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NON-NEGATIVE PERTURBATIONS OF NON-NEGATIVE SELF-ADJOINT OPERATORS

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In memory of my great teacher Mark Krein.

ABSTRACT. Let A be a non-negative self-adjoint operator in a Hilbert space $\mathcal{H}$ and A_0 be some densely defined closed restriction of A_0, $A_0 \subseteq A \neq A_0$. It is of interest to know whether A is the unique non-negative self-adjoint extensions of A_0 in $\mathcal{H}$. We give a natural criterion that this is the case and if it fails, we describe all non-negative extensions of A_0. The obtained results are applied to investigation of non-negative singular point perturbations of the Laplace and poly-harmonic operators in $\mathbb{L}_2(\mathbf{R}_n)$.

1. Introduction

In this paper we deal with a non-negative self-adjoint operator A in a Hilbert space$\mathcal{H}$, some densely defined not essentially self-adjoint restriction A_0 of A and again with self-adjoint extensions of A_0 in $\mathcal{H}$, which following [1] we call here *singular perturbations* of A. For quick getting onto the matter of the main problem let us compare the point perturbations of self-adjoint Laplace operators $-\Delta$ in three and two dimensions acting in $\mathbb{L}_2(\mathbf{R}_3)$ and $\mathbb{L}_2(\mathbf{R}_2)$, respectively, that is let us consider the restriction $-\Delta^0$ of $-\Delta$ onto the Sobolev subspaces $\mathbb{H}_2^2\left(\mathbf{R}_i \setminus \{0\}\right)$, $i = 3, 2$, and self-adjoint extensions $-\Delta_\alpha$, $\alpha \in \mathbf{R}$, of $-\Delta^0$ in $\mathbb{L}_2(\mathbf{R}_i)$ with domains

$$(1.1)\qquad \begin{aligned} \mathcal{D}_\alpha^{(3)} &:= \left\{ f : f \in \mathbb{H}_2^2\left(\mathbf{R}_3\right), \lim_{|\mathbf{x}|\downarrow 0} \left[\tfrac{d}{d|\mathbf{x}|}\left(|\mathbf{x}| f(\mathbf{x})\right) - \alpha |\mathbf{x}| f(\mathbf{x}) \right] = 0 \right\}, \\ \mathcal{D}_\alpha^{(2)} &:= \left\{ f : f \in \mathbb{H}_2^2\left(\mathbf{R}_2\right), \lim_{|\mathbf{x}|\downarrow 0} \left[\left(\tfrac{2\pi\alpha}{\ln|\mathbf{x}|} + 1 \right) f(\mathbf{x}) - \lim_{|\mathbf{x}'|\downarrow 0} \tfrac{\ln|\mathbf{x}|}{\ln|\mathbf{x}'|} f(\mathbf{x}') \right] = 0 \right\}. \end{aligned}$$

The self-adjoint operators $-\Delta_\alpha$ are just mentioned above singular perturbations of $-\Delta$. The resolvents $\left(-\Delta_\alpha - z\right)^{-1}$, $z \in \rho(-\Delta_\alpha)$, of the operators $-\Delta_\alpha$ act in the corresponding spaces $\mathbb{L}_2$ as integral operators with kernels (Green functions) [1],

$$(1.2)\qquad G^3_{\alpha,z}(\mathbf{x},\mathbf{x}') = \left\{ \begin{array}{c} G_z^{(0)}(\mathbf{x},\mathbf{x}') + (\alpha - i\sqrt{z}/4\pi)^{-1} G_z^{(0)}(\mathbf{x},0) G_z^{(0)}(0,\mathbf{x}'), \\ G_z^{(0)}(\mathbf{x},\mathbf{x}') = \frac{\exp i\sqrt{z}|\mathbf{x}-\mathbf{x}'|}{4\pi|\mathbf{x}-\mathbf{x}'|} \,(\text{three dimension}); \end{array} \right.$$

$$(1.3)\qquad G^2_{\alpha,z}(\mathbf{x},\mathbf{x}') = \left\{ \begin{array}{c} G_z^{(0)}(\mathbf{x},\mathbf{x}') + 2\pi(2\pi\alpha - \psi(1) + \ln\sqrt{z}/2i)^{-1} G_z^{(0)}(\mathbf{x},0) G_z^{(0)}(0,\mathbf{x}'), \\ G_z^{(0)}(\mathbf{x},\mathbf{x}') = (\frac{i}{4}) H_0^{(1)}(i\sqrt{z}|\mathbf{x}-\mathbf{x}'|) \,(\text{two dimension}). \end{array} \right.$$

By (1.2) the Green function $G_{\alpha,z}(\mathbf{x},\mathbf{x}')$ of the self-adjoint operator $-\Delta_\alpha$ in $\mathbb{L}_2(\mathbf{R}_3)$ is holomorphic on the half-axis $(-\infty, 0)$ for $\alpha \geq 0$ and has on this half-axis a simple pole

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for $\alpha < 0$. Hence in the case of three dimensions self-adjoint extensions $-\Delta_\alpha$ are non-negative for all ($\alpha \geq 0$) and non-positive for $\alpha < 0$.

Contrary to this by (1.3) in the case of two dimensions for any $\alpha \in \mathbf{R}$ the Green function $G_{\alpha,z}$ has a simple pole on the half-axis $(-\infty, 0)$. Hence all singular perturbations $-\Delta_\alpha$ of the two-dimensional Laplace operators have one negative eigenvalue. In other words the Laplace operator $-\Delta$ defined in a standard way is a unique non-negative self-adjoint extension in $\mathbb{L}_2(\mathbf{R}_2)$ of the symmetric operator $-\Delta^0$ in $\mathbb{L}_2(\mathbf{R}_2)$.[1]

In this note we try to reveal the underlying cause of such discrepancy. Recall that each densely defined non-negative symmetric operator has at least one non-negative canonical self-adjoint extension (Friedrichs extension). In a more general setting we try to understand here why in some cases the non-negative extension appears to be unique. Actually this questions is embedded into the framework of the general extension theory for semi-bounded symmetric operators developed in the famous paper of M. G. Krein [2]. Naturally, there is a criterion of uniqueness of non-negative extension in [2]. In the next Section, using only approaches of [2] we find another form of this criterion, directly facilitating the investigation of singular perturbations and, for cases where the conditions of these criteria fail, describe all non-negative singular perturbations of a given non-negative self-adjoint operator A associated with some its densely defined non-self-adjoint restriction A_0. In fact we give here a parametrization of the operator interval $[A_\mu, A_M]$ of all canonical non-negative self-adjoint extensions of a given densely defined non-negative operator. The third Section illustrates the obtained results by an example of singular perturbations of the Laplace and poly-harmonic operators in $\mathbb{L}_2(\mathbf{R}_n)$.

Note that very close results were obtained recently in a somewhat different way in [3] where, in terms of this note, the authors have described singular perturbations of the Friedrichs extension of a given densely defined non-negative operator and also gave an illustration with an example of singular perturbations of the Laplace operator in $\mathbb{L}_2(\mathbf{R}_3)$.

2. Uniqueness criterion and parametrization of non-negative singular perturbations

Let A be a non-negative self-adjoint operator acting in the Hilbert space $\mathcal{H}$ and A_0 be a densely defined closed operator, which is a restriction of A onto a subset $\mathcal{D}(A_0)$ of the domain $\mathcal{D}(A)$ of A. Let us consider the subspaces $\mathcal{M} := (I + A_0\mathcal{D}(A_0)$ and $\mathcal{N} := \mathcal{H} \ominus \mathcal{M}$. We will assume that

$$1)\ \mathcal{M} \neq \mathcal{H}, \quad 2)\ \mathcal{N} \cap \mathcal{D}(A) = \{0\}. \tag{2.1}$$

We call all self-adjoint extensions of A_0 in $\mathcal{H}$ other than the given A singular perturbations of A. It is of interest to know whether there are non-negative operators among singular perturbations of A. In this section we try to find a convenient criterion that such singular perturbations of A do not exist. In other words we look for a criterion that A is the only one non-negative operator among all self-adjoint extensions of A_0. Following the approach developed in the renowned paper of M. G. Krein [2] let us consider the operator from $K_0 : \mathcal{M} \to \mathcal{H}$ defined by the relations

$$f = (I + A_0)\, x, \quad K_0 f = \mathcal{A}_0 x, \quad x \in \mathcal{D}(A_0). \tag{2.2}$$

It is easy to see that K_0 is a non-negative contraction,

$$(K_0 f, f) \geq 0, \quad \|K_0 f\|^2 \leq \|f\|^2, \quad f \in \mathcal{M}. \tag{2.3}$$

Let A_1 be any non-negative self-adjoint extension of A_0 in $\mathcal{H}$. Then $K_1 := A_1 \left(A_1 + I\right)^{-1}$ is a non-negative operator, which is a contractive extension of K_0 from the domain $\mathcal{M}$ onto the whole $\mathcal{H}$, $K_1 f = K_0 f$, $f \in \mathcal{M}$.

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[1] The attention of the author to this phenomenon was drawn by Sergey Gredeskul.

On the other hand, for any contractive extension K_1 from $\mathcal{M}$ onto $\mathcal{H}$ such that the unity is not its eigenvalue the non-negative self-adjoint operator $A_1 = K_1\left(I-K_1\right)^{-1}$ is a self-adjoint extension of A_0 in $\mathcal{H}$. Therefore, A_0 *has a unique non-negative self-adjoint extension in* $\mathcal{H}$ *if and only if* K_0 *admits only one non-negative contractive extension onto the whole* $\mathcal{H}$*, no eigenvalue of which equals* 1*, that is,* $K = A(I+A)^{-1}$. So the uniqueness of A as a non-negative extension of A_0 is equivalent to uniqueness of K_0 as a non-negative contractive extension of K_0.

From now on we will denote by $\mathbf{G}$ the set consisting of A and all its singular perturbations and by $\mathbf{C}$ the set of non-negative contractions obtained from $\mathbf{G}$ by the transformation $A_1 \to A_1\left(A_1+I\right)^{-1}$, $A_1 \in \mathbf{G}$. Let us denote by $P_{\mathcal{M}}$ the orthogonal projector onto $\mathcal{M}$ in $\mathcal{H}$ and let $P_{\mathcal{N}} = I - P_{\mathcal{M}}$. With respect to representation of $\mathcal{H}$ as the orthogonal sum $\mathcal{M}\oplus\mathcal{N}$ we can represent each operator from $\mathbf{C}$ as 2×2 block operator matrix

$$K_X = \begin{pmatrix} T & \Gamma^* \\ \Gamma & X \end{pmatrix}. \tag{2.4}$$

Here

$$T = P_{\mathcal{M}}K_0|_{\mathcal{M}}, \quad \Gamma = P_{\mathcal{M}}K_0|_{\mathcal{M}}$$

and X is some non-negative contraction in $\mathcal{N}$, which distinguishes different elements from $\mathbf{C}$. Since each $K_X \in \mathbf{C}$ is non-negative and contractive, we have

$$T \geq 0, \quad T^2 + \Gamma^*\Gamma \leq I\,. \tag{2.5}$$

Note further that $K_X \in \mathbf{C}$ *is equivalent to*

$$K_X + \varepsilon I \geq 0, \quad (1+\varepsilon)I - K_X \geq 0 \tag{2.6}$$

for any $\varepsilon > 0$.

The block matrix representation of K_X and the Schur-Frobenius factorization formula transform (2.6) into the following block matrix inequalities:

$$\begin{aligned} &\begin{pmatrix} I & 0 \\ \Gamma(T+\varepsilon)^{-1} & I \end{pmatrix}\begin{pmatrix} T+\varepsilon & 0 \\ 0 & X+\varepsilon-\Gamma(T+\varepsilon)^{-1}\Gamma^* \end{pmatrix} \\ &\qquad\times\begin{pmatrix} I & (T+\varepsilon)^{-1}\Gamma^* \\ 0 & I \end{pmatrix} \geq 0, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\begin{pmatrix} I & 0 \\ -\Gamma(I+\varepsilon-T)^{-1} & I \end{pmatrix}\begin{pmatrix} 1+\varepsilon-T & 0 \\ 0 & 1+\varepsilon-X-\Gamma(1+\varepsilon-T)^{-1}\Gamma^* \end{pmatrix} \\ &\qquad\times\begin{pmatrix} I & -(1+\varepsilon-T)^{-1}\Gamma^* \\ 0 & I \end{pmatrix} \geq 0. \end{aligned} \tag{2.8}$$

By our assumptions $T \geq 0$ and $I - T \geq 0$. Therefore block matrix inequalities (2.7) and (2.8) are reduced to

$$\left\{ \begin{array}{c} X+\varepsilon I-\Gamma(T+\varepsilon I)^{-1}\Gamma^* \geq 0, \\ (1+\varepsilon)I - X - \Gamma[(1+\varepsilon)I-T]^{-1}\Gamma^* \geq 0,\ \varepsilon > 0. \end{array} \right. \tag{2.9}$$

Observe that the operator functions of ε in the left-hand sides of inequalities (2.9) are monotone. Setting

$$Y := X - \lim_{\varepsilon\downarrow 0}\Gamma(T+\varepsilon I)^{-1}\Gamma^*$$

we conclude from (2.9) that $K_X \in \mathbf{C}$ if and only if

$$0 \leq Y \leq I - \lim_{\varepsilon\downarrow 0}\left\{\Gamma(T+\varepsilon I)^{-1}\Gamma^* + \Gamma[(1+\varepsilon)I-T]^{-1}\Gamma^*\right\}. \tag{2.10}$$

Hence the equality

$$I - \lim_{\varepsilon\downarrow 0}\left\{\Gamma(T+\varepsilon I)^{-1}\Gamma^* + \Gamma[(1+\varepsilon)I-T]^{-1}\Gamma^*\right\} = 0 \tag{2.11}$$

is a criterion that there are no contractive non-negative extensions of K_0 in $\mathcal{H}$ other than K.

Let us express now (2.10) and (2.11) in terms of given K and A. To this end we use the following proposition.

Proposition 2.1. *Let L be a bounded invertible operator in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ given as $2 \times @$ block operator matrix*

$$L = \begin{pmatrix} R & U \\ V & S \end{pmatrix},$$

where R and S are invertible operators in $\mathcal{M}$ and $\mathcal{N}$, respectively, and U, V act between $\mathcal{M}$ and $\mathcal{N}$. If R is an invertible operator in $\mathcal{M}$, then

$$\begin{pmatrix} R^{-1} & 0 \\ 0 & 0 \end{pmatrix} = L^{-1} - L^{-1} P_{\mathcal{N}} \Lambda^{-1} P_{\mathcal{N}} L^{-1}, \quad \Lambda = P_{\mathcal{N}} L^{-1}|_{\mathcal{N}}. \tag{2.12}$$

Setting

$$\Lambda_{1,\varepsilon} = P_{\mathcal{N}}(K + \varepsilon I)^{-1}|_{\mathcal{N}}, \quad \Lambda_{2,\varepsilon} = P_{\mathcal{N}}[(1+\varepsilon)I - K]^{-1}|_{\mathcal{N}} \tag{2.13}$$

and applying (2.12) with $L = K + \varepsilon I$ and

$$\begin{array}{cc} R = T + \varepsilon I, & U = \Gamma^* = P_{\mathcal{M}} K|_{\mathcal{N}} = P_{\mathcal{M}}[K + \varepsilon I]|_{\mathcal{N}}, \\ V = \Gamma = P_{\mathcal{N}} K|_{\mathcal{M}} = P_{\mathcal{N}}[K + \varepsilon I]|_{\mathcal{M}}, & S = P_{\mathcal{N}} K|_{\mathcal{N}} + \varepsilon I, \end{array}$$

yields

$$\Gamma(T + \varepsilon I)^{-1}\Gamma^* = P_{\mathcal{N}} K|_{\mathcal{N}} + \varepsilon I - \Lambda_{1,\varepsilon}^{-1}.$$

In the same fashion we get

$$\Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* = P_{\mathcal{N}}[I - K]|_{\mathcal{N}} + \varepsilon I - \Lambda_{2,\varepsilon}^{-1}.$$

Hence

$$I - \lim_{\varepsilon \downarrow 0} \left(\Gamma(T + \varepsilon I)^{-1}\Gamma^* + \Gamma[(1+\varepsilon)I - T]^{-1}\Gamma^* \right) = \lim_{\varepsilon \downarrow 0} \Lambda_{1,\varepsilon}^{-1} + \lim_{\varepsilon \downarrow 0} \Lambda_{2,\varepsilon}^{-1}. \tag{2.14}$$

Combining (2.10), (2.11) and (2.14) results in the following theorem.

Theorem 2.2. *Let K be a non-negative contraction in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, K_0 is the restriction of K onto the subspace $\mathcal{M}(= \mathcal{M} \oplus \{0\})$ and*

$$G_1 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[K + \varepsilon I]|_{\mathcal{N}} \right)^{-1}, \quad G_2 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[I - K + \varepsilon I]|_{\mathcal{N}} \right)^{-1}.$$

Then the set $\mathbf{C}$ of all non-negative contractive extensions K_X of K_0 in $\mathcal{H}$ is described by the expression

$$K_X = \begin{pmatrix} P_{\mathcal{M}} K|_{\mathcal{M}} & P_{\mathcal{M}} K|_{\mathcal{N}} \\ P_{\mathcal{M}} K|_{\mathcal{N}} & X \end{pmatrix}, \tag{2.15}$$

where X runs over the set of all non-negative contractions in $\mathcal{N}$ satisfying the inequalities

$$P_{\mathcal{N}} K|_{\mathcal{N}} - G_1 \leq X \leq P_{\mathcal{N}} K|_{\mathcal{N}} + G_2. \tag{2.16}$$

In particular, K is a unique non-negative contractive extension of K_0 if and only if $G_1 = G_2 = 0$.

Remark 2.3. The set $\mathbf{C}$ of non-negative contractions of K_0 contains a minimal extension K_{X_μ} with $X_\mu = P_{\mathcal{N}} K|_{\mathcal{N}} - G_1$ in (2.15) and a maximal extension K_{X_M} with $X_M = P_{\mathcal{N}} K|_{\mathcal{N}} + G_2$ in (2.15). If $G_1 = 0\,(G_2 = 0)$, then K is a minimal (maximal) element of $\mathbf{C}$.

Theorem 2.2 can be formulated in terms of a non-negative self-adjoint operator A and its non-negative singular perturbations.

Theorem 2.4. *Let A be a non-negative self-adjoint operator in a Hilbert space $\mathcal{H}$, A_0 is a densely defined closed symmetric operator, which is a restriction of A onto a linear subset $\mathcal{D}(A_0) \subset \mathcal{D}(A)$ such that $\mathcal{N} = (I + A)\mathcal{D}(A_0) \neq \{0\}$ and let*

$$G_1 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[I + A][A + \varepsilon I]|_{\mathcal{N}}\right)^{-1}, \quad G_2 = \lim_{\varepsilon \downarrow 0} \left(P_{\mathcal{N}}[I + A][I + \varepsilon A]|_{\mathcal{N}}\right)^{-1}.$$

Then the set of all non-negative singular perturbations A_Y of A is described by the formula

$$\begin{cases} f = g - Y(I + A)g, \\ A_Y f = Ag + Y(I + A)g, \end{cases} \tag{2.17}$$

where $g \in \mathcal{D}(A)$ and Y runs over the set of non-negative contractions in $\mathcal{N}$ satisfying the inequalities

$$-G_1 \leq Y \leq G_2. \tag{2.18}$$

A has no singular non-negative perturbations if and only if $G_1 = G_2 = 0$.

Remark 2.5. The set of all non-negative singular perturbations of A contains a minimal perturbation A_μ with and a maximal perturbation A_M such that any non-negative perturbation A_X satisfies the inequalities

$$(I + A_M)^{-1} \leq A_Y \leq (I + A_\mu)^{-1}.$$

The corresponding values of the parameters Y in Theorem 2.4 are

$$\begin{aligned} Y_\mu &= -G_1, \\ Y_M &= G_2. \end{aligned} \tag{2.19}$$

If $G_1 = 0\,(G_2 = 0)$, then the minimal (maximal) perturbation coincides with A.

By simple calculation we get from (2.17) the following version of the M. G. Krein resolvent formula.

Proposition 2.6. *The set of resolvents of all non-negative singular perturbations A_Y of A is described by the M. G. Krein formula*

(2.20)

$$\begin{aligned} (A_Y - zI)^{-1} &= (A - zI)^{-1} \\ &\quad - (1 + z)(A + I)(A - zI)^{-1} Y \left[I + (1 + z) P_{\mathcal{N}}(A + I)(A - zI)^{-1} Y\right]^{-1} \\ &\quad \times P_{\mathcal{N}}(A + I)(A - zI)^{-1}, \end{aligned}$$

where Y runs over the contractions in $\mathcal{N}$ satisfying inequalities $-G_1 \leq Y \leq G_2$.

3. Application to some differential operators

Let us consider the multiplication operator A in $\mathbb{L}_2(\mathbf{R}_n)$ by a continuous function $\varphi(k)$, $k^2 = k_1^2 + \cdots + k_n^2$, such that $\varphi(k) > 0$ almost everywhere and

$$\int_0^\infty \frac{1}{(1 + \varphi(k))^2} k^{n-1}\, dk < \infty. \tag{3.1}$$

A is a non-negative self-adjoint operator,

$$\mathcal{D}(A) = \left\{ f : \int_{\mathbb{R}_n} |1 + \varphi(k)|^2 |f(\mathbf{k}|^2 d\mathbf{k} < \infty,\ f \in \mathbb{L}_2(\mathbf{R}_n) \right\}.$$

In the sequel $\hat{\delta}$ stands for the unbounded linear functional in $\mathbb{L}_2(\mathbf{R}_n)$, formally defined as follows:

$$\hat{\delta}(f) = \int_{\mathbb{R}_n} f(\mathbf{k}) d\mathbf{k}.$$

Note that the domain of $\hat{\delta}$ contains $\mathcal{D}(A)$. Let us denote by A_0 the restriction of A onto the linear set

$$\mathcal{D}_0(A) := \left\{ f : f \in \mathcal{D}(A),\ \hat{\delta}(f) = 0 \right\}. \tag{3.2}$$

The closure of $A_0 \neq A$ and

$$\mathcal{N} = (\mathbb{L}_2(\mathbf{R}_n) \ominus (I + A)\mathcal{D}_0(A)) = \left\{ \xi \cdot \frac{1}{1 + \varphi(k)},\ \xi \in \mathbf{C} \right\}.$$

Applying Theorem 2.4 yields

Proposition 3.1. *A is a unique non-negative self-adjoint extension of A_0, that is, A has no non-negative singular perturbations if and only if*

$$\int_0^\infty \frac{1}{\varphi(k)(1 + \varphi(k))} k^{n-1}\, dk = \infty \quad \text{and} \quad \int_0^\infty \frac{1}{(1 + \varphi(k))} k^{n-1}\, dk = \infty. \tag{3.3}$$

Put $\varphi(k) = k^2$ and let $n = 2$. Then the both integrals in Proposition 3.1 are divergent. Hence the restriction A_0 of the operator A of multiplication by k^2 in $\mathbb{L}_2(\mathbf{R}_2)$ onto the linear set (3.2) has a unique non-negative self-adjoint extension in $\mathbb{L}_2$. Note that the multiplication operator by k^2 in $\mathbb{L}_2(\mathbf{R}_n)$ is isomorphic to the self-adjoint Laplace operator $-\Delta$ in $\mathbb{L}_2(\mathbf{R}_n)$ and its considered here restriction A_0 is isomorphic to the restriction of $-\Delta$ onto the Sobolev subspace $\mathbb{H}_2^2(\mathbf{R}_n \setminus \{0\})$. It follows that the self-adjoint Laplace operator in $\mathbb{L}_2(\mathbf{R}_2)$ has no non-negative singular perturbations with support at one point of $\mathbf{R}_2$.

However, the non-negative singular perturbations of $-\Delta$ in $\mathbb{L}_2(\mathbf{R}_2)$ with support at two or more points do already exist. For example, let us consider there the restriction A_0 of the multiplication operator by k^2, for which the defect subspace $\mathcal{N}$ is one-dimensional and consists of functions collinear to

$$e_0(\mathbf{k}) = \frac{1 - \exp(-i(\mathbf{k} \cdot \mathbf{x}_0))}{1 + k^2}, \quad \mathbf{x}_0 \in \mathbf{R}_2.$$

In this case,

$$\|e_0\|^2 = \int_{\mathbf{R}_2} \frac{4 \sin^2 \frac{1}{2}(\mathbf{k} \cdot \mathbf{x}_0)}{(1 + k^2)^2} \cdot d\mathbf{k} < \infty,$$

$$\big((I + A)A^{-1} e_0, e_0\big) = \int_{\mathbf{R}_2} \frac{4 \sin^2 \frac{1}{2}(\mathbf{k} \cdot \mathbf{x}_0)}{k^2(1 + k^2)} \cdot d\mathbf{k} < \infty,$$

$$((I + A)e_0, e_0) = \int_{\mathbf{R}_2} \frac{4 \sin^2 \frac{1}{2}(\mathbf{k} \cdot \mathbf{x}_0)}{1 + k^2} \cdot d\mathbf{k} = \infty.$$

Hence $G_1 = \|e_0\|^2 \cdot ((I + A)e_0, e_0)^{-1} > 0$, but $G_2 = 0$. Hence, the considered restriction A_0 of the multiplication operator A by k^2 has non-negative self-adjoint extensions in $\mathbb{L}_2(\mathbf{R}_2)$ other than A, and A is a maximal element in the set of these extensions. It remains to note that A is isomorphic to the self-adjoint Laplace operator $-\Delta$ in $\mathbb{L}_2(\mathbf{R}_2)$, and A_0 is isomorphic to the restriction of this $-\Delta$ to the subset of functions $f(\mathbf{x})$ from $\mathcal{D}(-\Delta)$ satisfying the conditions

$$\lim_{|\mathbf{x}| \to 0} (\ln |\mathbf{x}|)^{-1} f(\mathbf{x}) - \lim_{|\mathbf{x} - \mathbf{x}_0| \to 0} (\ln |\mathbf{x} - \mathbf{x}_0|)^{-1} f(\mathbf{x}) = 0,$$

$$\lim_{|\mathbf{x}| \to 0} \left[f(\mathbf{x}) - \ln |\mathbf{x}| \lim_{|\mathbf{x}'| \to 0} (\ln |\mathbf{x}'|)^{-1} f(\mathbf{x}') \right] - \lim_{|\mathbf{x} - \mathbf{x}_0| \to 0} \left[f(\mathbf{x}) - \ln |\mathbf{x} - \mathbf{x}_0| \lim_{|\mathbf{x}' - \mathbf{x}_0| \to 0} (\ln |\mathbf{x}' - \mathbf{x}_0|)^{-1} f(\mathbf{x}') \right] = 0.$$

Put now as above $\varphi(k) = k^2$ and let $n = 3$. Then the first integral in Proposition 3.1 is convergent while the second one, as before, is divergent. Hence the restriction A_0 of the operator A of multiplication by k^2 in $\mathbb{L}_2(\mathbf{R}_3)$ onto the linear set (3.2) has infinitely many non-negative self-adjoint extension in $\mathbb{L}_2(\mathbf{R}_3)$. Hence, the self-adjoint Laplace operator in $\mathbb{L}_2(\mathbf{R}_3)$ has infinitely many non-negative singular perturbations with support at one point of $\mathbf{R}_3$ and the standard Laplace operator is a maximal element in the set of these perturbations.

As the next example we consider the multiplication operator A by k^{2l} in $\mathbb{L}_2(\mathbf{R}_n)$ assuming that $4l \leq n+1$. A is isomorphic to the polyharmonic operator $(-\Delta)^l$ in $\mathbb{L}_2(\mathbf{R}_n)$. Let us consider the restriction A_0 of A with the domain (3.2) that is non-negative symmetric operator which is isomorphic to the restriction of the polyharmonic operator $(-\Delta)^l$ onto the Sobolev subspace $\mathbb{H}^2_{2l}(\mathbf{R}_n \setminus \{0\})$. Applying Theorem 2.4 and Proposition 3.1 results in the following proposition.

Proposition 3.2. *If $n < 2l$ then there are infinitely many non-negative singular perturbations of $(-\Delta)^l$ associated with the one-point symmetric restriction A_0 and $(-\Delta)^l$ is a minimal element in the set of non-negative extensions of A_0 in $\mathbb{H}^2_{2l}(\mathbf{R}_n \setminus \{0\})$.*

If $n = 2l$ then $(-\Delta)^l$ has no such perturbations in $\mathbb{H}^2_{2l}(\mathbf{R}_n \setminus \{0\})$.

If $n > 2l$ then there is an infinite set of non-negative singular perturbations of $(-\Delta)^l$ associated with A_0 and, for them considered as non-negative extensions of A_0 in the set $\mathbb{H}^2_{2l}(\mathbf{R}_n \setminus \{0\})$, the operator $(-\Delta)^l$ is a maximal element.

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