

## INVERSE SPECTRAL PROBLEMS FOR COUPLED OSCILLATING SYSTEMS: RECONSTRUCTION FROM THREE SPECTRA

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*Dedicated to M. G. Krein, with admiration.*

**ABSTRACT.** We study an inverse spectral problem for a compound oscillating system consisting of a singular string and  $N$  masses joined by springs. The operator  $\mathcal{A}$  corresponding to this system acts in  $L_2(0,1) \times \mathbb{C}^N$  and is composed of a Sturm–Liouville operator in  $L_2(0,1)$  with a distributional potential and a Jacobi matrix in  $\mathbb{C}^N$  that are coupled in a special way. We solve the problem of reconstructing the system from three spectra—namely, from the spectrum of  $\mathcal{A}$  and the spectra of its decoupled parts. A complete description of possible spectra is given.

### 1. INTRODUCTION

The main aim of the present paper is to solve an inverse spectral problem for a class of oscillating systems composed of a singular string and  $N$  masses joined by springs. Mathematically such a system is described by a Sturm–Liouville operator  $S$  coupled in a special way to a Jacobi operator  $J$ .

Namely, assume that  $q$  is a real-valued distribution from  $W_2^{-1}(0,1)$  and denote by  $S$  a Sturm–Liouville operator in  $L_2(0,1)$  that is formally given by the differential expression

$$l := -\frac{d^2}{dx^2} + q$$

and the Robin or the Dirichlet boundary condition at the point  $x = 0$ . The precise definition of  $S$  is based on regularisation of  $l$  by quasi-derivatives [19, 20] and goes as follows. We fix a real-valued distributional primitive  $\sigma \in L_2(0,1)$  of  $q$  and rewrite  $ly$  as

$$l_\sigma y := -(y' - \sigma y)' - \sigma y'$$

on the natural domain

$$\mathcal{D}(l_\sigma) = \{y \in W_1^1(0,1) \mid y' - \sigma y \in W_1^1(0,1), l_\sigma y \in L_2(0,1)\}.$$

In what follows, we shall abbreviate the *quasi-derivative*  $y' - \sigma y$  to  $y_\sigma^{[1]}$  or simply to  $y^{[1]}$  when  $\sigma$  is fixed by the context. We define now the operator  $S$  by  $Sy = l_\sigma y$  on the domain

$$\mathcal{D}(S) = \{y \in \mathcal{D}(l_\sigma) \mid y^{[1]}(0) = hy(0)\}$$

for some  $h \in \mathbb{R} \cup \{\infty\}$ ,  $h = \infty$  corresponding to the Dirichlet boundary condition  $y(0) = 0$ .

Assume that  $J$  is a Jacobi matrix in  $\mathbb{C}^N$ ,  $N \in \mathbb{N}$ , i. e., that  $J$  in the standard basis  $e_1, \dots, e_N$  of  $\mathbb{C}^N$  is a symmetric matrix with real entries  $b_1, \dots, b_N$  on the main diagonal and positive entries  $a_1, \dots, a_{N-1}$  on the main sub- and super-diagonals.

Denote also by  $B$  the intertwining operator between  $L_2(0,1)$  and  $\mathbb{C}^N$  given on  $\mathcal{D}(S)$  by  $By = a_0 y^{[1]}(1)e_1$  for some  $a_0 > 0$ .

Finally, we consider the operator

$$(1.1) \quad \mathcal{A} := \begin{pmatrix} S & 0 \\ B & J \end{pmatrix}$$

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that acts in the product space  $\mathcal{H} := L_2(0, 1) \times \mathbb{C}^N$  on the domain

$$\mathcal{D}(\mathcal{A}) := \{(y, d)^t \in \mathcal{H} \mid y \in \mathcal{D}(S), d = (d(1), \dots, d(N)), y(1) = a_0 d(1)\}.$$

It is known [1] that  $\mathcal{A}$  is self-adjoint and bounded below in  $\mathcal{H}$  and has a simple discrete spectrum. Adding if necessary a sufficiently large constant to the potential  $q$  and to the numbers  $b_1, \dots, b_N$ , we can make the operator  $\mathcal{A}$  positive and shall assume this without loss of generality.

*Remark 1.1.* We observe that although the Sturm–Liouville differential expression  $l_\sigma$  is independent of the particular choice of the primitive  $\sigma$ , the quasi-derivative  $y^{[1]}$  in the interface condition and in the boundary condition for  $S$  if  $h$  is finite, and thus the whole operator  $\mathcal{A}$ , do depend on  $\sigma$ . We notice, however, that  $\mathcal{A}$  is invariant under the simultaneous change  $\sigma \mapsto \sigma + C$ ,  $h \mapsto h + C$ , and  $b_1 \mapsto b_1 + a_0 C$  for any real  $C$ . This invariance will be used in Section 4.

Along with  $\mathcal{A}$  we consider two operators,  $\mathcal{A}_0 := S_N \oplus J$  and  $\mathcal{A}_\infty := S_D \oplus J_{(1)}$ , where  $S_N$  and  $S_D$  are the restrictions of  $S$  by the “Neumann” boundary condition  $y^{[1]}(1) = 0$  and the Dirichlet boundary condition  $y(1) = 0$  respectively, and  $J_{(1)}$  is the Jacobi matrix obtained by removing the top row and the most-left column of  $J$ . The operators  $\mathcal{A}_0$  and  $\mathcal{A}_\infty$  formally correspond to two extreme cases of the coupling not allowed in  $\mathcal{A}$ : first with no coupling at all, and the second with infinite, i.e., rigid coupling. It is easily seen that  $\mathcal{A}$  and  $\mathcal{A}_0$  are self-adjoint extensions of the same symmetric operator with deficiency indices  $(1, 1)$  specifying the interface condition at the point  $x = 1$ , and the same holds for  $\mathcal{A}$  and  $\mathcal{A}_\infty$ . Therefore, as in the papers [8, 13, 16, 17], it is natural to study the question, to which extent  $\mathcal{A}$  is determined by the spectra of  $\mathcal{A}$  and  $\mathcal{A}_0$ , or those of  $\mathcal{A}$  and  $\mathcal{A}_\infty$ . As in the purely continuous case of a Sturm–Liouville operator [8, 13, 17] or of a purely discrete case of a Jacobi matrix [16], one has to know the spectra of  $S_N$  and  $J$  or those of  $S_D$  and  $J_{(1)}$  separately—and not just their union—in order to reconstruct  $\mathcal{A}$ .

Thus the **inverse spectral problem** we are going to solve is that of the reconstruction of the operator  $\mathcal{A}$  from the spectra of  $\mathcal{A}$ ,  $S_N$ , and  $J$  or from those of  $\mathcal{A}$ ,  $S_D$ , and  $J_{(1)}$ . It generalizes the inverse spectral problems by three spectra for the standard Sturm–Liouville operators or for Jacobi matrices treated in the above-cited papers and is related to the inverse spectral problem for Sturm–Liouville operators with rationally dependent boundary conditions, see [1, 3, 4, 5, 6].

We shall solve the above inverse problem by reducing it to that of reconstructing  $\mathcal{A}$  from its spectrum and the sequence of the corresponding norming constants. The latter problem was studied in detail in [1] (see also [3, 4, 5] for the related inverse problem for a Sturm–Liouville operator with rationally dependent boundary conditions), and this allows a complete description of the spectra for the operators involved. We shall prove that the operator  $\mathcal{A}$  is recovered uniquely if and only if the three spectra do not intersect. This establishes in this special case the conjecture raised in [8] for Sturm–Liouville operators, which was later proved in [13]; the case of finite Jacobi matrices was studied in [16].

The treatments of the Dirichlet boundary condition ( $h = \infty$ ) and the Robin boundary condition ( $h \in \mathbb{R}$ ) at the point  $x = 0$  are completely analogous, and we shall consider in detail only the Dirichlet case. In the next section we shall derive some useful formulae (e.g., for the resolvent of  $\mathcal{A}$  and the norming constants) that will be used in the subsequent analysis. In Sections 3 and 4 we reconstruct the operator  $\mathcal{A}$  from the spectra of  $\mathcal{A}$ ,  $S_D$ , and  $J_{(1)}$  and from the spectra of  $\mathcal{A}$ ,  $S_N$ , and  $J$  respectively.

*Notations.* Throughout the paper, the prime will denote the derivative in  $x \in [0, 1]$ , and the overdot will stand for differentiation in the complex variable  $\lambda$  or  $z$ . Given two strictly increasing (finite or infinite) sequences  $(a_n)$  and  $(b_n)$ , we shall denote by  $(c_n) := (a_n) \amalg (b_n)$  the non-decreasing sequence obtained by amalgamating the sequences

$(a_n)$  and  $(b_n)$  and listing the common elements twice. We shall write  $\sigma(T)$  for the spectrum of a linear operator  $T$  acting in a Hilbert space.

## 2. PRELIMINARIES

It is known [1] that the operator  $\mathcal{A}$  of (1.1) is self-adjoint, lower semi-bounded, and has discrete spectrum  $\lambda_1 < \lambda_2 < \dots$ ; we recall our standing and nonrestrictive assumption that  $\lambda_1 > 0$ .

For every nonzero  $\lambda \in \mathbb{C}$ , we define the “fundamental system of solutions”  $Y_-(\cdot, \lambda)$  and  $Y_+(\cdot, \lambda)$  corresponding to the eigenvalue problem  $\mathcal{A}Y = \lambda Y$ . Namely, the element  $Y_-(\cdot, \lambda) := (y_-(\cdot, \lambda), d_-(\cdot, \lambda))^t$  belongs to  $\mathcal{D}(\mathcal{A})$ , is normalised by the initial conditions  $y_-(0, \lambda) = 0$ ,  $y_-^{[1]}(0, \lambda) = \sqrt{\lambda}$ , and satisfies the relation  $\mathcal{A}Y = \lambda Y$  in the  $L_2(0, 1)$ -component and in the first  $N - 1$  components of  $\mathbb{C}^N$ . In other words, there is a unique  $c = c(\lambda) \in \mathbb{C}$  such that

$$(\mathcal{A} - \lambda)Y_-(\cdot, \lambda) = \begin{pmatrix} 0 \\ ce_N \end{pmatrix};$$

in particular,  $c(\lambda) = 0$  if and only if  $\lambda$  is in the spectrum of  $\mathcal{A}$ , in which case  $Y_-(\cdot, \lambda)$  is a corresponding eigenelement. The element  $Y_+(\cdot, \lambda) := (y_+(\cdot, \lambda), d_+(\cdot, \lambda))^t$  is normalized by the terminal condition  $d_+(N, \lambda) = 1$ , satisfies the system

$$\begin{aligned} ly_+ - \lambda y_+ &= 0, \\ a_0 y_+^{[1]}(1)e_1 + (J - \lambda)d_+ &= 0, \end{aligned}$$

and the interface condition  $y_+(1, \lambda) = a_0 d_+(1, \lambda)$ , but needn't satisfy the initial condition  $y_+(0, \lambda) = 0$ . Moreover,  $y_+(0, \lambda) = 0$  holds if and only if  $\lambda$  is in the spectrum of  $\mathcal{A}$ , in which case  $Y_+(\cdot, \lambda)$  is a corresponding eigenelement.

Using the elements  $Y_{\pm}(\cdot, \lambda)$ , it is possible to construct the Green function of the operator  $\mathcal{A}$  and to find the explicit form of its resolvent, similarly to such constructions for a Sturm–Liouville equation.

**Lemma 2.1.** *Assume that  $\lambda \in \mathbb{C}$  belongs to the resolvent set of the operators  $J$  and  $\mathcal{A}$  and that  $(g, v)^t$  is an arbitrary element of  $\mathcal{H}$ . Then the element*

$$\begin{pmatrix} y \\ d \end{pmatrix} := (\mathcal{A} - \lambda)^{-1} \begin{pmatrix} g \\ v \end{pmatrix}$$

is given by

$$\begin{aligned} y(x) &= \frac{y_-(x, \lambda)}{W(\lambda)} \left[ \int_x^1 y_+ g + (v, d_+(\cdot, \bar{\lambda}))_{\mathbb{C}^N} \right] + \frac{y_+(x, \lambda)}{W(\lambda)} \int_0^x y_- g, \\ d(k) &= (J - \lambda)^{-1} v(k) + \frac{d_+(k, \lambda)}{W(\lambda)} \left[ \int_0^1 y_- g + \frac{y_-^{[1]}(1, \lambda)}{y_+^{[1]}(1, \lambda)} (v, d_+(\cdot, \bar{\lambda}))_{\mathbb{C}^N} \right], \end{aligned}$$

where  $W(\lambda) := y_+(x, \lambda)y_-^{[1]}(x, \lambda) - y_+^{[1]}(x, \lambda)y_-(x, \lambda)$  is the Wronskian of the solutions  $y_+$  and  $y_-$ .

*Proof.* The function  $y$  solves the equation  $Sy = \lambda y + g$  and thus is equal to  $y_0 + \alpha y_-$ , with

$$y_0(x) := \frac{y_-(x, \lambda)}{W(\lambda)} \int_x^1 y_+ g + \frac{y_+(x, \lambda)}{W(\lambda)} \int_0^x y_- g$$

being a particular solution to the above non-homogeneous problem and  $\alpha$  some complex number. Since  $d_+(\cdot, \lambda) = -a_0 y_+^{[1]}(1, \lambda)(J - \lambda)^{-1} e_1$ , the relation

$$(2.1) \quad a_0 y_+^{[1]}(1)e_1 + (J - \lambda)d = v$$

implies that  $d = d_0 + \beta d_+(\cdot, \lambda)$  with  $d_0 := (J - \lambda)^{-1} v$  and some  $\beta \in \mathbb{C}$ .

The constants  $\alpha$  and  $\beta$  must be such that the interface condition  $y(1) = a_0 d(1)$  and relation (2.1) hold. By virtue of the relation

$$d_0(1) = ((J - \lambda)^{-1}v, e_1) = -\frac{(v, d_+(\cdot, \bar{\lambda}))_{\mathbb{C}^N}}{a_0 y_+^{[1]}(1, \lambda)}$$

the interface condition transforms into

$$\alpha y_-(1, \lambda) - \beta y_+(1, \lambda) = -\frac{y_+(1, \lambda)}{W(\lambda)} \int_0^1 y_- g - \frac{(v, d_+(\cdot, \bar{\lambda}))_{\mathbb{C}^N}}{y_+^{[1]}(1, \lambda)}.$$

Similarly, equation (2.1) can be recast as

$$\alpha y_-^{[1]}(1, \lambda) - \beta y_+^{[1]}(1, \lambda) = -\frac{y_+^{[1]}(1, \lambda)}{W(\lambda)} \int_0^1 y_- g.$$

The above two equations form a linear system for  $\alpha$  and  $\beta$ , solving which we find that

$$\alpha = \frac{(v, d_+(\cdot, \bar{\lambda}))_{\mathbb{C}^N}}{W(\lambda)}, \quad \beta = \frac{\int_0^1 y_- g}{W(\lambda)} + \frac{y_-^{[1]}(1, \lambda)}{y_+^{[1]}(1, \lambda)} \frac{(v, d_+(\cdot, \bar{\lambda}))_{\mathbb{C}^N}}{W(\lambda)},$$

and the required formula for  $(y, d)^\dagger$  follows.  $\square$

The sequence  $(Y_-(\cdot, \lambda_n))_{n \in \mathbb{N}}$  forms an orthogonal basis of the space  $\mathcal{H}$ . We denote by  $\alpha_n := \|Y_-(\cdot, \lambda_n)\|^{-2}$  the *norming constant* corresponding to the eigenvalue  $\lambda_n$ . A useful formula for the norming constants is given by the following lemma.

**Lemma 2.2.** *Assume that  $\lambda_n \in \sigma(\mathcal{A})$  is not in the spectrum of  $J$ . Then the corresponding norming constant  $\alpha_n := \|Y_-(\cdot, \lambda_n)\|^{-2}$  satisfies the equalities*

$$(2.2) \quad \alpha_n = -\frac{y_+^{[1]}(1, \lambda_n)}{\sqrt{\lambda_n} y_-^{[1]}(1, \lambda_n) \dot{y}_+(0, \lambda_n)}.$$

Similarly, if  $\lambda_n \in \sigma(\mathcal{A})$  is not in the spectrum of  $J_{(1)}$ , then

$$(2.3) \quad \alpha_n = -\frac{y_+(1, \lambda_n)}{\sqrt{\lambda_n} y_-(1, \lambda_n) \dot{y}_+(0, \lambda_n)}.$$

*Proof.* We take an arbitrary function  $g \in L_2(0, 1)$ , put  $G := (g, 0)^\dagger$ , and calculate the  $L_2$ -component  $\hat{g}$  of the element  $(\mathcal{A} - \lambda)^{-1}G$  in two ways. On the one hand, the resolution of identity of the operator  $\mathcal{A}$  gives

$$\hat{g}(x) = \sum_{k=1}^{\infty} \frac{\alpha_k (g, y_-(\cdot, \lambda_k))_{\mathbb{C}^N} y_-(x, \lambda_k)}{\lambda_k - \lambda}.$$

On the other hand, using Lemma 2.1, we find that

$$\hat{g}(x) = \frac{y_-(x, \lambda)}{W(\lambda)} \int_x^1 y_+ g + \frac{y_+(x, \lambda)}{W(\lambda)} \int_0^x y_- g.$$

Equating the residues at the point  $\lambda = \lambda_n$  and noting that the functions  $y_-(\cdot, \lambda_n)$  and  $y_+(\cdot, \lambda_n)$  are collinear and that  $\lambda_n$  is a simple zero of  $W$ , we conclude that

$$\alpha_n y_-(x, \lambda_n) (g, y_-(\cdot, \lambda_n))_{\mathbb{C}^N} = -\frac{y_+(x, \lambda_n)}{W(\lambda_n)} (g, y_-(\cdot, \lambda_n))_{\mathbb{C}^N},$$

or, on account of the relation  $W(\lambda) \equiv \sqrt{\lambda} y_+(0, \lambda)$ ,

$$\alpha_n = -\frac{y_+(x, \lambda_n)}{\sqrt{\lambda_n} y_-(x, \lambda_n)} \frac{1}{\dot{y}_+(0, \lambda_n)}.$$

Finally, the ratio  $y_+(x, \lambda_n)/y_-(x, \lambda_n)$  does not depend on  $x$ , and, moreover,

$$\frac{y_+(x, \lambda_n)}{y_-(x, \lambda_n)} = \frac{y_+(1, \lambda_n)}{y_-(1, \lambda_n)}$$

if  $y_-(1, \lambda_n) \neq 0$  and

$$\frac{y_+(x, \lambda_n)}{y_-(x, \lambda_n)} = \frac{y_+^{[1]}(1, \lambda_n)}{y_-^{[1]}(1, \lambda_n)},$$

if  $y_-^{[1]}(1, \lambda_n) \neq 0$ , and the required formulae follow. It remains to recall [1] that, for  $\lambda_n$  in the spectrum of  $\mathcal{A}$ , the equality  $y_-(1, \lambda_n) = 0$  holds if and only if  $\lambda_n$  is an eigenvalue of  $J$  and that  $y_-^{[1]}(1, \lambda_n) = 0$  if and only if  $\lambda_n$  is an eigenvalue of  $J_{(1)}$ .  $\square$

It is known (see [6] for the case  $q \in L_1(0, 1)$  and [1] for the case  $q \in W_2^{-1}(0, 1)$ ) that the eigenvalues  $(\lambda_n)$  and the corresponding norming constants  $(\alpha_n)$  determine the operator  $\mathcal{A}$  uniquely. Moreover, the cited papers give the algorithm of reconstruction of  $\mathcal{A}$  from these spectral data. The next proposition gives also the complete description of the spectral data, cf. [1, 6].

**Proposition 2.3.** *The eigenvalues  $(\lambda_n)$  of  $\mathcal{A}$  and the corresponding norming constants  $(\alpha_n)$  obey the asymptotics*

$$\lambda_n = [\pi(n - N) + \tilde{\lambda}_n]^2, \quad \alpha_n = 2 + \tilde{\alpha}_n,$$

where the sequences  $(\tilde{\lambda}_n)$  and  $(\tilde{\alpha}_n)$  belong to  $\ell_2$ .

Conversely, any sequences  $(\lambda_n)$  and  $(\alpha_n)$  of real numbers such that

- (a) the  $\lambda_n$  strictly increase and have the representation  $\lambda_n = [\pi(n - N) + \tilde{\lambda}_n]^2$  for some  $N \in \mathbb{N}$  and an  $\ell_2$ -sequence  $(\tilde{\lambda}_n)$ ;
- (b) the  $\alpha_n$  are positive and equal  $2 + \tilde{\alpha}_n$  for some  $\ell_2$ -sequence  $(\tilde{\alpha}_n)$

are the sequences of eigenvalues and the norming constants for a unique operator  $\mathcal{A}$  of the form (1.1).

In the following, we denote by  $\mu_{n,D}$  (resp.  $\mu_{n,N}$ ) the eigenvalues of the operator  $S_D$  (resp. of the operator  $S_N$ ), and by  $\nu_{1,J}, \dots, \nu_{N,J}$  (resp.  $\nu_1^1, \dots, \nu_{N-1}^1$ ) the eigenvalues of  $J$  (resp. of  $J_{(1)}$ ), all labeled in increasing order. It is well known that the operators  $S_D$  and  $S_N$  and the Jacobi matrices  $J$  and  $J_{(1)}$  have simple discrete spectra. We recall and derive next some properties of these spectra.

**Proposition 2.4.** ([10, 19, 20]). *There exist sequences  $(\tilde{\mu}_{n,D})$  and  $(\tilde{\mu}_{n,N})$  belonging to  $\ell_2(\mathbb{N})$  such that*

- (a)  $\mu_{n,D} = [\pi n + \tilde{\mu}_{n,D}]^2$ ;
- (b)  $\mu_{n,N} = [\pi(n - \frac{1}{2}) + \tilde{\mu}_{n,N}]^2$ .

We observe that the numbers  $\mu_{n,D}$  are zeros of the function  $y_-(1, \lambda)$  and  $\mu_{n,N}$ —those of  $y_-^{[1]}(1, \lambda)$ . Since both functions are exponential in  $\lambda$  of order  $\frac{1}{2}$ , they can be reconstructed from their zeros in the following way.

**Proposition 2.5.** ([11]). *The following equalities hold:*

$$y_-(1, \lambda) = \sqrt{\lambda} \prod_{k=1}^{\infty} \frac{\mu_{k,D} - \lambda}{\pi^2 k^2}, \quad y_-^{[1]}(1, \lambda) = \sqrt{\lambda} \prod_{k=1}^{\infty} \frac{\mu_{k,N} - \lambda}{\pi^2 (k - \frac{1}{2})^2}.$$

Simple considerations show that the functions  $y_+(1, \lambda)$  and  $y_+^{[1]}(1, \lambda)$  are related to the eigenvalues of  $J$  and  $J_{(1)}$  as follows.

**Lemma 2.6.** *The following equalities hold:*

$$y_+(1, \lambda) = \frac{a_0}{a_1 \cdots a_{N-1}} \prod_{k=1}^{N-1} (\lambda - \nu_k^1), \quad y_+^{[1]}(1, \lambda) = \frac{1}{a_0 a_1 \cdots a_{N-1}} \prod_{k=1}^N (\lambda - \nu_{k,J}).$$

*Proof.* To find the representation for  $y_+^{[1]}(1, \lambda)$ , it suffices to establish an analogous formula for  $d_+(1, \lambda)$ . Using the relation  $(J - \lambda)d_+(\cdot, \lambda) = -a_0 y_+^{[1]}(1, \lambda) e_1$  and the normalization  $d_+(N, \lambda) = 1$ , we find recursively that  $d_+(N - k, \lambda)$  is a polynomial in  $\lambda$  of degree  $k$  with leading coefficient  $(a_{N-1} \cdots a_{N-k})^{-1}$ . Therefore  $y_+^{[1]}(1, \lambda)$  is a polynomial in  $\lambda$  of degree  $N$  with leading coefficient  $(a_0 a_1 \cdots a_{N-1})^{-1}$ , and since it vanishes at the points  $\nu_{1,J}, \dots, \nu_{N,J}$ , the above formula follows.

Analogously  $y_+(1, \lambda) = a_0 d_+(1, \lambda)$  is a polynomial in  $\lambda$  of degree  $N - 1$  and with leading coefficient  $a_0/(a_1 \cdots a_{N-1})$ . Since  $d_+(1, \lambda)$  vanishes at the points  $\nu_1^1, \dots, \nu_{N-1}^1$ , the result follows.  $\square$

Finally, we find below an explicit expression for  $y_+(0, \lambda)$  in terms of the eigenvalues  $\lambda_n$  of the operator  $\mathcal{A}$ .

**Lemma 2.7.** *The following holds:*

$$y_+(0, \lambda) = -(a_0 a_1 \cdots a_{N-1})^{-1} \prod_{k=1}^N (\lambda - \lambda_k) \prod_{k=1}^{\infty} \frac{\lambda_{k+N} - \lambda}{\pi^2 k^2}.$$

*Proof.* In what follows,  $\lambda$  is an arbitrary nonzero complex number. We recall that  $y_+(x, \lambda)$  is a solution of the equation  $ly = \lambda y$  satisfying the terminal conditions  $y(1) = y_+(1, \lambda)$  and  $y^{[1]}(1) = y_+^{[1]}(1, \lambda)$ , whence

$$y_+(x, \lambda) = y_+(1, \lambda)u(x, \lambda) + y_+^{[1]}(1, \lambda)v(x, \lambda),$$

where  $u(\cdot, \lambda)$  and  $v(\cdot, \lambda)$  are solutions of the problems

$$\begin{cases} l_\sigma u = \lambda u, \\ u(1) = 1, \\ u^{[1]}(1) = 0, \end{cases} \quad \begin{cases} l_\sigma v = \lambda v, \\ v(1) = 0, \\ v^{[1]}(1) = 1. \end{cases}$$

Recalling [12] that  $u(x, \lambda)$  and  $v(x, \lambda)$  have the integral representations

$$\begin{aligned} u(x, \lambda) &= \cos \sqrt{\lambda}(x-1) + \int_x^1 k_1(x, t) \cos \sqrt{\lambda}(t-1) dt, \\ v(x, \lambda) &= \frac{\sin \sqrt{\lambda}(x-1)}{\sqrt{\lambda}} + \int_x^1 k_2(x, t) \frac{\sin \sqrt{\lambda}(t-1)}{\sqrt{\lambda}} dt \end{aligned}$$

for some upper-diagonal kernels  $k_j$  such that  $k(x, \cdot)$  belongs to  $L_2(0, 1)$  for every  $x \in [0, 1]$ , and that by Lemma 2.6  $y_+^{[1]}(1, \lambda)$  and  $y_+(1, \lambda)$  are polynomials in  $\lambda$  of degrees  $N$  and  $N - 1$  respectively, we find that

$$y_+(0, \lambda) = -\frac{\lambda^{N-\frac{1}{2}} \sin \sqrt{\lambda}}{a_0 a_1 \cdots a_{N-1}} [1 + o(1)]$$

as  $\lambda \rightarrow -\infty$ .

Since  $y_+(0, \lambda)$  is an entire function of  $\lambda$  of exponential type  $\frac{1}{2}$  and since its zeros coincide with the eigenvalues of  $\mathcal{A}$ , we conclude that

$$y_+(0, \lambda) = C_1 \prod_{k \in \mathbb{N}} \left(1 - \frac{\lambda}{\lambda_n}\right)$$

for some constant  $C_1 \in \mathbb{C}$ . Now we find that

$$\begin{aligned} -1 &= \lim_{\lambda \rightarrow -\infty} \frac{a_0 a_1 \cdots a_{N-1} y_+(0, \lambda)}{\lambda^{N-\frac{1}{2}} \sin \sqrt{\lambda}} \\ &= \lim_{\lambda \rightarrow -\infty} \frac{C_1 a_0 a_1 \cdots a_{N-1}}{\lambda^N} \prod_{k \in \mathbb{N}} \left(1 - \frac{\lambda}{\lambda_k}\right) / \prod_{k \in \mathbb{N}} \left(1 - \frac{\lambda}{\pi^2 k^2}\right) \\ &= \lim_{\lambda \rightarrow -\infty} \frac{C_1 a_0 a_1 \cdots a_{N-1}}{\lambda^N} \prod_{k=1}^N \left(1 - \frac{\lambda}{\lambda_k}\right) \prod_{k \in \mathbb{N}} \frac{\lambda_{k+N} - \lambda}{\lambda_{k+N}} \frac{\pi^2 k^2}{\pi^2 k^2 - \lambda} \\ &= (-1)^N \frac{C_1 a_0 a_1 \cdots a_{N-1}}{\lambda_1 \cdots \lambda_N} \prod_{k \in \mathbb{N}} \frac{\pi^2 k^2}{\lambda_{k+N}}, \end{aligned}$$

since the above products converge uniformly on  $\mathbb{C}$ , whence

$$\begin{aligned} y_+(0, \lambda) &= -\frac{(-1)^N \lambda_1 \cdots \lambda_N}{a_0 a_1 \cdots a_{N-1}} \prod_{k=1}^N \left(1 - \frac{\lambda}{\lambda_k}\right) \prod_{k=1}^{\infty} \frac{\lambda_{k+N} - \lambda}{\pi^2 k^2} \\ &= -(a_0 a_1 \cdots a_{N-1})^{-1} \prod_{k=1}^N (\lambda - \lambda_k) \prod_{k=1}^{\infty} \frac{\lambda_{k+N} - \lambda}{\pi^2 k^2}. \end{aligned}$$

The lemma is proved.  $\square$

We shall use several times the following statement about integral representations of some entire functions, cf. [15, Lemma 3.4.2].

**Proposition 2.8.** ([11]). *Assume that the numbers  $a_n$  and  $b_n$  are such that  $a_n = \pi n + \tilde{a}_n$  and  $b_n = \pi(n - \frac{1}{2}) + \tilde{b}_n$  with some  $\ell_2$ -sequences  $(\tilde{a}_n)$  and  $(\tilde{b}_n)$ . Put*

$$\phi(z) := \sqrt{z} \prod_{n \in \mathbb{N}} \frac{a_n^2 - z}{\pi^2 n^2}, \quad \psi(z) := \prod_{n \in \mathbb{N}} \frac{b_n^2 - z}{\pi^2 (n - \frac{1}{2})^2};$$

then there exist functions  $\tilde{\phi}$  and  $\tilde{\psi}$  in  $L_2(0, 1)$  such that

$$\phi(z) = \sin \sqrt{z} + \int_0^1 \tilde{\phi}(t) \sin \sqrt{zt} dt, \quad \psi(z) = \cos \sqrt{z} + \int_0^1 \tilde{\psi}(t) \cos \sqrt{zt} dt.$$

### 3. RECONSTRUCTION FROM $\mathcal{A}$ , $S_{\mathbb{D}}$ , AND $J_{(1)}$

Given an arbitrary operator matrix  $\mathcal{A}$  of the form (1.1), we denote by  $(\lambda_n)_{n \in \mathbb{N}}$ ,  $(\mu_{n, \mathbb{D}})_{n \in \mathbb{N}}$ , and  $(\nu_n^1)_{n=1}^{N-1}$  the eigenvalue sequences of  $\mathcal{A}$ , the operator  $S_{\mathbb{D}}$ , and the Jacobi matrix  $J_{(1)}$  respectively. Put also  $(\lambda'_n)_{n \in \mathbb{N}} := (\mu_{n, \mathbb{D}}) \amalg (\nu_n^1)_{n=1}^{N-1}$ , where the amalgamation operation  $\amalg$  was defined in the Introduction. An interesting property of the spectra involved is that every multiple element of  $(\lambda'_n)$  is an eigenvalue of  $\mathcal{A}$  and any eigenvalue of  $\mathcal{A}$  that belongs also to  $(\lambda'_n)$  occurs therein twice. In other words, the following statement holds true.

**Proposition 3.1.** ([1]).  $\sigma(\mathcal{A}) \cap \sigma(S_{\mathbb{D}}) = \sigma(\mathcal{A}) \cap \sigma(J_{(1)}) = \sigma(S_{\mathbb{D}}) \cap \sigma(J_{(1)})$ .

This allows us to establish the weak interlacing property of the sequences  $(\lambda_n)$  and  $(\lambda'_n)$  in the following sense.

**Lemma 3.2.** *The sequences  $(\lambda_n)$  and  $(\lambda'_n)$  weakly interlace, i.e.,  $\lambda_1 < \lambda'_1$  and for every  $n \in \mathbb{N}$  either  $\lambda'_n < \lambda_{n+1} < \lambda'_{n+1}$  or  $\lambda'_n = \lambda_{n+1} = \lambda'_{n+1}$ .*

*Proof.* Denote by  $\delta_1$  the Dirac delta-function at the point  $x = 1$  and put  $D_1 := (\delta_1, 0)^t$ . It is known that the domain of the Sturm–Liouville operator  $S$  is contained in  $W_2^1(0, 1)$ ; in particular, the functions  $y_{\pm}(\cdot, \lambda)$  belong to  $W_2^1(0, 1)$ . The explicit formula for the resolvent of the operator  $\mathcal{A}$  derived in Lemma 2.1 shows that the expression

$$f(\lambda) := \langle (\mathcal{A} - \lambda)^{-1} D_1, D_1 \rangle$$

makes sense for any  $\lambda$  not in the spectrum of  $\mathcal{A}$  and, moreover, that

$$f(\lambda) = \frac{y_+(1, \lambda) y_-(1, \lambda)}{W(\lambda)}.$$

Since  $f(\lambda)$  is a Nevanlinna function, its zeros and poles interlace. On the other hand, the zeros of  $f$  coincide with  $\lambda'_k$  and the poles with those  $\lambda_k$  which do not appear in  $(\lambda'_n)$ . In view of Proposition 3.1 and the known asymptotics of  $\lambda_n$  and  $\mu_{n, \mathbb{D}}$  this justifies the claim.  $\square$

The asymptotics of  $\lambda_n$  and  $\mu_{n, \mathbb{D}}$  shows that  $\lambda_{n+N} - \mu_{n, \mathbb{D}} = o(n)$  as  $n \rightarrow \infty$ . In fact, this result can be improved, cf. [5] for the case  $q \in L_1(0, 1)$ .

**Lemma 3.3.** *There exists an  $\ell_2$ -sequence  $(b_n)$  such that*

$$\lambda_{n+N} - \mu_{n,D} = 2a_0^2(1 + b_n).$$

*Proof.* It suffices to consider only large enough  $n$  such that  $\lambda_{n+N}$  is not in the spectrum of  $J_{(1)}$  and thus formula (2.3) for the norming constant  $\alpha_{n+N}$  holds. Using the representation of the functions  $y_+(0, \lambda)$  and  $y_+(1, \lambda)$ , we find that

$$(3.1) \quad \alpha_{n+N} = \frac{a_0^2 \prod_{k=1}^{N-1} (\lambda_{n+N} - \nu_k^1)}{y_-(1, \lambda_{n+N}) \prod_{k=1}^N (\lambda_{n+N} - \lambda_k)} \Big/ \frac{d}{d\lambda} \left( \sqrt{\lambda} \prod_{k \in \mathbb{N}} \frac{\lambda_{k+N} - \lambda}{\pi^2 k^2} \right) \Big|_{\lambda = \lambda_{n+N}}.$$

Due to the asymptotics of  $\lambda_n$  and Proposition 2.8 the function

$$\phi(\lambda) := \sqrt{\lambda} \prod_{k \in \mathbb{N}} \frac{\lambda_{k+N} - \lambda}{\pi^2 k^2}$$

can be represented in the form

$$\phi(\lambda) = \sin \sqrt{\lambda} + \int_0^1 f(t) \sin \sqrt{\lambda} t dt$$

for some  $f \in L_2(0, 1)$ , whence

$$\dot{\phi}(\lambda_{n+N}) = \frac{1}{2\sqrt{\lambda_{n+N}}} \left( \cos \sqrt{\lambda_{n+N}} + \int_0^1 t f(t) \cos \sqrt{\lambda_{n+N}} t dt \right).$$

By Propositions 2.5 and 2.8, there is also  $g \in L_2(0, 1)$  such that

$$\psi(\lambda) := y_-(1, \lambda) = \sin \sqrt{\lambda} + \int_0^1 g(t) \sin \sqrt{\lambda} t dt.$$

In view of the mean value theorem there are numbers  $\xi_n$  between  $\mu_{n,D}$  and  $\lambda_{n+N}$  such that

$$\begin{aligned} \psi(\lambda_{n+N}) &= (\lambda_{n+N} - \mu_{n,D}) \dot{\psi}(\xi_n) \\ &= \frac{\lambda_{n+N} - \mu_{n,D}}{2\sqrt{\xi_n}} \left( \cos \sqrt{\xi_n} + \int_0^1 t g(t) \cos \sqrt{\xi_n} t dt \right). \end{aligned}$$

Due to the asymptotics of  $\lambda_n$  and  $\xi_n$  the sequences  $(\cos \sqrt{\lambda_{n+N}} t)_{n \in \mathbb{N}}$  and  $(\cos \sqrt{\xi_n} t)_{n \in \mathbb{N}}$  form Riesz bases of  $L_2(0, 1)$  [9] and hence

$$\begin{aligned} \cos \sqrt{\lambda_{n+N}} + \int_0^1 t f(t) \cos \sqrt{\lambda_{n+N}} t dt &= (-1)^{n+N} (1 + c_n), \\ \cos \sqrt{\xi_n} + \int_0^1 t g(t) \cos \sqrt{\xi_n} t dt &= (-1)^{n+N} (1 + d_n) \end{aligned}$$

with square summable sequences  $(c_n)_{n \in \mathbb{N}}$  and  $(d_n)_{n \in \mathbb{N}}$ . Therefore (3.1) can be recast as

$$\alpha_{n+N} (1 + c_n) (1 + d_n) = \frac{4a_0^2}{\lambda_{n+N} - \mu_{n,D}} \frac{\sqrt{\lambda_{n+N} \xi_n}}{\lambda_{n+N} - \lambda_N} \prod_{k=1}^{N-1} \frac{\lambda_{n+N} - \nu_k^1}{\lambda_{n+N} - \lambda_k},$$

which, on account of the asymptotics of  $\alpha_n$  of Proposition 2.3, implies that

$$\frac{2a_0^2}{\lambda_{n+N} - \mu_{n,D}} = 1 + \hat{\alpha}_n, \quad (\hat{\alpha}_n) \in \ell_2,$$

and the result follows.  $\square$

**Definition 3.4.** We denote by  $\mathfrak{L}_N$  the set of all triples  $\Lambda := ((\lambda_n)_{n=1}^\infty, (\mu_n)_{n=1}^\infty, (\nu_n)_{n=1}^{N-1})$  of strictly monotone sequences such that the following holds:

- (1) there is an  $\ell_2$ -sequence  $(\tilde{\lambda}_n)$  such that  $\lambda_n = [\pi(n - N) + \tilde{\lambda}_n]^2$ ;
- (2) the sequences  $(\lambda_n)$  and  $(\lambda'_n) := (\mu_n) \amalg (\nu_n)$  weakly interlace in the sense of Lemma 3.2;
- (3) there exist  $\gamma_0 > 0$  and a sequence  $(\gamma_n) \in \ell_2$  such that  $\lambda_{k+N} - \mu_k = \gamma_0 + \gamma_k$ .

For a given  $\Lambda \in \mathfrak{L}_N$ , we denote by  $A_\Lambda$  the set of  $n \in \mathbb{N}$  such that  $\lambda_n = \lambda'_n$  and put  $B_\Lambda := \mathbb{N} \setminus A_\Lambda$ .



The results of this and the previous sections show that, for any operator  $\mathcal{A}$  of the form (1.1), the corresponding spectral triple  $((\lambda_n), (\mu_{n,D}), (\nu_n^1))$  forms an element of  $\mathfrak{L}_N$ . In the reverse direction, we shall prove that any element of  $\mathfrak{L}_N$  is the spectral triple of the above form.

**Theorem 3.5.** *For any  $\Lambda := ((\lambda_n), (\mu_n), (\nu_n)) \in \mathfrak{L}_N$  there exists an operator  $\mathcal{A}$  of the form (1.1) such that  $(\lambda_n)$ ,  $(\mu_n)$ , and  $(\nu_n)$  are the eigenvalues of the operators  $\mathcal{A}$ ,  $S_D$ , and  $J_{(1)}$  respectively. Such an operator  $\mathcal{A}$  is unique if and only if the set  $A_\Lambda$  is empty.*

*Proof.* We start with constructing the functions

$$\phi(z) := \sqrt{z} \prod_{n \in \mathbb{N}} \frac{\lambda_{n+N} - z}{\pi^2 n^2}, \quad \psi(z) := \sqrt{z} \prod_{n \in \mathbb{N}} \frac{\mu_n - z}{\pi^2 n^2},$$

and for  $n \in B_\Lambda$  put (cf. (3.1))

$$\beta_n := \frac{\gamma_0 \prod_{k=1}^{N-1} (\lambda_n - \nu_k)}{2\psi(\lambda_n)} \Big/ \frac{d}{d\lambda} \left( \phi(\lambda) \prod_{k=1}^N (\lambda - \lambda_k) \right) \Big|_{\lambda=\lambda_n}.$$

Due to the weak interlacing property of  $\Lambda$  the numbers  $\beta_n$  are positive and the proof of Lemma 3.3 shows that  $\beta_n = 2 + \tilde{\beta}_n$  for a sequence  $(\tilde{\beta}_n)$  belonging to  $\ell_2(B_\Lambda)$ .

Now we define the sequence  $(\alpha_n)$  with  $\alpha_n = \beta_n$  if  $n \in B_\Lambda$  and take  $\alpha_n$  to be an arbitrary positive number if  $n \in A_\Lambda$ . The sequences  $(\lambda_n)$  and  $(\alpha_n)$  satisfy all the requirements of Proposition 2.3 and thus there exists an operator  $\mathcal{A}$  of the form (1.1) whose eigenvalues and norming constants coincide respectively with  $(\lambda_n)$  and  $(\alpha_n)$ .

It remains to prove that the sequences  $(\mu_n)_{n \in \mathbb{N}}$  and  $(\nu_n)_{n=1}^{N-1}$  we have started with coincide with the eigenvalues  $(\mu_{n,D})_{n \in \mathbb{N}}$  and  $(\nu_n^1)_{n=1}^{N-1}$  of the related operator  $S_D$  and Jacobi matrix  $J_{(1)}$  respectively. Since for the norming constants  $\alpha_n$  with  $n \in B_\Lambda$  formula (2.3) holds, we conclude that, for such  $n$ ,

$$\frac{a_0^2 \prod_{k=1}^{N-1} (\lambda_n - \nu_k^1)}{y_-(1, \lambda_n)} = \frac{\gamma_0 \prod_{k=1}^{N-1} (\lambda_n - \nu_k)}{2\psi(\lambda_n)},$$

i.e., that

$$(3.2) \quad \frac{2a_0^2 \psi(\lambda_n)}{\sqrt{\lambda_n}} \prod_{k=1}^{N-1} (\lambda_n - \nu_k^1) - \frac{\gamma_0 y_-(1, \lambda_n)}{\sqrt{\lambda_n}} \prod_{k=1}^{N-1} (\lambda_n - \nu_k) = 0.$$

Recalling that  $\psi(\lambda_n) = 0 = \prod_{k=1}^{N-1} (\lambda_n - \nu_k)$  for  $n \in A_\Lambda$ , we conclude that equality (3.2) holds for all  $n \in \mathbb{N}$ . We observe that (3.2) takes the form  $\Phi(\lambda_n) = 0$ , where the function  $\Phi$  satisfies the relation

$$(3.3) \quad \Phi(z) = O(|z|^{N-3/2} e^{|\operatorname{Im} \sqrt{z}|})$$

as  $|z| \rightarrow \infty$ . We shall prove that  $\Phi \equiv 0$ .

Assume not, and observe that  $\Phi$  has then no zeros other than  $\lambda_n$ ,  $n \in \mathbb{N}$ . Indeed, in view of (3.3) Jensen's formula gives

$$(3.4) \quad \begin{aligned} \int_1^r \frac{n(t)}{t} dt &\leq \frac{1}{2\pi} \int_0^{2\pi} \log |\Phi(re^{i\theta})| d\theta + C_1 \\ &\leq (N - \frac{3}{2}) \log r + \frac{\sqrt{r}}{2\pi} \int_0^{2\pi} |\sin \theta/2| d\theta + C_2 \\ &= (N - \frac{3}{2}) \log r + \frac{2\sqrt{r}}{\pi} + C_2, \end{aligned}$$

where  $n(t)$  denotes the number of zeros of  $\Phi$  in the closed circle of radius  $t$  centered at the origin and  $C_1$  and  $C_2$  are some positive constants. On the other hand, if  $\Phi$  had at least one additional zero, then for any  $\varepsilon > 0$  and all sufficiently large  $t$  we would have

$$n(t) \geq \left[ \frac{\sqrt{t}}{\pi} - \varepsilon \right] + N + 1 \geq \frac{\sqrt{t}}{\pi} + N - \varepsilon,$$

which contradicts (3.4). Now  $\Phi$ , being of exponential type  $\frac{1}{2}$ , equals

$$\Phi(z) = C_3 \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right)$$

for some constant  $C_3$ . Using the canonical product for  $\sin \sqrt{z}$  and the asymptotics of  $\lambda_k$ , we conclude that

$$\lim_{z \rightarrow -\infty} \frac{\Phi(z)}{z^{N-\frac{1}{2}} \sin \sqrt{z}} =: C_4 \neq 0,$$

which contradicts (3.3).

Thus we have proved that  $\Phi \equiv 0$ , i.e., that

$$2a_0^2 \prod_{k \in \mathbb{N}} \frac{\mu_k - z}{\pi^2 k^2} \prod_{k=1}^{N-1} (z - \nu_k^1) \equiv \gamma_0 \prod_{k=1}^{\infty} \frac{\mu_{k,D} - z}{\pi^2 k^2} \prod_{k=1}^{N-1} (z - \nu_k).$$

It follows that every  $\nu_n$  that does not occur in  $(\mu_k)$  is an eigenvalue of  $J_{(1)}$  and, similarly, every  $\mu_n$  that does not occur in  $(\nu_k)$  is an eigenvalue of  $S_D$ . Since the sequences  $(\lambda_n)$  and  $(\mu_{n,D}) \amalg (\nu_n^1)$  weakly interlace in the sense of Lemma 3.2, and since the same is true of  $(\lambda_n)$  and  $(\lambda'_n)$ , simple considerations show that every multiple element of  $(\lambda'_n)$  belongs to the spectra of both  $S_D$  and  $J_{(1)}$ , cf. [13, Sect. 6]. Thus all  $\mu_n$  are eigenvalues of  $S_D$  and all  $\nu_n$ —those of  $J_{(1)}$ . Since neither  $J_{(1)}$  nor  $S_D$  can have other eigenvalues due to the size and asymptotics limitations respectively,  $\Lambda$  is the spectral triple for the operator  $\mathcal{A}$  found.

If the set  $A_\Lambda$  is empty, then the norming constants  $\alpha_n$  are uniquely determined by  $\Lambda$ , so that  $\mathcal{A}$  is unique in view of Proposition 2.3. If  $A_\Lambda$  is non-empty, then different choices of  $\alpha_n$  for  $n \in A_\Lambda$  lead to different operators  $\mathcal{A}$ . The proof is complete.  $\square$

*Remark 3.6.* It follows from the proof of Theorem 3.5 that the set of  $\Lambda$ -isospectral operators  $\mathcal{A}$  of the form (1.1) (i.e., the set of operators  $\mathcal{A}$  such that the spectra of  $\mathcal{A}$ ,  $S_D$ , and  $J_{(1)}$  form the prescribed triple  $\Lambda \in \mathfrak{L}_N$ ) is a manifold of dimension equal to the cardinality of the set  $A_\Lambda$ .

#### 4. RECONSTRUCTION FROM THE SPECTRA OF $\mathcal{A}$ , $S_N$ , AND $J$

Treatment of the inverse problem of reconstructing the operator  $\mathcal{A}$  from the spectra of the operators  $\mathcal{A}$ ,  $S_N$ , and the Jacobi matrix  $J$  parallels in general that of the inverse problem of Section 3. One essential difference is that the invariance of  $\mathcal{A}$  with respect to changing the primitive  $\sigma$  to  $\sigma + C$  and  $b_1$  to  $b_1 + a_0 C$  (mentioned in Remark 1.1) is important here as it changes the spectra of both the operator  $S_N$  and the Jacobi matrix  $J$ . Thus the more correct inverse problem should be not only to reconstruct the operator  $\mathcal{A}$  per se, but also to fix the appropriate quasi-derivative  $\sigma$  of the potential  $q$  and the corresponding Jacobi matrix  $J$ .

Given an arbitrary operator matrix  $\mathcal{A}$  of the form (1.1) (with fixed  $\sigma$ ), we denote by  $(\lambda_n)_{n \in \mathbb{N}}$ ,  $(\mu_{n,N})_{n \in \mathbb{N}}$ , and  $(\nu_{n,J})_{n=1}^N$  the eigenvalue sequences of  $\mathcal{A}$ ,  $S_N$ , and  $J$  respectively. Put also  $(\lambda'_n)_{n \in \mathbb{N}} := (\mu_{n,D}) \amalg (\nu_{n,J})_{n=1}^N$ . The above three spectra have the same intersection property as those of Section 3, namely

**Proposition 4.1.** ([1]).  $\sigma(\mathcal{A}) \cap \sigma(S_N) = \sigma(\mathcal{A}) \cap \sigma(J) = \sigma(S_N) \cap \sigma(J)$ .

**Lemma 4.2.** *The sequences  $(\lambda'_n)$  and  $(\lambda_n)$  weakly interlace, i.e., for every  $n \in \mathbb{N}$  either  $\lambda'_n < \lambda_n < \lambda'_{n+1}$  or  $\lambda'_n = \lambda_n = \lambda'_{n+1}$ .*

*Proof.* We observe that  $(\lambda'_n)$  is the sequence of eigenvalues of the operator  $\mathcal{A}_0 = S_N \oplus J$  counting multiplicities and that  $\mathcal{A}$  and  $\mathcal{A}_0$  are self-adjoint extensions of the symmetric operator  $\mathcal{A}'$ , which is the restriction of  $\mathcal{A}$  onto the domain

$$\mathcal{D}(\mathcal{A}') := \{(y, d)^t \in \mathcal{D}(\mathcal{A}) \mid y^{[1]}(1) = 0\}$$

and has deficiency indices  $(1, 1)$ . We denote by  $\mathcal{H}'$  a maximal subspace of  $\mathcal{D}(\mathcal{A}')$  that is invariant with respect to  $\mathcal{A}'$  and put  $\mathcal{H}'' := \mathcal{H} \ominus \mathcal{H}'$ . The restrictions of the operators  $\mathcal{A}$  and  $\mathcal{A}_0$  onto  $\mathcal{H}'$  coincide (with  $\mathcal{A}'$ ) and  $\dim \mathcal{H}' \leq N$  since if  $Y = (y, d)^t$  is an eigenvector of  $\mathcal{A}$  that belongs to  $\mathcal{H}'$ , then  $d$  is an eigenvector of  $J$ . It follows from [7] (see also [2, Ch. 1.2]) that the spectra of the restrictions of  $\mathcal{A}$  and  $\mathcal{A}_0$  onto the subspace  $\mathcal{H}''$  strictly interlace. Combining the two parts together, we see that either  $\lambda'_n \leq \lambda_n$  for all  $n \in \mathbb{N}$  or  $\lambda_n \leq \lambda'_n$  for all  $n \in \mathbb{N}$ ; however, the inequality  $\lambda_n \leq \lambda'_n$  is ruled out for all  $n$  sufficiently large by the asymptotics of  $\lambda_n$  and  $\mu_{n,N}$ , see Propositions 2.3 and 2.4. Taking into account the intersection property of Proposition 4.1, we conclude that the spectra weakly interlace in the specified sense.  $\square$

**Definition 4.3.** We denote by  $\mathfrak{L}'_N$  the set of all triples of strictly monotone sequences  $\Lambda := ((\lambda_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}}, (\nu_n)_{n=1}^N)$  satisfying the following properties:

- (1) there is an  $\ell_2$ -sequence  $(\tilde{\lambda}_n)$  such that  $\lambda_n = [\pi(n - N) + \tilde{\lambda}_n]^2$ ;
- (2) there is an  $\ell_2$ -sequence  $(\tilde{\mu}_n)$  such that  $\mu_n = [\pi(n - \frac{1}{2}) + \tilde{\mu}_n]^2$ ;
- (3) the sequences  $(\lambda_n)$  and  $(\lambda'_n) := (\mu_n) \amalg (\nu_n)$  weakly interlace in the sense of Lemma 4.2.

We denote by  $A_\Lambda$  the set of  $n \in \mathbb{N}$  such that  $\lambda_n = \lambda'_n$  and put  $B_\Lambda := \mathbb{N} \setminus A_\Lambda$ .

The results obtained so far show that, for any operator  $\mathcal{A}$  of the form (1.1), the corresponding spectral triple  $((\lambda_n), (\mu_{n,N}), (\nu_{n,j}))$  form an element of  $\mathfrak{L}'_N$ . In the reverse direction, we shall prove that any element of  $\mathfrak{L}'_N$  is the spectral triple of the above form. The approach lies in reducing the problem to that of reconstruction of  $\mathcal{A}$  from the eigenvalues and the norming constants. Lemmas 2.2, 2.6, and 2.7 imply that the three spectra determine uniquely the norming constants  $\alpha_n$  for  $n \in B_\Lambda$ . Hence, if a given triple  $\Lambda \in \mathfrak{L}'_N$  is composed of the spectra of some  $\mathcal{A}$  and its two parts, then the corresponding norming constants must be related to  $\Lambda$  via the formulae established in Section 2. As a preliminary, we show that any triple in  $\mathfrak{L}'_N$  produces in this way the numbers with correct asymptotics.

**Lemma 4.4.** *Assume that  $\Lambda = ((\lambda_n), (\mu_n), (\nu_n)) \in \mathfrak{L}'_N$  and define the functions  $\phi, \psi$ , and  $\chi$  by the formulae*

$$\begin{aligned} \phi(\lambda) &= \prod_{k=1}^N (\lambda - \lambda_k) \prod_{k=1}^{\infty} \frac{\lambda_{k+N} - \lambda}{\pi^2 k^2}, \\ \psi(\lambda) &= \sqrt{\lambda} \prod_{k=1}^{\infty} \frac{\mu_k - \lambda}{\pi^2 (k - \frac{1}{2})^2}, \\ \chi(\lambda) &= \prod_{k=1}^N (\lambda - \nu_k). \end{aligned}$$

Then the numbers

$$(4.1) \quad \beta_n := \frac{\chi(\lambda_n)}{\sqrt{\lambda_n} \phi(\lambda_n) \psi(\lambda_n)}, \quad n \in B_\Lambda,$$

have the asymptotics

$$\beta_n = 2 + \tilde{\beta}_n$$

where the sequence  $(\tilde{\beta}_n)$  belongs to  $\ell_2(B_\Lambda)$ .

*Proof.* It clearly suffices to prove that

$$\frac{1}{\beta_n} = \frac{\sqrt{\lambda_n} \phi(\lambda_n) \psi(\lambda_n)}{\chi(\lambda_n)} = \frac{1}{2} + \hat{\beta}_n$$

for some sequence  $(\hat{\beta}_n) \in \ell_2$ . In view of the asymptotics of  $(\lambda_k)$  and Proposition 2.8, there exists a function  $f \in L_2(0, 1)$  such that

$$\phi(\lambda) = \left( \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} + \int_0^1 f(t) \frac{\sin \sqrt{\lambda}t}{\sqrt{\lambda}} dt \right) \prod_{k=1}^N (\lambda - \lambda_k)$$

and thus, for  $n > N$ ,

$$\dot{\phi}(\lambda_n) = \frac{\cos \sqrt{\lambda_n} + \int_0^1 t f(t) \cos \sqrt{\lambda_n}t dt}{2\lambda_n} \prod_{k=1}^N (\lambda - \lambda_k).$$

Similarly, for some  $g \in L_2(0, 1)$  it holds

$$\psi(\lambda) = \sqrt{\lambda} \cos \sqrt{\lambda} + \sqrt{\lambda} \int_0^1 g(t) \cos \sqrt{\lambda}t dt.$$

Since the system  $\{\sin \sqrt{\lambda_n}t\}_{n>N}$  forms a Riesz basis of  $L_2(0, 1)$  [9], for any  $h \in L_2(0, 1)$  the sequence

$$\int_0^1 h(t) \cos \sqrt{\lambda_n}t dt, \quad n > N,$$

is square summable. The asymptotics of  $\lambda_n$  implies that  $\cos \sqrt{\lambda_n} = (-1)^{n+N}(1 + b_n)$ , where the sequence  $(b_n)$  is in  $\ell_2$ . Combining these relations, we arrive at the representation

$$\frac{\sqrt{\lambda_n} \dot{\phi}(\lambda_n) \psi(\lambda_n)}{\chi(\lambda_n)} = \frac{1}{2} (1 + d_n) \prod_{k=1}^N \frac{\lambda_n - \lambda_k}{\lambda_n - \nu_k}$$

with  $(d_n) \in \ell_2$ , which yields the result.  $\square$

**Theorem 4.5.** *For any  $\Lambda := ((\lambda_n), (\mu_n), (\nu_n)) \in \mathfrak{L}'_N$  there exist  $a_0 > 0$ , a function  $\sigma \in L_2(0, 1)$  and a Jacobi matrix  $J$  of size  $N$  such that  $(\lambda_n)$  is the spectrum of the corresponding operator  $\mathcal{A}$  in  $L_2(0, 1) \times \mathbb{C}^N$  of the form (1.1),  $(\mu_n)$  is the spectrum of the operator  $S_N$ , and  $(\nu_k)$  is the spectrum of the Jacobi matrix  $J$ . The operator  $\mathcal{A}$  is unique if and only if the set  $A_\Lambda$  is empty.*

*Proof.* We start with constructing the functions  $\phi, \psi$ , and  $\chi$  of Lemma 4.4 and defining the numbers  $\beta_n$  as in (4.1). Next, we put  $\alpha_n = \beta_n$  for  $n \in B_\Lambda$ , and take  $\alpha_n$  arbitrary positive for  $n \in A_\Lambda$ . According to Lemma 4.4,  $\alpha_n$  obey the asymptotics  $\alpha_n = 2 + \tilde{\alpha}_n$  with some  $(\tilde{\alpha}_n) \in \ell_2$ .

By Proposition 2.3, there exists an operator  $\mathcal{A}$  of the form (1.1), whose eigenvalues are  $\lambda_n$  and the corresponding norming constants are  $\alpha_n$ . We claim that one can fix a primitive of the potential  $q$  of the operator  $S$  and a Jacobi matrix  $J$  in the representation of  $\mathcal{A}$  in such a way that  $\mu_n$  are the eigenvalues of the operator  $S_N$  and  $\nu_n$  are the eigenvalues of  $J$ .

We take  $k^*$  such that  $\mu_{k^*}$  is not an eigenvalue of  $\mathcal{A}$  just found, fix the unique primitive  $\sigma$  of the potential  $q$  of the Sturm–Liouville operator  $S$  such that the relation  $(y'_- - \sigma y_-)(1, \mu_{k^*}) = 0$  holds, and determine the corresponding Jacobi matrix  $J$  giving the representation (1.1) of  $\mathcal{A}$ . We denote by  $\mu_{n,N}$  and  $\nu_{n,J}$  the eigenvalues of  $S_N$  and  $J$  and observe that the above choice of  $\sigma$  makes  $\mu_{k^*}$  an eigenvalue of  $S_N$ . Due to the construction of  $\beta_n$  and formula (2.2) for  $\alpha_n$ , we have the equality

$$\frac{\psi(\lambda_n)}{\chi(\lambda_n)} = \sqrt{\lambda_n} \prod_{k=1}^{\infty} \frac{\mu_{k,N} - \lambda_n}{\pi^2(k - \frac{1}{2})^2} \Big/ \prod_{k=1}^N (\lambda_n - \nu_{k,J})$$

for all  $n \in B_\Lambda$ . Recalling that  $\psi(\lambda_n) = \chi(\lambda_n) = 0$  for  $n \in A_\Lambda$ , we see that

$$\psi(\lambda_n) \prod_{k=1}^N (\lambda_n - \nu_{k,J}) = \sqrt{\lambda_n} \chi(\lambda_n) \prod_{k=1}^{\infty} \frac{\mu_{k,N} - \lambda_n}{\pi^2(k - \frac{1}{2})^2}$$

for all  $n \in \mathbb{N}$ .

Put

$$\Phi_1(z) := \frac{\psi(z)}{\sqrt{z}} \prod_{k=1}^N (z - \nu_{k,J}), \quad \Phi_2(z) := \chi(z) \prod_{k=1}^{\infty} \frac{\mu_{k,N} - z}{\pi^2(k - \frac{1}{2})^2};$$

then  $\Phi_1(\lambda_n) = \Phi_2(\lambda_n)$  for all  $n \in \mathbb{N}$ , and also  $\Phi_1(\mu_{k^*}) = \Phi_2(\mu_{k^*}) = 0$  (the latter relation follows from the fact that  $\mu_{k^*}$  is among  $\mu_{n,N}$  by the construction of  $S_N$ ). In view of Proposition 2.8 the functions  $\Phi_j$  have the form

$$\Phi_j(z) = p_j(z) \left( \cos \sqrt{z} + \int_0^1 g_j(t) \cos \sqrt{zt} dt \right)$$

for some monic polynomials  $p_j$  of degree  $N$  and some functions  $g_j \in L_2(0,1)$ ,  $j = 1, 2$ . It follows that  $\Phi := \Phi_1 - \Phi_2$  is an entire function of exponential type  $\frac{1}{2}$  with zeros  $\{\lambda_n\}_{n \in \mathbb{N}} \cup \{\mu_{k^*}\}$  such that

$$(4.2) \quad \Phi(z) = o(z^N e^{|\operatorname{Im} \sqrt{z}|})$$

as  $|z| \rightarrow \infty$ . Next we show as in the proof of Theorem 3.5 that  $\Phi \equiv 0$  by noticing that otherwise  $\Phi$  would have no zeros other than  $\lambda_n$ ,  $n \in \mathbb{N}$ , and  $\mu_{k^*}$ , and that the canonical product for  $\Phi$  then would contradict the estimate (4.2).

Thus  $\Phi_1 \equiv \Phi_2$ , which together with the weak interlacing property of  $(\lambda_n)$  and  $(\lambda'_n)$  as well as of  $(\lambda_n)$  and  $(\mu_{n,N}) \amalg (\nu_{n,J})$  shows that  $\mu_n = \mu_{n,N}$  for all  $n \in \mathbb{N}$  and that  $\nu_k = \nu_{k,J}$  for  $k = 1, \dots, N$ . Uniqueness statement follows from Proposition 2.3, and the proof is complete.  $\square$

We remark that the set of  $\Lambda$ -isospectral operators  $\mathcal{A}$  is again a manifold of dimension equal to the cardinality of the set  $A_\Lambda$ .

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