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ON THE GAUSS-MANIN CONNECTION IN CYCLIC HOMOLOGY

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In memory of Yu. L. Daletsky.

Abstract. Getzler constructed a flat connection in the periodic cyclic homology, called the Gauss-Manin connection. In this paper we define this connection, and its monodromy, at the level of the periodic cyclic complex.

The construction does not depend on an associator, and provides an explicit structure of a DG module over an auxiliary DG algebra. This paper is, to a large extent, an effort to clarify and streamline our work [4] with Yu. L. Daletsky.

1. Introduction

For an algebraic variety S over a commutative field k of characteristic zero, let A be a locally free $\mathcal{O}_S$-module which is an associative $\mathcal{O}_S$-algebra. In [6], Getzler constructed a flat connection in the $\mathcal{O}_S$-module $\mathrm{HC}^{\mathrm{per}}_\bullet(A)$, the periodic cyclic homology of A over the ring of scalars $\mathcal{O}_S$. This connection is called the Gauss-Manin connection. In this paper we define this connection, and its monodromy, at the level of the periodic cyclic complex $\mathrm{CC}^{\mathrm{per}}_\bullet(A)$.

Recall that for an associative algebra over a commutative unital ring K one can define the Hochschild chain complex $C_\bullet(A)$, the negative cyclic complex $\mathrm{CC}^-_\bullet(A)$, and the periodic cyclic complex $\mathrm{CC}^{\mathrm{per}}_\bullet(A)$, as well as the Hochschild cochain complex $C^\bullet(A)$ ([13], [18], [5]). The latter is a differential graded Lie algebra, or a DGLA, if one shifts the degree by one: $\mathfrak{g}^\bullet_A = C^{\bullet+1}(A)$. Recall that $\mathrm{CC}^-_\bullet(A) = (C_\bullet(A)[[u]], b + uB)$ is a complex of $K[[u]]$-modules. Here u is a formal variable of degree -2. We can view $\mathrm{CC}^-_\bullet(A)$ as a *cochain* complex if we reverse the grading. In particular, the *cohomological* degree of u is 2. The complex $\mathrm{CC}^-_\bullet(A)$ is known to be a DG module over the DGLA $\mathfrak{g}^\bullet_A$, the action of a cochain D given by the standard operator L_D (cf. [18] or 1.3.2 below).

Consider another formal variable, ϵ, of degree 1. Now consider the DGLA

$$\left(\mathfrak{g}^\bullet_A[u,\epsilon], \delta + u\frac{\partial}{\partial \epsilon}\right). \tag{1.0.1}$$

Theorem 1. *On* $\mathrm{CC}^-_\bullet(A)$*, there is a natural structure of an* L_∞ *module over* $(\mathfrak{g}^\bullet_A[u,\epsilon], \delta + u\frac{\partial}{\partial\epsilon})$*. This structure is* $K[[u]]$*-linear and* (u)*-adically continuous. The induced structure of an* L_∞ *module over* $\mathfrak{g}^\bullet_A$ *is the standard one.*

We recall that an L_∞ module structure, or, which is the same, an L_∞ morphism $\mathfrak{g}^\bullet_A[u,\epsilon] \to \mathrm{End}_{K[[u]]}(\mathrm{CC}^-_\bullet(A))$, can be defined in two equivalent ways. One definition expresses it as a sequence of DGLA morphisms

$$\mathfrak{g}^\bullet_A[u,\epsilon] \leftarrow \mathcal{L} \to \mathrm{End}_{K[[u]]}(\mathrm{CC}^-_\bullet(A)) \tag{1.0.2}$$

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where the morphism on the left is a quasi-isomorphism. Alternatively, one can define this L_∞ morphism as a collection of $K[[u]]$-linear maps

$$\phi_n : S^n(\mathfrak{g}_A^\bullet[u,\epsilon][1]) \to \mathrm{End}_{K[[u]]}(\mathrm{CC}_\bullet^-(A))[1] \tag{1.0.3}$$

satisfying certain quadratic equations. Using that, one can define Getzler's Gauss-Manin connection at the level of chains as a morphism

$$\Omega^\bullet(S, \mathrm{CC}_\bullet^{\mathrm{per}}(A)) \to \Omega^\bullet(S, \mathrm{CC}_\bullet^{\mathrm{per}}(A))$$

of total degree one such that

$$\omega \mapsto d\omega + \sum_{n=1}^{\infty} \frac{u^{-n}}{n!} \phi_n(\theta, \dots, \theta)$$

where θ is the $\mathfrak{g}_A^\bullet[u,\epsilon]$-valued one-form on S given by

$$\theta(X)(s) = L_X m_s \epsilon.$$

Here $s \in S$, m_s is the multiplication on the fiber A_s of A at the point s, and X is a tangent vector to S at s.

A few words about the proof of the main theorem. We define the L_∞ morphism by explicit formulas (Theorem 12 and Lemma 19), but the proof that they do satisfy L_∞ axioms is somewhat roundabout. Recall that the Hochschild cochain complex $C^\bullet(A)$, with the cup product, is a differential graded algebra (DGA). One can consider the negative cyclic complex $\mathrm{CC}_\bullet^-(C^\bullet(A))$ of this DGA. In [20] and [18], an A_∞ structure on this complex is constructed. The negative cyclic complex $\mathrm{CC}_\bullet^-(A)$ is an A_∞ module over this A_∞ algebra. From this, we deduce that $\mathrm{CC}_\bullet^-(A)$ is a DG module over some DGA which is related to the universal enveloping algebra $U(g_A[u,\epsilon])$ by a simple chain of quasi-isomorphisms.

A statement close to Theorem 1 was proven in [4]. Our proof substantially simplifies the proof given there. Note that a much stronger statement can be proven. Namely, $C^\bullet(A)$ is in fact a G_∞ algebra in the sense of Getzler-Jones [8] whose underlying L_∞ algebra is $\mathfrak{g}_A^\bullet$, ([19], [9]); moreover, the pair $(C^\bullet(A), C_\bullet(A))$ is a *homotopy calculus*, or a Calc_∞ algebra ([11], [20], [18]). The underlying L_∞ module structure on $\mathrm{CC}_\bullet^-(A)$ is the standard one. From this, Theorem 1 follows immediately. (The interpretation of the A_∞ algebra $\mathrm{CC}_\bullet^-(C^\bullet(A))$ in terms of the Calc_∞ structure is given in [20]). However, theorems from [11], [20], [18] are extremely inexplicit and the constructions are not canonical, i.e. dependent on a choice of a Drinfeld associator. Our construction here is much more canonical and explicit, though still not perfect in that regard. It does not depend on an associator; it provides an explicit structure of a DG module over an auxiliary DG algebra (denoted in this paper by $B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[u,\epsilon])$). Unfortunately, this auxiliary DGA is related to our DGLA somewhat inexplicitly.

Theorem 1 implies the existence on $\mathrm{CC}_\bullet^-(A)$ of a structure of an A_∞ module over $U(\mathfrak{g}_A^\bullet[u,\epsilon])$; the induced A_∞ module structure over $U(\mathfrak{g}_A^\bullet)$ is defined by the standard operators L_D. An explicit linear map

$$U(\mathfrak{g}_A^\bullet[u,\epsilon]) \otimes_{U(\mathfrak{g}_A^\bullet)} \mathrm{CC}_\bullet^-(A) \to \mathrm{CC}_\bullet^-(A)$$

was defined in [15]. It is likely to coincide with the first term of the above A_∞ module structure.

This paper is, to a large extent, an effort to clarify and streamline our work [4] with Yu. L. Daletsky. I greatly benefited from conversations with P. Bressler, K. Costello, V. Dolgushev, E. Getzler, M. Kontsevich, Y. Soibelman, and D. Tamarkin.

1.1. **The Hochschild cochain complex.** Let A be a graded algebra with unit over a commutative unital ring K of characteristic zero. A Hochschild d-cochain is a linear map $A^{\otimes d} \to A$. Put, for $d \geq 0$,

$$C^d(A) = C^d(A, A) = \mathrm{Hom}_K(\overline{A}^{\otimes d}, A)$$

where $\overline{A} = A/K \cdot 1$. Put

$$|D| = (\textit{degree of the linear map } D) + d\,.$$

Put for cochains D and E from $C^\bullet(A, A)$

$$(D \smile E)(a_1, \ldots, a_{d+e}) = (-1)^{|E| \sum_{i \le d}(|a_i|+1)} D(a_1, \ldots, a_d) E(a_{d+1}, \ldots, a_{d+e});$$
$$(D \circ E)(a_1, \ldots, a_{d+e-1}) = \sum_{j \ge 0} (-1)^{(|E|+1) \sum_{i=1}^{j}(|a_i|+1)} D(a_1, \ldots, a_j, E(a_{j+1}, \ldots, a_{j+e}), \ldots);$$
$$[D,\ E] = D \circ E - (-1)^{(|D|+1)(|E|+1)} E \circ D\,.$$

These operations define the graded associative algebra $(C^\bullet(A, A)\ , \smile)$ and the graded Lie algebra $(C^{\bullet+1}(A, A), [\ ,\ ])$ (cf. [2], [6]). Let

$$m(a_1, a_2) = (-1)^{\deg a_1}\ a_1 a_2;$$

this is a 2-cochain of A (not in C^2). Put

$$\delta D = [m, D];$$
$$(\delta D)(a_1, \ldots, a_{d+1}) = (-1)^{|a_1||D|+|D|+1} a_1 D(a_2, \ldots, a_{d+1}) + \sum_{j=1}^{d} (-1)^{|D|+1+\sum_{i=1}^{j}(|a_i|+1)} D(a_1, \ldots, a_j a_{j+1}, \ldots, a_{d+1}) + (-1)^{|D| \sum_{i=1}^{d}(|a_i|+1)} D(a_1, \ldots, a_d) a_{d+1}.$$

One has

$$\delta^2 = 0; \quad \delta(D \smile E) = \delta D \smile E + (-1)^{|\deg D|} D \smile \delta E\,;$$
$$\delta[D, E] = [\delta D, E] + (-1)^{|D|+1}\ [D, \delta E]$$

($\delta^2 = 0$ follows from $[m, m] = 0$).

Thus $C^\bullet(A, A)$ becomes a complex; we will denote it also by $C^\bullet(A)$. The cohomology of this complex is $H^\bullet(A, A)$ or the Hochschild cohomology. We denote it also by $H^\bullet(A)$. The $\smile$ product induces the Yoneda product on $H^\bullet(A, A) = Ext^\bullet_{A \otimes A^0}(A, A)$. The operation $[\ ,\ ]$ is the Gerstenhaber bracket [5].

If $(A,\ \partial)$ is a differential graded algebra then one can define the differential ∂ acting on A by

$$\partial D\ =\ [\partial, D]$$

Theorem 2. [5]. *The cup product and the Gerstenhaber bracket induce a Gerstenhaber algebra structure on* $H^\bullet(A)$.

For cochains D and D_i define a new Hochschild cochain by the following formula of Gerstenhaber ([5]) and Getzler ([6]):

$$D_0\{D_1, \ldots, D_m\}(a_1, \ldots, a_n) = \sum (-1)^{\sum_{k \le i_p}(|a_k|+1)(|D_p|+1)} D_0(a_1, \ldots, a_{i_1}, D_1(a_{i_1+1}, \ldots), \ldots, D_m(a_{i_m+1}, \ldots), \ldots).$$

Proposition 3. *One has*

$$(D\{E_1, \ldots, E_k\})\{F_1, \ldots, F_l\} = \sum (-1)^{\sum_{q \le i_p}(|E_p|+1)(|F_q|+1)} D\{F_1, \ldots, E_1\{F_{i_1+1}, \ldots,\}, \ldots, E_k\{F_{i_k+1}, \ldots,\}, \ldots,\}.$$

The above proposition can be restated as follows. For a cochain D let $D^{(k)}$ be the following k-cochain of the DGA $C^\bullet(A)$:

$$D^{(k)}(D_1, \ldots, D_k) = D\{D_1, \ldots, D_k\}.$$

Proposition 4. *The map*

$$D \mapsto \sum_{k \geq 0} D^{(k)}$$

is a morphism of differential graded algebras

$$C^{\bullet}(A) \to C^{\bullet}(C^{\bullet}(A)).$$

1.2. **Hochschild chains.** Let A be an associative unital dg algebra over a ground ring K. The differential on A is denoted by δ. Recall that by definition

$$\overline{A} = A/K \cdot 1\,.$$

Set

$$C_p(A, A) = C_p(A) = A \otimes \overline{A}^{\otimes p}.$$

Define the differentials $\delta : C_{\bullet}(A) \to C_{\bullet}(A)$, $b : C_{\bullet}(A) \to C_{\bullet-1}(A)$, $B : C_{\bullet}(A) \to C_{\bullet+1}(A)$ as follows:

$$\delta(a_0 \otimes \cdots \otimes a_p) = \sum_{i=1}^{p} (-1)^{\sum_{k<i}(|a_k|+1)+1}(a_0 \otimes \cdots \otimes \delta a_i \otimes \cdots \otimes a_p);$$

$$b(a_0 \otimes \ldots \otimes a_p) = \sum_{k=0}^{p-1}(-1)^{\sum_{i=0}^{k}(|a_i|+1)+1} a_0 \ldots \otimes a_k a_{k+1} \otimes \ldots a_p + (-1)^{|a_p|+(|a_p|+1)\sum_{i=0}^{p-1}(|a_i|+1)} a_p a_0 \otimes \ldots \otimes a_{p-1}; \tag{1.2.1}$$

$$B(a_0 \otimes \ldots \otimes a_p) = \sum_{k=0}^{p}(-1)^{\sum_{i\leq k}(|a_i|+1)\sum_{i \geq k}(|a_i|+1)} 1 \otimes a_{k+1} \otimes \ldots a_p \otimes a_0 \otimes \ldots \otimes a_k\,. \tag{1.2.2}$$

The complex $C_{\bullet}(A)$ is the total complex of the double complex with the differential $b+\delta$.

Let u be a formal variable of degree two. The complex $(C^{\bullet}(A)[[u]], b + \delta + uB)$ is called *the negative cyclic complex* of A.

One can define explicitly a product

$$\mathrm{sh} : C^{\bullet}(A) \otimes C^{\bullet}(A) \to C^{\bullet}(A) \tag{1.2.3}$$

and its extension

$$\mathrm{sh} + u\,\mathrm{sh}' : C^{\bullet}(A)[[u]] \otimes C^{\bullet}(A)[[u]] \to C^{\bullet}(A)[[u]] \tag{1.2.4}$$

[13]. When A is commutative, these are morphisms of complexes.

1.3. **Pairings between chains and cochains.** For a graded algebra A, for $D \in C^d(A, A)$, define

$$i_D(a_0 \otimes \ldots \otimes a_n) = (-1)^{|D||a_0|} a_0 D(a_1, \ldots, a_d) \otimes a_{d+1} \otimes \ldots \otimes a_n\,. \tag{1.3.1}$$

Proposition 5.

$$[b, i_D] = i_{\delta D}; \quad i_D i_E = (-1)^{|D||E|} i_{E \smile D}.$$

Now, put

$$L_D(a_0 \otimes \ldots \otimes a_n) = \sum_{k=1}^{n-d} \epsilon_k a_0 \otimes \ldots \otimes D(a_{k+1}, \ldots, a_{k+d}) \otimes \ldots \otimes a_n + \sum_{k=n+1-d}^{n} \eta_k D(a_{k+1}, \ldots, a_n, a_0, \ldots) \otimes \ldots \otimes a_k\,. \tag{1.3.2}$$

(The second sum in the above formula is taken over all cyclic permutations such that a_0 is inside D). The signs are given by

$$\epsilon_k = (|D| + 1)(|a_0| + \sum_{i=1}^{k}(|a_i| + 1))$$

and

$$\eta_k = |D| + \sum_{i\leq k}(|a_i|+1)\sum_{i\geq k}(|a_i|+1).$$

Proposition 6.

$$[L_D, L_E] = L_{[D,E]}; \quad [b, L_D] + L_{\delta D} = 0; \quad [L_D, B] = 0\,.$$

Now let us extend the above operations to the cyclic complex. Define

$$S_D(a_0\otimes\ldots\otimes a_n) = \sum_{j\geq 0;\ k\geq j+d} \epsilon_{jk} 1\otimes a_{k+1}\otimes\ldots a_0\otimes\ldots\otimes D(a_{j+1},\ldots,a_{j+d})\otimes\ldots\otimes a_k\,.$$

(The sum is taken over all cyclic permutations for which a_0 appears to the left of D). The signs are as follows:

$$\epsilon_{jk} = (|D|+1)(\sum_{i=k+1}^{n}(|a_i|+1) + |a_0| + \sum_{i=1}^{j}(|a_i|+1)).$$

As we will see later, all the above operations are partial cases of a unified algebraic structure for chains and cochains, cf. 3.1; the sign rule for this unified construction was explained in 3.

Proposition 7. [16].

$$[b+uB, i_D+uS_D] - i_{\delta D} - uS_{\delta D} = L_D.$$

The following statement implies that the differential graded Lie algebra $H^{\bullet+1}(A)[u,\epsilon]$ with the differential $u\frac{\partial}{\partial\epsilon}$ acts on the negative cyclic homology $\mathrm{HC}^-_\bullet(A)$. The extension of this action to the level of cochains will me the main result of this paper.

Proposition 8. [3]. *There exists a linear transformation $T(D,E)$ of the Hochschild chain complex, bilinear in D, $E\in C^\bullet(A,A)$, such that*

$$\begin{aligned}&[b+uB, T(D,E)] - T(\delta D, E) - (-1)^{|D|}T(D,\delta E)\\ &\quad = [L_D, i_E+uS_E] - (-1)^{|D|+1}(i_{[D,E]} + uS_{[D,E]}).\end{aligned}$$

2. The module structure on the negative cyclic complex

2.1. **Definitions.** For a monomial $Y = D_1\ldots D_n$ in $U(\mathfrak{g}_A^\bullet)$, set

$$\overline{Y} = (\ldots((D_1\circ D_2)\circ D_3)\ldots\circ D_n)\in C^\bullet(A). \tag{2.1.1}$$

By linearity, extend this to a map $U(\mathfrak{g}_A^\bullet)\to C^\bullet(A)$. It is easy to see, using induction on n and Proposition 3, that this map is well-defined [4].

Identify $S(\mathfrak{g}_A^\bullet)$ with $U(\mathfrak{g}_A^\bullet)$ as coalgebras via the Poincaré-Birkhoff-Witt map. The augmentation ideals $S(\mathfrak{g}_A^\bullet)^+$ and $U(\mathfrak{g}_A^\bullet)^+$ also get identified. By

$$Y\mapsto \sum Y_1^+\otimes\ldots\otimes Y_n^+ \tag{2.1.2}$$

denote the map

$$S(\mathfrak{g}_A^\bullet)^+\to (S(\mathfrak{g}_A^\bullet)^+)^{\otimes n} \tag{2.1.3}$$

defined as the n-fold coproduct, followed by the nth power of the projection from $S(\mathfrak{g}_A^\bullet)$ to $S(\mathfrak{g}_A^\bullet)^+$ along $K\cdot 1$. Similarly for $U(\mathfrak{g}_A^\bullet)$.

Definition 9. For $Y\in S(\mathfrak{g}_A^\bullet)^+$, define

$$\begin{aligned} i_Y(a_0\otimes\ldots\otimes a_n) &= (-1)^{|a_0||Y|}a_0\overline{Y}(a_1,\ldots,a_k)\otimes a_{k+1}\otimes\ldots\otimes a_n;\\ S_Y(a_0\otimes\ldots\otimes a_n) &= \sum_{n\geq 1}\sum_{i,j_1,\ldots,j_n} \pm 1\otimes a_{k+1}\otimes\ldots\otimes a_n\otimes a_0\otimes\ldots\\ &\qquad \otimes\overline{Y}_1^+(a_{j_1},\ldots)\otimes\ldots\otimes\overline{Y}_m^+(a_{j_m},\ldots)\otimes\ldots\otimes a_k\end{aligned}$$

where the sign is

$$\sum_{p=1}^{m}(|Y_p^+|+1)\Big(\sum_{i=k+1}^{n}(|a_i|+1)+|a_0|+\sum_{i=1}^{j_p-1}(|a_i|+1)\Big).$$

For $Y\in S(\mathfrak{g}_A^\bullet)^+$ and $D\in\mathfrak{g}_A^\bullet$, define

$$\begin{aligned}&T(D,Y)(a_0\otimes\ldots\otimes a_n)\\&\quad=\sum_{n\geq 1,k,j_1,\ldots,j_n}\pm D(a_{k+1},\ldots,a_0,\ldots,\overline{Y}_1^+(a_{j_1},\ldots),\ldots,\overline{Y}_m^+(a_{j_m},\ldots),\ldots,a_j)\otimes a_{j+1}\\&\qquad\qquad\otimes\ldots\otimes a_k\end{aligned}$$

where the sign is

$$(|D|+1)\Big(\sum_{p=1}^{m}(|Y_p^+|+1\Big)+\sum_{p=1}^{m}(|Y_p^+|+1)\Big(\sum_{i=k+1}^{n}(|a_i|+1)+|a_0|+\sum_{i=1}^{j_p-1}(|a_i|+1)\Big).$$

Now introduce the following differential graded algebras. Let $C(\mathfrak{g}_A^\bullet[u,\epsilon])$ be the standard Chevalley-Eilenberg chain complex of the DGLA $\mathfrak{g}_A^\bullet[u,\epsilon]$ over the ring of scalars $K[u]$. It carries the Chevalley-Eilenberg differential ∂ and the differentials δ and ∂_ϵ induced bu the corresponding differentials on $\mathfrak{g}_A^\bullet[u,\epsilon]$. Let $C_+(\mathfrak{g}_A^\bullet[u,\epsilon])$ be the augmentation co-ideal, i.e. the sum of all positive exterior powers of our DGLA. As in (2.1.3) above, the comultiplication defines maps

$$C_+(\mathfrak{g}_A^\bullet[u,\epsilon])\mapsto C_+(\mathfrak{g}_A^\bullet[u,\epsilon])^{\otimes n};$$
$$c\mapsto\sum c_1^+\otimes\ldots\otimes c_n^+.$$

Definition 10. Define the associative DGLA $B(\mathfrak{g}_A^\bullet[u,\epsilon])$ over $K[[u]]$ as the tensor algebra of $C_+(\mathfrak{g}_A^\bullet[u,\epsilon])$ with the differential d determined by

$$dc=(\delta+\partial)c-\frac{1}{2}\sum(-1)^{|c_1^+|}c_1^+c_2^++u\partial_\epsilon c.$$

Definition 11. Let the associative DGA $B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[u,\epsilon])$ over $K[[u]]$ be the tensor algebra of $C_+(\mathfrak{g}_A^\bullet[u,\epsilon])$ with the differential d determined by

$$dc=(\delta+\partial)c-\frac{1}{2}\sum(-1)^{|c_1^+|}c_1^+c_2^++u\sum_{n=1}^{\infty}\partial_\epsilon c_1^+\ldots\partial_\epsilon c_n^+.$$

Theorem 12. (*cf.* [4]). *The following formulas define an action of the DGA $B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[u,\epsilon])$ on $\mathrm{CC}_\bullet^-(A)$:*

$$\begin{aligned}D&\mapsto L_D;\\ \epsilon E_1\wedge\ldots\wedge\epsilon E_n&\mapsto i_Y+uS_Y\qquad &&\textit{for}\quad n\geq 1;\\ \epsilon E_1\wedge\ldots\wedge\epsilon E_n\wedge D&\mapsto T(D,Y)\qquad &&\textit{for}\quad n\geq 1;\\ \epsilon E_1\wedge\ldots\wedge\epsilon E_n\wedge D_1\wedge\ldots\wedge D_k&\mapsto 0 &&\textit{for}\quad k>1\,.\end{aligned}$$

Here $D,\ D_i,\ E_j\in\mathfrak{g}_A^\bullet$ and $Y=E_1\ldots E_n\in S(\mathfrak{g}_A^\bullet)^+$.

We will start the proof in section 3 below by recalling the A_∞ structure from [18], [20]. Then, in section 4, we will re-write the definitions in term of this A_∞ structure. The proof will follow from the definition of an A_∞ module.

3. The A_∞ algebra $C_\bullet(C^\bullet(A))$

In this section we will construct an A_∞ algebra structure on the negative cyclic complex of the DGA of Hochschild cochains of any algebra A. The negative cyclic complex of A itself will be a right A_∞ module over the above A_∞ algebra. Our construction is a direct generalization of the construction of Getzler and Jones [7] who constructed an A_∞

structure on the negative cyclic complex of any commutative algebra C. We adapt their definition to the case when C is *a brace algebra*, in particular the Hochschild cochain complex.

Note that all our constructions can be carried out for a unital A_∞ algebra A. The Hochschild and cyclic complexes of A_∞ algebras are introduced in [7]; as shown in [6], the Hochschild cochain complex becomes an A_∞ algebra; all the formulas in this section are good for the more general case. In fact they are easier to write using the A_∞ language, even if A is a usual algebra.

Recall [14], [17] that an A_∞ algebra is a graded vector space $\mathcal{C}$ together with a Hochschild cochain m of total degree 1,

$$m = \sum_{n=1}^{\infty} m_n$$

where $m_n \in C^n(\mathcal{C})$ and

$$[m, m] = 0.$$

Consider the Hochschild cochain complex of a graded algebra A as a differential graded associative algebra $(C^\bullet(A),\ \smile,\ \delta)$. Consider the Hochschild *chain* complex of this differential graded algebra. The total differential in this complex is $b+\delta$; the degree of a chain is given by

$$|D_0 \otimes \ldots \otimes D_n| = |D_0| + \sum_{i=1}^{n} (|D_i| + 1)$$

where D_i are Hochschild cochains.

The complex $C_\bullet(C^\bullet(A))$ contains the Hochschild cochain complex $C^\bullet(A)$ as a subcomplex (of zero-chains) and has the Hochschild chain complex $C_\bullet(A)$ as a quotient complex

$$C^\bullet(A) \xrightarrow{i} C_\bullet(C^\bullet(A)) \xrightarrow{\pi} C_\bullet(A)$$

(this sequence is not by any means exact). The projection on the right splits if A is commutative. If not, $C_\bullet(A)$ is naturally a graded subspace but not a subcomplex.

Theorem 13. *There is an A_∞ structure $\mathbf{m}$ on $C_\bullet(C^\bullet(A))[[u]]$ such that:*

- *All $\mathbf{m}_n$ are $k[[u]]$-linear, (u)-adically continuous.*
- $\mathbf{m}_1 = b + \delta + uB$.

  For $x,\ y \in C_\bullet(A)$:
- $(-1)^{|x|}\mathbf{m}_2(x, y) = (\mathrm{sh} + u\,\mathrm{sh}')(x, y)$.

  For $D,\ E \in C^\bullet(A)$:
- $(-1)^{|D|}\mathbf{m}_2(D, E) = D \smile E$,
- $\mathbf{m}_2(1 \otimes D,\ 1 \otimes E) + (-1)^{|D||E|}\mathbf{m}_2(1 \otimes E,\ 1 \otimes D) = (-1)^{|D|} 1 \otimes [D,\ E]$,
- $\mathbf{m}_2(D,\ 1 \otimes E) + (-1)^{(|D|+1)|E|}\mathbf{m}_2(1 \otimes E,\ D) = (-1)^{|D|+1}[D,\ E]$.

Here is an explicit description of the above A_∞ structure. We define for $n \geq 2$

$$\mathbf{m}_n = \mathbf{m}_n^{(1)} + u\mathbf{m}_n^{(2)}$$

where, for

$$a^{(k)} = D_0^{(k)} \otimes \ldots \otimes D_{N_k}^{(k)},$$

$$\mathbf{m}_n^{(1)}(a^{(1)}, \ldots, a^{(n)}) = \sum \pm m_k\{\underline{\ldots}, D_0^{(0)}\{\underline{\ldots}\}, \underline{\ldots}, D_0^{(n)}\{\underline{\ldots}\}\underline{\ldots}\} \otimes \underline{\ldots}$$

The space designated by $\_$ is filled with $D_i^{(j)},\ i > 0$, in such a way that:

- the cyclic order of each group $D_0^{(k)}, \ldots, D_{N_k}^{(k)}$ is preserved;

- any cochain $D_j^{(i)}$ may contain some of its neighbors on the right inside the braces, provided that all of these neighbors are of the form $D_q^{(p)}$ with $p<i$. The sign convention: any permutation contributes to the sign; the parity of $D_j^{(i)}$ is always $|D_j^{(i)}|+1$

$$\mathbf{m}_n^{(2)}(a^{(1)},\ldots,a^{(n)})=\sum \pm 1\otimes \underline{\ldots}\otimes D_0^{(0)}\{\underline{\ldots}\}\otimes\underline{\ldots}\otimes D_0^{(n)}\{\underline{\ldots}\}\otimes\underline{\ldots}$$

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- the cyclic order of each group $D_0^{(k)},\ldots,D_{N_k}^{(k)}$ is preserved;
- any cochain $D_j^{(i)}$ may contain some of its neighbors on the right inside the braces, provided that all of these neighbors are of the form $D_q^{(p)}$ with $p<i$. The sign convention: any permutation contributes to the sign; the parity of $D_j^{(i)}$ is always $|D_j^{(i)}|+1$.

Remark 14. Let A be a commutative algebra. Then $C_\bullet(A)[[u]]$ is not only a subcomplex but an A_∞ subalgebra of $C_\bullet(C^\bullet(A))[[u]]$. The A_∞ structure on $C_\bullet(A)[[u]]$ is the one from [7].

Proof of the Theorem. First let us prove that $\mathbf{m}^{(1)}$ is an A_∞ structure on $C_\bullet(C^\bullet(A))$. Decompose it into the sum $\delta+\widetilde{\mathbf{m}}^{(1)}$ where δ is the differential induced by the differential on $C^\bullet(A)$. We want to prove that $[\delta,\widetilde{\mathbf{m}}^{(1)}]+\frac{1}{2}[\widetilde{\mathbf{m}}^{(1)},\widetilde{\mathbf{m}}^{(1)}]=0$. We first compute $\frac{1}{2}[\widetilde{\mathbf{m}}^{(1)},\widetilde{\mathbf{m}}^{(1)}]$. It consists of the following terms:

(1) $m\{\ldots D_0^{(1)}\ldots m\{\ldots D_0^{(i+1)}\ldots D_0^{(j)}\ldots\}\ldots D_0^{(n)}\ldots\}\otimes\ldots$
where the only elements allowed inside the inner $m\{\ldots\}$ are $D_p^{(q)}$ with $i+1\le q\le j$;

(2) $m\{\ldots D_0^{(1)}\ldots m\{\ldots\}\ldots D_0^{(n)}\ldots\}\otimes\ldots$
where the only elements allowed inside the inner $m\{\ldots\}$ are $D_p^{(q)}$ for one and only q (these are the contributions of the term $\widetilde{\mathbf{m}}^{(1)}(a^{(1)},\ldots,ba^{(q)},\ldots,a^{(n)})$;

(3) $m\{\ldots D_0^{(1)}\ldots D_0^{(n)}\ldots\}\otimes\ldots\otimes m\{\ldots\}\otimes\ldots$
with the only requirement that the second $m\{\ldots\}$ should contain elements $D_p^{(q)}$ and $D_{p'}^{(q')}$ with $q\neq q'$.(The terms in which the second $m\{\ldots\}$ contains $D_p^{(q)}$ where all q's are the same cancel out: they enter twice, as contributions from $b\widetilde{\mathbf{m}}^{(1)}(a^{(1)},\ldots,a^{(q)},\ldots,a^{(n)}$ and from $\widetilde{\mathbf{m}}^{(1)}(a^{(1)},\ldots,ba^{(1)},\ldots,a^{(n)})$.

The collections of terms (1) and (2) differ from

(0) $\frac{1}{2}[m,m]\{\ldots D_0^{(1)}\ldots\ldots D_0^{(n)}\ldots\}\otimes\ldots$
by the sum of all the following terms:

($1'$) terms as in (1), but with a requirement that in the inside $m\{\ldots\}$ an element $D_p^{(q)}$ must me present such that $q\le i$ or $q>j$;

($2'$) terms as in (1), but with a requirement that the inside $m\{\ldots\}$ must contain elements $D_p^{(q)}$ and $D_{p'}^{(q')}$ with $q\neq q'$.

Assume for a moment that $D_p^{(q)}$ are elements of a commutative algebra (or, more generally, of a C_∞ algebra, i.e. a homotopy commutative algebra). Then there is no δ and $\widetilde{\mathbf{m}}^{(1)}=\mathbf{m}^{(1)}$. But the terms ($1'$) and ($2'$) all cancel out, as well as (3). Indeed, they all involve $m\{\ldots\}$ with some shuffles inside, and m is zero on all shuffles. (the last statement is obvious for a commutative algebra, and is exactly the definition of a C_∞ algebra).

Now, we are in a more complex situation where $D_p^{(q)}$ are Hochschild cochains (or, more generally, elements of a *brace algebra*). Recall that all the formulas above assume that cochains $D_p^{(q)}$ may contain their neighbors on the right inside the braces. We claim that

(A) the terms (1′), (2′) and (3), together with (0), cancel out with the terms constituting $[\delta, \widetilde{\mathbf{m}}^{(1)}]$.

To see this, recall from [12] the following description of brace operations. To any rooted planar tree with marked vertices one can associate an operation on Hochschild cochains. The operation

$$D\{\ldots E_1\{\ldots\{Z_{1,1},\ldots,Z_{1,k_1}\},\ldots\}\ldots E_n\{\ldots\{Z_{n,1},\ldots,Z_{n,k_n}\}\ldots\}\ldots\}$$

corresponds to a tree where D is at the root, E_i are connected to D by edges, and so on, with Z_{ij} being external vertices. The edge connecting D to E_i is to the left from the edge connecting D to E_j for $i < j$, etc. Furthermore, one is allowed to replace some of the cochains D, E_i, etc. by the cochain m defining the A_∞ structure. In this case we leave the vertex unmarked, and regard the result as an operation whose input are cochains marking the remaining vertices (at least one vertex should remain marked).

For a planar rooted tree T with marked vertices, denote the corresponding operation by $\mathbf{O}_T$. The following corollary from Proposition 3 was proven in [12]:

$$[\delta, \mathbf{O}_T] = \sum_{T'} \pm \mathbf{O}_{T'}$$

where T' are all the trees from which T can be obtained by contracting an edge. One of the vertices of this new edge of T' inherits the marking from the vertex to which it gets contracted; the other vertex of that edge remains unmarked. There is one restriction: the unmarked vertex of T' must have more than one outgoing edge. Using this description, it is easy to see that the claim (A) is true.

Now let us prove that

$$[\delta, \widetilde{\mathbf{m}}^{(2)}] + \widetilde{\mathbf{m}}^{(1)} \circ \mathbf{m}^{(2)} + \mathbf{m}^{(2)} \circ \widetilde{\mathbf{m}}^{(1)} = 0.$$

The summand $\mathbf{m}^{(2)} \circ \widetilde{\mathbf{m}}^{(1)}$ contributes both terms

(1) $D_0^{(1)} \otimes \cdots \otimes D_0^{(2)} \otimes \cdots \otimes D_0^{(n)} \otimes \ldots$

(2) $D_0^{(n)} \otimes \cdots \otimes D_0^{(1)} \otimes \cdots \otimes D_0^{(n-1)} \otimes \ldots$

twice, causing them to cancel out. Indeed, $b\mathbf{m}^{(2)}(a^{(1)}, \ldots, a^{(1)})$ contributes both (1) and (2); $\widetilde{\mathbf{m}}^{(1)}(a^{(1)}, \mathbf{m}^{(2)}(a^{(2)}, \ldots, a^{(n)}))$ contributes (1), and $\widetilde{\mathbf{m}}^{(1)}(\mathbf{m}^{(2)}(a^{(1)}, \ldots, a^{(n-1)}), a^{(n)})$ contributes (2).

(3) $1 \otimes \ldots D_0^{(1)} \otimes \cdots \otimes m\{D_0^{(i+1)} \ldots D_0^{(j)}\} \otimes \ldots \otimes D_0^{(n)} \otimes \ldots$

where $j \geq i$. The summand $\widetilde{\mathbf{m}}^{(1)} \circ \mathbf{m}^{(2)}$ consists of terms.

(4) Same as (3), but with the only elements allowed inside the $m\{\ldots\}$ being $D_p^{(q)}$ with $i + 1 \leq q \leq j$.

(5) $1 \otimes \ldots D_0^{(1)} \otimes \cdots \otimes m\{\ldots\} \otimes \ldots \otimes D_0^{(n)} \otimes \ldots$

where the only elements allowed inside the $m\{\ldots\}$ are $D_p^{(q)}$ for one and only q. The sum of the terms (3), (4), (5) is equal to zero by the same reasoning as in the end of the proof of $[\widetilde{\mathbf{m}}^{(1)}, \widetilde{\mathbf{m}}^{(1)}] = 0$.

3.1. **The A_∞ module structure on Hochschild chains.** Recall the definition of A_∞ modules over A_∞ algebras. First, note that for a graded space $\mathcal{M}$, the Gerstenhaber bracket [,] can be extended to the space

$$\mathrm{Hom}(\overline{\mathcal{C}}^{\otimes \bullet}, \mathcal{C}) \oplus \mathrm{Hom}(\mathcal{M} \otimes \overline{\mathcal{C}}^{\otimes \bullet}, \mathcal{M}).$$

For a graded k-module $\mathcal{M}$, a structure of an A_∞ module over an A_∞ algebra $\mathcal{C}$ on $\mathcal{M}$ is a cochain

$$\mu = \sum_{n=1}^{\infty} \mu_n, \qquad \mu_n \in \mathrm{Hom}(\mathcal{M} \otimes \overline{\mathcal{C}}^{\otimes n-1}, \mathcal{M})$$

such that

$$[m + \mu, m + \mu] = 0.$$

Theorem 15. *On $C_\bullet(A)[[u]]$, there exists a structure of an A_∞ module over the A_∞ algebra $C_\bullet(C^\bullet(A))[[u]]$ such that:*

- *All μ_n are $k[[u]]$-linear, (u)-adically continuous.*
- *$\mu_1 = b + uB$ on $C_\bullet(A)[[u]]$.*
  For $a \in C_\bullet(A)[[u]]$:
- *$\mu_2(a, D) = (-1)^{|a||D|+|a|}(i_D + uS_D)a$,*
- *$\mu_2(a, 1 \otimes D) = (-1)^{|a||D|}L_D a$.*
  For $a,\ x \in C_\bullet(A)[[u]]$: $(-1)^{|a|}\mu_2(a, x) = (\mathrm{sh} + u\,\mathrm{sh}')(a, x)$.

To obtain formulas for the structure of an A_∞ module from Theorem 15, one has to assume that, in the formulas for the A_∞ structure from Theorem 13, all $D_j^{(1)}$ are elements of A; then one has to replace braces { } by the usual parentheses ( ) symbolizing evaluation of a multi-linear map at elements of A. The proof is identical to the one for the A_∞ algebra case.

4. Proof of Theorem 12

We start with two key properties of the A_∞ structures from section 3.

Proposition 16. *Both $\mathbf{m}_k(c_1, \ldots, c_k)$ and $\mu_k(c_1, \ldots, c_k)$ are equal to zero if one of the arguments c_i, $i < k$, is of the form $1 \otimes \ldots$*

Proposition 17. *For $D_i \in \mathfrak{g}_A^\bullet$, $1 \leq i \geq N$, let $Y = D_1 \ldots D_N \in U(\mathfrak{g}_A^\bullet)$. For the A_∞ algebra from Theorem 13, put*

$$x \bullet y = (-1)^{|x|}\mathbf{m}_2(x, y).$$

Then

$$(1 \otimes D_1) \bullet \ldots \bullet (1 \otimes D_N) = \sum_{n \geq 1} 1 \otimes \overline{Y}_1^+ \otimes \ldots \otimes \overline{Y}_n^+.$$

By virtue of Proposition 16, the order of parentheses in the left hand side of the above formula is irrelevant.

Proposition 16 follows immediately from the definitions, Proposition 17 can be easily obtained by induction on N.

Now let us rewrite the operators from Theorem 12 in terms of the A_∞ structures. We replace the left module by a right module by the usual rule $x \cdot a = (-1)^{|a||x|} a \cdot x$.

For $n \geq 1$,

$$x \cdot (\epsilon E_1 \wedge \ldots \wedge \epsilon E_n) = \sum_{n \geq 1} (-1)^{|x|}\mu_{n+1}(x, \overline{Y}_1^+, \ldots, \overline{Y}_n^+);$$

for $n \geq 0$,

$$x \cdot (\epsilon E_1 \wedge \ldots \wedge \epsilon E_n \wedge D) = \sum_{n \geq 1} (-1)^{|x|}\mu_{n+2}(x, \overline{Y}_1^+, \ldots, \overline{Y}_n^+, 1 \otimes D);$$

for $k > 1$,

$$x \cdot (\epsilon E_1 \wedge \ldots \wedge \epsilon E_n \wedge D_1 \wedge \ldots \wedge D_k) = 0.$$

Here $D,\ D_i,\ E_j \in \mathfrak{g}_A^\bullet$ and $Y = E_1 \ldots E_n \in S(\mathfrak{g}_A^\bullet)^+$.

Lemma 18. *For $Y \in S(\mathfrak{g}_A^\bullet)^+$ and $D \in \mathfrak{g}_A^\bullet$,*

$$\overline{(\mathrm{ad}_D Y)} = \sum D\{\overline{Y}_1^+, \overline{Y}_2^+, \ldots, \overline{Y}_n^+\} - (-1)^{(|D|+1)(|Y|+1)}\overline{Y}\{D\}.$$

In particular, for $Y \in S(\mathfrak{g}_A^\bullet)^+$,

$$\overline{(\delta Y)} = \delta\overline{Y} + \sum m_2\{\overline{Y}_1^+, \overline{Y}_2^+\} = \delta\overline{Y} + \sum (-1)^{|Y_1^+|+1}(\overline{Y}_1^+ \smile \overline{Y}_2^+).$$

Indeed, let $Y = E_1 \ldots E_n$. For $n = 2$, the lemma follows from Proposition 3; it general, it is obtained from the same proposition by induction on n.

To prove the theorem, we have to show that

$$\begin{aligned} &-(b+uB)(x\cdot c)+((b+uB)x)\cdot c)+(-1)^{|x|}x\cdot\partial c+(-1)^{|x|}x\cdot\delta c\\ &+\sum(-1)^{|x|+|c_1^+|+1}x\cdot c_1^+\cdot c_2^+ +(-1)^{|x|}u\sum\frac{1}{n!}x\cdot\partial_\epsilon c_1\cdot\ldots\cdot\partial_\epsilon c_n=0. \end{aligned} \tag{4.0.1}$$

Let us start by applying the A_∞ identity

$$\sum\sum_{0\le i\le n}\pm\mu_1(\mu_{n+3}(x,\overline{Y}_1^+,\ldots,\overline{Y}_i^+,1\otimes D,\overline{Y}_{i+1}^+,\ldots,\overline{Y}_n^+,T)))+\ldots=0$$

where $T=1\otimes F_1\otimes\ldots\otimes F_m$ is a cycle with respect to b (and, automatically, to B). By virtue of Proposition 16, all the terms containing $1\otimes D$ in the middle vanish. The only surviving terms produce the identity

$$\begin{aligned} &\sum\pm\mu_{n-i+1}(\mu_{i+2}(x,\overline{Y}_1^+,\ldots,\overline{Y}_i^+,1\otimes D),\overline{Y}_{i+1}^+,\ldots,\overline{Y}_n^+,T))\\ &\quad+\sum\pm\mu_{n+1}(x,\overline{Y}_1^+,\ldots,\overline{Y}_i^+\{D\},\ldots,\overline{Y}_n^+,T)\\ &\quad+\sum\pm\mu_{n+2-j}(x,\overline{Y}_1^+,\ldots,D\{\overline{Y}_{i+1}^+,\ldots,\overline{Y}_{i+j}^+\},\ldots,\overline{Y}_n^+,T)=0. \end{aligned}$$

When $T=1\otimes F$, $F\in\mathfrak{g}_A^\bullet$, we obtain, using the first part of Lemma 18, the identity (4.0.1) for $c=\epsilon E_1\wedge\ldots\wedge\epsilon E_n\wedge D\wedge F$. An identical computation without a T at the end yields (4.0.1) for $c=\epsilon E_1\wedge\ldots\wedge\epsilon E_n\wedge D$. Now apply the A_∞ identity

$$\sum\pm\mu_1(\mu_{n+1}(x,\overline{Y}_1^+,\ldots,\overline{Y}_n^+))+\ldots=0.$$

We obtain

$$\begin{aligned} &\sum\pm\mu_1(\mu_{n+1}(x,\overline{Y}_1^+,\ldots,\overline{Y}_n^+))+\sum\pm\mu_{n+1}(\mu_1(x),\overline{Y}_1^+,\ldots,\overline{Y}_n^+)\\ &\quad+\sum\pm\mu_{n+1}(x,\overline{Y}_1^+,\ldots,m_1(\overline{Y}_i^+),\ldots,\overline{Y}_n^+)\\ &\quad+\sum\pm\mu_{n+1}(x,\overline{Y}_1^+,\ldots,m_2\{\overline{Y}_i^+,\overline{Y}_{i+1}^+\},\ldots,\overline{Y}_n^+)\\ &\quad+\sum\pm\mu_{n-i+1}(\mu_{i+1}(x,\overline{Y}_1^+,\ldots,\overline{Y}_i^+),\overline{Y}_{i+1}^+,\ldots,\overline{Y}_n^+)\\ &\quad+\sum\pm\mu_{i+2}(x,\overline{Y}_1^+,\ldots,\overline{Y}_i^+,1\otimes\overline{Y}_{i+1}^+\otimes\ldots\otimes\overline{Y}_n^+)+=0. \end{aligned}$$

The first two sums in the above formula correspond to the first two terms in (4.0.1); the second two sums, by virtue of the second part of Lemma 18, corresponds to the third term of (4.0.1);the fourth term of (4.0.1) is in our case equal to zero. The fifth sum corresponds to the fifth term of (4.0.1). Now, consider the last sum in the above formula. Use Proposition 17, and apply the computation right after (4.0.1) in the case when $T=\sum 1\otimes\overline{Y}_{i+2}^+\otimes\ldots\otimes\overline{Y}_n^+$ and $D=1\otimes\overline{Y}_{i+1}^+$. Then proceed by induction on i. We see that the sixth sum in the formula corresponds to the sixth term of (4.0.1).

4.1. **End of the proof.** It remains to pass from $B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[\epsilon,u])$ to $U(\mathfrak{g}_A^\bullet[\epsilon,u])$.

Lemma 19. *The formulas*

$$D\to D;$$
$$\epsilon E_1\wedge\ldots\wedge\epsilon E_n\mapsto\frac{1}{n!}\sum_{\sigma\in S_n}\frac{1}{n!}(\epsilon E_{\sigma_1})E_{\sigma_2}\ldots E_{\sigma_n};$$
$$D_1\wedge\ldots D_k\wedge\epsilon E_1\wedge\ldots\wedge\epsilon E_n\mapsto 0$$

for $k>1$ or $k=1$, $n\ge 1$ define a quasi-isomorphism of DGAs

$$B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[\epsilon,u])\to U(\mathfrak{g}_A^\bullet[\epsilon,u]).$$

Proof. The fact that the above map is a morphism of DGAs follows from an easy direct computation. To show that this is a quasi-isomorphism, consider the increasing filtration by powers of ϵ. At the level of graduate quotients, $B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[\epsilon,u])$ becomes the standard free resolution of $(U(\mathfrak{g}_A^\bullet[\epsilon,u]),\delta)$, and the morphism is the standard map from the resolution to the algebra, therefore a quasi-isomorphism. The statement now follows from the comparison argument for spectral sequences.

To summarize, we have constructed explicitly a DGA $B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[\epsilon, u])$ and the morphisms of DGAs

$$U(\mathfrak{g}_A^\bullet[\epsilon, u]) \leftarrow B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[\epsilon, u]) \to \mathrm{End}_{K[[u]]}(\mathrm{CC}_\bullet^-(A))$$

where the morphism on the left is a quasi-isomorphism. This yields an A_∞ morphism

$$U(\mathfrak{g}_A^\bullet[\epsilon, u]) \to \mathrm{End}_{K[[u]]}(\mathrm{CC}_\bullet^-(A))$$

and therefore an L_∞ morphism

$$\mathfrak{g}_A^\bullet[\epsilon, u] \to \mathrm{End}_{K[[u]]}(\mathrm{CC}_\bullet^-(A)).$$

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