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THE INVESTIGATION OF A GENERALIZED MOMENT PROBLEM ASSOCIATED WITH CORRELATION MEASURES

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This paper is dedicated to great mathematicians M. Krein and I. Gel'fand whose papers have made it possible for this article to be written.

Abstract. The classical power moment problem can be viewed as a theory of spectral representations of a positive functional on some classical commutative algebra with involution. We generalize this approach to the case where the algebra is a special commutative algebra of functions on the space of multiple finite configurations.

If the above-mentioned functional is generated by a measure on the space of usual finite configurations then this measure is a correlation measure for a probability spectral measure on the space of infinite configurations. The latter measure is practically arbitrary, so that we have a connection between this complicated measure and its correlation measure defined on more simple objects that are finite configurations. The paper gives an answer to the following question: when this latter measure is a correlation measure for a complicated measure on infinite configurations? (Such measures are essential objects of statistical mechanics).

0. Introduction

A vast number of works was devoted to studying and developing the classical moment problem, starting from pioneer works of T. Stieltjes (the end of XIX century). Recall that the classical moment problem deals with a possibility to represent a given sequence $r = (r_n)_{n=0}^\infty$ of real numbers r_n in the following form:

$$r_n = \int_{\mathbb{R}} \lambda^n d\mu(\lambda), \quad n \in \mathbb{N}_0 := \{0, 1, \dots\}, \tag{0.1}$$

where μ is a Borel measure on the axis $\mathbb{R}$. The answer is very simple: the representation for given $r = (r_n)_{n=0}^\infty$ takes place iff the following positivity condition holds:

$$\sum_{j,k=0}^{\infty} r_{j+k} \xi_j \overline{\xi_k} \geq 0 \tag{0.2}$$

for an arbitrary finite sequence of complex numbers $(\xi_n)_{n=0}^\infty$.

A number amount of works was devoted to the question about possibility to replace the index n in (0.1) with a multi-index $n = (n_1, \dots, n_p)$, λ^n with $\lambda_1^{n_1} \dots \lambda_p^{n_p}$, and $\mathbb{R}$ with $\mathbb{R}^p$, where $p \leq \infty$, which would make this a multidimensional moment problem; to replace r_n with $r_{n,m}$, $n, m \in \mathbb{N}_0$, $\mathbb{R}$ with $\mathbb{C}$, and λ^n with $z^n \overline{z}^m$ to obtain a complex moment problem, in particular, a trigonometric moment problem, etc. For us it will be essential to generalize the problem where one replaces λ^n in (0.1) with eigenvectors $\varphi_n(\lambda)$ of some differential, difference, or a more general operator A, or with a commuting family of such type operators (note that $\varphi(\lambda) = (1, \lambda, \lambda^2, \dots)$ is a generalized eigenvector of a trivial shift operator).

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On the other hand, the spectral theory of classical Jacobi matrices and the theory of orthogonal polynomials are deeply connected with the classical moment problem. Different generalizations of Jacobi matrices and orthogonal polynomials are connected with the above-mentioned generalizations of the moment problem.

This part of the spectral theory is described, for example, in [1, 4, 21, 12, 8, 7, 45, 10, 11] and the references cited there.

This paper is also connected, on the one hand, with the moment problem (0.1) and, on the other hand, with some tasks of mathematical statistical physics. Formally speaking, we construct a representation of type (0.1) but for a commuting family A of selfadjoint operators acting in a Hilbert space constructed by using a positive functional r (instead of a sequence $(r_n)_{n=0}^{\infty}$). These operators are generated by the standard creation and neutral operators from the Fock space theory.

In the simplest (and the most essential) case this functional r has the form

$$r(f) = \int\limits_{\Gamma_{X,0}} f(\xi)\, d\rho(\xi), \tag{0.3}$$

where $\Gamma_{X,0}$ is the space of all usual finite configurations $\xi = (x_1, \dots, x_n)$ ($n \in \mathbb{N}$ is arbitrary) consisting of from distinct points x_j belonging to a Riemann manifold X. If the functional r from (0.3) is positive definite in the sense of type (0.2) (but connected with a so-called Yu. G. Kondratiev–T. Kuna convolution [28, 29, 33]) and the measure ρ on $\Gamma_{X,0}$ has a certain growth behavior for $x_j \to \infty$, $j \in \mathbb{N}$, then there exists a representation of type (0.1) with some joint generalized eigenvectors of the family A and a Borel probability measure μ on the space Γ_X.

It is possible to count these eigenvectors. But it is very interesting and unexpected that the corresponding unitary Fourier transform I, acting from the linear space of functions on the space $\Gamma_{X,0}$ of finite configurations onto the linear space of functions on the space Γ_X of infinite configurations $\gamma = (x_1, x_2, \dots)$ ($x_j \in X$ are distinct), has a very simple and well-known form, $I = K$, where K is the classical A. Lenard transform introduced very easily [34, 35, 36].

So, the Parseval equality (i. e., the generalization of representation (0.1)) has now the following form: for a sufficiently broad space $\mathrm{Fun}(\Gamma_{X,0})$ of functions $f(\xi)$, $\xi \in \Gamma_{X,0}$,

$$\int\limits_{\Gamma_{X,0}} f(\xi)\, d\rho(\xi) = r(f) = \int\limits_{\Gamma_X} (Kf)(\eta)\, d\mu(\eta). \tag{0.4}$$

Equality (0.4) shows the following. Introduce the pairing between functions and measures in the usual manner via integration, and denote by $*$ the corresponding operation of adjointness. Then (0.4) means

$$\rho = K^* \mu\,. \tag{0.5}$$

Summarizing we can assert that for a measure μ on the space Γ_X of infinite configurations we can construct a measure ρ on the more simple space $\Gamma_{X,0}$ of finite configurations by applying to μ the operator adjoint to the Lenard operator K. Vice versa, every measure μ on Γ_X can be constructed from a measure ρ on $\Gamma_{X,0}$ by taking $\mu = K\rho$ (recall that the operator $I = K$ is invertible, moreover, it is unitary if regarded as an operator between the corresponding Hilbert spaces).

The above reasoning and the conditions on ρ given by representation (0.4) (the corresponding positiveness of ρ of type (0.2) given by (0.3) and the behavior of ρ at infinity) have some relation to statistical mechanics. Namely, the time behavior of a statistical system is described by a complicated differential equation

$$(\mathcal{L}M)(\gamma) = F(\gamma) \tag{0.6}$$

satisfied by functions $M(\gamma)$ on the space Γ_X of infinite configurations with a standard and fixed measure $d\gamma$. Introducing the measure $d\mu(\gamma) = M(\gamma)\, d\gamma$ we can describe the behavior of the statistical system by a corresponding variant of equation (0.6) for the measure μ.

But in statistical mechanics, as a rule (see, e. g., [22, 44, 42]), it is acceptable to pass from (0.6) to corresponding equations for correlation functions $R(\xi)$ for $M(\gamma)$. These functions are defined on the more simple space $\Gamma_{X,0}$ of ordinary finite configurations; so, these functions have a more simple structure, it is possible to understand every such function as an infinite sequence of functions $f_n(x_1, \ldots, x_n)$, $n \in \mathbb{N}_0$ ($f_0 \in \mathbb{C}$).

The equation of type (0.6) for $R(\xi)$ (more exactly, the infinite system of equations for f_n) is more simple, and in some cases it is possible to find the corresponding solution (or to investigate it to a sufficient extend).

But here there is the following problem. If we introduce the measure $d\rho(\xi) = R(\xi)d\xi$, where $d\xi$ is the Lebesgue (or Riemann) measure with $\xi = (x_1, \ldots, x_n)$, then the measures ρ and μ are connected by equality (0.5) being the definition of correlation functions. In reality, physically interesting is the solution $M(\gamma)$ of equation (0.6) rather than the solution $R(\xi)$ of the auxiliary equation for the correlation function. So, for this solution $R(\xi)$ or for the measure ρ it is necessary to check additionally that representation (0.4) holds, i. e., the required positive definiteness and behavior at infinity take place.

We stress once more what this situation means in the simplest case of the classical moment problem (0.1). We have an equation for the measure μ in (0.1) and we have to find its solution μ^0. We can rewrite this equation in terms of the corresponding moments $r = (r_n)_{n=0}^{\infty}$ (assuming that this is possible), and we can find a solution $r^0 = (r_n^0)_{n=0}^{\infty}$ of the last equation for the moments. But it is not evident that the latter sequence r^0 is a moment sequence: to claim this it is necessary to check its positivity (0.2). Therefore, generally speaking, the question about finding a solution of the equation for measure μ is open.

This paper is organized in the following manner. In Section 1 we introduce basic spaces $\ddot{\Gamma}_{X,0}$ ($\Gamma_{X,0}$) of multiple (usual) configurations. These spaces, as well as probability measures on them, appear naturally in several topics of mathematics and physics. Let us only mention the theory of point processes [27, 26]; a modern account of these objects can be found in [33, 41]. The notion of Yu. G. Kondratiev–T. Kuna convolution [28, 29, 33], which is essential for us, is also introduced in this section.

The main mathematical instruments yielding the results of this paper are given in Sections 2 and 3. They consist in using the spectral theory of selfadjoint operators and commuting families of them, together with the theory of their generalized eigenfunctions for integral representations of moments, positive definite functions and kernels, etc. Here we only note that this method goes back to old works of M. G. Krein [31, 32]. Later, in 1956, Yu. M. Berezansky [3] has developed these works, by using the results about generalized eigenvectors of I. M. Gelfand and A. G. Kostyuchenko [24] and of Yu. M. Berezansky [2], to a representation of positively definite kernels. The corresponding theory and its different generalizations can be found in many works, see the books [4, 25, 38, 39, 23, 12] containing references to corresponding articles; see also [14, 15, 16, 37]. Note that the measure μ in representations of type (0.4) is the spectral measure of a corresponding family A of operators.

Section 4 is devoted to a calculation of the Fourier transform in a sufficiently general case, where the functional r is generated by a measure ρ on $\ddot{\Gamma}_{X,0}$, i. e., is of type (0.3) but with integration over the space $\ddot{\Gamma}_{X,0}$ of multiple configurations (recall that $\ddot{\Gamma}_{X,0} \supset \Gamma_{X,0}$). Here we have used some results from [4, 30, 33, 41] and the theory of generalized functions of infinitely many variables related to generalized translation operators, developed by Yu. M. Berezansky, Yu. G. Kondratiev, and V. A. Tesko ; refs. [19, 20] give a survey of these results. Note that in this section the following question is also investigated: in the general case of a positive functional r, the spectral measure μ in a representation of type (0.3) is given on the space $\mathcal{D}'$ of generalized functions generated by the classical space $\mathcal{D} = \mathcal{D}(X)$ of test functions. But for the case of integration over $\ddot{\Gamma}_{X,0}$, the spectral measure μ is concentrated on charges on the space X (of course, these charges belong to the space $\mathcal{D}'$). Some converse results also hold.

Section 5 is devoted to introducing and investigating a so-called K-transform being a very simple and natural linear operator mapping functions on the space $\Gamma_{X,0}$ of the usual configurations onto the space Γ_X of infinite configurations. This K-transform was introduced by A. Lenard [34, 35, 36] and investigated in these papers and in [28, 29, 33, 41].

In the last Section 6 we prove that under some restriction on the growth of the measure ρ in the integral (0.1) at infinity, the essential representation (0.4) holds. Namely, in this case one can talk about a connection between the initial measure μ and its correlation measure ρ (see above).

Note that the first results about representation of type (0.4) were described in [28, 29, 33] by using the R. Minlos theorem concerning positive definite functions of infinite many variables [40]. A general approach to this and the above mentioned problems, using generalized eigenvector expansion, was given in [13, 9]. This approach has reduced to more general and precise results.

This paper is a complete exposition, with proofs and more precise formulations, of the short article of Yu. M. Berezansky [9] devoted to the 90th birthday of I. M. Gelfand.

1. Initial spaces. The Kondratiev–Kuna convolution

Let X be a connected oriented C^∞ (non-compact) Riemann manifold. We denote by $\mathcal{D}$ the classic space $C^\infty_{\text{fin}}(X)$ of all functions which are real, infinitely differentiable on X, and finite (i. e., with a compact support); a corresponding topology turns $\mathcal{D}$ into a nuclear space (see, e. g., [18, 30, 17]). We will always denote the complexification of a real space with the subscript c. So, $\mathcal{D}_c$ is the complexification of $\mathcal{D}$.

Let $\mathcal{F}_0(\mathcal{D}) := \mathbb{C}$ and $\mathcal{F}_n(\mathcal{D}) := \mathcal{D}_c^{\widehat{\otimes} n}$, $n \in \mathbb{N}$, where $\widehat{\otimes}$ denotes the symmetric tensor product; $\mathcal{D}_c^{\widehat{\otimes} n}$ is equal to the space of all complex, symmetric, finite, C^∞-functions on X^n. Then we construct a Fock-type space. Namely, let

$$\mathcal{F}_{\text{fin}}(\mathcal{D}) := \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\mathcal{D}) \tag{1.1}$$

be the topological direct sum of the spaces $\mathcal{F}_n(\mathcal{D})$, where every element $f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$ is a *finite* sequence $f = (f_n)_{n=0}^\infty$, where $f_n \in \mathcal{F}_n(\mathcal{D})$. The topology of $\mathcal{F}_{\text{fin}}(\mathcal{D})$ is given by the following convergence: $\mathcal{F}_{\text{fin}}(\mathcal{D}) \ni f^{(m)} = (f_n^{(m)})_{n=0}^\infty \to f = (f_n)_{n=0}^\infty \in \mathcal{F}_{\text{fin}}(\mathcal{D})$, $m \to \infty$, where $f^{(m)}$ are uniformly finite with respect to (w. r. t.) n (i. e., there exists $k \in \mathbb{N}$ such that $f_n^{(m)} = 0$ if $n > k$, $m \in \mathbb{N}$) and, for every $n \in \mathbb{Z}_+$, $f_n^{(m)} \to f_n$ in the topology of the space $\mathcal{F}_n(\mathcal{D})$. Note that the linear topological space $\mathcal{F}_{\text{fin}}(\mathcal{D})$ is nuclear, being a topological direct sum of nuclear spaces (see, e. g., [12, 18]).

It will be convenient for us to interpret elements of the space (1.1) as functions on a certain set: the set of multiple configurations. Namely, a multiple configuration of order $n \in \mathbb{N}$ is, by definition, a (non-ordered) set $\xi_n = [x_1, \ldots, x_n]$ of points $x_1, \ldots, x_n \in X$ (equal points can be among them). In other words, the set $\ddot{\Gamma}_X^{(n)}$ of such configurations is equal to the factor space X^n/S_n, where S_n is the group of all permutations of the set $\{1, \ldots, n\}$ which acts, in a natural way, on points $(x_1, \ldots, x_n) \in X^n$. The topology of $\ddot{\Gamma}_X^{(n)}$ is generated by that of X^n as the factor topology.

Put $\ddot{\Gamma}_X^{(0)} := \{\emptyset\}$ and construct the following set:

$$\ddot{\Gamma}_{X,0} := \bigsqcup_{n=0}^{\infty} \ddot{\Gamma}_X^{(n)}. \tag{1.2}$$

So, every element of $\ddot{\Gamma}_{X,0}$ is a multiple configuration in any natural order or the empty set (a "configuration" of zero points).

Let $\Lambda \Subset X$ be a subset of X, and let the topology of Λ be induced by that of X. Replacing above X with Λ we get the spaces $\ddot{\Gamma}_\Lambda^{(n)}$ and $\ddot{\Gamma}_{\Lambda,0}$ of multiple configurations.

Below we will also need the space $\Gamma_{X,0}$ of usual configurations. It is defined similarly to (1.2) as

$$\Gamma_{X,0} := \bigsqcup_{n=0}^{\infty} \Gamma_X^{(n)}, \tag{1.3}$$

where, for every $n \in \mathbb{N}$, $\Gamma_X^{(n)}$ is a set of all $\xi_n = [x_1, \dots, x_n] \in \ddot{\Gamma}_X^{(n)}$ such that all x_j are *distinct*, $\Gamma_X^{(0)} := \{\emptyset\}$. So, $\Gamma_{X,0} \subset \ddot{\Gamma}_{X,0}$. The spaces $\Gamma_\Lambda^{(n)}$ and $\Gamma_{\Lambda,0}$ are introduced in a similar way.

Now it is easy to understand that elements of the space (1.1) can be treated as functions on the space $\ddot{\Gamma}_{X,0}$, i. e., one can embed $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ into the space $\mathrm{Fun}(\ddot{\Gamma}_{X,0})$ of all complex-valued functions on $\ddot{\Gamma}_{X,0}$. Namely, let $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$. Every f_n is a symmetric, smooth, finite, complex-valued function on X^n, $f_n(x_1, \dots, x_n)$. Its symmetry allows us to treat it as a function of a point $[x_1, \dots, x_n]$, $f_n(x_1, \dots, x_n) = f_n([x_1, \dots, x_n])$. Then one can embed $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ into $\mathrm{Fun}(\ddot{\Gamma}_{X,0})$ due to (1.1) and (1.2).

Of course, from this point of view, $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ is only a part of $\mathrm{Fun}(\ddot{\Gamma}_{X,0})$; we will not need an explicit description of this part (although one can get it). Note also that some vectors from $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ can be also treated as a function on $\Gamma_{X,0} \subset \ddot{\Gamma}_{X,0}$. This vector has to be such that $f_n(x_1, \dots, x_n) = 0$ if, for at least one pair of numbers j and k, $x_j = x_k$. Then one can consider that f is a function on $\Gamma_{X,0}$ which was extended to all $\ddot{\Gamma}_{X,0}$ by setting zero on $\ddot{\Gamma}_{X,0} \setminus \Gamma_{X,0}$.

Let us pass to a definition of the main notion of this paper, namely, a convolution introduced by Yu. G. Kondratiev and T. Kuna [28, 29, 33], see also [13, 41]. This convolution $\star$ acts in the space $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ and turns it into a commutative topological nuclear algebra, but it is convenient to define $\star$ treating $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ as a part of $\mathrm{Fun}(\ddot{\Gamma}_{X,0})$.

Everywhere below we will use the same letter for an element from $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ and for its interpretation as a function on $\ddot{\Gamma}_{X,0}$. So,

$$\mathcal{F}_{\mathrm{fin}}(\mathcal{D}) \ni f = (f_n)_{n=0}^{\infty} = f(\xi), \quad \xi \in \ddot{\Gamma}_{X,0}, \quad f_n = f \restriction \ddot{\Gamma}_X^{(n)}. \tag{1.4}$$

For every $f, g \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ we define the convolution $\star$ by the formula

$$\forall \xi \in \ddot{\Gamma}_{X,0} \quad (f \star g)(\xi) = \sum_{\xi' \cup \xi'' \cup \xi''' = \xi} f(\xi' \cup \xi'') g(\xi'' \cup \xi'''), \tag{1.5}$$

where the sum is taken over all representations of the configuration ξ (belonging to the space $\ddot{\Gamma}_X^{(n)}$) as a set sum of three *disjoint* configurations ξ', ξ'', ξ''' ($\emptyset$ can be present among them).

The sum in (1.5) is finite. It is known that $f \star g$ is also belongs to $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$; the convolution $\star$ is commutative, associative, additive, homogeneous, and continuous w. r. t. both variables (see [28, 29, 33] and also [13, 41]). So, $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ with $\star$ is a commutative topological nuclear algebra $\mathcal{A}$. This algebra has a unit e, $e(\xi) = 1$ if $\xi = \emptyset$ and $e(\xi) = 0$ if $\ddot{\Gamma}_{X,0} \ni \xi \neq \emptyset$.

2. Positive functionals and a construction of a family of commutative selfadjoint operators

In the algebra $\mathcal{A} = \mathcal{F}_{\text{fin}}(\mathcal{D})$ one introduces a natural involution, $\mathcal{A} \ni f = f(\xi) \mapsto \overline{f} := \overline{f(\xi)} \in \mathcal{A}$. Note that, obviously, due to (1.5), $\forall f, g \in \mathcal{A}\ \ \overline{f} \star \overline{g} = \overline{f \star g}$. A continuous linear functional $r \in \mathcal{A}' = \mathcal{F}'_{\text{fin}}(\mathcal{D})$ is called positive if

$$\forall f \in \mathcal{A} \quad r(f \star \overline{f}) \geq 0. \tag{2.1}$$

Any positive functional r generates the following quasi-scalar product $(\cdot, \cdot)_{\mathcal{A}_r}$ in $\mathcal{A}$:

$$\forall f, g \in \mathcal{A} \quad (f, g)_{\mathcal{A}_r} := r(f \star \overline{g}). \tag{2.2}$$

Identifying every $f \in \mathcal{A}$ such that $r(f \star \overline{f}) = 0$ with zero, considering the corresponding classes of $f \in \mathcal{A}$, and completing the space of these classes, we construct a Hilbert space $\mathcal{H}_r$. Let $\{f\}$ be the class containing $f \in \mathcal{A}$, and let $\{\mathcal{A}\}$ be the space of all such classes. Then $\{\mathcal{A}\} \subset \mathcal{H}_r$ and $\{\mathcal{A}\}$ is dense in $\mathcal{H}_r$.

An important example of a positive functional r is the one generated by a σ-finite Borel measure ρ on the space $\ddot{\Gamma}_{X,0}$ (note that a σ-finite measure is a measure which is finite on every compact subset of $\ddot{\Gamma}_X^{(n)}$, $n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$)

$$\forall f \in \mathcal{A} \quad r(f) = \int\limits_{\ddot{\Gamma}_{X,0}} f(\xi)\, d\rho(\xi) = \sum_{n=0}^{\infty} \int\limits_{\ddot{\Gamma}_X^{(n)}} f(\xi)\, d\rho(\xi). \tag{2.3}$$

Convergence in $\mathcal{A} = \mathcal{F}_{\text{fin}}(\mathcal{D})$ is such that this functional is continuous. Since it is positive, the following inequality holds:

$$\forall f \in \mathcal{A} \quad r(f \star \overline{f}) = \int\limits_{\ddot{\Gamma}_{X,0}} (f \star \overline{f})(\xi)\, d\rho(\xi) \geq 0. \tag{2.4}$$

Remark 2.1. It is useful to consider *positive* functionals r of kind (2.3), where ρ is a finite, complex-valued Borel measure on $\ddot{\Gamma}_{X,0}$ (a charge). We will refer to such functionals as ones generated by charges.

The main aim of this paper is to construct an integral representation of the positive functional r using joint generalized eigenfunctions of a family of selfadjoint commuting operators acting in the Hilbert space $\mathcal{H}_r$ and generated by the convolution $\star$; see Theorem 3.1 and (3.17), (3.18).

Operators of this family are introduced as follows. One can consider a function $\varphi \in \mathcal{D} \subset \mathcal{F}_1(\mathcal{D})$ as a (real-valued) function of a point $\xi \in \ddot{\Gamma}_X^{(1)} \subset \ddot{\Gamma}_{X,0}$, belonging to the algebra $\mathcal{A}$. The operation $\mathcal{A} \ni f \mapsto \varphi \star f \in \mathcal{A}$ is Hermitian in the quasi-scalar product (2.2),

$$\forall f, g \in \mathcal{A} \quad (\varphi \star f,\ g)_{\mathcal{A}_r} = r(\varphi \star f \star \overline{g}) = r(f \star \overline{(\varphi \star g)}) = (f,\ \varphi \star g)_{\mathcal{A}_r}.$$

Therefore (see, e. g., [4, 12]) this operation can be considered as acting in the set of the corresponding classes, $\{\mathcal{A}\} \ni \{f\} \mapsto \{\varphi \star f\} \in \{\mathcal{A}\}$. So, we have introduced a Hermitian operator $A(\varphi)$ defined densely in $\mathcal{H}_r$,

$$\forall f \in \mathcal{A} \quad \mathrm{Dom}(A(\varphi)) = \{\mathcal{A}\} \ni \{f\} \mapsto A(\varphi)\{f\} := \{\varphi \star f\} \in \{\mathcal{A}\}. \tag{2.5}$$

Any two such operators $A(\varphi)$, $A(\psi)$ ($\varphi, \psi \in \mathcal{D}$) commute formally, $A(\varphi)\{\mathcal{A}\} \subset \{\mathcal{A}\} = \mathrm{Dom}(A(\psi))$, $A(\psi)\{\mathcal{A}\} \subset \{\mathcal{A}\} = \mathrm{Dom}(A(\varphi))$, and for every $\{f\} \in \{\mathcal{A}\}$, according to (2.5),

$$A(\varphi)A(\psi)\{f\} = A(\varphi)\{\psi \star f\} = \{\varphi \star \psi \star f\} = \{\psi \star \varphi \star f\} = A(\psi)A(\varphi)\{f\}.$$

Then how to check whether the set of all closures $\tilde{A}(\varphi)$ of $A(\varphi)$ is a family of selfadjoint (strongly) commuting operators? Now a sufficient condition for this fact is the following (see [12], Ch. 5, Theorem 1.15, [5, 18]): there exists $z \in \mathbb{C} \setminus \mathbb{R}$ such that for each $\varphi, \psi \in \mathcal{D}$ there exists a total set of vectors which are quasi-analytic for the operators $A(\varphi)$, $A(\psi)$, $A(\varphi) \restriction (A(\psi) - z\mathbf{1})\{\mathcal{A}\}$. (Recall that, for an operator A acting in a Hilbert space H, a vector $f \in H$ is called quasi-analytic if $f \in \bigcap_{n=1}^{\infty} \mathrm{Dom}(A^n)$ and the class $C\{m_n\}$ with $m_n = \|A^n f\|_H$ is quasi-analytic.)

According to (2.5), for every $\varphi \in \mathcal{D}$, $\{\mathcal{A}\} = \bigcap_{n=1}^{\infty} \mathrm{Dom}((A(\varphi))^n)$. In what follows we demand the following condition to hold:

(i) *There exists a linear set $D \subset \{\mathcal{A}\}$ such that: 1) D is invariant w. r. t. every operator $A(\varphi)$ ($\varphi \in \mathcal{D}$); 2) D is total in $\mathcal{H}_r$; 3) every vector $\{f\} \in D$ is quasi-analytic for every operator $A(\varphi)$ ($\varphi \in \mathcal{D}$), i. e., the class*

$$C\{\|(A(\varphi))^n\{f\}\|_{\mathcal{H}_r}\} \tag{2.6}$$

is quasi-analytic.

It follows from condition (i) that $(\tilde{A}(\varphi))_{\varphi\in\mathcal{D}}$ is a family of selfadjoint commuting operators. Indeed, due to the fact that D is invariant for $A(\psi)$ ($\psi \in D$) we have $(A(\psi) - z\mathbf{1})D \subset D$ for $\operatorname{Im} z \neq 0$, and thus the condition (2.6) provides the conditions of the above mentioned theorem from [12].

For some positive functionals r it is possible to formulate more explicit conditions that imply condition (i). We will consider only the most important case where the functional r is generated by a measure ρ according to (2.3).

Theorem 2.1. *Let for a measure ρ from (2.3) the following condition hold: for every compact $\Lambda \subset X$ and for every $k \in \mathbb{N}$, the class $C\{m_n\}$, where*

$$\forall n \in \mathbb{N}_0 \quad m_n = \left(\sum_{\ell=0}^{2k} \left(\frac{(\ell+2n)!}{\ell!} \sum_{j=0}^{2n} \rho\left(\ddot{\Gamma}_{\Lambda}^{(\ell+j)}\right) \right) \right)^{\frac{1}{2}}, \tag{2.7}$$

is quasi-analytic. Then (i) *is true and $(\tilde{A}(\varphi))_{\varphi\in\mathcal{D}}$ is a family of selfadjoint commuting operators in the space $\mathcal{H}_r$.*

Proof. Note at first that the convolution (1.5) is well defined for all complex-valued functions $f(\xi)$, $g(\xi)$, $\xi \in \ddot{\Gamma}_{X,0}$, vanishing for $\xi \in \bigsqcup_{k=\ell}^{\infty} \ddot{\Gamma}_X^{(k)}$ for a sufficiently large $\ell \in \mathbb{N}$ and is Borel, bounded, and finite on every $\ddot{\Gamma}_X^{(k)}$, $k \in \{0, \dots, \ell\}$ (i. e., the corresponding $f_k(x_1, \dots, x_k)$, $g_k(x_1, \dots, x_k)$ are finite on X^k in the usual sense).

Considering such functions as elements of a Fock space of type (1.1) we pass to its extension, the space of finite sequences $f = (f_n)_{n=0}^{\infty}$, where $f_n = f_n(x_1, \dots, x_n)$ is a complex-valued Borel, bounded, finite, symmetric function on X^n. The functional r generated by a given measure ρ according to (2.3) is well defined on this space. Now we introduce again a quasi-scalar product using (2.2); it is easy to check that the corresponding Hilbert space coincides with $\mathcal{H}_r$ introduced above (for the proof it is necessary to note that convergence in $L^2\left(\bigsqcup_{n=0}^{m} \Gamma_X^{(n)}, d\rho(\xi)\right)$, where $m < \infty$ is fixed, implies convergence in $\mathcal{H}_r$).

In what follows, κ_α always denotes the characteristic function of the set α.

Then let us consider the function

$$(\kappa_\Lambda^{\star n} \star \kappa_{\ddot{\Gamma}_\Lambda^{(k)}})(\xi), \quad \xi \in \ddot{\Gamma}_{X,0}, \tag{2.8}$$

where Λ is a compact subset of X, $n, k \in \mathbb{N}$. Note that $\Lambda = \ddot{\Gamma}_\Lambda^{(1)}$.

Put at first $n = 1$. According to (1.5),

$$(\kappa_\Lambda \star \kappa_{\ddot{\Gamma}_\Lambda^{(k)}})(\xi) = \sum_{\xi' \cup \xi'' \cup \xi''' = \xi} \kappa_{\ddot{\Gamma}_\Lambda^{(k)}}(\xi' \cup \xi'') \kappa_{\ddot{\Gamma}_\Lambda^{(1)}}(\xi'' \cup \xi'''). \tag{2.9}$$

Since κ_Λ is a function on $\ddot{\Gamma}_X^{(1)}$, i. e., it depends on a configuration $[x]$ of order 1, ξ'' and ξ''' in (2.9) can be of kind $\emptyset$ or $[x_j]$ only. Since $\kappa_{\ddot{\Gamma}_\Lambda^{(k)}}$ depends on $[x_1, \dots, x_k]$, ξ in (2.9) can be from $\ddot{\Gamma}_X^{(k)}$ or $\ddot{\Gamma}_X^{(k+1)}$ only, and ξ'' is of kind $[x_j]$ or $\emptyset$, respectively. As a result, we have

$$(2.10)\qquad \begin{aligned}&(\kappa_\Lambda \star \kappa_{\ddot\Gamma_\Lambda^{(k)}})(\xi) = k\kappa_{\ddot\Gamma_\Lambda^{(k)}}(\xi), \quad \xi \in \ddot\Gamma_X^{(k)};\\ &(\kappa_\Lambda \star \kappa_{\ddot\Gamma_\Lambda^{(k)}})(\xi) = (k+1)\kappa_{\ddot\Gamma_\Lambda^{(k+1)}}(\xi), \quad \xi \in \ddot\Gamma_X^{(k+1)}.\end{aligned}$$

Let $k \in \mathbb{N}$ be fixed. Then we use (2.10) for performing subsequent convolutions with κ_Λ obtaining (2.8) at the n-th step. It is clear that $(\kappa_\Lambda^{\star n} \star \kappa_{\ddot\Gamma_\Lambda^{(k)}})(\xi)$ is a function on $\bigsqcup_{j=0}^{n} \ddot\Gamma_X^{(k+j)}$ and on every $\ddot\Gamma_X^{(k+j)}$ it is equal to $(C_{k,n,j}\kappa_{\ddot\Gamma_\Lambda^{(k+j)}})(\xi)$ with a certain positive coefficient $C_{k,n,j}$. It follows from (2.10) that these coefficients can be found by the following recurrence formulas:

$$(2.11)\qquad \begin{gathered}\forall n \in \mathbb{N}, \quad \forall j \in \{0,\ \ldots,\ n+1\}\\ C_{k,n+1,j} = (k+j)(C_{k,n,j-1} + C_{k,n,j}), \quad C_{k,n+1,n+1} = (k+n+1)C_{k,n,n},\\ C_{k,1,0} = k, \quad C_{k,1,1} = k+1, \quad C_{k,n,-1} = 0.\end{gathered}$$

Using (2.11) one obtains the following estimate:

$$C_{k,n+1,j} \le 2(k+n+1) \max_{\lambda \in \{0,\ldots,n\}} C_{k,n,\lambda}.$$

Then, taking also into account that, for $j \in \{0,1\}$, $C_{k,1,j} \le k+1$ and moving step by step, one can write

$$\forall n \in \mathbb{N}, \quad \forall j \in \{0,\ \ldots,\ n\} \quad C_{k,n,j} \le \frac{2^{n-1}(k+n)!}{k!}.$$

So, we can write for every $\xi \in \ddot\Gamma_{X,0}$, $n,k \in \mathbb{N}$,

$$(2.12)\qquad (\kappa_\Lambda^{\star n} \star \kappa_{\ddot\Gamma_\Lambda^{(k)}})(\xi) \le \frac{2^{n-1}(k+n)!}{k!}.$$

We have estimated the expression (2.8).

Then let us estimate the norm of (2.8), i. e., $\|\kappa_\Lambda^{\star n} \star \kappa_{\ddot\Gamma_\Lambda^{(k)}}\|_{\mathcal{H}_r}$. To this end, we note at first something about the following function: $(\kappa_{\ddot\Gamma_\Lambda^{(k)}} \star \kappa_{\ddot\Gamma_\Lambda^{(k)}})(\xi)$. According to (1.5), it does not equal 0 on $\bigsqcup_{\ell=k}^{2k} \ddot\Gamma_X^{(\ell)}$ only, and on each $\ddot\Gamma_X^{(\ell)}$ it is equal to $M_{k,\ell}\kappa_{\ddot\Gamma_\Lambda^{(\ell)}}(\xi)$ with some non-negative constants $M_{k,\ell}$. Let $M_k := \max_{\ell \in \{k,\ldots,2k\}} M_{k,\ell}$. According to (2.2), (2.3), and (2.12) we have the following estimate:

$$(2.13)\qquad \begin{aligned}\|\kappa_\Lambda^{\star n} \star \kappa_{\ddot\Gamma_\Lambda^{(k)}}\|_{\mathcal{H}_r}^2 &= \int_{\ddot\Gamma_{X,0}} (\kappa_\Lambda^{\star 2n} \star \kappa_{\ddot\Gamma_\Lambda^{(k)}} \star \kappa_{\ddot\Gamma_\Lambda^{(k)}})(\xi)\, d\rho(\xi)\\ &= \sum_{\ell=k}^{2k} \left(M_{k,\ell} \int_{\ddot\Gamma_{X,0}} (\kappa_\Lambda^{\star 2n} \star \kappa_{\ddot\Gamma_\Lambda^{(\ell)}})(\xi)\, d\rho(\xi) \right)\\ &\le \sum_{\ell=k}^{2k} \left(\frac{M_{k,\ell} \cdot 2^{2n-1}(\ell+2n)!}{\ell!} \sum_{j=0}^{2n} \int_{\ddot\Gamma_\Lambda^{(\ell+j)}} d\rho(\xi) \right)\\ &\le 2^{2n-1} M_k \sum_{\ell=0}^{2k} \left(\frac{(\ell+2n)!}{\ell!} \sum_{j=0}^{2n} \rho\left(\ddot\Gamma_\Lambda^{(\ell+j)}\right) \right) =: R_{\Lambda,k,n}^2.\end{aligned}$$

For any bounded function f on $\ddot\Gamma_{X,0}$, it is obvious that

$$(2.14)\qquad \forall \xi \in \ddot\Gamma_{X,0} \quad |f(\xi)| \le \sup_{\xi \in \ddot\Gamma_{X,0}} |f(\xi)| \kappa_{\operatorname{supp} f}(\xi).$$

Take any functions $f_1, \dots, f_m$ from $\mathcal{F}_{\rm fin}(\mathcal{D})$. Using (2.14) and (1.5) and taking into account that $\operatorname{supp}(f_1 \star \dots \star f_m)$ can be defined completely by $\operatorname{supp} f_1, \dots, \operatorname{supp} f_m$, it is easy to understand that

$$\begin{aligned} &\forall \xi \in \ddot{\Gamma}_{X,0} \\ &|(f_1 \star \dots \star f_m)(\xi)| \le (|f_1| \star \dots \star |f_m|)(\xi) \\ &\qquad \le \sup_{\xi \in \ddot{\Gamma}_{X,0}} |f_1(\xi)| \cdots \sup_{\xi \in \ddot{\Gamma}_{X,0}} |f_m(\xi)| (\kappa_{\operatorname{supp} f_1} \star \dots \star \kappa_{\operatorname{supp} f_m})(\xi). \end{aligned} \tag{2.15}$$

Moreover, now we will need the following fact: let f and g be functions from $\mathcal{F}_{\rm fin}(\mathcal{D})$, and let $\forall \xi \in \ddot{\Gamma}_{X,0}\ \ g(\xi) \ge 0$; then

$$|f(\xi)| \le g(\xi) \Longrightarrow \|f\|_{\mathcal{H}_r} \le \|g\|_{\mathcal{H}_r}. \tag{2.16}$$

Indeed, taking into account (1.5), (2.2), and (2.4), we can write

$$\|f\|^2_{\mathcal{H}_r} = \int_{\ddot{\Gamma}_{X,0}} (f \star \overline{f})(\xi)\, d\rho(\xi) \le \int_{\ddot{\Gamma}_{X,0}} (|f| \star |f|)(\xi)\, d\rho(\xi) \le \int_{\ddot{\Gamma}_{X,0}} (g \star g)(\xi)\, d\rho(\xi) = \|g\|^2_{\mathcal{H}_r}.$$

Let $\varphi \in \mathcal{D}$ and $f \in \mathcal{F}_k(\mathcal{D})$. Choose a compact set $\Lambda \subset X$ such that $\Lambda \supset \operatorname{supp} \varphi$ and $\ddot{\Gamma}^{(k)}_\Lambda \supset \operatorname{supp} f$. Then using (2.16), (2.15), and (2.13) we obtain for each $k, n \in \mathbb{N}$ that

$$\begin{aligned} \|\varphi^{\star n} \star f\|_{\mathcal{H}_r} &\le \|\, |\varphi|^{\star n} \star |f|\, \|_{\mathcal{H}_r} \le (\sup_{x \in \Lambda} |\varphi(x)|)^n (\sup_{\xi \in \ddot{\Gamma}^{(k)}_\Lambda} |f(\xi)|) \|\kappa^{\star n}_\Lambda \star \kappa_{\ddot{\Gamma}^{(k)}_\Lambda}\|_{\mathcal{H}_r} \\ &\le (K(\varphi, f))^{n+1} R_{\Lambda,k,n} \end{aligned}$$

with a constant $K(\varphi, f)$ depending on φ and f. In other words, for each $k, n \in \mathbb{N}$,

$$\|(A(\varphi))^n \{f\}\|_{\mathcal{H}_r} \le (K(\varphi, f))^{n+1} R_{\Lambda,k,n}. \tag{2.17}$$

The inequalities (2.13) and (2.17) hold also for $k = 0$ because it is not difficult to conclude that all the constructions from (2.8) until (2.17) are correct for $k = 0$.

Now the proof can be finished in a simple way. Let $D = \{\mathcal{A}\}$, then D has the first property from (i) (concerning the invariance). Let $\{f\} \in D$, then $f \in \mathcal{F}_{\rm fin}(\mathcal{D})$, and so $f = \sum_{j=0}^k f_j$, $\{f\} = \sum_{j=0}^k \{f_j\}$, where f_j is a function on $\ddot{\Gamma}^{(j)}_X$. Choose a compact set $\Lambda \subset X$ such that $\Lambda \supset \operatorname{supp} \varphi$ and $\forall j \in \{0, \dots, k\}$, $\ddot{\Gamma}^{(j)}_\Lambda \supset \operatorname{supp} f_j$. Then we conclude from (2.17) (using also the fact that, evidently, $R_{\Lambda,j,n}$ are not decreasing w. r. t. j) that

$$\begin{aligned} \|(A(\varphi))^n \{f\}\|_{\mathcal{H}_r} &\le \sum_{j=0}^k \|(A(\varphi))^n \{f_j\}\|_{\mathcal{H}_r} \le \sum_{j=0}^k (K(\varphi, f))^{n+1} R_{\Lambda,j,n} \\ &\le (k+1)(K(\varphi, f))^{n+1} R_{\Lambda,k,n}. \end{aligned}$$

Now it is sufficient to note that the class $C\{m_n\}$, where m_n is given by (2.7), is quasi-analytic if and only if $C\{(k+1)(K(\varphi, f))^{n+1} R_{\Lambda,k,n}\}$ is a quasi-analytic class (see (2.7) and (2.13)). □

Remark 2.2. It is easy to see that, by making more accurate estimations, one can replace the right-hand side of (2.7) with

$$\max_{\lambda \in \{0, \dots, k\}} \left(\sum_{\ell=\lambda}^{2\lambda} \left(\frac{(\ell + 2n)!}{\ell!} \sum_{j=0}^{2n} \rho \left(\ddot{\Gamma}^{(\ell+j)}_\Lambda \right) \right) \right)^{\frac{1}{2}}. \tag{2.18}$$

Since formulas (2.7) and (2.18) are so complicated, we will formulate now a simpler sufficient condition for (i).

Corollary 2.1. *Let, for the measure ρ from (2.3), the following condition hold: for every compact $\Lambda \subset X$ there exists a constant C_Λ such that*

$$\forall n \in \mathbb{N} \quad \rho(\ddot{\Gamma}^{(n)}_\Lambda) \le C^n_\Lambda. \tag{2.19}$$

Then (i) *is true and $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$ is a family of selfadjoint commuting operators in the space $\mathcal{H}_r$.*

Proof. It is clear that for the proof one can take $C_\Lambda \geq 2$ without losing the generality. We will assume this inequality to hold throughout proof.

According to Theorem 2.1, it is sufficient to check that the class $C\{m_n\}$ is quasi-analytic, where $C\{m_n\}$ is given by (2.7). To do this, we will make at first some estimations of (2.7) using (2.19),

$$\begin{aligned} m_n &= \left(\sum_{\ell=0}^{2k}\left(\frac{(\ell+2n)!}{\ell!}\sum_{j=0}^{2n}\rho\left(\ddot{\Gamma}_\Lambda^{(\ell+j)}\right)\right)\right)^{\frac{1}{2}} \leq \left(\sum_{\ell=0}^{2k}\left(\frac{(\ell+2n)!}{\ell!}\sum_{j=0}^{2n}C_\Lambda^{\ell+j}\right)\right)^{\frac{1}{2}} \\ &= \left(\sum_{\ell=0}^{2k}\frac{(\ell+2n)!}{\ell!}\cdot\frac{C_\Lambda^{\ell+2n+1}-C_\Lambda^{\ell}}{C_\Lambda-1}\right)^{\frac{1}{2}} \leq \left(\sum_{\ell=0}^{2k}C_\Lambda^{\ell+2n+1}(\ell+2n)!\right)^{\frac{1}{2}} \\ &= C_\Lambda^{n+\frac{1}{2}}\left(\sum_{\ell=0}^{2k}C_\Lambda^{\ell}(\ell+2n)!\right)^{\frac{1}{2}} \leq C_\Lambda^{n+\frac{1}{2}}\left((2k+1)C_\Lambda^{2k}(2k+2n)!\right)^{\frac{1}{2}} \\ &\leq C_\Lambda^{2n}\sqrt{2k+1}C_\Lambda^{k}\sqrt{(2n+2k)!} =: C_\Lambda^{2n}b_n. \end{aligned}$$

Now it is sufficient to prove that the class $C\{b_n\}$ is quasi-analytic. In order to do this, we will use the following known fact (see, e. g., [4]): if

$$\sum_{n=1}^{\infty}\frac{1}{b_n^{\frac{1}{n}}} = \infty, \tag{2.20}$$

then the class $C\{b_n\}$ is quasi-analytic.

So, we will show that the series from (2.20) is divergent. For a simplification of the calculations below, we introduce the following notations:

$$a_n := \frac{1}{b_n^{\frac{1}{n}}}, \quad N := \sqrt{2k+1}C_\Lambda^{k}.$$

Thus $b_n = N\sqrt{(2n+2k)!}$, N does not depend on n, and the series from (2.20) can be written as

$$\sum_{n=1}^{\infty}a_n = \sum_{n=1}^{\infty}\frac{1}{N^{\frac{1}{n}}((2n+2k)!)^{\frac{1}{2n}}}.$$

Let us use the Raabe rule. It is sufficient to prove that

$$\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right) > 1. \tag{2.21}$$

For this aim we will make the following calculations:

$$\begin{aligned} n\left(1-\frac{a_{n+1}}{a_n}\right) &= n\left(1-\frac{N^{\frac{1}{n}}((2n+2k)!)^{\frac{1}{2n}}}{N^{\frac{1}{n+1}}((2n+2k+2)!)^{\frac{1}{2n}}}\right) \\ &= n\left(1-\frac{N^{\frac{1}{n(n+1)}}}{((2n+2k+1)(2n+2k+2))^{\frac{1}{2n}}}\right) \\ &= n\cdot\frac{((2n+2k+1)(2n+2k+2))^{\frac{1}{2n}}-N^{\frac{1}{n(n+1)}}}{((2n+2k+1)(2n+2k+2))^{\frac{1}{2n}}} \\ &= n\cdot\frac{N^{-\frac{1}{n(n+1)}}((2n+2k+1)(2n+2k+2))^{\frac{1}{2n}}-1}{N^{-\frac{1}{n(n+1)}}((2n+2k+1)(2n+2k+2))^{\frac{1}{2n}}} \\ &= n\cdot\frac{e^{\frac{1}{n}\log\left(N^{-\frac{1}{n+1}}\sqrt{(2n+2k+1)(2n+2k+2)}\right)}-1}{N^{-\frac{1}{n(n+1)}}((2n+2k+1)(2n+2k+2))^{\frac{1}{2n}}}. \end{aligned}$$

It is easy to calculate (e. g., using logarithm) that the denominator of the last fraction tends to 1 if $n \to \infty$. Therefore it is sufficient to calculate the limit of the following expression:

$$n\left(e^{\frac{1}{n}\log\left(N^{-\frac{1}{n+1}}\sqrt{(2n+2k+1)(2n+2k+2)}\right)}-1\right)$$
$$=\frac{e^{\frac{1}{n}\log\left(N^{-\frac{1}{n+1}}\sqrt{(2n+2k+1)(2n+2k+2)}\right)}-1}{\frac{1}{n}}$$
$$=\frac{e^{\frac{1}{n}\log\left(N^{-\frac{1}{n+1}}\sqrt{(2n+2k+1)(2n+2k+2)}\right)}-1}{\frac{1}{n}\log\left(N^{-\frac{1}{n+1}}\sqrt{(2n+2k+1)(2n+2k+2)}\right)}$$
$$\times\log\left(N^{-\frac{1}{n+1}}\sqrt{(2n+2k+1)(2n+2k+2)}\right).$$

Using a known fact that $\lim\limits_{x\to 0}\frac{e^x-1}{x}=1$, we conclude that the left-hand side of (2.21) equals

$$\lim_{n\to\infty}\log\left(N^{-\frac{1}{n+1}}\sqrt{(2n+2k+1)(2n+2k+2)}\right)=\infty>1,$$

so that the corollary is proved. □

Remark 2.3. Since the proof of Corollary 2.1 is based on information about divergence of a series, it is easy to see that in (2.19) one can write $\forall n\in\mathbb{N}_q:=\{q,q+1,\dots\}$ instead of $\forall n\in\mathbb{N}$ with any non-negative integer q.

3. The decomposition in generalized joint eigenvectors and the integral representation of a positive functional

So, we will investigated the family $(\tilde{A}(\varphi))_{\varphi\in\mathcal{D}}$ of selfadjoint commuting operators in the space $\mathcal{H}_r$, introduced in Section 2. Below we will construct a decomposition in their generalized joint eigenvectors.

At first it is necessary to recall some results concerning weighted Fock spaces constructed similarly to (1.1) (see, e. g., [18, 12, 16]). It is known that $\mathcal{D}$ is the projective limit of real Sobolev spaces, $H_\tau=W^{\tau_1}_{2,\mathrm{Re}}(X,\tau_2(x)\,\mathrm{dm}(x))$, where $\tau=(\tau_1,\tau_2(x)),\ \tau_1\in\mathbb{N}_0$, $\tau_2(x)\ge 1$ is a C^∞ weight, m is a Riemann measure on X. The projective limit (uncountable) is taken over the set T of all such τ. Note that for every $\tau\in T$ there exists $\tau'=(\tau_1',\tau_2'(x))\in T,\ \ \tau_1'>\tau_1,\ \ \forall x\in X\ \ \tau_2'(x)\ge\tau_2(x)$ (we will write $\tau'>\tau$) such that the embedding $H_{\tau'}\subset H_\tau$ is quasi-nuclear (i. e., a Hilbert–Schmidt one).

Let $p=(p_n)_{n=0}^\infty$, where $\forall n\in\mathbb{N}_0\ \ p_n>0$, be a number weight. Let $\mathcal{F}(H_\tau,p)$ be the weighted Fock space consisting of sequences $f=(f_n)_{n=0}^\infty$, where $f_n\in H_{\tau,c}^{\widehat{\otimes}n}=:\mathcal{F}_n(H_\tau)$, such that

$$\forall f,g\in\mathcal{F}(H_\tau,p)\quad \|f\|^2_{\mathcal{F}(H_\tau,p)}=\sum_{n=0}^\infty\|f_n\|^2_{\mathcal{F}_n(H_\tau)}p_n<\infty,$$
$$(f,g)_{\mathcal{F}(H_\tau,p)}=\sum_{n=0}^\infty(f_n,g_n)_{\mathcal{F}_n(H_\tau)}p_n. \tag{3.1}$$

It is known [4, 15] that for every $\tau\in T$ and a number weight $p\ge 1$ (i. e., each $p_n\ge 1$) there exists $\tau'\in T$ and a weight $p'=(p_n')_{n=0}^\infty$, $p_n'\ge p_n$, such that the embedding $\mathcal{F}(H_{\tau'},p')\subset\mathcal{F}(H_\tau,p)$ is quasi-nuclear (moreover, τ' is the same as was denoted by this symbol above). Let us also stress that the space (1.1) $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ is a projective limit of the spaces $\mathcal{F}(H_\tau,p)$ with arbitrary $\tau\in T$ and $p\ge 1$.

Due to the above, it is necessary to construct a chain (rigging) of the usual Fock spaces, $\mathcal{F}(H) := \mathcal{F}(H,p)$, where $H = L^2_{\mathrm{Re}}(X, \mathrm{dm}(x))$ and $p = (1, 1, \ldots)$ [12, 16]. Namely,

$$\begin{aligned}(3.2)\qquad \operatorname{ind}\lim_{\tau\in T,\ p\geq 1} \mathcal{F}(H_{-\tau}, p^{-1}) = (\mathcal{F}_{\mathrm{fin}}(\mathcal{D}))' \supset \mathcal{F}(H_{-\tau}, p^{-1}) \supset \mathcal{F}(H) \\ \supset \mathcal{F}(H_\tau, p) \supset \mathcal{F}_{\mathrm{fin}}(\mathcal{D}) = \operatorname{pr}\lim_{\tau\in T,\ p\geq 1} \mathcal{F}(H_\tau, p).\end{aligned}$$

Here $H_{-\tau}$ is the negative space w. r. t. the zero space H and the positive space H_τ; $p^{-1} = (p_n^{-1})_{n=0}^\infty$; $\mathcal{F}(H_{-\tau}, p^{-1})$ is the negative space w. r. t. $\mathcal{F}(H_\tau, p)$ and $\mathcal{F}(H)$. Note that if we talk about a functional, it is always a linear one (rather than anti-linear).

Let us consider the general positive functional r introduced at the beginning of Section 2. This functional is continuous on the space $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$, which coincides with a projective limit of the weighted Fock spaces introduced above, $\mathcal{F}_{\mathrm{fin}}(\mathcal{D}) = \operatorname{pr}\lim_{\tau\in T,\ p\geq 1} \mathcal{F}(H_\tau, p)$. Therefore ,it is continuous on some Fock space $\mathcal{F}(H_\tau, p)$, i. e., $r \in \mathcal{F}(H_{-\tau}, p^{-1})$ with some τ and p (see, e. g., [18]). It is easy to understand that the continuity of $\star$ in the space $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ (being the projective limit of $\mathcal{F}(H_\tau, p)$) gives a possibility, for any $\tau' \in T$ and any $p' \geq 1$, to find sufficiently large τ'' and $p'' \geq p'$ (i. e., $\forall n \in \mathbb{N}_0\ \ p''_n \geq p'_n$) such that $\forall f, g \in \mathcal{F}(H_{\tau''}, p'')$, $f \star g$ exists, belongs to the space $\mathcal{F}(H_{\tau'}, p')$, and depends continuously on f, g,

$$(3.3)\qquad \exists C > 0 \quad \forall f, g \in \mathcal{F}(H_{\tau''}, p'') \quad \|f \star g\|_{\mathcal{F}(H_{\tau'}, p')} \leq C\|f\|_{\mathcal{F}(H_{\tau''}, p'')}\|g\|_{\mathcal{F}(H_{\tau''}, p'')}$$

(for the proof it is necessary to use the definition of the topology of the projective limit.) Let again τ''' and $p''' \geq p''$ be so "large" that the embedding $\mathcal{F}(H_{\tau'''}, p''') \subset \mathcal{F}(H_{\tau''}, p'')$ is quasi-nuclear. *We fix these τ''' and p''' and will denote them by τ^0 and p^0.*

Let us now consider the Hilbert space $\mathcal{H}_r$ introduced at the beginning of Section 2. For simplicity, we will assume that the scalar product (2.2) is non-degenerated, every class $\{f\}$ consists of one vector f, $f \in \mathcal{A} = \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$. Note that it is possible to consider the general case using results from [4], Ch. 8, Section 1, and [12], Ch. 5, Section 5.

The continuity of r on the space $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ means that there exists $C_1 > 0$ for which $\forall f \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})\quad |r(f)| \leq C_1\|f\|_{\mathcal{F}(H_{\tau'}, p')}$ with some $\tau' \in T$ and $p' \geq 1$. Using (2.2), this inequality, (3.3), and the choice of τ'', p'', τ^0, p^0 described above, we conclude the following: $\exists C_2 > 0\ \ \forall f \in \mathcal{F}(H_{\tau^0}, p^0)$

$$(3.4)\quad \|\{f\}\|^2_{\mathcal{H}_r} = r(f \star \overline{f}) \leq C_1\|f \star \overline{f}\|_{\mathcal{F}(H_{\tau'}, p')} \leq C_1 C\|f\|^2_{\mathcal{F}(H_{\tau''}, p'')} \leq C_2\|f\|^2_{\mathcal{F}(H_{\tau^0}, p^0)}.$$

This estimate means that there exists the following dense and continuous embedding:

$$\mathcal{H}_r \supset \mathcal{F}(H_{\tau''}, p'') \supset \mathcal{F}(H_{\tau^0}, p^0).$$

The last embedding is quasi-nuclear due to the choice of τ^0 and p^0; therefore the embedding $\mathcal{F}(H_{\tau^0}, p^0) \subset \mathcal{H}_r$ is also quasi-nuclear.

Using the possibility to pass to the spaces consisting of classes $\{f\}$, we can assert that the following chain of spaces is constructed:

$$(3.5)\qquad \{\mathcal{F}(H_{\tau^0}, p^0)\}_- \supset \mathcal{H}_r \supset \{\mathcal{F}(H_{\tau^0}, p^0)\} \supset \{\mathcal{A}\} = \{\mathcal{F}_{\mathrm{fin}}(\mathcal{D})\}.$$

Here $\{\mathcal{F}(H_{\tau^0}, p^0)\}$ is a positive space consisting of classes $\{f\}$, where $f \in \mathcal{F}(H_{\tau^0}, p^0)$; it belongs to $\mathcal{H}_r$ because the convolution $\star$ of vectors from $\mathcal{F}(H_{\tau^0}, p^0)$ is defined and belongs to the space $\mathcal{F}(H_{\tau'}, p')$ on which the functional r is defined.

The embedding $\{\mathcal{F}(H_{\tau^0}, p^0)\} \subset \mathcal{H}_r$ is quasi-nuclear because the embedding $\mathcal{F}(H_{\tau^0}, p^0) \subset \mathcal{F}(H_{\tau''}, p'')$ is quasi-nuclear. Passing to the corresponding classes is not an essential operation for this (we also stress that $\mathcal{H}_r$ is a completion of $\mathcal{F}(H_{\tau''}, p'')$ w. r. t. $(\cdot,\cdot)_{\mathcal{A}_r}$).

So, we have constructed the chain (3.5) required for applying the projective spectral theorem to the family $(\tilde{A}(\varphi))_{\varphi\in\mathcal{D}}$ [12, 4, 16, 43].

It is difficult to describe the negative space from (3.5) because the space $\mathcal{H}_r$ is complicated. Thus we will apply a simple procedure that was already used repeatedly in analogous situations [4]. Namely, as well as (3.5) we construct another chain: the following chain of Fock spaces:

$$\begin{aligned}(3.6)\qquad &\operatorname{ind}\lim_{\tau\in T,\tau\geq\tau^0,p\geq p^0}\mathcal{F}(H_{-\tau},p^{-1})=(\mathcal{F}_{\mathrm{fin}}(\mathcal{D}))'\supset\mathcal{F}(H_{-\tau^0},(p^0)^{-1})\supset\mathcal{F}(H)\\ &\supset\mathcal{F}(H_{\tau^0},p^0)\supset\mathcal{F}_{\mathrm{fin}}(\mathcal{D})=\operatorname{pr}\lim_{\tau\in T,\tau\geq\tau^0,p\geq p^0}\mathcal{F}(H_\tau,p).\end{aligned}$$

There is the same positive space $\mathcal{F}(H_{\tau^0},p^0)$ in the chains (3.5) and (3.6) (the only difference is that there are classes $\{f\}$ instead of f in (3.5)), and therefore (see [8], Lemma 2.2) the corresponding negative spaces $\{\mathcal{F}(H_{\tau^0},p^0)\}_-$ and $\mathcal{F}(H_{-\tau^0},(p^0)^{-1})$ are unitary isomorphic. More exactly, there exists a unitary operator $U:\{\mathcal{F}(H_{\tau^0},p^0)\}_-\to\mathcal{F}(H_{-\tau^0},(p^0)^{-1})$ such that

$$(3.7)\qquad \forall\alpha\in\{\mathcal{F}(H_{\tau^0},p^0)\}_-,\quad\forall f\in\mathcal{F}(H_{-\tau^0},(p^0)^{-1})\quad(U\alpha,f)_{\mathcal{F}(H)}=(\alpha,\{f\})_{\mathcal{H}_r}.$$

We will use this remark for calculating generalized eigenvectors for the family $(\tilde{A}(\varphi))_{\varphi\in\mathcal{D}}$.

So, let us pass to the operators $A(\varphi)$ defined by (2.5), where $\varphi\in\mathcal{D}$ is a real-valued function of a point $\xi\in\ddot{\Gamma}_X^{(1)}\subset\ddot{\Gamma}_{X,0}$. At first we will consider these operators acting in $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})=\mathcal{A}$. Using the equalities $A(\varphi)f=\varphi\star f$, (1.5) and reasoning as in the proof of Theorem 2.1 (in particular, taking into account the formulas of type (2.10) for φ instead of κ_Λ), one can write

$$(3.8)\qquad \forall\varphi\in\mathcal{D}\quad A(\varphi)=A^+(\varphi)+A^0(\varphi).$$

Here $A^+(\varphi)$ and $A^0(\varphi)$ are respectively the standard creation and neutral operators acting by

$$(3.9)\qquad \forall n\in\mathbb{N}_0\quad A^+(\varphi)\psi^{\otimes n}=(n+1)\varphi\widehat{\otimes}\psi^{\otimes n},\quad A^0(\varphi)\psi^{\otimes n}=n(\varphi\psi)\widehat{\otimes}\psi^{\otimes(n-1)}.$$

Here $\psi\in\mathcal{D}_c$ and thus $\psi^{\otimes n}\in\mathcal{D}_c^{\widehat{\otimes}n}=\mathcal{F}_n(\mathcal{D})$, $(\varphi\psi)(x)=\varphi(x)\psi(x)$, $\psi^{\otimes(-1)}:=0$.

The operators $A^+(\varphi)$ and $A^0(\varphi)$ depend linearly on $\varphi\in\mathcal{D}$ because of (3.9). Therefore, $A(\varphi)$ also depends linearly on $\varphi\in\mathcal{D}$. Moreover (3.8) and (3.9) imply that $\forall n\in\mathbb{N}_0$, $\operatorname{Ran}(A(\varphi)\restriction\mathcal{F}_n(H))\subset\mathcal{F}_n(H)\oplus\mathcal{F}_{n+1}(H)$, and the mapping $\mathcal{D}\ni\varphi\mapsto(A(\varphi)\restriction\mathcal{F}_n(H))f\in\mathcal{F}(H)$ is continuous with each $f\in\mathcal{D}_c^{\widehat{\otimes}n}$ (we use the natural notation $\mathcal{F}_n(H):=H_c^{\widehat{\otimes}n}$).

Passing to the action of the operators $A(\varphi)$ on the classes $\{f\}$ we get again that they depend on $\varphi\in\mathcal{D}$ linearly. So, the operators of the family $(A(\varphi))_{\varphi\in\mathcal{D}}$ are invariant w. r. t. their domain $\mathcal{A}\subset\mathcal{H}_r$ and depend on $\varphi\in\mathcal{D}$ linearly. *We will assume that the condition (i) is true*; then the operators $\tilde{A}(\varphi)$, $\varphi\in\mathcal{D}$, are selfadjoint in $\mathcal{H}_r$ and commuting.

So, we consider the operators of the family $(\tilde{A}(\varphi))_{\varphi\in\mathcal{D}}$ on the space $\mathcal{H}_r$ from the chain of spaces (3.4). This chain is a standard one connected with the operators $\tilde{A}(\varphi)$: these operators act continuously on the space $\{\mathcal{A}\}$ and the embedding $\{\{\mathcal{F}(H_{\tau^0},p^0)\}\}\subset\mathcal{H}_r$ is quasi-nuclear. Note at first that the family $(\tilde{A}(\varphi))_{\varphi\in\mathcal{D}}$ has a strong cyclic vector.

Lemma 3.1. *There exists a vector $\{\Omega\}\in\{\mathcal{A}\}$ such that the set of vectors*

$$\{A^{m_1}(\varphi_1)\dots A^{m_p}(\varphi_p)\Omega\mid\varphi_1,\dots,\varphi_p\in\mathcal{D},\ m_1,\dots,m_p\in\mathbb{N}_0,\ p\in\mathbb{N}_0\}$$

(for $p=0$ we put $A^{m_1}(\varphi_1)\dots A^{m_p}(\varphi_p)\Omega:=\Omega$) is total in $\mathcal{F}(H_{\tau^0},p^0)$ (such a vector is called cyclic).

Proof. In fact this proof is reduced to Lemma 2.1 from [6] about the vector $\Omega=(1,0,0,\dots)$ which is cyclic for a Jacobi field in a Fock space. Note at first that the proof of this lemma remains to hold if, in the matrices of a Jacobi field (2.2) from the above-mentioned paper [6], the elements $c_j(\varphi)$ from the upper diagonal are replaced with zeros (i. e., $A_-(\varphi)=0$ in (2.8) from [6]). Applying such a modification of this lemma to our case we conclude that the set

$$\{A^{m_1}(\varphi_1)\dots A^{m_p}(\varphi_p)\Omega\mid\varphi_1,\dots,\varphi_p\in\mathcal{D},\ m_1,\dots,m_p\in\mathbb{N}_0,\ p\in\mathbb{N}_0\}$$

is total in $\mathcal{F}(H_{\tau^0}, p^0)$ (now $A(\varphi)$ is the operator (3.8) acting in $\mathcal{F}(H_{\tau^0}, p^0)$). The totality of this set in the space $\mathcal{H}_r$ follows from density and continuity of the embedding $\mathcal{F}(H_{\tau^0}, p^0) \subset \mathcal{H}_r$. □

Now we pass to applying the projective spectral theorem formulated as Theorem 1.6 in Ch. 4 of [12] (see also Theorem 1.1 in [6] or Theorem 1.1 in [16] and [8]).

Note at first that the operators $\tilde{A}(\varphi)$ depend on $\varphi \in \mathcal{D}$ linearly. Thus their generalized joint eigenvectors $\alpha(\omega)$ are indexed with generalized functions $\omega \in \mathcal{D}'$ and

$$\forall \varphi \in \mathcal{D}, \quad \forall f \in \mathcal{F}(H_{\tau^0}, p^0) \quad (\alpha(\omega), A(\varphi)\{f\})_{\mathcal{H}_r} = \langle \omega, \varphi \rangle \, (\alpha(\omega), \{f\})_{\mathcal{H}_r}, \tag{3.10}$$

where $\langle \omega, \varphi \rangle = \langle \varphi, \omega \rangle$ is the result of acting ω on φ; also note that, in particular, one can take $f \in \mathcal{A}$. Then using (3.7) we put

$$P(\omega) = U(\alpha(\omega)) \in \mathcal{F}(H_{-\tau^0}, (p^0)^{-1}) \subset (\mathcal{F}_{\mathrm{fin}}(\mathcal{D}))'. \tag{3.11}$$

So, $P(\omega)$ is a generalized joint eigenvector in the sense of the chain (3.6). According to (3.7) and (3.11), the equality (3.10) turns into the following:

$$\begin{gathered} \forall \omega \in \mathcal{D}', \quad \forall \varphi \in \mathcal{D}, \quad \forall f \in \mathcal{A} = \mathcal{F}_{\mathrm{fin}}(\mathcal{D}) \\ (P(\omega), A(\varphi) f)_{\mathcal{F}(H)} = \langle \omega, \varphi \rangle \, (P(\omega), f)_{\mathcal{F}(H)}. \end{gathered} \tag{3.12}$$

Since $P(\omega) \in (\mathcal{F}_{\mathrm{fin}}(\mathcal{D}))'$, we can write

$$\begin{gathered} P(\omega) = (P_n(\omega))_{n=0}^{\infty}, \quad P_n(\omega) \in (\mathcal{D}^{\widehat{\otimes} n})' =: (\mathcal{D}')^{\widehat{\otimes} n}; \\ \forall \omega \in \mathcal{D}', \quad \forall f \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D}) \quad (P(\omega), f)_{\mathcal{F}(H)} = \sum_{n=0}^{\infty} (P_n(\omega), f_n)_{\mathcal{F}_n(H)} \end{gathered} \tag{3.13}$$

(due to the fact that the operators $A(\varphi)$ are real, $P(\omega)$, $P_n(\omega)$ are also real). Note also that in (3.13) one can take f from the wider space $\mathcal{F}(H_{\tau^0}, p^0) \supset \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$. Multiplying $P(\omega)$ by a required function of ω it is always possible to achieve the equality $P_0(\omega) = 1$ for every $\omega \in \mathcal{D}$. Of course, $P_n(\omega)$ are analogues of the first type polynomials in the theory of Jacobi matrices and fields.

So, due to the above mentioned projective spectral theorem and (3.13) we can claim the following.

Theorem 3.1. *Let the assumption (i) for the family $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$ be fulfilled. Then the family $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$ generates a Fourier transform I given by*

$$\begin{aligned} &\mathcal{F}_{\mathrm{fin}}(\mathcal{D}) \ni f = (f_n)_{n=0}^{\infty} \\ &\quad \mapsto (If)(\omega) =: \widehat{f}(\omega) = (f, P(\omega))_{\mathcal{F}(H)} = \sum_{n=0}^{\infty} (f_n, P_n(\omega))_{\mathcal{F}_n(H)} \in L^2(\mathcal{D}', d\mu(\omega)). \end{aligned} \tag{3.14}$$

Here μ is the spectral measure of the family being a probability Borel measure on the space $\mathcal{D}'$. The closure of the operator I by continuity is a unitary operator between the spaces $\mathcal{H}_r$ and $L^2(\mathcal{D}', d\mu(\omega))$, it turns each operator $\tilde{A}(\varphi)$ into the operator of multiplication by the function $\langle \omega, \varphi \rangle$.

Remark 3.1. In (3.14) one can take $f \in \mathcal{F}(H_{\tau^0}, p^0) \supset \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ and the series will converge; it is clear from the above that $P_n(\omega) \in (H_{-\tau^0}, (p^0)^{-1})^{\widehat{\otimes} n}$ and $P(\omega) \in \mathcal{F}(H_{-\tau^0}, (p^0)^{-1})$. Moreover, the measure μ is concentrated on the space $H_{-\tau^0} \subset \mathcal{D}'$ from the chain (see [12], Ch. 4, Theorem 1.6, [8], Section 1, Theorem 1.1),

$$\mathcal{D}' \supset H_{-\tau^0} \supset H = L^2(X, \mathrm{dm}(x_1)) \supset H_{\tau^0} \supset \mathcal{D}. \tag{3.15}$$

For every $n \in \mathbb{N}$ and every $\varphi_1, \dots, \varphi_n \in \mathcal{D}$, the moment $\langle \omega, \varphi_1 \rangle \dots \langle \omega, \varphi_n \rangle$ belongs to $L^2(\mathcal{D}', d\mu(\omega))$.

The Parseval equality, connected with Theorem 3.1, is the following:

$$(\{f\},\{g\})_{\mathcal{H}_r} = r(f \star \overline{g}) = \int\limits_{\mathcal{D}'} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}\, d\mu(\omega). \tag{3.16}$$

Let us take $g = (1, 0, 0, \ldots) = \Omega$ in this Parseval equality. Then, according to (3.13), $\widehat{g}(\omega) = 1$. In the sense of functions on $\ddot{\Gamma}_{X,0}$, g is an identity in the algebra $\mathcal{A}$. Thus (3.16) implies

$$r(f) = \int\limits_{\mathcal{D}'} \widehat{f}(\omega)\, d\mu(\omega) = \int\limits_{\mathcal{D}'} \Big(\sum_{n=0}^{\infty} (f_n, P_n(\omega))_{\mathcal{F}_n(H)}\Big) d\mu(\omega). \tag{3.17}$$

So, we have obtained an integral representation of the functional r.

If $f = f_n \in \mathcal{F}_n(\mathcal{D}) \subset \mathcal{F}(\mathcal{D})$ in (3.17), with a fixed $n \in \mathbb{N}_0$, then every term of the sum from (3.17), except for $(f_n, P_n(\omega))_{\mathcal{F}_n(H)}$, vanishes and thus

$$\forall n \in \mathbb{N}_0 \quad r(f_n) = \int\limits_{\mathcal{D}'} (f_n, P_n(\omega))_{\mathcal{F}_n(H)} d\mu(\omega). \tag{3.18}$$

4. Calculation of the Fourier transform

Let us calculate the “polynomials” $P_n(\omega)$ from the definition (3.14) of the Fourier transform.

We will use the notations standard in the theory of Jacobi matrices and fields [4, 6, 7],

$$A(\varphi) = A^{+}(\varphi) + A^{0}(\varphi) = \begin{pmatrix} b_0(\varphi) & 0 & 0 & 0 & \ldots \\ a_0(\varphi) & b_1(\varphi) & 0 & 0 & \ldots \\ 0 & a_1(\varphi) & b_2(\varphi) & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{4.1}$$

where $a_n(\varphi) : \mathcal{F}_n(H) \to \mathcal{F}_{n+1}(H)$ and $b_n(\varphi) : \mathcal{F}_n(H) \to \mathcal{F}_n(H)$ (it is selfadjoint) are real creation and neutral operators defined in accordance with (3.9); * and $^+$ are used, respectively, for the adjoint operator in $\mathcal{F}(H)$ and for the adjoint operator w. r. t. the zero space of the corresponding chain. It is known that the annihilation operator $a_n^*(\varphi) : \mathcal{F}_{n+1}(H) \to \mathcal{F}_n(H)$ acts as follows:

$$a_n^*(\varphi)\psi^{\otimes(n+1)} = (n+1)(\varphi, \psi)_H \psi^{\otimes n}. \tag{4.2}$$

Taking into account (3.12) and (3.13) we get

$$\begin{aligned} &\forall \varphi \in \mathcal{D}, \quad \forall f \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D}) \\ &(P(\omega),\ A(\varphi)f)_{\mathcal{F}(H)} = \sum_{n=0}^{\infty} (P_n(\omega),\ a_{n-1}(\varphi)f_{n-1} + b_n(\varphi)f_n)_{\mathcal{F}_n(H)} \\ &\quad = \sum_{n=0}^{\infty} ((a_n^*(\varphi))^{+} P_{n+1}(\omega) + (b_n(\varphi))^{+} P_n(\omega),\ f_n)_{\mathcal{F}_n(H)} \\ &\quad = \langle \omega, \varphi \rangle\, (P(\omega),\ f)_{\mathcal{F}(H)} = \sum_{n=0}^{\infty} (\langle \omega, \varphi \rangle\, P_n(\omega),\ f_n)_{\mathcal{F}_n(H)}. \end{aligned} \tag{4.3}$$

Since f in (4.3) is arbitrary, we get the following *recurrence relation for* $P_n(\omega)$:

$$\begin{aligned} &\forall \varphi \in \mathcal{D}, \quad \forall \omega \in \mathcal{D}', \quad \forall n \in \mathbb{N}_0 \\ &(a_n^*(\varphi))^{+} P_{n+1}(\omega) = \langle \omega, \varphi \rangle\, P_n(\omega) - (b_n(\varphi))^{+} P_n(\omega), \quad P_0(\omega) = 1, \end{aligned} \tag{4.4}$$

(here $+$ denotes the conjugation w. r. t. the chain (3.15)).

It is possible to write (4.4) in another manner. Since $P_n(\omega) \in (\mathcal{D}')^{\widehat{\otimes} n}$ (i. e., it is symmetric and real), in order to find $P_n(\omega)$ it is sufficient to know $(P_n(\omega), \varphi^{\otimes n})_{\mathcal{F}_n(H)}$

for every $\varphi \in \mathcal{D}$. Let us apply (4.3) for $f = (f_k)_{k=0}^{\infty}$, where $f_n = \varphi^{\otimes n}$ and every other $f_k = 0$. Then (4.3) turns into

$$(P_{n+1}(\omega),\ a_n(\varphi)\varphi^{\otimes n})_{\mathcal{F}_{n+1}(H)} + (P_n(\omega),\ b_n(\varphi)\varphi^{\otimes n})_{\mathcal{F}_n(H)} = \langle\omega,\varphi\rangle\,(P_n(\omega),\ \varphi^{\otimes n})_{\mathcal{F}_n(H)}. \tag{4.5}$$

Taking into account the formulas (3.9) for $a_n(\varphi)$ and $b_n(\varphi)$ we rewrite (4.5) as follows:

$$\begin{aligned}&\forall\varphi\in\mathcal{D},\quad \forall n\in\mathbb{N}_0\\ &(P_{n+1}(\omega),\ \varphi^{\otimes(n+1)})_{\mathcal{F}_{n+1}(H)}\\ &\quad=\frac{1}{n+1}\Big((P_n(\omega)\otimes\omega,\ \varphi^{\otimes(n+1)})_{\mathcal{F}_{n+1}(H)} - (P_n(\omega),\ n\varphi^2\otimes\varphi^{\otimes(n-1)})_{\mathcal{F}_n(H)}\Big),\\ &P_0(\omega)=1.\end{aligned} \tag{4.6}$$

Note also that, according to (4.6), $P_1(\omega) = \omega$. Formula (4.6) *is just another form of* (4.4) *which was mentioned at the beginning of this paragraph.*

Moreover the polynomials $P_n(\omega)$ can be found as coefficients of the power decomposition of a certain function. We have the following theorem.

Theorem 4.1. *For any $\omega \in \mathcal{D}'$, consider the function*

$$e^{\langle \log(1+\varphi),\ \omega\rangle}, \tag{4.7}$$

where $\varphi \in \mathcal{D}$ and $\forall x \in X$, $\varphi(x) > -1$. It is analytic w. r. t. φ in a neighborhood $U(0)$ of 0 *from $\mathcal{D}_c$, and thus can be decomposed into a series w. r. t. tensor powers $\varphi^{\otimes n}$. It is claimed that the coefficients of this decomposition are just $P_n(\omega)$, i. e.,*

$$e^{\langle \log(1+\varphi),\ \omega\rangle} = \sum_{n=0}^{\infty}\left\langle \varphi^{\otimes n},\ P_n(\omega)\right\rangle_{\mathcal{F}_n(H)}. \tag{4.8}$$

Proof. We will use the decomposition of the character $\chi(\omega,\varphi) = \exp\langle\log(1+\varphi),\omega\rangle$ into a series of the corresponding Delsarte characters $\chi_n(\omega)$,

$$\exp\langle\log(1+\varphi),\omega\rangle = \chi(\omega,\varphi) = \sum_{n=0}^{\infty}\frac{1}{n!}\left\langle\varphi^{\otimes n},\ \chi_n(\omega)\right\rangle, \tag{4.9}$$

used in the theory of Poisson analysis ([20], formula (3.1)). We replace here the notation x from the negative space $S_{-2}(\mathbb{R}^1)$ (w. r. t. the positive Sobolev space $W_2^2(\mathbb{R}^1,\ (1+t^2)^2 d\sigma(t)))$ to a vector $\omega \in \mathcal{D}'$, and $\lambda \in W^2_{2,\mathbb{C}}(\mathbb{R}^1)$ to $\varphi \in \mathcal{D}$. Such a replacement, as it is easy to understand, is possible. Note that φ has finite norm in $W^2_{2,\mathbb{C}}(\mathbb{R}^1)$ in (3.1) from [20] and the vector $\omega \in \mathcal{D}'$ is a generalized function on $\mathcal{D} = C^{\infty}_{\mathrm{fin}}(X)$. The function $\chi_n(\omega)$ is the action $\chi_n \in \mathcal{D}$ on $\omega \in \mathcal{D}'$.

So, we can rewrite the formula, following (9.2) from [20], as follows:

$$\left\langle\chi_n(\omega),\ \varphi^{\otimes n}\right\rangle = \sum_{\ell=0}^{n-1}(-1)^{n-\ell-1}\frac{(n-1)!}{\ell!}\left\langle\varphi^{n-\ell},\omega\right\rangle\left\langle\varphi^{\otimes\ell},\chi_\ell(\omega)\right\rangle. \tag{4.10}$$

Here $\varphi^{n-\ell}$ is the usual product, $(\varphi(\omega))^{n-\ell}$, $\omega \in \mathcal{D}'$; $\left\langle\varphi^{n-\ell},\omega\right\rangle$ is the action of $\omega \in \mathcal{D}'$ on this function.

To derive (4.8) from (4.9), it is necessary to prove that

$$\frac{1}{n!}\chi_n(\omega) = P_n(\omega),\quad \omega\in\mathcal{D}',\quad n\in\mathbb{N}_0. \tag{4.11}$$

We will use induction. We have that (4.11) takes place for $n = 0$ (both expressions in (4.11) are equal to 1). Let us assume that identity (4.10) takes place for $n = 0, \ldots, m$ and prove that it is correct for $n = m + 1$.

Using (4.6) and (4.11) we conclude

$$
\begin{aligned}
&(P_{m+1}(\omega),\ \varphi^{\otimes(m+1)})_{\mathcal{F}_{m+1}(H)}\\
&= \frac{1}{m+1}\Bigg(\Big(\frac{1}{m!}\chi_m(\omega)\otimes\omega, \varphi^{\otimes(m+1)} \Big)_{\mathcal{F}_{m+1}(H)}\\
&- \frac{1}{m!}\Big(\chi_m(\omega),\ m\varphi^2\otimes\varphi^{\otimes(m-1)}\Big)_{\mathcal{F}_m(H)}\Bigg)\\
&- \frac{1}{(m+1)!}\Big[(\chi_m(\omega)\otimes\omega,\ \varphi^{\otimes(m+1)})_{\mathcal{F}_{m+1}(H)}\\
&- m(\chi_m(\omega),\ \varphi^2\otimes\varphi^{\otimes(m-1)})_{\mathcal{F}_m(H)}\Big].
\end{aligned}
\tag{4.12}
$$

From (4.10) for $n = m+1$ and $\ell = m$, and $\ell = 0, 1, \dots, m-1$ we get

$$
\begin{aligned}
&\Big\langle \chi_{m+1}(\omega),\ \varphi^{\otimes(m+1)} \Big\rangle = (-1)^{m+1-m-1}\frac{m!}{m!}\langle \varphi,\omega\rangle \langle \varphi^{\otimes m},\ \chi_m(\omega)\rangle\\
&+ \sum_{\ell=0}^{m-1} (-1)^{m+1-\ell-1}\frac{m!}{\ell!}\Big\langle \varphi^2\otimes\varphi^{\otimes(\ell-1)},\ \chi_\ell(\omega)\Big\rangle\\
&= \Big\langle \varphi^{\otimes(m+1)},\ \omega\otimes\chi_m(\omega)\Big\rangle\\
&- m\sum_{\ell=0}^{m-1} (-1)^{m-\ell-1}\frac{(m-1)!}{\ell!}\Big\langle \varphi^2\otimes\varphi^{\otimes(\ell-1)},\ \chi_\ell(\omega)\Big\rangle\\
&= \Big\langle \chi_m(\omega)\otimes\omega,\ \varphi^{\otimes(m+1)}\Big\rangle - m\Big\langle \chi_m(\omega),\ \varphi^2\otimes\varphi^{\otimes(m-1)}\Big\rangle
\end{aligned}
\tag{4.13}
$$

(we used formula (4.10) for χ_m in the last equality). We see that the last expression is the same as the expression in the square brackets in (4.12). Therefore (4.12) and (4.13) give

$$
(P_{m+1}(\omega),\ \varphi^{\otimes(m+1)})_{\mathcal{F}_{m+1}(H)} = \frac{1}{(m+1)!}\Big\langle \chi_{m+1}(\omega), \varphi^{\otimes(m+1)}\Big\rangle, \quad \varphi\in\mathcal{D}.
$$

□

Now let us clarify where the measure μ from (3.17) is concentrated if the positive functional r is generated by a measure ρ on $\ddot{\Gamma}_{X,0}$ according to (2.3). Recall that a finite complex-valued Borel measure on a space Q is called a charge on Q.

Theorem 4.2. *Let a positive functional r be generated by a measure ρ on $\ddot{\Gamma}_{X,0}$. Then, for the corresponding spectral measure μ, the set of $\omega\in\mathcal{D}'$ generated by charges on X is a set of full measure μ.*

On the contrary, let the above mentioned set be a set of full measure μ, where μ is the spectral measure connected with a functional r. Then the positive functional r is generated by a charge on $\ddot{\Gamma}_{X,0}$.

Proof. According to Remark 3.1 it is possible to consider the negative Hilbert space $H_{-\tau^0}$ from the chain (3.15), instead of the space $\mathcal{D}'$. Let $H_{-\tau^0} = \alpha\cup\beta$, $\alpha\cap\beta=\emptyset$, where α is the set of all ω generated by charges on X and β be the rest of $H_{-\tau^0}$. Apply (3.18) to the case where $n=1$. Since $P_1(\omega)=\omega$, we have

$$
\forall f_1\in\mathcal{D}\quad r(f_1) = \int_{\mathcal{D}'} \langle f_1,\omega\rangle\, d\mu(\omega) = \int_{\alpha} \langle f_1,\omega\rangle\, d\mu(\omega) + \int_{\beta} \langle f_1,\omega\rangle\, d\mu(\omega).
\tag{4.14}
$$

Suppose that r is generated by a measure ρ but $\mu(\beta)>0$. Denote by $\beta_S\subset\beta$ the support of the restriction $\mu\restriction\beta$ of the measure μ to β. So, β_S is a closed set of full measure, $\mu(\beta_S)=\mu(\beta)>0$ (for the corresponding facts about the support for a Borel measure on $H_{-\tau^0}$, see [5], Ch. 2, Section 1, and [12], Ch. 3, Section 1).

Let us fix a positive continuous function $\psi(x) \le 1$ on X integrable w. r. t. the measure $\rho \restriction X$. Since $H_{-\tau^0}$ is a separable space, there exists a countable set $\{\omega_1, \omega_2, \ldots\}$, dense in β_S, of points from β_S. Consider ω_1 and construct a sequence $(f_1^{(1,k)})_{k=1}^\infty$ of functions from H_{τ^0} such that $f_1^{(1,k)}(x) \to 0$ $(k \to \infty)$ uniformly w. r. t. $x \in X$, $|f_1^{(1,k)}(x)| \le \psi(x)$ (for every $x \in X$) and, for a certain number $\varepsilon_1 > 0$,

$$\forall k \in \mathbb{N} \quad \left\langle f_1^{(1,k)}, \omega_1 \right\rangle := (f_1^{(1,k)}, \omega_1)_H \ge \varepsilon_1. \tag{4.15}$$

Such a construction is possible; since ω_1 is not a charge, there exists a sequence $(g_1^{(k)})_{k=1}^\infty$, $g_1^{(k)} \in \mathcal{D}$, such that $g_1^{(k)}(x) \to 0$ $(k \to \infty)$ uniformly, $|g_1^{(k)}(x)| \le \psi(x)$, $x \in X$, and $\left\langle g_1^{(k)}(x), \omega_1 \right\rangle$ does not tend to 0 $(k \to \infty)$. Therefore there exist $\varepsilon_1 > 0$ and a subsequence $(f_1^{(1,k)})_{k=1}^\infty \subset (g_1^{(k)})_{k=1}^\infty$ such that $\left|\left\langle f_1^{(1,k)}, \omega_1 \right\rangle\right| \ge \varepsilon_1$. Changing the sign of $f_1^{(1,k)}(x)$ we obtain inequality (4.15).

Analogously, taking ω_2 we construct a sequence $(f_1^{(2,k)})_{k=1}^\infty$ of functions from H_{τ^0} such that $f_1^{(2,k)}(x) \to 0$ $(k \to \infty)$ uniformly w. r. t. $x \in X$, $|f_1^{(2,k)}(x)| \le \psi(x)$ (for every $x \in X$) and, for a certain number $\varepsilon_2 > 0$,

$$\forall k \in \mathbb{N} \quad \left\langle f_1^{(2,k)}, \omega_2 \right\rangle \ge \varepsilon_2. \tag{4.16}$$

Do the same for the points ω_3, ω_4, $\ldots$; it is possible to assume that $\varepsilon_1 > \varepsilon_2 > \cdots$ and $\varepsilon_n \to 0$ $(n \to \infty)$. Now we use the diagonal procedure. Consider the diagonal sequence $(f_1^{(k,k)})_{k=1}^\infty$. The functions $f_1^{(k,k)}(x)$ are uniformly bounded (by the function $\psi(x)$) and tend to 0 uniformly w. r. t. $x \in X$. Moreover for every $m \in \mathbb{N}$ there exists ε_m such that

$$\forall k \in \mathbb{N} \quad \left\langle f_1^{(k,k)}, \omega_m \right\rangle \ge \varepsilon_m > 0. \tag{4.17}$$

The function $H_{-\tau^0} \ni \omega \mapsto \left\langle f_1^{(k,k)}, \omega \right\rangle \in \mathbb{R}$ is continuous in the topology of $H_{-\tau^0}$. Therefore each point ω_m has a neighborhood $U(\omega_m)$ such that (4.17) implies

$$\forall \omega \in U(\omega_m), \quad \forall k \in \mathbb{N} \quad \left\langle f_1^{(k,k)}, \omega \right\rangle \ge \frac{\varepsilon_m}{2} > 0. \tag{4.18}$$

The neighborhoods $U(\omega_m) \subset \beta_S$ and, therefore, have positive measure μ. Inequalities (4.18) imply that there exists a continuous non-negative function $g(\omega)$ given on the closed set β_S such that

$$\forall \omega \in \beta_S, \quad \forall k \in \mathbb{N} \quad \left\langle f_1^{(k,k)}, \omega \right\rangle \ge g(\omega) \ge 0 \tag{4.19}$$

and

$$\forall \omega \in U(\omega_m), \quad \forall m \in \mathbb{N} \quad g(\omega) \ge c_m > 0.$$

Writing (4.14) for $f_1 = f_1^{(k,k)}$ and replacing β with β_S we get

$$\begin{aligned} r(f_1^{(k,k)}) &= \int_X f_1^{(k,k)}(x)\, d\rho(x) = \int_\alpha \left\langle f_1^{(k,k)}, \omega \right\rangle d\mu(\omega) + \int_{\beta_S} \left\langle f_1^{(k,k)}, \omega \right\rangle d\mu(\omega) \\ &= \int_\alpha \left(\int_X f_1^{(k,k)}(x)\, d\sigma_\omega(x) \right) d\mu(\omega) + \int_{\beta_S} \left\langle f_1^{(k,k)}, \omega \right\rangle d\mu(\omega) \\ &\ge \int_\alpha \left(\int_X f_1^{(k,k)}(x)\, d\sigma_\omega(x) \right) d\mu(\omega) + \int_{\beta_S} g(\omega)\, d\mu(\omega), \end{aligned} \tag{4.20}$$

where σ_ω is the charge on X corresponding to $\omega \in \alpha$. Formula (4.20) means that

$$\forall k \in \mathbb{N} \quad \int_X f_1^{(k,k)}(x)\, d\rho(x) \ge \int_\alpha \left(\int_X f_1^{(k,k)}(x)\, d\sigma_\omega(x) \right) d\mu(\omega) + \int_{\beta_S} g(\omega)\, d\mu(\omega). \tag{4.21}$$

Since $f_1^{(k,k)}(x)$ tends to 0 uniformly and is majorized by the function $\psi(x)$, which is integrable on X, the left-hand integral in (4.21) tends to 0 too.

Then, due to the same convergence and the fact that $\psi(x) \le 1$, we can write

$$\forall \omega \in \alpha \quad \int_X f_1^{(k,k)}(x)\, d\sigma_\omega(x) \to 0 \quad (k \to \infty). \tag{4.22}$$

Our nearest aim is to show that the first integral in the right-hand side of (4.21) also tends to 0 if $k \to \infty$ (together with (4.22) this means that it is possible to pass to the limit under the integral sign).

All above constructed functions $f_1^{(\ell,k)}$, $\ell, k \in \mathbb{N}$, can be chosen in such a way that $\forall \ell, k \in \mathbb{N}$ $\|f_1^{(\ell,k)}\|_{H_{\tau 0}} \le C_3$ with some $C_3 > 0$. Therefore, we can consider that $\forall k \in \mathbb{N}$ $\|f_1^{(k,k)}\|_{H_{\tau 0}} \le C_3$ in (4.22).

But, because of (3.14), (4.6) and the fact that $f_1^{(k,k)} \in \mathcal{F}_1(\mathcal{D})$,

$$\int_X f_1^{(k,k)}(x)\, d\sigma_\omega(x) = \left\langle f_1^{(k,k)}, \omega \right\rangle = (I f_1^{(k,k)})(\omega) = \widehat{f}_1^{(k,k)}(\omega). \tag{4.23}$$

The operator I between $\mathcal{H}_r$ and $L^2(\mathcal{D}', d\mu(\omega))$ is unitary, which is due to (3.14) and (3.4) for each natural k,

$$\begin{aligned} \int_{\mathcal{D}'} \Big| \int_X f_1^{(k,k)}(x)\, d\sigma_\omega(x) \Big|^2 d\mu(\omega) &= \int_{\mathcal{D}'} \Big| (\widehat{f}_1^{(k,k)})(\omega) \Big|^2 d\mu(\omega) = \|f_1^{(k,k)}\|^2_{\mathcal{H}_r} \\ &\le C_2 \|f_1^{(k,k)}\|^2_{\mathcal{F}(H_{\tau 0}, p^0)} = C_2 \|f_1^{(k,k)}\|^2_{\mathcal{F}_1(H_{\tau 0}, p_1^0)} = C_2 p_1^0 \|f_1^{(k,k)}\|^2_{H_{\tau 0}} \le C_2 C_3^2 p_1^0. \end{aligned} \tag{4.24}$$

Since the integrals in (4.24) are bounded w. r. t. $k \in \mathbb{N}$, one can proceed to the limit under the considered integral for $k \to \infty$ and conclude that this limit is equal to 0.

As the result we get from (4.21) that

$$\int_{\beta_S} g(\omega)\, d\mu(\omega) = 0.$$

But this is impossible, since the inequality (4.19) shows that the last integral is more than or equal to $\sum_{m=1}^{\infty} c_m \mu(U(\omega_m)) > 0$ (because $U(\omega_m)$ is an open subset of the support β_S and, therefore, $\mu(U(\omega_m)) > 0$). The first part of the theorem is proved. So, we have shown the following. Let $\mathcal{D}'_{\text{ch}}$ be the set of all functionals $\omega \in \mathcal{D}'$ generated by charges on $\mathcal{D}$. If r is generated by a measure, then $\mathcal{D}'_{\text{ch}}$ is a set of full measure w. r. t. μ.

Let us prove a proposition converse to Theorem 4.2; if $\mathcal{D}'_{\text{ch}}$ is a set of full measure w. r. t. μ, then r is generated by a charge. Now one can rewrite (3.17) and (3.18) replacing $\mathcal{D}'$ with $\mathcal{D}'_{\text{ch}}$. Applying (3.18) with $n = 1$ and taking into account that $P_1(\omega) = \omega$ we get

$$\forall f_1 \in \mathcal{D} \subset \mathcal{D}_c = \mathcal{F}_1(\mathcal{D}) \quad r(f_1) = \int_{\mathcal{D}'_{\text{ch}}} (f_1, \omega)_{\mathcal{F}_1(H)} d\mu(\omega), \tag{4.25}$$

where ω is generated by a charge σ_ω. It follows from (4.25) that

$$\forall f_1 \in \mathcal{D} \quad |r(f_1)| \le \max_{x_1 \in X} |f_1(x_1)| (\operatorname{Var} \sigma_\omega)(X) \tag{4.26}$$

(recall that μ is a probability measure). According to (4.6)

$$\forall \varphi \in \mathcal{D} \quad (P_2(\omega),\ \varphi^{\otimes 2})_{\mathcal{F}_2(H)} = \frac{1}{2} \left((\omega \widehat{\otimes} \omega,\ \varphi^{\otimes 2})_{\mathcal{F}_2(H)} - (\omega,\ \varphi^2)_{H_c} \right), \tag{4.27}$$

where, obviously, $\omega \widehat{\otimes} \omega = \omega \otimes \omega = \omega^{\otimes 2}$ is also generated by a charge denoted by $\sigma_{\omega^{\otimes 2}}$. It follows from (4.27) that

$$\begin{aligned} &\forall \varphi \in \mathcal{D} \\ &\big| (P_2(\omega),\ \varphi^{\otimes 2})_{\mathcal{F}_2(H)} \big| \\ &\quad \le \frac{1}{2} \left(\max_{x_1, x_2 \in X} |\varphi^{\otimes 2}(x_1, x_2)| (\operatorname{Var} \sigma_{\omega^{\otimes 2}})(X^2) + \max_{x_1 \in X} |\varphi^2(x_1)| (\operatorname{Var} \sigma_\omega)(X) \right) \\ &\quad \le c_2 \max_{x_1, x_2 \in X} |\varphi^{\otimes 2}(x_1, x_2)| \end{aligned} \tag{4.28}$$

with a constant $c_2 > 0$.

Taking the sum of equalities of type (4.27) we conclude that an analogous equality holds for every sum $f_2 = \sum_{j=1}^{m} \varphi_j^{\otimes 2}$, where $\varphi_j \in \mathcal{D}$, $m \in \mathbb{N}$ (i. e., for every linear combination of functions of the kind $\varphi^{\otimes 2}$). Using this equality we can get an inequality of type (4.28) by replacing $\varphi^{\otimes 2}$ with f_2. Note that the constant c_2 does not depend on m.

The set of all linear combinations of functions of the form $\varphi^{\otimes 2}(x_1, x_2)$ is dense in the topology of $\mathcal{D}^{\widehat{\otimes} 2}$. This topology is stronger than the uniform topology of the space of continuous finite functions of point $(x_1, x_2) \in X^2$. Therefore it follows by the limit procedure from the above-mentioned generalization of (4.28) that

$$\forall f_2 \in \mathcal{D}^{\widehat{\otimes} 2} \quad \left|(P_2(\omega),\ f_2)_{\mathcal{F}_2(H)}\right| \leq c_2 \max_{x_1, x_2 \in X} |f_2(x_1, x_2)|. \tag{4.29}$$

The estimations (4.26) and (4.29) imply that $P_1(\omega)$ (which is equal to ω) and $P_2(\omega)$ are generated by charges (due to the Riesz theorem).

One can continue to get similar estimations for $P_3(\omega)$, $P_4(\omega)$, ..., i. e., we obtain inequalities of type (4.29).

Indeed, suppose that $P_n(\omega)$ is generated by a charge. Then $P_n(\omega)\widehat{\otimes}\omega$ is generated by a charge too, and analogously to the case $n = 1$ from (4.6) and (4.30) one can conclude

$$\forall f_n \in \mathcal{D}^{\widehat{\otimes} n} \quad \left|(P_n(\omega),\ f_n)_{\mathcal{F}_n(H)}\right| \leq c_n \max_{x_1, \ldots, x_n \in X} |f_n(x_1, \ldots, x_n)|. \tag{4.30}$$

There exists a finite constant $c_{n+1} > 0$ such that

$$\begin{aligned} &\forall f_{n+1} \in \mathcal{D}^{\widehat{\otimes}(n+1)} \\ &\left|(P_{n+1}(\omega),\ f_{n+1})_{\mathcal{F}_{n+1}(H)}\right| \leq c_{n+1} \max_{x_1, \ldots, x_{n+1} \in X} |f_{n+1}(x_1, \ldots, x_{n+1})|. \end{aligned} \tag{4.31}$$

The estimation (4.31) implies that $P_{n+1}(\omega)$ is generated by a charge.

So, using (3.18) and (4.31), we conclude that

$$\begin{aligned} &\forall n \in \mathbb{N}, \quad \forall f_n \in \mathcal{D}^{\widehat{\otimes} n} \\ &|r(f_n)| = \left| \int_{\mathcal{D}'_{\mathrm{ch}}} (f_n, P_n(\omega))_{\mathcal{F}_n(H)} d\mu(\omega) \right| \\ &\quad \leq c_n \max_{x_1, \ldots, x_n \in X} |f_n(x_1, \ldots, x_n)| \int_{\mathcal{D}'_{\mathrm{ch}}} d\mu(\omega) = c_n \max_{x_1, \ldots, x_n \in X} |f_n(x_1, \ldots, x_n)| \end{aligned} \tag{4.32}$$

(recall that μ is a probability measure).

The estimation (4.32) shows that the functional $\mathcal{D}^{\widehat{\otimes} n} \ni f_n \mapsto r(f_n) \in \mathbb{C}$ is generated by a charge, that is, because of (2.3), we can say the same about r on $\mathcal{A}$. □

Corollary 4.1. *Let a positive functional r on $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ be such that its restriction to $\mathcal{F}_1(\mathcal{D})$ be generated by a σ-finite Borel measure. Then the functional r is generated by a charge.*

Proof. Indeed, in the proof of the first part of Theorem 4.2 we have only used the fact that the restriction of r to $\mathcal{F}_1(\mathcal{D})$ is generated by a measure. Therefore, in the case under consideration, μ is concentrated on elements that coincide as charges on X. Then, according to the second part of this theorem, r is generated by a charge on $\ddot{\Gamma}_{X,0}$. □

5. The space of infinite configurations and the Lenard transform

The space of infinite configurations Γ_X over X (or the configuration space) is defined as a set of all locally finite usual configurations in X, i. e.,

$$\Gamma_X = \left\{\gamma \subset X \,\middle|\, |\gamma \cap \Lambda| < \infty \text{ for every compact } \Lambda \subset \mathrm{X}\right\}, \tag{5.1}$$

where $|\alpha|$ denotes cardinality of the set α; we stress that usual finite configurations belong to Γ_X, i. e., $\forall n \in \mathbb{N}\;\; \Gamma_X \supset \Gamma_X^{(n)}$. It will often be convenient to identify γ with a σ-finite Borel measure on X of the kind

$$\sigma_\gamma := \sum_{x\in\gamma} \delta_x, \tag{5.2}$$

where δ_x is a unit measure concentrated at the point x (the δ-function concentrated in x). From the other hand, each measure (5.2) generates a linear continuous functional ω_γ on the space $\mathcal{D}$,

$$\mathcal{D} \ni \varphi \mapsto \omega_\gamma(\varphi) = \int_X \varphi(x)\, d\sigma_\gamma(x) = \sum_{x\in\gamma} \varphi(x) \in \mathbb{C}. \tag{5.3}$$

Because φ is finite and, it follows from condition (5.1) that mapping (5.3) is, actually, a linear continuous functional on $\mathcal{D}$, i. e., $\omega_\gamma \in \mathcal{D}'$. So, *identifying γ with ω_γ we get the inclusion $\Gamma_X \subset \mathcal{D}'$.*

Now, let us consider the space $\Gamma_{X,0}$ of usual finite configurations (1.3). It is easy to see that $\Gamma_{X,0}$ is a Borel set in the space $\ddot{\Gamma}_{X,0}$ of all multiple configurations. Therefore it is possible to consider σ-finite Borel measures ρ on $\ddot{\Gamma}_{X,0}$ such that

$$\rho(\ddot{\Gamma}_{X,0} \setminus \Gamma_{X,0}) = 0. \tag{5.4}$$

We will consider below measures ρ on $\ddot{\Gamma}_{X,0}$ satisfying property (5.4) only. Of course, it is possible to treat them also as σ-finite Borel measures on Γ_X, and we will not introduce any new notation for this interpretation.

According to the general rule (2.3), one constructs a functional r. We will suppose that it is positive and thus Theorem 3.1, representations (3.17), (3.18), and Theorem 4.2 hold. Due to the last theorem, it is possible to assume that the spectral measure μ is a Borel probability measure concentrated on elements in $\mathcal{D}'$ that are generated by charges. Therefore in (3.14), (3.17), and (3.18) one can replace $\mathcal{D}'$ with $\mathcal{D}'_{\mathrm{ch}} \subset \mathcal{D}'$. Note that the functional (5.3) is generated by charge (5.2) and thus it belongs to $\mathcal{D}'_{\mathrm{ch}}$.

Now our aim is to show that if (5.4) and a certain additional condition imposed on ρ hold, then the set of all functionals of the form (5.3) is a set of full measure μ on $\mathcal{D}'_{\mathrm{ch}}$. This fact will imply that in this case one can replace $\mathcal{D}'_{\mathrm{ch}}$ with Γ_X in the above-mentioned integrals.

The polynomials $P_n(\omega)$ can be calculated in a simple way in the case $\omega = \gamma \in \Gamma_X \subset \mathcal{D}'$.

Lemma 5.1. *The following formula holds:*

$$\forall \gamma \in \Gamma_X \subset \mathcal{D}', \quad \forall n \in \mathbb{N} \quad P_n(\gamma) = \sum_{\xi\subset\gamma,\ |\xi|=n} \widehat{\otimes}_{x\in\xi}\delta_x, \quad P_0(\gamma) = 1. \tag{5.5}$$

Proof. For $n = 1$, formula (5.5) is obvious (recall that $P_1(\gamma) = \gamma$). Let us assume that it holds for $n \in \mathbb{N}$ and prove it for $n+1$. Let $\varphi \in \mathcal{D}$. According to (4.6) we have

$$\begin{aligned}
&(P_{n+1}(\gamma),\ \varphi^{\otimes(n+1)})_{\mathcal{F}_{n+1}(H)} \\
&\quad= \frac{1}{n+1}\Big((P_n(\gamma)\widehat{\otimes}\gamma,\ \varphi^{\otimes(n+1)})_{\mathcal{F}_{n+1}(H)} - (P_n(\gamma),\ n\varphi^2\widehat{\otimes}\varphi^{\otimes(n-1)})_{\mathcal{F}_n(H)}\Big) \\
&\quad= \frac{1}{n+1}\Big((P_n(\gamma),\ \varphi^{\otimes n})_{\mathcal{F}_n(H)}\,\langle\gamma,\varphi\rangle - (P_n(\gamma),\ n\varphi^2\widehat{\otimes}\varphi^{\otimes(n-1)})_{\mathcal{F}_n(H)}\Big) \\
&\quad= \frac{1}{n+1}\Bigg(\bigg(\sum_{\xi\subset\gamma,\ |\xi|=n} \widehat{\otimes}_{x\in\xi}\delta_x,\ \varphi^{\otimes n}\bigg)_{\mathcal{F}_n(H)} \langle\gamma,\varphi\rangle \\
&\quad\quad - \bigg(\sum_{\xi\subset\gamma,\ |\xi|=n} \widehat{\otimes}_{x\in\xi}\delta_x,\ n\varphi^2\widehat{\otimes}\varphi^{\otimes(n-1)}\bigg)_{\mathcal{F}_n(H)}\Bigg)
\end{aligned}$$

$$= \frac{1}{n+1}\bigg(\Big(\sum_{\xi\subset\gamma,\,|\xi|=n}\Big(\prod_{x\in\xi}\varphi(x)\Big)\Big)\Big(\sum_{x\in\gamma}\varphi(x)\Big)$$
$$- \sum_{\xi\subset\gamma,\,|\xi|=n}\Big(\sum_{y\in\xi}\Big(\varphi^2(y)\prod_{x\in\xi\setminus\{y\}}\varphi(x)\Big)\Big)\bigg)$$
$$= \frac{1}{n+1}\sum_{\xi\subset\gamma,\,|\xi|=n}\bigg(\Big(\prod_{x\in\xi}\varphi(x)\Big)\Big(\sum_{x\in\gamma}\varphi(x)\Big)-\sum_{y\in\xi}\Big(\varphi^2(y)\prod_{x\in\xi\setminus\{y\}}\varphi(x)\Big)\bigg)$$
$$= \frac{1}{n+1}\sum_{\xi\subset\gamma,\,|\xi|=n}\bigg(\sum_{y\in\gamma}\Big(\varphi(y)\prod_{x\in\xi}\varphi(x)\Big)-\sum_{y\in\xi}\Big(\varphi(y)\prod_{x\in\xi}\varphi(x)\Big)\bigg)$$
$$= \frac{1}{n+1}\sum_{\xi\subset\gamma,\,|\xi|=n}\bigg(\sum_{y\in\gamma\setminus\xi}\Big(\varphi(y)\prod_{x\in\xi}\varphi(x)\Big)\bigg) = \sum_{\xi\subset\gamma,\,|\xi|=n+1}\Big(\prod_{x\in\xi}\varphi(x)\Big)$$
$$= \Big(\sum_{\xi\subset\gamma,\,|\xi|=n+1}\widehat{\otimes}_{x\in\xi}\delta_x,\quad \varphi^{\otimes(n+1)}\Big)_{\mathcal{F}_{n+1}(H)}.$$

Since here $\varphi\in\mathcal{D}$ is arbitrary, we conclude that (5.5) is true for $n+1$, and the lemma is proved by induction. □

Now let us consider a transform that is important for our purposes, which is the Lenard transform. It is a mapping K that acts from $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ (the vectors of this space are treated as functions) to $\mathrm{Fun}(\Gamma_X)$ according to the following formula:

$$\mathcal{F}_{\mathrm{fin}}(\mathcal{D})\ni f\mapsto (Kf)(\gamma)=\sum_{\xi\subset\gamma} f(\xi)\in\mathbb{C}, \tag{5.6}$$

where ξ are usual finite configurations contained as subsets in γ (the case $\xi=\gamma$ is also possible). The sum in (5.6) is a finite number, and has compact support as a function, since $f(\xi)=0$ for $\xi\in\ddot{\Gamma}_{X,0}\setminus\bigsqcup_{j=0}^{n}\ddot{\Gamma}_{\Lambda}^{(j)}$ if the compact set $\Lambda\subset X$ and the number $n\in\mathbb{N}_0$ are sufficiently large. This follows from the fact that the corresponding vector $f=(f_n)_{n=0}^{\infty}\in\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ is finite and its components f_n are finite functions of $(x_1,\dots,x_n)\in X^n$. We stress that (5.6) includes values of $f(\xi)$ only on the usual (not multiple) configurations and on $\xi=\emptyset$.

We will quote two important properties of the mapping K established in [34, 35, 36, 28, 29, 33, 41].

Proposition 5.1. *For every $f,g\in\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ and every $\gamma\in\Gamma_X$,*

$$(K(f\star g))(\gamma)=(Kf)(\gamma)(Kg)(\gamma). \tag{5.7}$$

For $f\in\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$, $(Kf)(\gamma)$ is a function on all infinite configurations $\gamma\in\Gamma_X$ and, in particular, every usual configuration $\eta\in\Gamma_X^{(n)}\subset\Gamma_X$, $n\in\mathbb{N}$, and also $\emptyset$ can be its argument. It turns out that the function $f(\xi)$, $\xi\in\Gamma_{X,0}$, can be restored from these values $(Kf)(\eta)$. Moreover, let any function $\Gamma_{X,0}\ni\eta=[y_1,\dots,y_n]\mapsto F(\eta)\in\mathbb{C}$, $n\in\mathbb{N}$, $\emptyset\mapsto F(\emptyset)$ be given, such that $F(\eta)=F([y_1,\dots,y_n])$ is a symmetric infinitely differentiable finite function of point $(y_1,\dots,y_n)\in X^n$. Then one can find a function $f\in\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ such that, for every $\eta\in\Gamma_{X,0}$, $(Kf)(\eta)=F(\eta)$. So, the inverse transform K^{-1} exists in the just explained sense. The following proposition gives a formula for it.

Proposition 5.2. *For any above-mentioned function $F(\eta)$ over $\Gamma_{X,0}$, the following formulas hold:*

$$\begin{gathered}\forall\xi\in\Gamma_{X,0}\quad (K^{-1}F)(\xi)=\sum_{\eta\subset\xi}(-1)^{|\xi\setminus\eta|}F(\eta),\\ \forall\eta\in\Gamma_{X,0}\quad (K(K^{-1}F))(\eta)=F(\eta).\end{gathered}\tag{5.8}$$

Then using (5.8) we will easily prove the following lemma.

Lemma 5.2. *Let $\Lambda\subset X$ be a compact set. Then for any above-mentioned function $F(\eta)$ the following estimate holds:*

$$\forall n\in\mathbb{N},\quad \forall\xi\in\Gamma_\Lambda^{(n)}\quad |(K^{-1}F)(\xi)|\le 2^n\max_{\eta\in\bigsqcup_{j=0}^{n}\Gamma_\Lambda^{(j)}}|F(\eta)|.\tag{5.9}$$

Proof. For $n\in\mathbb{N},\ \xi\in\Gamma_\Lambda^{(n)}$ we have, according to (5.8), that

$$\begin{aligned}|(K^{-1}F)(\xi)|&=\Big|\sum_{\eta\subset\xi}(-1)^{|\xi\setminus\eta|}F(\eta)\Big|\\ &\le\max_{\eta\in\bigsqcup_{j=0}^{n}\Gamma_\Lambda^{(j)}}|F(\eta)|\sum_{\eta\subset\xi}|(-1)^{|\xi\setminus\eta|}|\le 2^n\max_{\eta\in\bigsqcup_{j=0}^{n}\Gamma_\Lambda^{(j)}}|F(\eta)|.\end{aligned}$$

We used that, for $\eta\in\Gamma_\Lambda^{(n)}$, each configuration $\eta\subset\xi=[x_1,\dots,x_n]$ (and also $\eta=\emptyset$) belongs to $\bigsqcup_{j=0}^{n}\Gamma_\Lambda^{(j)}$ and the number of these configurations equals the number of all subsets of the set $\{x_1,\dots,x_n\}$ ($x_1,\dots,x_n\in X$ are distinct), i. e., 2^n. □

The following fact is important.

Proposition 5.3. *For the Fourier transform $(If)(\omega)$ ($f\in\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$, see (3.14)) in the case $\omega=\gamma\in\Gamma_X\subset\mathcal{D}'$, the following formula holds:*

$$\forall\gamma\in\Gamma_X\quad (If)(\gamma)=(Kf)(\gamma).\tag{5.10}$$

So, the Fourier transform $(If)(\omega)$ in points $\omega=\gamma\in\Gamma_X\subset\mathcal{D}'$ can easily be calculated by using formula (5.6) in the general case.

Proof. This proposition is a simple consequence of Lemma 5.1. Indeed, let $f=(f_n)_{n=0}^\infty\in\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$. Then, according to (3.14) and (5.5),

$$\begin{aligned}(If)(\gamma)&=\sum_{n=0}^\infty(f_n,P_n(\gamma))_{\mathcal{F}_n(H)}=f_0+\sum_{n=1}^\infty\Big(f_n,\sum_{\xi\subset\gamma,\ |\xi|=n}\widehat{\otimes}_{x\in\xi}\delta_x\Big)_{\mathcal{F}_n(H)}\\ &=f(\emptyset)+\sum_{\xi\subset\gamma,\ |\xi|>0}f(\xi)=(Kf)(\gamma).\end{aligned}$$

□

6. Representation of a positive functional generated by a measure on the space of usual configurations

In this section we will show that in the case where a positive functional r is generated (according to (2.3)) by a measure ρ concentrated on *usual* configurations (condition (5.4)), the spectral measure μ is concentrated on $\Gamma_X\subset\mathcal{D}'$. Thus, in this case, the equality (5.10) gives that $I=K$. To verify this, we will impose certain additional conditions on the growth of the measure $\rho\restriction\Gamma_X^{(n)}$ as $n\to\infty$.

At first we will establish some auxiliary facts.

Let us construct the following linear functional from the above-mentioned measure ρ:

$$\operatorname{Ran}K\ni F\mapsto \ell(F)=\int_{\Gamma_{X,0}}(K^{-1}F)(\xi)\,d\rho(\xi)\in\mathbb{C}.\tag{6.1}$$

Let $\Lambda \subset X$ be a fixed compact set with an infinitely differentiable border. Construct the following compact set for each fixed $n \in \mathbb{N}$:

$$\Gamma_{\Lambda,n} := \bigsqcup_{j=0}^{n} \Gamma_{\Lambda}^{(j)}, \tag{6.2}$$

where $\Gamma_{\Lambda}^{(0)} := \emptyset$. According to the results in Section 5, each function $\Gamma_{\Lambda,n} \ni \eta \mapsto F(\eta) \in \mathbb{C}$, for which $F([y_1, \ldots, y_j])$ is an infinitely differentiable symmetric function of point $(y_1, \ldots, y_j) \in \Lambda^j$, vanishing outside Λ^j, belongs to $\operatorname{Ran} K$ (see a reasoning before Proposition 5.2). Denote this class of functions by $C^{\infty}_{\text{fin}}(\Gamma_{\Lambda,n})$.

Since $C^{\infty}_{\text{fin}}(\Gamma_{\Lambda,n}) \subset \operatorname{Ran} K$, the functional ℓ (6.1) is defined on $F \in C^{\infty}_{\text{fin}}(\Gamma_{\Lambda,n})$. We will get now its integral representation. According to (5.8), the values $(K^{-1}F)(\xi)$ for $\xi \in \Gamma_X^{(k)}$, $k = 1, \ldots, n$, are defined by the values $F(\eta)$ with $\eta \in \Gamma_{\Lambda,k}$ and are zero outside $\Gamma_{\Lambda}^{(k)}$. Therefore the estimate (5.9) gives

$$\begin{gathered} \forall k = 1, \ldots, n, \quad \forall \xi \in \Gamma_{\Lambda}^{(k)} \\ |(K^{-1}F)(\xi)| \le 2^k \max_{\eta \in \Gamma_{\Lambda,k}} |F(\eta)| \le 2^k \max_{\eta \in \Gamma_{\Lambda,n}} |F(\eta)|, \\ \forall \xi \in \Gamma_X^{(k)} \setminus \Gamma_{\Lambda}^{(k)} \quad (K^{-1}F)(\xi) = 0. \end{gathered} \tag{6.3}$$

It follows from (6.1) and (6.3) that

$$\begin{aligned} &\forall F \in C^{\infty}_{\text{fin}}(\Gamma_{\Lambda,n}) \\ &|\ell(F)| = \Bigl| \sum_{k=0}^{n} \int_{\Gamma_X^{(k)}} (K^{-1}F)(\xi)\, d\rho(\xi) \Bigr| = \Bigl| \sum_{k=0}^{n} \int_{\Gamma_{\Lambda}^{(k)}} (K^{-1}F)(\xi)\, d\rho(\xi) \Bigr| \\ &\qquad \le \max_{\eta \in \Gamma_{\Lambda,n}} |F(\eta)| \sum_{k=0}^{n} \int_{\Gamma_{\Lambda}^{(k)}} 2^k\, d\rho(\xi) = \max_{\eta \in \Gamma_{\Lambda,n}} |F(\eta)| \sum_{k=0}^{n} 2^k \rho(\Gamma_{\Lambda}^{(k)}). \end{aligned} \tag{6.4}$$

The inequality (6.4) shows that the functional ℓ is continuous w. r. t. the norm of the space $C(\Gamma_{\Lambda,n})$ of continuous functions. Therefore *there exists an integral representation of this functional with a Borel charge $\nu_{\Lambda,n}$ on the space $\Gamma_{\Lambda,n}$*,

$$\forall F \in C^{\infty}_{\text{fin}}(\Gamma_{\Lambda,n}) \quad \ell(F) = \int_{\Gamma_{\Lambda,n}} F(\eta)\, d\nu_{\Lambda,n}(\eta), \quad (\operatorname{Var} \nu_{\Lambda,n})(\Gamma_{\Lambda,n}) \le \sum_{k=0}^{n} 2^k \rho(\Gamma_{\Lambda}^{(k)}) \tag{6.5}$$

(in (6.5) $\Lambda \subset X$ is an arbitrary compact set with a smooth border, $n \in \mathbb{N}$).

Note that the charge $\nu_{\Lambda,n}$ is not defined uniquely by values of ℓ on $C^{\infty}_{\text{fin}}(\Gamma_{\Lambda,n})$, since this class is not dense in the space $C(\Gamma_{\Lambda,n})$; functions from $C^{\infty}_{\text{fin}}(\Gamma_{\Lambda,n})$ vanish with all their derivatives on the border of the compact set Λ. But in what follows *we will consider only the charges $\nu_{\Lambda,n}$, for which $\operatorname{Var} \nu_{\Lambda,n}$ vanishes on the border of the compact set Λ.* Such charges will be defined by ℓ uniquely.

We will extend the representation (6.5) of the functional (6.1) to a more general $F \subset \operatorname{Ran} K$.

Let $m > n$, $m \in \mathbb{N}$. The class $C^{\infty}_{\text{fin}}(\Gamma_{\Lambda,n})$ embeds naturally into the class $C^{\infty}_{\text{fin}}(\Gamma_{\Lambda,m})$; for every function from $C^{\infty}_{\text{fin}}(\Gamma_{\Lambda,n})$ it is necessary to take all its values on $\bigsqcup_{j=n+1}^{m} \Gamma_{\Lambda}^{(j)}$ equal to 0. Therefore, one can write

$$\begin{gathered} \forall F \in C^{\infty}_{\text{fin}}(\Gamma_{\Lambda,n}) \subset C^{\infty}_{\text{fin}}(\Gamma_{\Lambda,m}) \\ \int_{\Gamma_{\Lambda,m}} F(\eta)\, d\nu_{\Lambda,m}(\eta) = \int_{\Gamma_{\Lambda,n}} F(\eta)\, d\nu_{\Lambda,n}(\eta) = \ell(F). \end{gathered} \tag{6.6}$$

Taking into account the fact that F is arbitrary in (6.6) and also the above-mentioned agreement about values of $\nu_{\Lambda,n}$ on the border of Λ, we conclude that $\nu_{\Lambda,m} \restriction \Gamma_{\Lambda,n} = \nu_{\Lambda,n}$. So, we have constructed a sequence of charges $\nu_{\Lambda,n}$ on $\Gamma_{\Lambda,n}$, $n \in \mathbb{N}$, such that for $m > n$ $\nu_{\Lambda,m} \restriction \Gamma_{\Lambda,n} = \nu_{\Lambda,n}$.

Now we will assume that

$$\sum_{k=0}^{\infty} 2^k \rho(\Gamma_\Lambda^{(k)}) < \infty. \tag{6.7}$$

According to the estimate from (6.5) the condition (6.7) means that

$$\forall m \in \mathbb{N} \quad (\operatorname{Var} \nu_{\Lambda,m})(\Gamma_{\Lambda,m}) \le \sum_{k=0}^{\infty} 2^k \rho(\Gamma_\Lambda^{(k)}) < \infty. \tag{6.8}$$

Let us define, on the space $\Gamma_{\Lambda,0} = \bigsqcup_{k=0}^{\infty} \Gamma_\Lambda^{(k)} \subset \Gamma_{X,0}$, a function $\nu_\Lambda(\alpha)$ of sets, where α is a Borel set from $\Gamma_{\Lambda,n}$ with some $n \in \mathbb{N}$ (n depends on α). Namely, we put

$$\nu_\Lambda(\alpha) = \lim_{m\to\infty} \nu_{\Lambda,m}(\alpha); \tag{6.9}$$

if $m \ge n$ then $\nu_{\Lambda,m}(\alpha) = \nu_{\Lambda,n}(\alpha)$, therefore the sequence in (6.9) is stationary. Due to (6.8) this set function can be extended to a finite charge ν_Λ on all Borel sets from $\Gamma_{\Lambda,0}$ (see, e. g., [18]).

The weak topology of $\mathcal{D}'$ induces the corresponding topology in $\Gamma_\Lambda \subset \Gamma_X \subset \mathcal{D}'$. Let a function $\Gamma_\Lambda \ni \gamma \mapsto F(\gamma) \in \mathbb{C}$ be continuous in this topology. Then its restriction to $\Gamma_{\Lambda,n}$ is also continuous and thus there exists the integral $\int_{\Gamma_{\Lambda,n}} F(\gamma) d\nu_{\Lambda,n}(\gamma)$. Therefore, assuming in addition that moreover $F(\gamma)$ is bounded, we see that there also exists the following integral:

$$\int_{\Gamma_\Lambda} F(\gamma)\, d\nu_\Lambda(\gamma) := \int_{\Gamma_{\Lambda,0}} F(\gamma)\, d\nu_\Lambda(\gamma) = \lim_{n\to\infty} \int_{\Gamma_{\Lambda,n}} F(\gamma)\, d\nu_{\Lambda,n}(\gamma). \tag{6.10}$$

If $f \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ then (5.6) implies that $(Kf)(\gamma) = F(\gamma) \subset \operatorname{Ran} K$ is continuous in the topology of Γ_Λ and thus there exists the integral (6.10) for this function $F(\gamma)$ (if it is bounded). But according to (6.6) there is the constant $\ell(F)$ under the limit in (6.10), i. e., the left integral in (6.10) is equal to $\ell(F)$.

So, let $F(\gamma)$ be bounded, and let $(K^{-1}F)(\xi) = f(\xi)$ (and also $F(\gamma)$ because of (5.6)) vanish for $\xi \in \Gamma_X^{(n)} \setminus \Gamma_\Lambda^{(n)}$, $n \in \mathbb{N}$. Then according to (6.1) and (6.10) we can write

$$\int_{\Gamma_{X,0}} (K^{-1}F)(\xi)\, d\rho(\xi) = \ell(F) = \int_{\Gamma_\Lambda} F(\gamma)\, d\nu_\Lambda(\gamma). \tag{6.11}$$

In other words, for $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$, f_n vanish outside Λ^n, and if $(Kf)(\gamma) = F(\gamma)$ is bounded then (6.11) gives

$$\int_{\Gamma_{X,0}} f(\xi)\, d\rho(\xi) = \int_{\Gamma_\Lambda} (Kf)(\gamma)\, d\nu_\Lambda(\gamma). \tag{6.12}$$

Let $\Lambda' \supset \Lambda$ be a compact subset of X, larger than Λ. Then we can write an equality of type (6.12) with Λ' instead of Λ for $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$, connected with Λ'. It is easy to see that the charge $\nu_{\Lambda'}$ is an extension of ν_Λ from Γ_Λ to $\Gamma_{\Lambda'}$. Note also that we can regard every charge ν_Λ as a Borel charge on the entire space Γ_X, supported on Γ_Λ.

Note also that for every $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ its coordinates $f_n(\xi)$ vanish if $\xi \in \Gamma_X^{(n)} \setminus \Gamma_\Lambda^{(n)}$, where $\Lambda \subset X$ is compact.

It is easy to conclude from the above made remarks that there exists a Borel charge ν on Γ_X such that for an arbitrary $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ the following equality holds (see also (3.17) and (5.4)):

$$\int_{\Gamma_{X,0}} f(\xi)\, d\rho(\xi) = \int_{\Gamma_X} (Kf)(\gamma)\, d\nu(\gamma) = r(f). \tag{6.13}$$

According to (5.10) we can get from (6.13) the following formula:

$$\forall f \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D}) \quad r(f) = \int_{\Gamma_X} (If)(\gamma)\, d\nu(\gamma). \tag{6.14}$$

Now we can present the following essential theorem.

Theorem 6.1. *Suppose that the measure ρ from (2.3) is supported on the space of usual configurations (that is, $\rho(\ddot{\Gamma}_{X,0} \setminus \Gamma_{X,0}) = 0$, (5.4)) and satisfies the following condition: for each compact set $\Lambda \subset X$ the series (6.7) is convergent,*

$$\sum_{k=0}^{\infty} 2^k \rho(\Gamma_\Lambda^{(k)}) < \infty. \tag{6.15}$$

Then the space $\Gamma_X \subset \mathcal{D}'$ has full spectral measure μ, the Fourier operator I (3.14) is equal to the Lenard operator K (5.6) (i. e., more exactly, $\forall f \in \mathcal{F}_{\rm fin}(\mathcal{D}) \quad (If)(\omega) = (Kf)(\omega), \quad \omega = \gamma \in \Gamma_X$), and the spectral representation (3.17), (3.18) has now the following form: $\forall f \in \mathcal{F}_{\rm fin}(\mathcal{D})$,

$$\int_{\Gamma_{X,0}} f(\xi)\, d\rho(\xi) = r(f) = \int_{\Gamma_X} (Kf)(\gamma)\, d\mu(\gamma), \tag{6.16}$$

where μ is a probability Borel measure on the space Γ_X.

Proof. Using definition (2.4), condition (5.4), and equality (5.10), we obtain from (6.14) the following equality:

$$\begin{aligned} &\forall f, g \in \mathcal{F}_{\rm fin}(\mathcal{D}) \\ &(f,g)_{\mathcal{H}_r} = r(f \star \overline{g}) = \int_{\Gamma_X} (I(f \star \overline{g}))(\gamma)\, d\nu(\gamma) = \int_{\Gamma_X} (K(f \star \overline{g}))(\gamma)\, d\nu(\gamma) \\ &\qquad = \int_{\Gamma_X} (Kf)(\gamma)\overline{(Kg)(\gamma)}\, d\nu(\gamma) = \int_{\Gamma_X} (If)(\gamma)\overline{(Ig)(\gamma)}\, d\nu(\gamma) \end{aligned} \tag{6.17}$$

(here Proposition 5.1 was used).

Proposition 5.2 shows that $K\mathcal{F}_{\rm fin}(\mathcal{D})$ contains all infinitely differentiable finite functions of an arbitrary number of different variables $y_1, \dots, y_n$, $n \in \mathbb{N}$. Therefore the equality

$$\forall f \in \mathcal{F}_{\rm fin}(\mathcal{D}) \quad \int_{\Gamma_X} |(Kf)(\gamma)|^2 d\nu(\gamma) = (f,f)_{\mathcal{H}_r} \geq 0 \tag{6.18}$$

obtained from (6.17) shows that the charge ν is in fact a Borel measure on Γ_X. This measure is a probability measure. Indeed, take $f(\xi)$, $\xi \in \Gamma_{X,0}$, such that $f(\emptyset) = 1$ and $f(\xi) = 0$ if $\xi \neq \emptyset$ (i. e., $f = e$). Then, according to (5.6), $(Kf)(\gamma) = 1$ and $(f,f)_{\mathcal{H}_r} = 1$. Then it follows from (6.18) that $\nu(\Gamma_X) = 1$.

We conclude from (6.17) that $\forall f, g \in \mathcal{F}_{\rm fin}(\mathcal{D})$,

$$(f,g)_{\mathcal{H}_r} = \int_{\Gamma_X} (If)(\gamma)\overline{(Ig)(\gamma)}\, d\nu(\gamma)$$

and I is the Fourier transform from (3.14). Using [4, 12, 18] it is possible to state that ν is equal to the spectral measure of the family $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$, i. e., $\nu = \mu$.

Other assertions of this theorem are clear. □

So, as result, formulas (3.17), (3.18), and (6.16) are solutions of the investigated moment problem: the first two formulas correspond to the general case, and the last one corresponds to the case where $\rho(\ddot{\Gamma}_{X,0} \setminus \Gamma_{X,0}) = 0$ and (6.15) holds.

The corresponding positivity has the form (2.4). The estimate of growth is formulated in Theorem 2.1. To obtain representation (6.16), it is necessary to additionally demand for condition (6.15) to hold.

Remark 6.1. Formulas (3.17) and (6.16) show that, in an evident way, $r = I^* \mu$ and r can be treated as a "correlation functional" of the measure μ on $\mathcal{D}'$ or on Γ_X. Then the results of Theorems 3.1 and 6.1 provide sufficient conditions for a functional $r \in (\mathcal{F}_{\rm fin}(\mathcal{D}))'$ to be the correlation functional for a measure μ on $\mathcal{D}'$ or on Γ_X.

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