^{-1}(\alpha)\|_{\mathcal{L}(\mathfrak{H})} \cdot \|\omega\|^2_{\mathfrak{H}} = \|\alpha\|_{C^{\mathbb{C}}(M)}.$$

Moreover, it is non-negative, $\forall \alpha \in C^{\mathbb{C}}(M) :\ \alpha(t) \geqslant 0\ \forall t \in M$ it holds $\ell(\alpha) \geqslant 0$. The proof of this statement is conducted as follows. Let $\alpha \in C^{\mathbb{C}}(M) : \alpha(t) \geqslant 0\ \forall t \in M$. Choose a concrete linear bounded functional $t_0 \in M :\ t_0(A) = (A\omega, \omega)_{\mathfrak{H}}\ \forall A \in \mathcal{A}$. Recall that the Gelfand transformation is defined by $(g(A))t = t(A),\ \forall t \in M, A \in \mathcal{A}$, so

$$0 \leqslant \alpha(t_0) = t_0(g^{-1}(\alpha(\cdot))) = (g^{-1}(\alpha(\cdot))\omega, \omega)_{\mathfrak{H}} = \ell(\alpha), \quad \text{thus} \quad \ell(\alpha) \geqslant 0.$$

By the Riesz theorem, the functional ℓ can be presented as

$$\ell(\alpha) = \int_M \alpha(t)\, d\nu(t), \quad \alpha \in C^{\mathbb{C}}(M),$$

where $\nu : \mathfrak{B} \longrightarrow \mathbb{R}^+$ is Borel measure. This is a probability measure, because

$$\nu(M) = \int_M 1\, \nu(t) = \ell(1) = (g^{-1}(1)\omega, \omega)_{\mathfrak{H}} = (\mathrm{Id}\, \omega, \omega)_{\mathfrak{H}} = \|\omega\|^2_{\mathfrak{H}} = 1.$$

Proposition 4.3. *$\forall A, B \in \mathcal{A}$ the following equality takes place:*

$$(A\omega, B\omega)_{\mathfrak{H}} = (g(A), g(B))_{L^2_{\mathbb{C}}(M,d\nu)}.$$

Proof.

$$\begin{aligned}(A\omega, B\omega)_{\mathfrak{H}} &= (B^*A\omega, \omega)_{\mathfrak{H}} = \ell(g(B^*A)) = \ell(g(B^*)g(A)) = \ell\big(\overline{g(B)}g(A)\big) \\ &= \int_M \big(g(A)\big)(t)\overline{\big(g(B)\big)(t)}\, d\nu(t) = \big(g(A), g(B)\big)_{L^2_{\mathbb{C}}(M,d\nu)}.\end{aligned}$$

□

Proposition 4.4. *The set $\{A\omega | A \in \mathcal{A}\}$ is dense in $\mathfrak{H}$.*

Proof. It is sufficient to prove that the set $\{E_a(\Delta)\omega | a \in \mathfrak{L},\ \Delta \in \mathfrak{B}(\mathbb{C})\}$ is dense in $\mathfrak{H}$. Let $f \perp E_a(\Delta)\omega\ \forall a \in \mathfrak{L},\ \forall \Delta \in \mathfrak{B}(\mathbb{C})$, then

$$(a * \omega, f)_{\mathfrak{H}} = (Y_a\omega, f)_{\mathfrak{H}} = \int_{\mathbb{C}} \lambda\, d(E_a(\lambda)\omega, f)_{\mathfrak{H}} = 0.$$

Thus $f \perp a * \omega\ \forall a \in \mathfrak{L}$, but this set is dense in $\mathfrak{H}$, so $f = 0$. □

Corollary 4.5. There exists an isometric isomorphism $G : \mathfrak{H} \xrightarrow{\sim} L^2_{\mathbb{C}}(M, d\nu)$ with the property

$$G(A\omega) = g(A) \quad \forall A \in \mathcal{A}. \tag{7}$$

Proof. Define G on the dense (due to Proposition 4.4) set $\mathfrak{A} = \{A\omega | A \in \mathcal{A}\} \subset \mathfrak{H}$ using the formula (7).

Due to the properties of the Gelfand-Naimark isomorphism g, we have $\mathrm{Ran}\, g = C^{\mathbb{C}}(M)$. And from (7) it is obvious that $\mathrm{Ran}\, g = \mathrm{Ran}\, G$. So $\mathrm{Ran}\, G = G(\mathfrak{A}) = C^{\mathbb{C}}(M)$.

Due to Proposition 4.3, G is an isometry and can be continued to an isometry $\tilde{G}$ with the domain $\mathrm{Dom}\, \tilde{G} = \mathfrak{H}$. Since the range $G(\mathfrak{A}) = C^{\mathbb{C}}(M)$ is dense in $L^2_{\mathbb{C}}(M, d\nu)$, we obtain $\mathrm{Ran}\, \tilde{G} = L^2_{\mathbb{C}}(M, d\nu)$. Thus $\tilde{G}$ is an isometric isomorphism between the spaces $\mathfrak{H}$ and $L^2_{\mathbb{C}}(M, d\nu)$. We shall preserve the old notation G for this isometry. □

Proposition 4.6. *Under the action of the isometry G, each operator $A \in \mathcal{A}$ is mapped into the operator of multiplication in $L^2_{\mathbb{C}}(M, d\nu)$ by a continuous function $\big(g(A)\big)(\cdot)$,*

$$\big(G(Af)\big)(\cdot) = \big(g(A)\big)(\cdot) \cdot \big(Gf\big)(\cdot), \quad \forall f \in \mathfrak{H}, \quad \forall A \in \mathcal{A}.$$

Proof. Fix any $f \in \mathfrak{H}$. Due to Proposition 4.4, $\exists A_n \in \mathcal{A}: \ A_n\omega \overset{\|\cdot\|_{\mathfrak{H}}}{\longrightarrow} f$. All the following limits are taken with respect to the norm $\|\cdot\|_{L^2_{\mathbb{C}}(M,d\nu)}$,

$$\begin{aligned} G(Af) &= \lim_{n\to\infty} G(AA_n\omega) = \lim_{n\to\infty} g(AA_n) = \lim_{n\to\infty} g(A)g(A_n) \\ &= g(A)\lim_{n\to\infty} g(A_n) = g(A)\lim_{n\to\infty} G(A_n\omega) = g(A)(Gf). \end{aligned}$$

Note that the equality $\lim_{n\to\infty} g(A)g(A_n) = g(A)\lim_{n\to\infty} g(A_n)$ follows from compactness of M,

$$\begin{aligned} &\|g(A)g(A_m) - g(A)\lim_{n\to\infty} g(A_n)\|_{L^2_{\mathbb{C}}(M,d\nu)} \\ &\qquad \leqslant \max_M |g(A)| \cdot \|g(A_m) - \lim_{n\to\infty} g(A_n)\|_{L^2_{\mathbb{C}}(M,d\nu)} \overset{m\to\infty}{\longrightarrow} 0. \end{aligned}$$

□

Corollary 4.7. Take in the previous statement $A = E_a(\Delta)$ for some fixed $a \in \mathfrak{L}$, $\Delta \in \mathfrak{B}(\mathbb{C})$,

$$\big(G(E_a(\Delta)f)\big)(\cdot) = \big(g(E_a(\Delta))\big)(\cdot)\cdot\big(Gf\big)(\cdot).$$

This formula corresponds to the statement **b)** of our theorem. Making another step, we conclude that due to the former equality being valid for *every* $\Delta \in \mathfrak{B}(\mathbb{C})$ we have the equality of the measures,

$$dG(E_a(\cdot)f) = (dg(E_a(\cdot)))(Gf),$$

and we are able to determine the image of the operator of multiplication Y_a under the action of the isometry G,

$$\begin{aligned} (G(Y_af))(\cdot) &= G\bigg(\int_{\mathbb{C}} \lambda\, dE_a(\lambda)f\bigg) = \int_{\mathbb{C}} \lambda\, dG\big(E_a(\lambda)f\big) = \bigg(\int_{\mathbb{C}} \lambda\, dg\big(E_a(\lambda)\big)\bigg)\cdot(Gf) \\ &= \bigg[g\bigg(\int_{\mathbb{C}} \lambda\, dE_a(\lambda)\bigg)(\cdot)\bigg]\cdot(Gf)(\cdot) = (Gf)g(Y_a). \end{aligned} \tag{8}$$

This formula is valid because both G and g are linear and continuous. Thus Y_a belongs to the C^*-algebra $\mathcal{A}$ and is mapped into the operator of multiplication by a continuous function $g(Y_a)(\cdot)$. □

5. Replacement of the base chain

Now we are able to replace the abstract algebraic chain (5) with a chain of functional spaces and make sure that this construction in essence is still quite general.

First we should build complexifications of the spaces H and L. Denote them by $\mathfrak{H} = H_{\mathbb{C}}$ and $\mathfrak{L} = L_{\mathbb{C}}$. In accordance with the just proved theorem and (8) we can map $H_{\mathbb{C}}$ into the space $L^2_{\mathbb{C}}(M,d\nu)$ and $L_{\mathbb{C}}$ into the subset $\{g(Y_a) \mid a \in L_{\mathbb{C}}\} \subset C^{\mathbb{C}}(M)$. By taking real subspaces we obtain an isometric isomorphism between the real parts of these spaces, $H \simeq L^2_{\mathrm{Re}}(M,d\nu)$, $L \hookrightarrow C^{\mathrm{Re}}(M)$. Recall that the real part of an algebra with involution is the set of elements that are invariant with respect to this involution.

The main purpose of the replacement of the base chain is that the abstract multiplication turns into the usual scalar multiplication. This allows us to use a wide range of well-known properties of L^2 instead of dealing with quite obscure properties of the abstract H.

By restricting the isometric isomorphism G from H onto its dense subspace H_+ we obtain a bijection between H_+ and some subset $\hat{H}_+ \subset L^2_{\mathrm{Re}}(M,d\nu)$. Since $H_+ \subset L$ and all images of elements of L are continuous functions we can conclude that $\hat{H}_+$ consists of continuous functions. Taking, by definition, $\|G(a)\|_{\hat{H}_+} = \|a\|_{H_+}$ $\forall a \in H_+$ we obtain a new Hilbert space $(\hat{H}_+, \|\cdot\|_{\hat{H}_+})_{\mathbb{R}}$, isometric to H_+. In the same way we build $\hat{H}_- \simeq H_-$.

As a result, the original chain (5) turns into

$$\hat{H}_- \supset L^2_{\mathrm{Re}}(M,d\nu) \supset C^{\mathrm{Re}}(M) \supset \hat{H}_+. \tag{9}$$

We finish this section with the proof of an essential fact that plays an important role in the whole paper and is not obvious.

Theorem 5.1. *The space $L^2_{\mathrm{Re}}(M, d\nu)$ is separable.*

Proof. We shall use the following well-known fact ([14, Chapter 11.4, Theorem 4]):

Let $(X, \mathcal{F}, \lambda)$ be a space with a σ-finite measure, $\mathcal{F} = \sigma a(\mathcal{K})$, $\mathcal{K}$ the semiring of subsets of X. (Here we denote $\sigma a(\mathcal{T})$ the sigma-algebra spanned over the class of sets $\mathcal{T}$.) *Then for any $f \in L^p_{\mathrm{Re}}(X, \mathcal{F}, \lambda)$ and $\forall \varepsilon > 0$ there exists a function $h(x) = \sum_{i=1}^{j} c_i \kappa_{A_i}(t)$, $t \in X$, $A_i \in \mathcal{K}, c_i \in \mathbb{R}$ such that $\|f - h\|_{L^p} < \varepsilon$.*

Corollary 5.2. If the semiring $\mathcal{K}$ consists of a countable set of subsets then the set $\{\sum_{i=1}^{j} c_i \kappa_{A_i}(t) : c_i \in \mathbb{Q},\ t \in X,\ A_i \in \mathcal{K},\ j \in \mathbb{N}\}$ is countable and dense in $L^p_{\mathrm{Re}}(X, \mathcal{F}, \lambda)$. Thus L^p is separable.

In $\mathbb{C}$, the set $\mathcal{K} = \{(a, b] \times (c, d] : a, b, c, d \in \mathbb{Q}\}$ is a countable semiring. It is easy to prove that the Borel σ-algebra $\mathfrak{B}(\mathbb{C}) = \sigma a(\mathcal{K})$. Denote by $\mathcal{G}(\mathbb{C})$ the class of open sets in $\mathbb{C}$. By definition, $\mathfrak{B}(\mathbb{C}) = \sigma a(\mathcal{G}(\mathbb{C}))$.

The topology of M was built as the weak topology, the open sets from M are preimages of open subsets from $\mathbb{C}$ for the functions $g(A) : M \longrightarrow \mathbb{C},\ \forall A \in \mathcal{A}$

$$\mathcal{G}(M) = \{(g(A))^{-1}(U) :\ U \in \mathcal{G}(\mathbb{C}),\ A \in \mathcal{A}\} = \{(g(A))^{-1}(\mathcal{G}(\mathbb{C})),\ A \in \mathcal{A}\}.$$

By definition, $\mathfrak{B}(M) = \sigma a(\mathcal{G}(M))$. It's not difficult to check that

$$\mathfrak{B}(M) = \sigma a(\{(g(A))^{-1}(\mathcal{K}),\ A \in \mathcal{A}\}).$$

From the properties of preimages, it follows that each class of sets $(g(A))^{-1}(\mathcal{K}),\ A \in \mathcal{A}$ is a countable semiring. There is only left to prove that it is sufficient to take only a countable part of $A \in \mathcal{A}$ to build $\mathfrak{B}(M)$.

If the algebra $\mathcal{A}$ is separable (it is really so because L is separable from the very beginning) the problem is solved. Let $(A_n)_{n=1}^{\infty}$ be a dense countable set of elements from $\mathcal{A}$. Take any $A \in \mathcal{A}$ and choose a corresponding subsequence A_{n_j} that approximates it, $A_{n_j} \longrightarrow A$. The following equality takes place:

$$\max_{t \in M} |g(A)(t) - g(A_{n_j})(t)| = \|g(A) - g(A_{n_j})\|_{C(M)} = \|A - A_{n_j}\|_{\mathcal{A}} \overset{j \to \infty}{\longrightarrow} 0.$$

This means that images of $g(A)$ and $g(A_{n_j})$ tend to coincidence.

Let $\varepsilon_k = \frac{1}{m(k)+1},\ k \in \mathbb{N}$. Here the function $m : \mathbb{N} \longrightarrow \mathbb{N}$ is defined recursively, $m(1) = 1$, $m(k+1) = 2m(k)(m(k)+1) - 1$. There exists $M_k \in \mathbb{N}$ such that

$$\forall j \geqslant M_k \quad \|g(A) - g(A_{n_j})\|_{C(M)} < \frac{1}{m(k)+1}.$$

Take two rectangles $(a_k, b_k] \times (c_k, d_k] \subset (a, b] \times (c, d]$, where

$$a_k = a + \frac{1}{m(k)}, \quad b_k = b - \frac{1}{m(k)}, \quad c_k = c + \frac{1}{m(k)}, \quad d_k = d - \frac{1}{m(k)}.$$

Denote $T_{j,k} = (g^{-1}(A_{n_j}))((a_k, b_k] \times (c_k, d_k])$. We have $\max_{t \in T_{j,k}} |g(A)(t) - g(A_{n_j})(t)| \leqslant \max_{t \in M} |g(A)(t) - g(A_{n_j})(t)| = \|g(A) - g(A_{n_j})\|_{C(M)} < \frac{1}{m(k)+1}$. But $(g(A_{n_j}))(T_{j,k}) = (a_k, b_k] \times (c_k, d_k] \subset (a, b] \times (c, d]$. Moreover, by the construction,

$$\min_{\substack{r \in (a_k, b_k] \times (c_k, d_k] \\ s \in \partial(a, b] \times (c, d]}} |r - s| = \frac{1}{m(k)} > \frac{1}{m(k)+1} > \max_{t \in T_{j,k}} |g(A)(t) - g(A_{n_j})(t)|.$$

We can conclude that $\forall j \geqslant M_k$

$$g(A)(T_{j,k}) \subset (a, b] \times (c, d],$$

$$\min_{\substack{r \in g(A)(T_{j,k}) \\ s \in \partial(a,b] \times (c,d]}} |r-s| \geqslant \frac{1}{m(k)} - \frac{1}{m(k)+1} = \frac{1}{m(k)(m(k)+1)}.$$

And if we take next $\varepsilon_{k+1} = \frac{1}{m(k+1)+1} = \frac{1}{2\cdot m(k)(m(k)+1)}$ and build the corresponding set $g(A)(T_{M_k,k})$ we will have the following inclusion $g(A)(T_{M_k,k}) \subset g(A)(T_{M_{k+1},k+1}) \subset (a,b] \times (c,d]$. So we have a monotone sequence of sets that span $(a,b] \times (c,d]$ from inside. Thus $(g(A))^{-1}((a,b] \times (c,d]) = \bigcup_{k\in\mathbb{N}} g(A)(T_{M_k,k})$.

We have shown that any element from $\{(g(A))^{-1}(\mathcal{K}),\ A \in \mathcal{A}\}$ can be represented as a countable union of elements of $\{(g(A_n))^{-1}(\mathcal{K}),\ n \in \mathbb{N}\}$. That's why

$$\sigma a(\{(g(A))^{-1}(\mathcal{K}),\ A \in \mathcal{A}\}) \subset \sigma a(\{(g(A_n))^{-1}(\mathcal{K}),\ n \in \mathbb{N}\}).$$

The opposite inclusion is obvious, because $A_n \in \mathcal{A}$. Thus

$$\mathfrak{B}(M) = \sigma a(\{(g(A_n))^{-1}(\mathcal{K}),\ n \in \mathbb{N}\}).$$

The family in braces is countable and derive σ-algebra (it can be checked that the ring $K = k(\mathcal{K})$) is also countable). So the linear combinations with rational coefficients of indicators built over sets from $\mathcal{K}$ with values in $\mathbb{Q}$ will form a dense subset in $L^2_{\mathrm{Re}}(M, d\nu)$. Thus L^2 is separable. $\square$

6. Spectral measure for a finite-dimensional subspace

The main purpose now is to adapt ideas from [10] to the chain (9) and then to apply the obtained results to the original chain (5). The main difference between the case under discussion and the case described in [10] is that the zero Hilbert space L^2 is built over a compact set M instead of the real axis.

Consider some fixed finite partition of the compact set M into Borel sets $\gamma_1, \dots, \gamma_d$. Consider $\kappa_j(t) = \kappa_{\gamma_j}(t),\ t \in M, j = 1, \dots, d$ and build a d-dimensional subspace $\hat{H}_d \subset L^2_{\mathrm{Re}}(M, d\nu)$ of functions $\varphi(t) = \sum_{j=1}^d \varphi_j \kappa_j(t)$. By associating each function $\varphi(\cdot) \in \hat{H}_d$ with a d-tuple of real numbers $(\varphi_1, \dots, \varphi_d) \in \mathbb{R}^d$ the space $\hat{H}_d$ can be interpreted as the space $\mathbb{R}^d$.

The initial Jacobi field $J = (J(\varphi))_{\varphi \in H_+}$ that acts on the Fock space $\mathcal{F}(H)$ is mapped under G into the Jacobi field $\hat{J} = (\hat{J}(\varphi))_{\varphi \in \hat{H}_+}$ that acts on the Fock space $\hat{\mathcal{F}} = \mathcal{F}(L^2_{\mathrm{Re}}(M, d\nu))$. Note that $\hat{H}_d \not\subset \hat{H}_+$ because the indicators are not continuous. But the spaces $\hat{H}_d$ and $\hat{H}_+$ have the same multiplication (the ordinary pointwise complex multiplication, that is the image of the algebraic multiplication under G).

Consider two families of Jacobi matrices. Construct $\hat{J}^\circ_d = (\hat{J}(\varphi))_{\varphi \in \hat{H}_d}$ that act in the Fock space $\hat{\mathcal{F}}$. Consider then the same set of matrices $\hat{J}_d$ acting as restriction of $\hat{J}(\varphi) \in \hat{J}$ to the subspace $\mathcal{F}(\hat{H}_d) \subset \hat{\mathcal{F}}$. It is possible to calculate the spectral measure of $\hat{J}_d$ and then prove that it is common for both families $\hat{J}_d$ and $\hat{J}^\circ_d$. The next step will be passing to the limit with respect to d to carry the obtained result from $\hat{J}^\circ_d$ over $\hat{J}$. At last we shall go back to the original Jacobi field J by applying G^{-1}.

Construct an orthonormal basis of space $\hat{H}_d$,

$$e_j = e_j(t) = \frac{\kappa_{\gamma_j}(t)}{\sqrt{\nu(\gamma_j)}}, \quad j = 1, \dots, d, \quad t \in M,$$

the orthonormal basis of n-particle subspace $\mathcal{F}_n(\hat{H}_d)$,

$$e_\alpha = \varepsilon_\alpha e_1^{\otimes \alpha_1} \widehat{\otimes} e_2^{\otimes \alpha_2} \widehat{\otimes} \cdots \widehat{\otimes} e_d^{\otimes \alpha_d}, \quad \varepsilon_\alpha = \sqrt{\frac{|\alpha|!}{\alpha_1! \alpha_2! \cdots \alpha_d!}},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{Z}^d_{+,\mathrm{fin}}$, $\quad |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d = n$, $\quad e_j^{\otimes 0} = 1$ and the corresponding BON (Basis of Occupation Numbers) of $\mathcal{F}(\hat{H}_d)$,

$$\tilde{e}_\alpha = (0, \dots, 0, \underset{|\alpha|}{e_\alpha}, 0, \dots).$$

Construct $J(\varphi) \in \hat{J}_d$ as follows:

$$J_+(e_i)\tilde{e}_\alpha = J_+(e_i)\begin{pmatrix} 0 \\ . \\ 0 \\ e_\alpha \\ 0 \\ 0 \\ . \end{pmatrix} = \begin{pmatrix} 0 \\ . \\ 0 \\ 0 \\ \sqrt{|\alpha|+1}\, e_i \widehat{\otimes} e_\alpha \\ 0 \\ . \end{pmatrix} = \sqrt{\alpha_i + 1}\,\tilde{e}_{\alpha + 1_i}$$

because $\sqrt{|\alpha|+1}\,\varepsilon_\alpha = \sqrt{|\alpha|+1}\sqrt{\frac{1\cdot 2\cdots|\alpha|}{\alpha_1!\alpha_2!\dots\alpha_d!}} = \sqrt{\frac{(|\alpha|+1)!}{\alpha_1!\alpha_2!\dots\alpha_d!}} = \sqrt{\frac{(|\alpha|+1)!}{\alpha_1!\alpha_2!\dots(1\cdot 2\cdots\alpha_i)\dots\alpha_d!}}$ $= \sqrt{\alpha_i+1}\sqrt{\frac{(|\alpha|+1)!}{\alpha_1!\alpha_2!\dots(1\cdot 2\cdots\alpha_i\cdot(\alpha_i+1))\dots\alpha_d!}} = \sqrt{\alpha_i+1}\,\varepsilon_{(\alpha_1,\dots,\alpha_i+1,\dots\alpha_d)} \overset{df}{=} \sqrt{\alpha_i+1}\,\varepsilon_{\alpha+1_i}$. It is quite simple to find the action of $J_-(\varphi) = (J_+(\varphi))^*$.

For every $f = \sum_{n=0}^{\infty}\sum_{\beta:|\beta|=n} f_\beta \tilde{e}_\beta \in \mathcal{F}(\hat{H}_d)$ the following equality is true:

$$\begin{aligned}(f, J_-(e_i)\tilde{e}_\alpha) &= \sum_{n=0}^{\infty}\sum_{\beta:|\beta|=n} f_\beta (J_+(e_i)\tilde{e}_\beta, \tilde{e}_\alpha) = \sum_{n=0}^{\infty}\sum_{\beta:|\beta|=n} f_\beta(\sqrt{\beta_i+1}\,\tilde{e}_{\beta+1_i}, \tilde{e}_\alpha) \\ &= f_{\alpha-1_i}\sqrt{(\alpha_i-1)+1}(\tilde{e}_{(\alpha-1_i)+1_i}, \tilde{e}_\alpha) \\ &= f_{\alpha-1_i}\sqrt{\alpha_i}\cdot 1 = f_{\alpha-1_i}\sqrt{\alpha_i}(\tilde{e}_{\alpha-1_i}, \tilde{e}_{\alpha-1_i}) \\ &= \sum_{n=0}^{\infty}\sum_{\beta:|\beta|=n}\sqrt{\alpha_i} f_\beta(\tilde{e}_\beta, \tilde{e}_{\alpha-1_i}) = (f, \sqrt{\alpha_i}\,\tilde{e}_{\alpha-1_i}).\end{aligned}$$

Thus $J_-(e_i)\tilde{e}_\alpha = \sqrt{\alpha_i}\,\tilde{e}_{\alpha-1_i}$. Analogously, $B(e_i)\tilde{e}_\alpha = q_j\alpha_j\tilde{e}_\alpha$ where $q_j = (\nu(\gamma_j))^{-1/2}$. As a result we have

$$J(e_j)\tilde{e}_\alpha = \sqrt{\alpha_j+1}\,\tilde{e}_{\alpha+1_j} + q_j\alpha_j\tilde{e}_\alpha + \sqrt{\alpha_j}\,\tilde{e}_{\alpha-1_j}.$$

Let $(H_j)_{j=1}^d$ be a sequence of some complex infinite-dimensional separable Hilbert spaces, $(k_i^{(j)})_{i=0}^\infty$ a set of their orthonormal bases. Build the tensor product $\mathcal{H}^{(d)} = \bigotimes_{j=1}^d H_j$ as the Hilbert space spanned by the formal products as an orthonormal basis,

$$k^{(1)}_{\alpha_1} \otimes \dots \otimes k^{(d)}_{\alpha_d} = k_\alpha, \quad \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d_+.$$

Build the isomorphism

$$\begin{array}{rcl} K : \mathcal{H}^{(d)} & \longrightarrow & \mathcal{F}(\hat{H}_d) \\ k_\alpha & \longmapsto & \tilde{e}_\alpha \end{array}$$

and the corresponding operator $\mathcal{J}(e_j) = K^{-1} \circ \hat{J}(e_j) \circ K$:

$$\begin{array}{ccc} \mathcal{H}^{(d)} & \xrightarrow{\mathcal{J}(e_j)} & \mathcal{H}^{(d)} \\ \downarrow K & & K\downarrow \\ \mathcal{F}(\hat{H}_d) & \xrightarrow{\hat{J}(e_j)} & \mathcal{F}(\hat{H}_d) \end{array}$$

So we obtain the following expression:

$$\mathcal{J}(e_j)k_\alpha = \sqrt{\alpha_j+1}\,k_{\alpha+1_j} + q_j\alpha_j k_\alpha + \sqrt{\alpha_j}\,k_{\alpha-1_j}.$$

Note that $k_{\alpha\pm 1_j} = k^{(1)}_{\alpha_1} \otimes \ldots \otimes k^{(j-1)}_{\alpha_{j-1}} \otimes k^{(j)}_{\alpha_j \pm 1} \otimes k^{(j+1)}_{\alpha_{j+1}} \otimes \ldots \otimes k^{(d)}_{\alpha_d}$. So the operator $\mathcal{A}(e_j)$ (that is the closure of the operator generated by the matrix $\mathcal{J}(e_j)$ in $\mathcal{H}^{(d)}$) is not identity only in the space H_j and this operator can be represented as a tensor product of operators,

$$\mathcal{A}(e_j) = \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes L_j \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{d} : \mathrm{Dom}(\mathcal{A}(e_j)) \longrightarrow \mathcal{H}^{(d)}.$$

Here the operator L_j has the following form:

$$L_j = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 1q_j & \sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 2q_j & \sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 3q_j & \sqrt{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In the article [10] it is proved that the spectral measure of the operator

$$M_j = \begin{pmatrix} 0 & \sqrt{h}\sqrt{1} & 0 & 0 & 0 & \cdots \\ \sqrt{h}\sqrt{1} & 1 & \sqrt{h}\sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{h}\sqrt{2} & 2 & \sqrt{h}\sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{h}\sqrt{3} & 3 & \sqrt{h}\sqrt{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the measure $\pi^c_h(\{k-h\}) = \frac{h^k}{k!}e^{-h}, k \in \mathbb{Z}_+$. Denote by $M_j = q_j^{-1}L_j$. Then we have the following spectral measure:

$$\sigma_j(x_j) = \pi^c_{q_j^{-2}}(x_j), \quad x_j \in \mathbb{R}.$$

Here we use the following fact: if $\rho(\lambda)$ is a spectral measure of a self-adjoint operator A, then for $c \in \mathbb{R}, c \neq 0$ the measure $\rho(c^{-1}\lambda)$ is a spectral measure of the operator cA. In our case, $L_j = q_j M_j$, thus for spectral measures of the operators L_j, we obtain the following expression:

$$\rho_j(x_j) = \pi^c_{q_j^{-2}}(q_j^{-1}x_j), \quad x_j \in \mathbb{R}.$$

Finally, the spectral measure of the Jacobi field $(\mathcal{A}(e_j))_{j=1}^d$ and the spectral measure of the field $\hat{J}_d$ is the following product:

$$\rho(x) = \rho_1(x_1) \times \cdots \times \rho_d(x_d), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Calculate the Fourier transform for the obtained measure. The characteristic function of the measure $\pi^c_h(sx)$ is

$$\int_{\mathbb{R}} e^{ix\lambda} d\pi^c_h(sx) = \sum_{k=0}^{\infty} e^{i\lambda\left(\frac{k-h}{s}\right)} \frac{e^{-h}h^k}{k!} = \exp\left\{h\left(e^{\frac{i\lambda}{s}} - 1 - \frac{i\lambda}{s}\right)\right\}, \quad \lambda \in \mathbb{R}, \quad h > 0.$$

It is not difficult to find the characteristic function of the measure $\rho_j(x_j)$,

$$\int_{\mathbb{R}} e^{ix_j\varphi_j} d\rho_j(x_j) = \int_{\mathbb{R}} e^{ix_j\varphi_j} d\pi^c_{q_j^{-2}}(q_j^{-1}x_j) = \exp\left\{q_j^{-2}\left(e^{i\varphi_j q_j} - 1 - i\varphi_j q_j\right)\right\}.$$

Note that

$$\int_{\mathbb{R}^d} e^{i(x,\varphi)} d\rho(x) = \int_{\mathbb{R}^d} \exp\left(i\sum_{j=1}^d x_j\varphi_j\right) d\rho_1(x_1)\cdots d\rho_d(x_d) = \prod_{j=1}^d \int_{\mathbb{R}} e^{ix_j\varphi_j} d\rho_j(x_j).$$

the Fourier transform of the spectral measures has the following form:

$$\begin{aligned}\int_{\mathbb{R}^d} e^{i(x,\varphi)} d\rho(x) &= \prod_{j=1}^{d} \exp\left\{q_j^{-2}\left(e^{i\varphi_j q_j} - 1 - i\varphi_j q_j\right)\right\} \\ &= \exp\left\{\sum_{j=1}^{d} q_j^{-2}\left(e^{i\varphi_j q_j} - 1 - i\varphi_j q_j\right)\right\} \\ &= \exp\left\{\int_M \left(e^{i\varphi(t)} - 1 - i\varphi(t)\right) d\nu(t)\right\}\end{aligned} \tag{10}$$

The last formula is obtained in the following way. Build the function $\varphi(\cdot)$ as a linear combination of indicator functions, $\varphi(t) = \sum_{j=1}^{d} \varphi_j \frac{1}{\sqrt{\nu(\gamma_j)}} \kappa_{\gamma_j}(t) = \sum_{j=1}^{d} \varphi_j q_j \kappa_j(t)$. Then rewrite the sum in braces in an integral form using the measure ν.

7. The spectral measure of $\hat{J}_d^{\circ}$

In this section we prove that the spectral measures of $\hat{J}_d$ and $\hat{J}_d^{\circ}$ are identical.

Let P_d be the projector from $L^2_{\mathbb{C}}(M, d\nu)$ into $\hat{H}_d^{\mathbb{C}}$ (here $\mathbb{C}$ denotes the complexification),

$$\forall \varphi \in L^2_{\mathbb{C}}(M, d\nu), \quad P_d\varphi = \sum_{k=1}^{d} (\varphi, e_k) e_k. \tag{11}$$

Then the operator $P_d^{(n)}$ acts from $\mathcal{F}_n(L^2_{\mathrm{Re}}(M, d\nu))$ into $\mathcal{F}_n(\hat{H}_d)$. Let

$$\forall f \in \mathcal{F}_n(L^2_{\mathrm{Re}}(M, d\nu)), \quad f = \sum_{\substack{\alpha \in \mathbb{Z}^{\infty}_{+,\mathrm{fin}} \\ |\alpha| = n}} f_\alpha [\varepsilon_\alpha e_1^{\otimes \alpha_1} \widehat{\otimes} \dots \widehat{\otimes} e_d^{\otimes \alpha_d} \widehat{\otimes} \dots].$$

Here the part of vectors that is denoted as the last dot sequence belongs to the orthogonal complement $\mathcal{F}_n(L^2_{\mathrm{Re}}(M, d\nu)) \ominus \mathcal{F}_n(\hat{H}_d)$. Then for $P_d^{(n)}$ we have the following expression:

$$P_d^{(n)} f = \sum_{\substack{\alpha \in \mathbb{Z}^{\infty}_{+,\mathrm{fin}} \\ |\alpha| = n}} f_\alpha [\varepsilon_\alpha e_1^{\otimes \alpha_1} \widehat{\otimes} \dots \widehat{\otimes} e_d^{\otimes \alpha_d}] \in \mathcal{F}_m(\hat{H}_d), \quad m \leqslant n.$$

Construct the following projector P :

$$P = \bigoplus_{n=0}^{\infty} P_d^{(n)} : \mathcal{F}(L^2_{\mathrm{Re}}(M, d\nu)) \longrightarrow \mathcal{F}(\hat{H}_d).$$

Proposition 7.1.

$$\forall \varphi \in \hat{H}_d, \quad \forall f \in \mathcal{F}_{\mathrm{fin}}(L^2_{\mathrm{Re}}(M, d\nu)) \quad J(\varphi) P f = P J(\varphi) f.$$

Proof. Recall that $J(\varphi) = J_+(\varphi) + B(\varphi) + J_-(\varphi)$. So the equality can be checked for each component of $J(\varphi)$ separately.

$J_+(\varphi):\ \forall \varphi \in \hat{H}_d,\ \forall f_n \in \mathcal{F}_n(L^2_{\mathrm{Re}}(M, d\nu))$

$$P_d^{(n+1)}(a_n(\varphi) f_n) = P_d^{(n+1)}(\sqrt{n+1}\varphi \widehat{\otimes} f_n) = \sqrt{n+1}\varphi \widehat{\otimes} (P_d^{(n)} f_n) = a_n(\varphi)(P_d^{(n)} f_n).$$

To obtain the same result for $J_-(\varphi)$, it is sufficient to form the adjoint of the equality just proved, $P_d^{(n+1)} a_n(\varphi) = a_n(\varphi) P_d^{(n)}$

$$a_n^*(\varphi) P_d^{(n+1)} = P_d^{(n)} a_n^*(\varphi).$$

There is only left to prove the same equality for $B(\varphi)$. It is sufficient to show that

$$\forall \varphi \in \hat{H}_d, \psi \in L^2_{\mathrm{Re}}(M, d\nu) \quad P_d(\varphi \cdot \psi) = \varphi \cdot P_d \psi.$$

This can be done by an explicit calculation using (11) and the property of orthogonality of e_j. $\square$

Now we can prove that the spectral measures of $\hat{J}_d$ and $\hat{J}_d^\circ$ are identical.

Proposition 7.2. *The family $\hat{J}_d^\circ$ has the spectral measure*

$$\rho(x) = \rho_1(x_1) \times \cdots \times \rho_d(x_d), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

where

$$\rho_j(x_j) = \pi^c_{q_j^{-2}}(q_j^{-1} x_j).$$

Proof. Take the Jacobi matrix $J(\varphi)$, $\varphi \in \hat{H}_d$. This matrix generates an operator $A(\varphi) : \mathcal{F}_{\text{fin}}(L^2_{\text{Re}}(M, d\nu)) \longrightarrow \mathcal{F}_{\text{fin}}(L^2_{\text{Re}}(M, d\nu))$. By taking the closure of this operator in the space $\mathcal{F}(L^2_{\text{Re}}(M, d\nu))$ obtain the operator $\tilde{A}(\varphi)$. By taking closure of $A(\varphi)$ in $\mathcal{F}(\hat{H}_d)$ we obtain the operator $\tilde{A}'(\varphi)$. Since $\mathcal{F}(\hat{H}_d)$ is a subspace in $\mathcal{F}(L^2_{\text{Re}}(M, d\nu))$ it is obvious that $\text{Dom}\tilde{A}'(\varphi) \subset \tilde{A}(\varphi)$. On finite vectors both operators act identically, so the following equality is true:

$$\tilde{A}(\varphi)Pf = \tilde{A}'(\varphi)Pf, \quad f \in \text{Dom}(A(\varphi)).$$

Note that for $f \in \text{Dom}(\tilde{A}(\varphi))$, we have $Pf \in \text{Dom}(\tilde{A}'(\varphi)) \subset \text{Dom}(\tilde{A}(\varphi))$.

Denote by E_φ the resolution of identity of $\tilde{A}(\varphi)$ and consider the measure $E_\varphi(\cdot)P$. Its range consists of projectors in the space $\mathcal{F}(\hat{H}_d)$. $E_\varphi(\cdot)P$ is some resolution of identity, because the operators P and $\tilde{A}(\varphi)$ strongly commute (see the previous statement).

Since $\forall f \in \text{Dom}(\tilde{A}'(\varphi)) \subset \mathcal{F}(\hat{H}_d)$

$$\tilde{A}'(\varphi)f = \tilde{A}'(\varphi)Pf = \tilde{A}(\varphi)Pf = \int_{\mathbb{R}} \lambda dE_\varphi(\lambda)Pf,$$

the measure $E'_\varphi(\cdot) = E_\varphi(\cdot)P$ is a resolution of identity for $\tilde{A}'(\varphi)$. The corresponding spectral measure ρ'_φ was found in the previous section. Since $\Omega \in \mathcal{F}(\hat{H}_d)$ we have $P\Omega = \Omega$.

From the procedure of construction of the joint resolution of identity for commuting self-adjoint operators (see [10, Sec. 3]) we have $\forall \Delta \in \mathfrak{B}(\mathbb{R}^d)$

$$\rho'_\varphi(\Delta) = (E'_\varphi(\Delta)\Omega, \Omega)_{\mathcal{F}(\hat{H}_d)} = (E_\varphi(\Delta)P\Omega, \Omega)_{\mathcal{F}(\hat{H}_d)} = (E_\varphi(\Delta)\Omega, \Omega)_{\mathcal{F}(L^2_{\text{Re}}(M,d\nu))} = \rho_\varphi(\Delta).$$

Here $\rho_\varphi(\cdot)$ is the spectral measure of the operator $\tilde{A}(\varphi)$. Thus both spectral measures are identical. $\square$

8. Passing to the limit

We have found the spectral measure for the family $\hat{J}_d^\circ = (\hat{J}(\varphi))_{\varphi \in \hat{H}_d}$. Here the operators generated by $\hat{J}(\varphi)$ act in the Fock space $\hat{\mathcal{F}} = \mathcal{F}(L^2_{\text{Re}}(M, d\nu))$. In this section this result will be extended to the family $\hat{J} = (\hat{J}(\varphi))_{\varphi \in \hat{H}_+}$ that acts on the Fock space $\hat{\mathcal{F}} = \mathcal{F}(L^2_{\text{Re}}(M, d\nu))$.

Here is the final result:

The spectral measure ρ of the field $\hat{J}$ is a probability measure on $\mathfrak{B}(\hat{H}_-)$ for which the Fourier transform has the form

$$\int_{\hat{H}_-} e^{i\langle x, \varphi \rangle} d\rho(x) = \exp\left(\int_M (e^{i\varphi(t)} - 1 - i\varphi(t))\, d\nu(t) \right), \quad \varphi \in \hat{H}_+.$$

The angle brackets $\langle \cdot, \cdot \rangle$ denote a pairing between the positive $\hat{H}_+$ and the negative $\hat{H}_-$ spaces.

We shall use the notations from the previous section. Each operator of the family $\tilde{A} = (\tilde{A}(\varphi))_{\varphi \in \hat{H}_+}$ can be represented as a spectral integral in two ways: using the standard resolution of identity and using the corollary of the spectral projection theorem (see [10, p. 125]),

$$\tilde{A}(\varphi) = \int_{\widehat{H}_-} \langle \lambda, \varphi \rangle \, dE(\lambda) = \int_{\mathbb{R}} \lambda \, dE_\varphi(\lambda).$$

Here $E(\cdot)$ is the joint resolution of identity of the family $\tilde{A}$ (see [10, Section 3]). Thus $\forall \varphi \in \widehat{H}_+$ the following equality takes place:

$$e^{i\tilde{A}(\varphi)} f = \int_{\widehat{H}_-} e^{i\langle \lambda, \varphi \rangle} dE(\lambda) f = \int_{\mathbb{R}} e^{i\lambda} dE_\varphi(\lambda) f.$$

The Fourier transform for the spectral measure $\hat{\rho}$ of the field $\tilde{A}$ has the following form:

$$\begin{aligned}\int_{\widehat{H}_-} e^{i\langle x, \varphi \rangle} d\hat{\rho}(x) &= \left(\int_{\widehat{H}_-} e^{i\langle \lambda, \varphi \rangle} dE(\lambda)\Omega, \Omega \right)_{\mathcal{F}(L^2_{\mathrm{Re}}(M, d\nu))} \\ &= \left(\int_{\mathbb{R}} e^{i\lambda} dE_\varphi(\lambda)\Omega, \Omega \right)_{\mathcal{F}(L^2_{\mathrm{Re}}(M, d\nu))} = \left(e^{i\tilde{A}(\varphi)}\Omega, \Omega \right)_{\mathcal{F}(L^2_{\mathrm{Re}}(M, d\nu))}.\end{aligned}$$

Since $\hat{H}_+ \subset C^{\mathrm{Re}}(M)$, each function $\varphi \in \hat{H}_+$ can be uniformly approximated by step functions φ_n from the space $L^2_{\mathrm{Re}}(M, d\nu)$ (see [13, p. 78]).

Build the corresponding operators $\tilde{A}(\varphi_n)$. In [10, p. 136] it is proved that, in the weak sense,

$$E_{\varphi_n} \longrightarrow E_\varphi, \quad n \to \infty.$$

Note that this proof uses the fact that ν is a finite measure. Continue the chain of transformations for the Fourier transform of the spectral measure,

$$\begin{aligned}\left(e^{i\tilde{A}(\varphi)}\Omega, \Omega \right)_{\mathcal{F}(L^2_{\mathrm{Re}}(M, d\nu))} &= \lim_{n\to\infty} \left(e^{i\tilde{A}(\varphi_n)}\Omega, \Omega \right)_{\mathcal{F}(L^2_{\mathrm{Re}}(M, d\nu))} \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} e^{i\lambda} d \left(E_{\varphi_n}(\lambda)\Omega, \Omega \right)_{\mathcal{F}(L^2_{\mathrm{Re}}(M, d\nu))} \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} e^{i\lambda} d\rho_{\varphi_n}(\lambda).\end{aligned}$$

Associate the step function

$$\varphi_n(t) = \sum_{k=1}^{d_n} \chi_k \cdot 1_{t\in\gamma_k}(t)$$

with the vector $(\chi_1, \ldots, \chi_{d_n}) \in \mathbb{R}^{d_n}$. Then the operators $\tilde{A}(\varphi_n)$ belong to the family $(\tilde{A}(\varphi))_{\varphi\in\mathbb{R}^{d_n}}$. Transform the last integral to the form that is convenient for applying the corresponding result for the finite-dimensional case (formula (10)),

$$\begin{aligned}\int_{\mathbb{R}} e^{i\lambda} d\rho_{\varphi_n}(\lambda) &= \left(\int_{\mathbb{R}} e^{i\lambda} dE_{\varphi_n}(\lambda)\Omega, \Omega \right)_{\mathcal{F}(L^2_{\mathrm{Re}}(M, d\nu))} \\ &= \left(\int_{\mathbb{R}^{d_m}} e^{i(x,\varphi_n)_{\mathbb{R}^{d_n}}} \, dE_n(x)\Omega, \Omega \right)_{\mathcal{F}(L^2_{\mathrm{Re}}(M, d\nu))} = \int_{\mathbb{R}^{d_n}} e^{i(x,\varphi_n)_{\mathbb{R}^{d_n}}} \, d\hat{\rho}_{d_n}(x),\end{aligned}$$

where $\hat{\rho}_{d_n}(\delta) = (E_n(\delta)\Omega, \Omega)_{\mathcal{F}(L^2_{\mathrm{Re}}(M, d\nu))}$, $\delta \in \mathfrak{B}(\mathbb{R}^{d_n})$ is the spectral measure of the family $(\tilde{A}(\varphi))_{\varphi\in\mathbb{R}^{d_n}} = \tilde{A}_{d_n}$. After applying (10) we finally obtain

$$\begin{aligned}\int_{\widehat{H}_-} e^{i\langle x, \varphi \rangle} \, d\hat{\rho}(x) &= \lim_{n\to\infty} \exp\left(\int_M (e^{i\varphi_n(t)} - 1 - i\varphi_n(t)) \, d\nu(t) \right) \\ &= \exp\left(\int_M (e^{i\varphi(t)} - 1 - i\varphi(t)) \, d\nu(t) \right).\end{aligned}$$

The last equality is obtained using the classical Lebesgue theorem: $\varphi \in \hat{H}_+ \subset L^2_{\mathrm{Re}}(M, d\nu)$, φ_n are integrable and tend pointwise to φ. Here we have even uniform convergence. φ is a continuous function on the compact set M, so this function is bounded and the family φ_n is uniformly bounded: $\exists C \in \mathbb{R} : \forall n \in \mathbb{N}, \forall t \in M :\ |\varphi_n(t) < C|$. Here C is a constant that can be used as majorant in the Lebesgue theorem. It is integrable because the measure ν is finite. The proof is finished.

The last thing left is to build the preimage of the spectral measure $d\rho$ onto the space H_-. Denote $\mathfrak{U} = G^{-1}(\mathfrak{B}(\hat{H}_-)) = \{G^{-1}(\Delta) : \Delta \in \mathfrak{B}(\hat{H}_-)\}$ and the corresponding measure image $\sigma(\delta) \stackrel{df}{=} \rho(G(\delta))\quad \forall \delta \in \mathfrak{U}$. In accordance with the theorem about change of variables in spectral integrals obtain the final result.

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