NON-NEGATIVE PERTURBATIONS OF NON-NEGATIVE SELF-ADJOINT OPERATORS

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In memory of my great teacher Mark Krein.

Abstract. Let $A$ be a non-negative self-adjoint operator in a Hilbert space $\mathcal{H}$ and $A_0$ be some densely defined closed restriction of $A_0$, $A_0 \subseteq A \neq A_0$. It is of interest to know whether $A$ is the unique non-negative self-adjoint extensions of $A_0$ in $\mathcal{H}$. We give a natural criterion that this is the case and if it fails, we describe all non-negative extensions of $A_0$. The obtained results are applied to investigation of non-negative singular point perturbations of the Laplace and poly-harmonic operators in $L_2(\mathbb{R})$.

1. Introduction

In this paper we deal with a non-negative self-adjoint operator $A$ in a Hilbert space $\mathcal{H}$, some densely defined not essentially self-adjoint restriction $A_0$ of $A$ and again with self-adjoint extensions of $A_0$ in $\mathcal{H}$, which following [1] we call here singular perturbations of $A$. For quick getting onto the matter of the main problem let us compare the point perturbations of self-adjoint Laplace operators $-\Delta$ in three and two dimensions acting in $L_2(\mathbb{R}_3)$ and $L_2(\mathbb{R}_2)$, respectively, that is let us consider the restriction $-\Delta_0^\alpha$ of $-\Delta$ onto the Sobolev subspaces $H_2^2(\mathbb{R}_i \setminus \{0\})$, $i = 3, 2$, and self-adjoint extensions $-\Delta_\alpha$, $\alpha \in \mathbb{R}$, of $-\Delta^0$ in $L_2(\mathbb{R}_i)$ with domains

$$D_0^3 := \left\{ f : f \in H_2^2(\mathbb{R}_3), \lim_{|x| \to 0} \frac{d}{|x|} \left( (|x| f(x)) - \alpha |x| f(x) \right) = 0 \right\},$$

$$D_\alpha^2 := \left\{ f : f \in H_2^2(\mathbb{R}_2), \lim_{|x| \to 0} \left( \frac{2\alpha}{|x|^2} + 1 \right) f(x) - \lim_{|x'| \to |x|} \frac{\ln |x|}{|x'|} f(x') = 0 \right\}.$$

The self-adjoint operators $-\Delta_\alpha$ are just mentioned above singular perturbations of $-\Delta$. The resolvents $(-\Delta_\alpha - z)^{-1}$, $z \in \rho(-\Delta_\alpha)$, of the operators $-\Delta_\alpha$ act in the corresponding spaces $L_2$ as integral operators with kernels (Green functions) [1],

$$G_{\alpha, z}^3(x, x') = \begin{cases} G_{z}^{(0)}(0, x') + (\alpha - i\sqrt{z/4\pi})^{-1}G_{z}^{(0)}(0, x') & \text{(three dimension);} \\ G_{z}^{(0)}(0, x') & \text{two dimension}. \end{cases}$$

By (1.2) the Green function $G_{\alpha, z}(x, x')$ of the self-adjoint operator $-\Delta_\alpha$ in $L_2(\mathbb{R}_3)$ is holomorphic on the half-axis $(-\infty, 0]$ for $\alpha \geq 0$ and has on this half-axis a simple pole

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for \( \alpha < 0 \). Hence in the case of three dimensions self-adjoint extensions \(-\Delta_\alpha\) are non-negative for all \( \alpha \geq 0 \) and non-positive for \( \alpha < 0 \).

Contrary to this by (1.3) in the case of two dimensions for any \( \alpha \in \mathbb{R} \) the Green function \( G_{\alpha,z} \) has a simple pole on the half-axis \((-\infty,0)\). Hence all singular perturbations \(-\Delta_\alpha\) of the two-dimensional Laplace operators have one negative eigenvalue. In other words the Laplace operator \(-\Delta\) defined in a standard way is a unique non-negative self-adjoint extension in \( L^2(\mathbb{R}^2) \) of the symmetric operator \(-\Delta^0\) in \( L^2(\mathbb{R}^2) \).

In this note we try to reveal the underlying cause of such discrepancy. Recall that each densely defined non-negative symmetric operator has at least one non-negative canonical self-adjoint extension (Friedrichs extension). In a more general setting we try to understand here why in some cases the non-negative extension appears to be unique. Actually this questions is embedded into the framework of the general extension theory for semi-bounded symmetric operators developed in the famous paper of M. G. Krein [2]. Naturally, there is a criterion of uniqueness of non-negative extension in [2]. In the next Section, using only approaches of [2] we find another form of this criterion, directly facilitating the investigation of singular perturbations and, for cases where the conditions of these criteria fail, describe all non-negative singular perturbations of a given non-negative self-adjoint operator \( A \) associated with some its densely defined non-self-adjoint restriction \( A_0 \). In fact we give here a parametrization of the operator interval \([A_\mu, A_M]\) of all canonical non-negative self-adjoint extensions of a given densely defined non-negative operator. The third Section illustrates the obtained results by an example of singular perturbations of the Laplace and poly-harmonic operators in \( L^2(\mathbb{R}^n) \).

Note that very close results were obtained recently in a somewhat different way in [3].

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2. Uniqueness criterion and parametrization of non-negative singular perturbations

Let \( A \) be a non-negative self-adjoint operator acting in the Hilbert space \( \mathcal{H} \) and \( A_0 \) be a densely defined closed operator, which is a restriction of \( A \) onto a subset \( \mathcal{D}(A_0) \) of the domain \( \mathcal{D}(A) \) of \( A \). Let us consider the subspaces \( \mathcal{M} := \{I + A_0 \mathcal{D}(A_0)\} \) and \( \mathcal{N} := \mathcal{H} \ominus \mathcal{M} \). We will assume that

\[
1) \mathcal{M} \neq \mathcal{H}, \quad 2) \mathcal{N} \cap \mathcal{D}(A) = \{0\}.
\]

We call all self-adjoint extensions of \( A_0 \) in \( \mathcal{H} \) other than the given \( A \) singular perturbations of \( A \). It is of interest to know whether there are non-negative operators among singular perturbations of \( A \). In this section we try to find a convenient criterion that such singular perturbations of \( A \) do not exist. In other words we look for a criterion that \( A \) is the only one non-negative operator among all self-adjoint extensions of \( A_0 \). Following the approach developed in the renowned paper of M. G. Krein [2] let us consider the operator from \( K_0 : \mathcal{M} \to \mathcal{H} \) defined by the relations

\[
f = (I + A_0) x, \quad K_0 f = A_0 x, \quad x \in \mathcal{D}(A_0).
\]

It is easy to see that \( K_0 \) is a non-negative contraction,

\[
(K_0 f, f) \geq 0, \quad \|K_0 f\|^2 \leq \|f\|^2, \quad f \in \mathcal{M}.
\]

Let \( A_1 \) be any non-negative self-adjoint extension of \( A_0 \) in \( \mathcal{H} \). Then \( K_1 := A_1 (A_1 + I)^{-1} \) is a non-negative extension of \( K_0 \) from the domain \( \mathcal{M} \) onto the whole \( \mathcal{H} \), \( K_1 f = K_0 f, \ f \in \mathcal{M} \).

\[1\text{The attention of the author to this phenomenon was drawn by Sergey Gredeskul.}\]
On the other hand, for any contractive extension $K_1$ from $\mathcal{M}$ onto $\mathcal{H}$ such that the unity is not its eigenvalue the non-negative self-adjoint operator $A_1 = K_1 (I - K_1)^{-1}$ is a self-adjoint extension of $A_0$ in $\mathcal{H}$. Therefore, $A_0$ has a unique non-negative self-adjoint extension in $\mathcal{H}$ if and only if $K_0$ admits only one non-negative contractive extension onto the whole $\mathcal{H}$, no eigenvalue of which equals 1, that is, $K = A(I + A)^{-1}$. So the uniqueness of $A$ as a non-negative extension of $A_0$ is equivalent to uniqueness of $K_0$ as a non-negative contractive extension of $K_0$.

From now on we will denote by $\mathbf{G}$ the set consisting of $A$ and all its singular perturbations and by $\mathbf{C}$ the set of non-negative contractions obtained from $\mathbf{G}$ by the transformation $A_1 \rightarrow A_1 (A_1 + I)^{-1}$, $A_1 \in \mathbf{G}$. Let us denote by $P_\mathcal{M}$ the orthogonal projector onto $\mathcal{M}$ in $\mathcal{H}$ and let $P_\mathcal{N} = I - P_\mathcal{M}$. With respect to representation of $\mathcal{H}$ as the orthogonal sum $\mathcal{M} \oplus \mathcal{N}$ we can represent each operator from $\mathbf{C}$ as $2 \times 2$ block operator matrix

$$(2.4) \quad K_X = \begin{pmatrix} T & \Gamma^* \\ \Gamma & X \end{pmatrix}.$$  

Here $T = P_\mathcal{M}K_0|_\mathcal{M}$, $\Gamma = P_\mathcal{M}K_0|_\mathcal{M}$ and $X$ is some non-negative contraction in $\mathcal{N}$, which distinguishes different elements from $\mathbf{C}$. Since each $K_X \in \mathbf{C}$ is non-negative and contractive, we have

$$(2.5) \quad T \geq 0, \quad T^2 + \Gamma^* \Gamma \leq I.$$  

Note further that $K_X \in \mathbf{C}$ is equivalent to

$$(2.6) \quad K_X + \varepsilon I \geq 0, \quad (1 + \varepsilon)I - K_X \geq 0$$

for any $\varepsilon > 0$.

The block matrix representation of $K_X$ and the Schur-Frobenius factorization formula transform (2.6) into the following block matrix inequalities:

$$(2.7) \quad \begin{pmatrix} I & 0 \\ \Gamma(T + \varepsilon)^{-1} & I \end{pmatrix} \begin{pmatrix} T + \varepsilon & 0 \\ 0 & X + \varepsilon - \Gamma(T + \varepsilon)^{-1}\Gamma^* \end{pmatrix} \begin{pmatrix} I & (T + \varepsilon)^{-1}\Gamma^* \\ 0 & I \end{pmatrix} \geq 0,$$

$$(2.8) \quad \begin{pmatrix} I & 0 \\ -\Gamma(I + \varepsilon - T)^{-1} & I \end{pmatrix} \begin{pmatrix} 1 + \varepsilon - T & 0 \\ 0 & 1 + \varepsilon - X - \Gamma(1 + \varepsilon - T)^{-1}\Gamma^* \end{pmatrix} \begin{pmatrix} I & -(1 + \varepsilon - T)^{-1}\Gamma^* \\ 0 & I \end{pmatrix} \geq 0.$$  

By our assumptions $T \geq 0$ and $I - T \geq 0$. Therefore block matrix inequalities (2.7) and (2.8) are reduced to

$$(2.9) \quad \begin{cases} X + \varepsilon I - \Gamma(T + \varepsilon I)^{-1}\Gamma^* \geq 0, \\
(1 + \varepsilon)I - X - \Gamma[(1 + \varepsilon)I - T]^{-1}\Gamma^* \geq 0, \quad \varepsilon > 0.
\end{cases}$$

Observe that the operator functions of $\varepsilon$ in the left-hand sides of inequalities (2.9) are monotone. Setting

$$Y := X - \lim_{\varepsilon \downarrow 0} \Gamma(T + \varepsilon I)^{-1}\Gamma^*$$

we conclude from (2.9) that $K_X \in \mathbf{C}$ if and only if

$$(2.10) \quad 0 \leq Y \leq I - \lim_{\varepsilon \downarrow 0} \{ \Gamma(T + \varepsilon I)^{-1}\Gamma^* + \Gamma[(1 + \varepsilon)I - T]^{-1}\Gamma^* \}.$$  

Hence the equality

$$(2.11) \quad I - \lim_{\varepsilon \downarrow 0} \{ \Gamma(T + \varepsilon I)^{-1}\Gamma^* + \Gamma[(1 + \varepsilon)I - T]^{-1}\Gamma^* \} = 0$$
is a criterion that there are no contractive non-negative extensions of $K_0$ in $\mathcal{H}$ other than $K$.

Let us express now (2.10) and (2.11) in terms of given $K$ and $A$. To this end we use the following proposition.

**Proposition 2.1.** Let $L$ be a bounded invertible operator in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ given as $2 \times 2$ block operator matrix

$$L = \begin{pmatrix} R & U \\ V & S \end{pmatrix},$$

where $R$ and $S$ are invertible operators in $\mathcal{M}$ and $\mathcal{N}$, respectively, and $U, V$ act between $\mathcal{M}$ and $\mathcal{N}$. If $R$ is an invertible operator in $\mathcal{M}$, then

$$L^{-1} = \begin{pmatrix} R^{-1} & 0 \\ 0 & 0 \end{pmatrix} \cdot L^{-1} = L^{-1} - L^{-1} P_N L^{-1}, \quad \Lambda = P_N L^{-1} |_N.

Setting

$$\Lambda_{1, \varepsilon} = P_N (K + \varepsilon I)^{-1} |_N, \quad \Lambda_{2, \varepsilon} = P_N [(1 + \varepsilon)I - K]^{-1} |_N$$

and applying (2.12) with $L = K + \varepsilon I$ and

$$R = T + \varepsilon I, \quad V = \Gamma = P_N K |_M = P_N [K + \varepsilon I] |_M, \quad U = \Gamma^* = P_M K |_N = P_M [K + \varepsilon I] |_N, \quad S = P_N K |_N + \varepsilon I,$

yields

$$\Gamma (T + \varepsilon I)^{-1} \Gamma^* = P_N K |_N + \varepsilon I - \Lambda_{1, \varepsilon}^{-1}.$$ 

In the same fashion we get

$$\Gamma [(1 + \varepsilon)I - T]^{-1} \Gamma^* = P_N [(I - K) |_N + \varepsilon I - \Lambda_{2, \varepsilon}^{-1}].$$

Hence

$$I - \lim_{\varepsilon \to 0} \left( \Gamma (T + \varepsilon I)^{-1} \Gamma^* + \Gamma [(1 + \varepsilon)I - T]^{-1} \Gamma^* \right) = \lim_{\varepsilon \to 0} \Lambda_{1, \varepsilon}^{-1} + \lim_{\varepsilon \to 0} \Lambda_{2, \varepsilon}^{-1}.$$ 

Combining (2.10), (2.11) and (2.14) results in the following theorem.

**Theorem 2.2.** Let $K$ be a non-negative contraction in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, $K_0$ is the restriction of $K$ onto the subspace $\mathcal{M} (= \mathcal{M} \oplus \{0\})$ and

$$G_1 = \lim_{\varepsilon \to 0} (P_N [K + \varepsilon I] |_N)^{-1}, \quad G_2 = \lim_{\varepsilon \to 0} (P_N [I - K + \varepsilon I] |_N)^{-1}.$$ 

Then the set $\mathcal{C}$ of all non-negative contractive extensions $K_X$ of $K_0$ in $\mathcal{H}$ is described by the expression

$$(2.15) \quad K_X = \begin{pmatrix} P_M K |_M & P_M K |_N \\ P_M K |_N & X \end{pmatrix},$$

where $X$ runs over the set of all non-negative contractions in $\mathcal{N}$ satisfying the inequalities

$$(2.16) \quad P_N K |_N - G_1 \leq X \leq P_N K |_N + G_2.$$ 

In particular, $K$ is a unique non-negative contractive extension of $K_0$ if and only if $G_1 = G_2 = 0$.

**Remark 2.3.** The set $\mathcal{C}$ of non-negative contractions of $K_0$ contains a minimal extension $K_{X_M}$ with $X_M = P_N K |_N - G_1$ in (2.15) and a maximal extension $K_{X_M}$ with $X_M = P_N K |_N + G_2$ in (2.15). If $G_1 = 0 (G_2 = 0)$, then $K$ is a minimal (maximal) element of $\mathcal{C}$.

Theorem 2.2 can be formulated in terms of a non-negative self-adjoint operator $A$ and its non-negative singular perturbations.
Theorem 2.4. Let $A$ be a non-negative self-adjoint operator in a Hilbert space $H$, $A_0$ is a densely defined closed symmetric operator, which is a restriction of $A$ onto a linear subset $D(A_0) \subset D(A)$ such that $\mathcal{N} = (I + A)D(A_0) \neq \{0\}$ and let

$$G_1 = \lim_{\varepsilon \downarrow 0} (P_{\mathcal{N}}[I + A][A + \varepsilon I]_{\mathcal{N}})^{-1}, \quad G_2 = \lim_{\varepsilon \downarrow 0} (P_{\mathcal{N}}[I + A][I + \varepsilon A]_{\mathcal{N}})^{-1}.$$ 

Then the set of all non-negative singular perturbations $A_Y$ of $A$ is described by the formula

$$(2.17) \begin{cases} f = g - Y(I + A)g, \\ A_Y f = Ag + Y(I + A)g, \end{cases}$$

where $g \in D(A)$ and $Y$ runs over the set of non-negative contractions in $\mathcal{N}$ satisfying the inequalities

$$(2.18) -G_1 \leq Y \leq G_2.$$ 

A has no singular non-negative perturbations if and only if $G_1 = G_2 = 0$.

Remark 2.5. The set of all non-negative singular perturbations of $A$ contains a minimal perturbation $A_\mu$ with and a maximal perturbation $A_M$ such that any non-negative perturbation $A_X$ satisfies the inequalities

$$(2.19) (I + A_M)^{-1} \leq A_Y \leq (I + A_\mu)^{-1}.$$ 

The corresponding values of the parameters $Y$ in Theorem 2.4 are

$$Y_\mu = -G_1, \quad Y_M = G_2.$$ 

If $G_1 = 0 (G_2 = 0)$, then the minimal (maximal) perturbation coincides with $A$.

By simple calculation we get from (2.17) the following version of the M. G. Krein resolvent formula.

Proposition 2.6. The set of resolvents of all non-negative singular perturbations $A_Y$ of $A$ is described by the M. G. Krein formula

$$(2.20) \quad (A_Y - zI)^{-1} = (A - zI)^{-1} - (1 + z)(A + I)(A - zI)^{-1}Y \left[ I + (1 + z)P_{\mathcal{N}}(A + I)(A - zI)^{-1}Y \right]^{-1} \times P_{\mathcal{N}}(A + I)(A - zI)^{-1},$$

where $Y$ runs over the contractions in $\mathcal{N}$ satisfying inequalities $-G_1 \leq Y \leq G_2$.

3. Application to some differential operators

Let us consider the multiplication operator $A$ in $L_2(\mathbb{R}_n)$ by a continuous function $\varphi(k)$, $k^2 = k_1^2 + \cdots + k_n^2$, such that $\varphi(k) > 0$ almost everywhere and

$$(3.1) \quad \int_0^\infty \frac{1}{(1 + \varphi(k))^2} k^{n-1} dk < \infty.$$ 

$A$ is a non-negative self-adjoint operator,

$$D(A) = \left\{ f : \int_{\mathbb{R}_n} |1 + \varphi(k)|^2 |f(k)|^2 dk < \infty, \ f \in L_2(\mathbb{R}_n) \right\}.$$ 

In the sequel $\hat{\delta}$ stands for the unbounded linear functional in $L_2(\mathbb{R}_n)$, formally defined as follows:

$$\hat{\delta}(f) = \int_{\mathbb{R}_n} f(k) dk.$$
Hence the restriction $A$ and

$$D_0(A) := \left\{ f : f \in D(A), \hat{\delta}(f) = 0 \right\}.$$  

The closure of $A_0 \neq A$ and

$$\mathcal{N} = (\mathbb{L}_2(\mathbb{R}_n) \ominus (I + A)D_0(A)) = \left\{ \xi \cdot \frac{1}{1 + \varphi(k)}, \xi \in \mathbb{C} \right\}.$$  

Applying Theorem 2.4 yields

**Proposition 3.1.** $A$ is a unique non-negative self-adjoint extension of $A_0$, that is, $A$ has no non-negative singular perturbations if and only if

$$\int_0^\infty \frac{1}{\varphi(k)(1 + \varphi(k))} k^{n-1} dk = \infty \quad \text{and} \quad \int_0^\infty \frac{1}{(1 + \varphi(k))} k^{n-1} dk = \infty.$$  

Put $\varphi(k) = k^2$ and let $n = 2$. Then the both integrals in Proposition 3.1 are divergent. Hence the restriction $A_0$ of the operator $A$ of multiplication by $k^2$ in $\mathbb{L}_2(\mathbb{R}_2)$ onto the linear set (3.2) has a unique non-negative self-adjoint extension in $\mathbb{L}_2$. Note that the multiplication operator by $k^2$ in $\mathbb{L}_2(\mathbb{R}_n)$ is isomorphic to the self-adjoint Laplace operator $-\Delta$ in $\mathbb{L}_2(\mathbb{R}_n)$ and its considered here restriction $A_0$ is isomorphic to the restriction of $-\Delta$ onto the Sobolev subspace $\mathbb{H}_n^2(\mathbb{R}_n \setminus \{0\})$. It follows that the self-adjoint Laplace operator in $\mathbb{L}_2(\mathbb{R}_2)$ has no non-negative singular perturbations with support at one point of $\mathbb{R}_2$.

However, the non-negative singular perturbations of $-\Delta$ in $\mathbb{L}_2(\mathbb{R}_2)$ with support at two or more points do already exist. For example, let us consider there the restriction $A_0$ of the multiplication operator by $k^2$, for which the defect subspace $\mathcal{N}$ is one-dimensional and consists of functions collinear to

$$e_0(k) = \frac{1 - \exp(-i(k \cdot x_0))}{1 + k^2}, \quad x_0 \in \mathbb{R}_2.$$  

In this case,

$$\|e_0\|^2 = \int_{\mathbb{R}_2} \frac{4 \sin^2 \frac{1}{2}(k \cdot x_0)}{(1 + k^2)^2} \cdot dk < \infty,$$

$$((I + A)A^{-1}e_0, e_0) = \int_{\mathbb{R}_2} \frac{4 \sin^2 \frac{1}{2}(k \cdot x_0)}{k^2(1 + k^2)} \cdot dk < \infty,$$

$$((I + A)e_0, e_0) = \int_{\mathbb{R}_2} \frac{4 \sin^2 \frac{1}{2}(k \cdot x_0)}{1 + k^2} \cdot dk = \infty.$$  

Hence $G_1 = \|e_0\|^2 \cdot ((I + A)e_0, e_0)^{-1} > 0$, but $G_2 = 0$. Hence, the considered restriction $A_0$ of the multiplication operator $A$ by $k^2$ has non-negative self-adjoint extensions in $\mathbb{L}_2(\mathbb{R}_2)$ other than $A$, and $A$ is a maximal element in the set of these extensions. It remains to note that $A$ is isomorphic to the self-adjoint Laplace operator $-\Delta$ in $\mathbb{L}_2(\mathbb{R}_2)$, and $A_0$ is isomorphic to the restriction of this $-\Delta$ to the subset of functions $f(x)$ from $D(-\Delta)$ satisfying the conditions

$$\lim_{|x| \to 0} (\ln |x|)^{-1} f(x) - \lim_{|x - x_0| \to 0} (\ln |x - x_0|)^{-1} f(x) = 0,$$

$$\lim_{|x| \to 0} \left[ f(x) - \ln |x| \lim_{|x'| \to 0} (\ln |x'|)^{-1} f(x') \right] - \lim_{|x - x_0| \to 0} \left[ f(x) - \ln |x - x_0| \lim_{|x'| - x_0| \to 0} (\ln |x' - x_0|)^{-1} f(x') \right] = 0.$$  

Put now as above $\varphi(k) = k^2$ and let $n = 3$. Then the first integral in Proposition 3.1 is convergent while the second one, as before, is divergent. Hence the restriction $A_0$ of the
operator $A$ of multiplication by $k^2$ in $L^2(\mathbb{R}_3)$ onto the linear set (3.2) has infinitely many non-negative self-adjoint extension in $L^2(\mathbb{R}_3)$. Hence, the self-adjoint Laplace operator in $L^2(\mathbb{R}_3)$ has infinitely many non-negative singular perturbations with support at one point of $\mathbb{R}_3$ and the standard Laplace operator is a maximal element in the set of these perturbations.

As the next example we consider the multiplication operator $A$ by $k^2 l$ in $L^2(\mathbb{R}_n)$ assuming that $4l \leq n + 1$. $A$ is isomorphic to the polyharmonic operator $(-\Delta)^l$ in $L^2(\mathbb{R}_n)$. Let us consider the restriction $A_0$ of $A$ with the domain (3.2) that is non-negative symmetric operator which is isomorphic to the restriction of the polyharmonic operator $(-\Delta)^l$ onto the Sobolev subspace $H^{2l}_2(\mathbb{R}_n \setminus \{0\})$. Applying Theorem 2.4 and Proposition 3.1 results in the following proposition.

**Proposition 3.2.** If $n < 2l$ then there are infinitely many non-negative singular perturbations of $(-\Delta)^l$ associated with the one-point symmetric restriction $A_0$ and $(-\Delta)^l$ is a minimal element in the set of non-negative extensions of $A_0$ in $H^{2l}_2(\mathbb{R}_n \setminus \{0\})$.

If $n = 2l$ then $(-\Delta)^l$ has no such perturbations in $H^{2l}_2(\mathbb{R}_n \setminus \{0\})$.

If $n > 2l$ then there is an infinite set of non-negative singular perturbations of $(-\Delta)^l$ associated with $A_0$ and, for them considered as non-negative extensions of $A_0$ in the set $H^{2l}_2(\mathbb{R}_n \setminus \{0\})$, the operator $(-\Delta)^l$ is a maximal element.

**References**


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