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ON WHITNEY CONSTANTS FOR DIFFERENTIABLE FUNCTIONS

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Abstract. Some estimates of the constants in Whitney inequality for the differentiable functions are obtained.

1. Introduction

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{P}_n$ be the space of algebraic polynomials of total degree at most $n \in \mathbb{N}_0$, $C\,[a,b]$ the space of the real valued continuous functions on the closed interval $[a,b]$ equipped with the uniform norm,

$$\|f\|_{C[a,b]} := \max_{x\in[a,b]} |f(x)|\,,$$

and $C^r\,[a,b]$, $r \in \mathbb{N}_0$, be the set all r-times continuously differentiable functions $f \in C\,[a,b]$, $C^0\,[a,b] := C\,[a,b]$. The deviation of $f \in C\,[a,b]$ from $\mathbb{P}_n$ is defined by

$$E_n\left(f,[a,b]\right) := \inf_{P_n\in\mathbb{P}_n} \|f - P_n\|_{C[a,b]}\,.$$

The purpose of the paper is to estimate the constants $W(k,r)$, $k \in \mathbb{N}$, in the well known Whitney Inequality: *if* $f \in C^r\,[a,b]$, *then*

$$E_{k+r-1}\left(f,[a,b]\right) \leqslant W(k,r)\left(\frac{b-a}{k}\right)^r \omega_k\left(\frac{b-a}{k}, f^{(r)}, [a,b]\right),$$

where

$$\omega_k\left(t,g,[a,b]\right) = \sup_{0<h\leqslant t}\ \sup_{x\in[a,b-kh]} \left|\Delta_h^k g(x)\right|$$

is the k-th modulus of smoothness of the function g, and

$$\Delta_h^k g(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} g(x+jh)$$

is an m-th finite difference of g.

Many mathematicians have tried to estimate the Whitney constants: see, say, [1–8] for the references. Burkill [1] obtained the only known precise result: $W(2,0) = 1/2$. Whitney [2] proved that $W(k,0) < \infty$ for each k and gave numerical estimates for $W(k,0)$ when $k \leqslant 5$. In 1982, Sendov [3] conjectured that $W(k,0) \leqslant 1$ for all k. However, this conjecture has been proved only for "small" k's: Whitney [2] for $k = 3$, Kryakin [4] for $k = 4$ and Zhelnov [5,6] for $k = 5, 6, 7, 8$. In general case, the most recent result is due to Gilewicz, Kryakin and Shevchuk [7] who proved that

$$W(k,0) \leqslant 2 + \frac{1}{e^2}, \quad k \in \mathbb{N}.$$

It follows from Lemma 1, below, which belongs to Zhuk and Natanson [8] that

$$W(k,1) \leqslant \frac{1}{e\sigma_k}, \quad k \in \mathbb{N},$$

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where $\sigma_k = 1 + 1/2 + \cdots + 1/k$. For $r \geqslant 2$ the estimates of $W(k, r)$ are not known, except those, that readily follow from the estimates $W(k, 0)$ and $W(k, 1)$.

The main results of the paper are the following.

Theorem 1. *We have*

$$W(k, 2) \leqslant \left(\frac{2}{e\sigma_{k+1}} \right)^2, \quad k \in \mathbb{N}.$$

Theorem 1 follows from Theorem 2.

Theorem 2. *For any $f \in C^2[-1, 1]$, there is a polynomial $P_{k+1} \in \mathbb{P}_{k+1}$ such that*

$$|f(x) - P_{k+1}(x)| \leqslant \frac{k^k}{2^{k+1}k!} \left(\left(1 - x^2\right) |\Pi(x)| + \frac{1}{2^{k+1}} \right) \omega_k \left(\frac{2}{k}, f'', [-1, 1] \right), \quad x \in [-1, 1],$$

where $\Pi(x) := \prod_{j=0}^{k} (x + 1 - 2j/k)$.

Remark 1. The method of proof of Theorem 1 carries over to the case $r = 3$ and $r = 4$, so that one can obtain the inequality:

$$W(k, r) \leqslant \left(\frac{r}{e\sigma_{k+r-1}} \right)^r, \quad k \in \mathbb{N}.$$

Theorem 3. *For $r \in \mathbb{N}$, we have*

$$W(1, r) \leq \frac{1}{r! 2^{2r+1} \cos \frac{\pi}{2(r+1)}}.$$

Remark 2. Similar arguments in the proof of Theorem 3 provide

$$W(2, r) \leq \frac{1}{r! 2^{r^*} \cos^2 \frac{\pi}{2r^*}},$$

where $r^* = 2[(r + 1)/2] + 1$, where $[a]$ stands for the integral part of a.

We prove Theorems 1–3 in section 3.

2. Auxiliary Results

In this section we shall give some auxiliary facts and notations which we will need in the proofs of the theorems. First let us give the following lemma which we will generalize in the end of this section.

Lemma 1. [8, Lemma 3]. *Let f be an absolutely continuous function on $[a, b]$ and $x_j = a + jh$, $j = 0, 1, 2, \ldots, k$, $h = \frac{b-a}{k}$. Then*

$$f(x) - L(x; f; x_0, x_1, \ldots, x_k) = \frac{\prod\limits_{j=0}^{k} (x - x_j)}{h^k k!} \int\limits_0^1 \Delta_{uh}^k f'(au + x(1 - u))\, du.$$

Let $k \in \mathbb{N}$ and $\{y_j\}_{j=0}^k$ be a collection of distinct points $y_j \in [a, b]$. Recall, the divided difference of a function $g : [a, b] \to \mathbb{R}$ is defined by

$$[y_0, y_1, \ldots, y_k; g] = \sum_{j=0}^{k} \frac{g(y_j)}{\prod_{i=0, i\neq j}^{k} (y_j - y_i)}.$$

Denote by $L(x; g; y_0, y_1, \ldots, y_k)$ the Lagrange polynomial of degree $\leqslant k$, that interpolates the function g at the points y_j. Then, as well known

$$g(x) - L(x; g; y_0, y_1, \ldots, y_k) = [x, y_0, y_1, \ldots, y_k; g] \prod_{j=0}^{k} (x - y_j), \quad x \neq y_j, \quad j = \overline{0, k}.$$

Now, let $n \in \mathbb{N}$ and $\{x_i\}_{i=0}^n$ be a collection of points $x_i \in [a,b]$ that may coincide. Let $\{y_j\}_{j=0}^k$ be a collection of distinct points $y_j \in [a,b]$ such that each of $n+1$ points x_i coincides with one of the points y_j. Let a point y_j coincides exactly with s_j points x_i, then the number $p_j = s_j - 1$ is called "multiplicity" of the point y_j. Clearly, $\sum_{j=0}^k s_j = n+1$, that is $\sum_{j=0}^k p_j = n-k$. Let a function $g \in C[a,b]$ have p_j first derivatives at a neighborhood of each point y_j. The Lagrange–Hermite divided difference of order n of the function g at the points x_i, $i = 0,1,\dots,n$, is defined by

$$[x_0, x_1, \dots, x_n; g] := \left(\prod_{j=0}^{k} \frac{1}{p_j!}\right) \frac{\partial^{n-k}}{\partial y_0^{p_0} \partial y_1^{p_1} \cdots \partial y_k^{p_k}} [y_0, y_1, \dots, y_k; g].$$

For $n=0$ set $[x_0; g] = g(x_0)$. The Lagrange–Hermite divided differences possess the same properties as the ordinary divided differences. Say, if $x_0 \neq x_n$, then

$$[x_0, x_1, \dots, x_n; g] = \frac{[x_1, x_2, \dots, x_n; g] - [x_0, x_1, \dots, x_{n-1}; g]}{(x_n - x_0)}, \tag{1}$$

if c is a constant and $g(x) = (x-c)f(x)$ then

$$[x_0, x_1, \dots, x_n; g] = [x_1, x_2, \dots, x_n; f] + (x_0 - c)[x_0, x_1, \dots, x_n; f], \tag{2}$$

and then, let $L(x; g; x_0, x_1, \dots, x_n)$ be the Lagrange–Hermite polynomial of degree $\leqslant n$, that interpolates the function g at the points $y_0, y_1, \dots, y_k$ and interpolates all first p_j derivatives of g at the each point y_j. Then

$$g(x) - L(x; g; x_0, x_1, \dots, x_k) = [x, x_0, x_1, \dots, x_k; g] \prod_{j=0}^{k} (x - x_j), \qquad x \neq x_j, \quad j = \overline{0,n}. \tag{3}$$

Lemma 2. *Let $r \in \mathbb{N}_0$, $n \in \mathbb{N}$, $r \le n$ and $\{x_i\}_{i=0}^n$ be an arbitrary collection of points $x_i \in [a,b]$. If, a function $f \in C[a,b]$ has the the $r-1$-st absolutely continuous derivative on $[a,b]$, then*

$$[x_0, x_1, \dots, x_n; f] = [x_r, x_{r+1}, \dots, x_n; f_r], \tag{4}$$

where $f_0(x) := f(x)$, $\quad f_1(x) := \int\limits_0^1 f'(xt + (1-t)x_0)dt$ *and, for* $r > 1$,

$$f_r(x) := \int\limits_0^1 \int\limits_0^{t_1} \cdots \int\limits_0^{t_{r-1}} f^{(r)}(xt_r + (t_{r-1} - t_r)x_{r-1} + \cdots + (t_1 - t_2)x_1 + (1-t_1)x_0) \times dt_r \cdots dt_2 dt_1.$$

Proof. The proof is by the induction on r. If $r=0$, then the equality (4) is trivial. We assume that it has been established for a number $r-1$, that is

$$[x_0, x_1, \dots, x_n; f] = [x_{r-1}, x_r, \dots, x_n; f_{r-1}]. \tag{5}$$

Since

$$f_{r-1}(x) = (x-c)f_r(x) + c_1$$

where $c = x_{r-1}$ and $c_1 = f_{r-1}(x_{r-1})$, then for the number r, (4) follows from (2) and (5). $\square$

The same arguments provide

Lemma 3. *Let $r_0 \in \mathbb{N}$, $n \in \mathbb{N}$, $r_0 \le n$, and $\{x_i\}_{i=0}^n$ be a collection of points $x_i \in [a,b]$ such that at most $r_0 + 1$ points x_i may coincide. If a function $f \in C[a,b]$ has the $r-1$-st absolutely continuous derivative on $[a,b]$, then (4) holds for each $r = 0, 1, \dots, r_0$.*

The following lemma generalizes Lemma 1 (which is for $r = 1$) and follows immediately from previous lemma and equality (3).

Lemma 4. *Let $r \in \mathbb{N}$, $n \in \mathbb{N}$, $r \leq n$ and $\{x_i\}_{i=1}^n$ be a collection of points $x_i \in [a,b]$ such that for $j = r, r+1, \ldots, n$*

$$x_j := x_r + h(j-r),$$

where $h = 2/(n-r)$. If $f \in C^r[a,b]$, then

$$f(x) - L(x; f; x_1, x_2, \ldots, x_n) = \frac{\prod_{j=1}^{n}(x - x_j)}{h^{n-r}(n-r)!} \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{r-1}} \Delta_{ht_r}^k g(x_r) dt_r \cdots dt_2 dt_1,$$

where $g(u) = f^{(r)}(ut_r + (t_{r-1} - t_r)x_{r-1} + \cdots + (t_1 - t_2)x_1 + (1 - t_1)x)$.

3. Proofs of Theorems

From now on $[a,b] := [-1,1]$. For simplicity of notations, we write $\|g\|$ and $\omega_k(g)$ instead of $\|g\|_{C[-1,1]}$ and $\omega_k(2/k, g, [-1,1])$, respectively.

Proof of Theorem 2. Let $x_j = -1 + 2j/k$ and L_{k+2} be the Hermite–Lagrange interpolation polynomial of degree $\leqslant k+2$, which interpolates f at the points $x_0, x_0, x_1, x_2, \ldots, x_{k-2}, x_{k-1}, x_k, x_k$, that is $L_{k+2}(x_j) = f(x_j)$ for $j = 0, 1, \ldots, k$, $L'_{k+2}(x_0) = f'(x_0)$ and $L'_{k+2}(x_k) = f'(x_k)$. By Newton's formula, the coefficient A_{k+2} of x^{k+2} in the polynomial L_{k+2} is

$$A_{k+2} = [x_0, x_0, x_1, x_2, \ldots, x_{k+2}, x_{k-1}, x_k, x_k; f]. \tag{6}$$

Consider the polynomial

$$P_{k+1}(x) = L_{k+2}(x) - \frac{A_{k+2}}{2^{k+1}} T_{k+2}(x)$$

of degree $\leqslant k+1$, where $T_{k+2}(x) = \cos((k+2)\arccos x)$ is Chebyshev polynomial. The polynomial P_{k+1} is the desired one in Theorem 2. Indeed, since $\|T_{k+2}\| = 1$, we conclude that

$$|f(x) - P_{k+1}(x)| \leqslant |f(x) - L_{k+2}(x)| + \frac{|A_{k+2}|}{2^{k+1}} := i_1 + i_2. \tag{7}$$

First we estimate i_2. Using (6) and Lemma 3, we obtain

$$A_{k+2} = \int_0^1 \int_0^{t_1} [x_0, x_1, \ldots, x_k; g]\, dt_2 dt_1,$$

where $g(u) = f''(ut_2 + (t_1 - t_2)x_k + (1 - t_1)x_0)$. Since

$$\begin{aligned} |[x_0, x_1, \ldots, x_k; g]| &= \left| \frac{k^k}{2^k k!} \Delta_{2/k}^k g(x_0) \right| \leqslant \frac{k^k}{2^k k!} \omega_k\left(\frac{2}{k}, g, [-1,1]\right) \\ &= \frac{k^k}{2^k k!} \omega_k\left(\frac{2}{k} t_2, f'', [-1,1]\right) \leqslant \frac{k^k}{2^k k!} \omega_k(f''), \end{aligned}$$

then

$$i_2 = \frac{|A_{k+2}|}{2^{k+1}} \leqslant \frac{k^k}{2^{2k+1} k!} \int_0^1 \int_0^{t_1} \omega_k(f'')\, dt_2 dt_1 = \frac{k^k}{4^{k+1} k!} \omega_k(f''). \tag{8}$$

Let us now estimate i_1. By (1), (3) and the Lemma 3, we obtain

$$\begin{aligned}
f(x) - L_{k+2}(x) &= \left(x^2-1\right)\Pi(x)\left[x_0, x_0, x_1, \ldots, x_{k-1}, x_k, x_k, x; f\right] \\
&= \frac{\left(x^2-1\right)\Pi(x)}{2} \\
&\times \left(\left[x_0, x_1, \ldots, x_{k-1}, x_k, x_k, x; f\right] - \left[x_0, x_0, x_1, \ldots, x_{k-1}, x_k, x; f\right]\right) \\
&= \frac{\left(x^2-1\right)\Pi(x)}{2} \\
&\times \left(\int_0^1\int_0^{t_1}\left[x_0, x_1, \ldots, x_k; g_1\right] dt_2 dt_1 - \int_0^1\int_0^{t_1}\left[x_0, x_1, \ldots, x_k; g_{-1}\right] dt_2 dt_1\right),
\end{aligned}$$

where $g_l(u) = f''(ut_2 + (t_1 - t_2)x + (1 - t_1)l)$, $l = -1, 1$. Hence

$$\begin{aligned}
&\left|f(x) - L_{k+2}(x)\right| \\
&\le \frac{\left(1-x^2\right)\left|\Pi(x)\right|}{2}\int_0^1\int_0^{t_1}\left(\left|\left[x_0, x_1, \ldots, x_k; g_1\right]\right| + \left|\left[x_0, x_1, \ldots, x_k; g_{-1}\right]\right|\right) dt_2 dt_1. \qquad (9)
\end{aligned}$$

As in the estimation of i_2, the inequality (9) gives

$$i_1 = \left|f(x) - L_{k+2}(f, x)\right| \leqslant \frac{k^k\left(1-x^2\right)\left|\Pi(x)\right|}{2^{k+1}k!}\omega_k\left(f''\right).$$

After adding this inequality to (8), by (7), we obtain the desired assertion. □

Proof of Theorem 1. In order to prove Theorem 1 it is enough to check the inequality

$$\frac{k^k}{2^{k+1}k!}\left(\left(1-x^2\right)\left|\Pi(x)\right| + \frac{1}{2^{k+1}}\right) \leqslant \left(\frac{4}{ek\sigma_{k+1}}\right)^2, \quad x \in [-1, 1]. \qquad (10)$$

We first prove the estimate

$$\frac{k^k\left|\left(x^2-1\right)\Pi(x)\right|}{2^{k+1}k!} \leqslant \left(\frac{4}{e\left(k\sigma_k+1\right)}\right)^2, \quad x \in [-1, 1]. \qquad (11)$$

Indeed, put $h = 2/k$. Since, for $-1 + h \le y \le -h/2$ and $k \ge 4$,

$$\left|\frac{\left((y+h)^2-1)\right)\Pi(y+h)}{(y^2-1))\Pi(y)}\right| = \frac{(y+h+1)\left(1-(y+h)^2\right)}{(1-y)\left(1-y^2\right)} \le 1,$$

then

$$\max_{x\in[-1,1]}\left|\left(x^2-1)\right)\Pi(x)\right| = \max_{x\in[-1,-1+2h]}\left|\left(x^2-1)\right)\Pi(x)\right|.$$

If $-1 < x < -1 + 2/k$ and $u = k(x+1)/2$ then $0 < u < 1$ and

$$\begin{aligned}
\frac{k^k}{2^{k+1}k!}\left|\left(x^2-1\right)\Pi(x)\right| &= \frac{4u^2\left(1-\frac{u}{1}\right)\left(1-\frac{u}{2}\right)\cdots\left(1-\frac{u}{k-1}\right)\left(1-\frac{u}{k}\right)^2}{k^2} \\
&\leqslant \frac{4u^2\left[1-\frac{(\sigma_k+1/k)u}{k+1}\right]^{k+1}}{k^2} \leqslant \frac{4u^2e^{-(\sigma_k+1/k)u}}{k^2} \\
&\leqslant \left(\frac{4}{\left(k\sigma_k+1\right)e}\right)^2.
\end{aligned}$$

On the other hand, applying similar arguments to the case $-1 + h < x < -1 + 2h$, we obtain (11). Now, taking into account (11) and the inequality $k! \geqslant k^ke^{-k}\sqrt{2\pi k}$ which follows from Stirling's formula , we get

$$W(k, 2) \leqslant \left(\frac{2}{e\left(\sigma_k+1/k\right)}\right)^2 + \frac{\sqrt{k}e^k}{4^{k+2}\sqrt{2\pi}}.$$

It is easy to check that

$$\left(\frac{2}{e\,(\sigma_k+1/k)}\right)^2+\frac{\sqrt{k}e^k}{4^{k+2}\sqrt{2\pi}}\leqslant\left(\frac{2}{e\sigma_{k+1}}\right)^2,$$

say, for $k\geqslant 33$. Thus, (10) is proved for $k\geqslant 33$. For $1\le k\le 4$ and $10\le k<33$ we check (10) by the direct calculations. For $4<k<10$ we also use the direct calculations, replacing "$1/2^{k+1}$" by "$\left(1/2^{k+1}\right)|T_{k+2}(x)|$" in (10). □

Proof of Theorem 3. Denote by $y_j=\cos\frac{(2j+1)\pi}{2(r+1)}$, $j=0,1,\dots,r$, the zeros of the Chebyshev polynomial T_{r-1}. Let L_r be the Lagrange interpolation polynomial of f at the points y_j. Then it follows from (3) and Lemma 3, that

$$\begin{aligned}|f(x)-L_r(x)|&=\left|\frac{T_{r+1}(x)}{2^r}[x,x_0,x_1,\dots,x_r;f]\right|\\&\le\frac{1}{2^r}|[x_0,x_r;f_r]|=\frac{1}{2^r(x_0-x_r)}|f_r(x_0)-f_r(x_r)|\\&\le\frac{1}{2^{r+1}\cos\frac{\pi}{2(r+1)}}\left|\int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}}\omega\left((x_0-x_r)t_r,f^{(r)},[-1,1]\right)dt_r\cdots dt_2dt_1\right|\\&\le\frac{1}{r!2^{r+1}}\cos\frac{\pi}{2(r+1)}\omega\left(f^{(r)}\right),\end{aligned}$$

where

$$f_r(u):=\int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}}f^{(r)}\big(ut_r+(t_{r-1}-t_r)x_{r-1}+\dots+(t_1-t_2)x_1+(1-t_1)x\big)\times dt_r\cdots dt_2dt_1.$$

□

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