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ON STABILITY AND INSTABILITY OF SMALL MOTIONS OF HYDRODYNAMICAL SYSTEMS

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Dedicated to the 100th anniversary of Mark Krein.

Abstract. We give a short survey of an operator approach to some linear evolution and spectral problems of hydrodynamics: small motions and normal oscillations of a heavy or capillary fluid, a partially filled cavity in a moving or immovable vessel. The main attention is given to the problem of stability and instability of these hydromechanical systems with an infinite number of degrees of freedom.

1. Introduction

In the report, we give a short survey of an operator approach to some evolution and spectral problems of hydrodynamics. A large contribution to investigations of continuous media motions by using methods of functional analysis was made in the works of M. G. Krein and G. Langer [1, 2], S. L. Sobolev [3], S. G. Krein and his students [4]–[6], and others. The theories of Pontryagin spaces, M. Krein spaces, the spectral theory of operator pencils, semigroup theory are widely used for studying these problems (see, for instance, monographs [7]–[10]).

The paper is devoted to problems on stability and instability of small motions of some hydromechanical system, i.e., systems with an infinite number of degrees of freedom. We study the cases of an ideal or a viscous fluid when gravity forces and surface tension (capillary forces) must be taken into account (low-gravity conditions).

In Section 2, we consider a classical problem on small oscillations of a pendulum with a cavity partially filled with a heavy ideal fluid. We prove the theorem on correct solvability of the problem (in some classes of Hilbert spaces), give a variation principle for eigenvalues of the spectral problem and formulate conditions for stability or instability of the hydromechanical system. Section 3 is devoted to the problem on small oscillations of an ideal capillary fluid in an immovable vessel. We consider the same problems for the case. Further, in Section 4, we study a two-dimensional problem for a pendulum with a capillary ideal fluid.

The second part of the survey deals with similar problems for a viscous fluid. In Section 5, we prove a theorem on correct solvability of the initial boundary value problem on small motions of a capillary viscous fluid uniformly rotating in a vessel, consider normal oscillation, prove a so-called Abel–Lidsky basis property for eigen- and associated elements of the corresponding spectral problem and prove the assertion that is called the inversion of the Lagrange theorem on stability.

The same problems are considered in Section 6 for the case of two-dimensional pendulum with a cavity partially filled with a viscous capillary fluid. At last, in Section 7, for problem of normal convective motions (i.e., for a spectral problem) we obtain conditions

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sufficient for stability (the first case) or instability (the second one) of normal convective motions.

2. A pendulum with a cavity partially filled with an ideal heavy incompressible fluid

This problem was, may be, the first one, where (after the famous paper [3]) methods of spectral theory of linear operators acting in the space with an indefinite metric were applied for investigating hydrodynamical problem. The first publication [11] was in 1957 (see also [12]); here we follow the work [13] (see also [14]).

2.1. Let us assume that a rigid body $\Omega_0 \subset \mathbb{R}^3$ (a pendulum) is fixed at a certain point O and performs small motions relative to that point. The body has a cavity partially filled with an ideal incompressible fluid. We assume that, in the nonperturbed state, the fluid fills the region Ω and has a boundary $\partial\Omega$ consisting of a rigid wall S and an equilibrium surface Γ that is orthogonal to the gravitation acceleration $\vec{g}$. We introduce the stationary coordinate system $Oy_1y_2y_3$ with the axis Oy_3 directed against the vector $\vec{g}$, and the nonstationary system $Ox_1x_2x_3$ rigidly connected to the body. We assume also that, in the nonperturbed state, the systems $Oy_1y_2y_3$ and $Ox_1x_2x_3$ coincide.

Then, in the coordinate system $Ox_1x_2x_3$, the problem on small motions of the investigated hydrodynamical system can be formulated in the following form (see, for instance, [13]):

$$\rho\frac{\partial\vec{u}}{\partial t} + \rho\left(\frac{d\vec{\omega}}{dt}\times\vec{r}\right) + \nabla p = \rho\vec{f}, \quad \operatorname{div}\vec{u} = 0 \quad (\text{in } \Omega), \tag{2.1}$$

$$u_n := \vec{u}\cdot\vec{n} = 0 \quad (\text{on } S), \qquad \int_\Gamma \zeta\, d\Gamma = 0, \tag{2.2}$$

$$p = \rho g(\zeta + (\vec{\delta}\times\vec{r})\cdot\vec{e}_3), \qquad \frac{\partial\zeta}{\partial t} = u_3 \quad (\text{on } \Gamma), \qquad \frac{d\vec{\delta}}{dt} - \vec{\omega} = 0, \tag{2.3}$$

$$\vec{J}\frac{d\vec{\omega}}{dt} + \rho\int_\Omega\left(\vec{r}\times\frac{\partial\vec{u}}{\partial t}\right)d\Omega + mgl\,(\delta_1\vec{e}_1 + \delta_2\vec{e}_2) - \rho g\int_\Gamma(\vec{e}_3\times\vec{r})\,\zeta\,d\Gamma = \vec{M}(t), \tag{2.4}$$

$$\begin{gathered}\vec{u}(0,x) = \vec{u}^0(x), \quad x\in\Omega; \quad \zeta(0,x_1,x_2) = \zeta^0(x_1,x_2), \quad (x_1,x_2)\in\Gamma;\\ \vec{\delta}(0) = \vec{\delta}^0, \quad \vec{\omega}(0) = \vec{\omega}^0.\end{gathered} \tag{2.5}$$

Here $\rho > 0$ is a constant density of the fluid, $p = p(t,x)$ is the dynamic pressure, $\vec{u} = \vec{u}(t,x)$ is the relative velocity field, $\vec{\omega} = \vec{\omega}(t)$ is the angular velocity, $\vec{\delta} = \vec{\delta}(t)$ is the angular displacement of the rigid body relatively stationary coordinate system $Oy_1y_2y_3$, $\vec{f} = \vec{f}(t,x)$ is a small field of external mass forces. Further, $\vec{n}$ is the unique external normal to $\partial\Omega$, $\zeta = \zeta(t,x_1,x_2)$ is the displacement field of the moving free surface relatively to the equilibrium surface Γ, $\vec{J}$ is the inertia tensor of the system in the nonperturbed state, $l > 0$ is a distance between the point O and the mass center C of the entire system, $\vec{M}(t)$ is the total moment (relatively to O) of small forces influencing the system, $m > 0$ is the mass of the whole system.

For classic solutions to problem (2.1)–(2.5) the law of full energy balance hold,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big\{\rho\int_\Omega|\vec{u}|^2\,d\Omega + 2\rho\int_\Omega(\vec{\omega}\times\vec{r})\cdot\vec{u}\,d\Omega| + \left(\vec{J}\vec{\omega}\right)\cdot\vec{\omega} + gml(|\delta_1|^2 + |\delta_2|^2)\\ &\qquad + g\rho\int_\Gamma|\zeta|^2\,d\Gamma + 2g\rho\int_\Gamma(\vec{\delta}\times\vec{r})\cdot\vec{e}_3\zeta\,d\Gamma\Big\} = \rho\int_\Omega\vec{f}\cdot\vec{u}\,d\Omega + \vec{M}\cdot\vec{\omega},\end{aligned} \tag{2.6}$$

where in the brackets we have the twice full energy of the system and in the right-hand side—the capacity of the external forces.

2.2. We will assume that the unknown functions $\vec{u}(t,x)$, $\nabla p(t,x)$, $\zeta(t,x_1,x_2)$ are functions of the variable t with values in Hilbert spaces, and we will use the orthogonal decomposition (see, for instance, [7, p. 106], or [8, p. 118]),

$$\vec{L}_2(\Omega)=\vec{J}_0(\Omega)\oplus\vec{G}_{h,S}(\Omega)\oplus\vec{G}_{0,\Gamma}(\Omega),\quad \vec{G}_{0,\Gamma}(\Omega):=\Big\{\nabla\varphi\in\vec{L}_2(\Omega):\varphi=0\ (\text{on }\Gamma)\Big\},$$

$$\vec{G}_{h,S}(\Omega):=\Big\{\nabla\Phi\in\vec{L}_2(\Omega):\ \Delta\Phi=0\ (\text{in }\Omega),\ \frac{\partial\Phi}{\partial n}=0\ (\text{on }S),\ \int_\Gamma\Phi\,d\Gamma=0\Big\},\tag{2.7}$$

$$\vec{J}_0(\Omega):=\Big\{\vec{w}\in\vec{L}_2(\Omega):\ \operatorname{div}\vec{w}=0\ (\text{in }\Omega),\ \vec{w}\cdot\vec{n}=0\ (\text{on }\partial\Omega)\Big\}.$$

Let $\theta:\ L_2(\Gamma)\longrightarrow L_{2,\Gamma}:=L_2(\Gamma)\ominus\{1_\Gamma\}$ be the orthoprojection and $G:\ H^{1/2}(\Gamma)\cap L_{2,\Gamma}\longrightarrow\vec{G}_{h,S}(\Omega)$ be the operator of the boundary value problem

$$\Delta\widetilde{p}=0\quad(\text{in }\Omega),\quad \frac{\partial\widetilde{p}}{\partial n}=0\quad(\text{on }S),\quad \widetilde{p}=\psi\quad(\text{on }\Gamma),\quad \int_\Gamma\psi\,d\Gamma=0,\tag{2.8}$$

i.e., $\nabla\widetilde{p}=G\psi$. Denote also

$$z=(z_1,z_2)^t,\quad z_1:=(\vec{w};\nabla\Phi;\vec{\omega})\in\vec{J}_0(\Omega)\oplus\vec{G}_{h,S}(\Omega)\oplus\mathbb{R}^3=:\mathcal{H}_1,$$
$$z_2=(\zeta;P_2\vec{\delta}),\quad P_2\vec{\delta}=\sum_{k=1}^{2}\delta_k\vec{e}_k,\quad z_2\in L_{2,\Gamma}\oplus\mathbb{R}^2=:\mathcal{H}_2.\tag{2.9}$$

Then problem (2.1)–(2.5) can be rewritten in the form (see [13])

$$A\frac{dz}{dt}+Bz=f(t),\quad z(0)=z^0,\quad f(t)=(f_1(t);0)^t,\quad f_1(t)=(\rho P_0\vec{f},\rho P_{h,S}\vec{f},\vec{M}),\tag{2.10}$$

where

$$A=\operatorname{diag}(A_1;gA_2)=A^*\in\mathcal{L}(\mathcal{H}),\quad \mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2,\tag{2.11}$$

is the full energy operator of the hydromechanical system,

$$B=g\begin{pmatrix}0 & B_{12}\\ B_{21} & 0\end{pmatrix}=-B^*,\quad \mathcal{D}(B)=\mathcal{D}(B_{21})\oplus\mathcal{D}(B_{12}),\tag{2.12}$$

is the operator of an exchange between kinetic and potential energies of the system.

Lemma 2.1. *The following properties of the operator coefficients in* (2.10)–(2.11) *hold:*
i) $0\ll A_1\in\mathcal{L}(\mathcal{H}_1)$ *and it is an operator of kinetic energy;*
ii) $A_2:\ \mathcal{H}_2\longrightarrow\mathcal{H}_2$ *is a bounded from below operator of potential energy,* $A_2\in\mathcal{L}(\mathcal{H}_2)$. *If the conditions*

$$\Delta_1>0,\quad \Delta_2>0,\quad \Delta_1:=ml-\rho\alpha_{11},\quad \Delta_2:=(ml-\rho\alpha_{11})(ml-\rho\alpha_{22})-2\rho\alpha_{12}^2,$$
$$\alpha_{ij}:=\int_\Gamma(\theta x_i)x_j\,d\Gamma,\quad i,j=1,2,\tag{2.13}$$

hold, then $A_2\gg0$. *If* $\Delta_2\neq0$, *then* A_2 *has a bounded inverse operator,* $A_2^{-1}\in\mathcal{L}(\mathcal{H}_2)$, *and the rank of indefiniteness* k *of the operator* A_2 *is equal to* $k=1$ *or* $k=2$.

Lemma 2.2. *The operator* B *is an unbounded closed skew-self-adjoint operator on the domain*

$$\mathcal{D}(B)=\mathcal{D}(B_{21})\oplus\mathcal{D}(B_{12}),\quad \mathcal{D}(B_{21})=H_\Gamma^{1/2}\oplus\mathbb{R}^2,$$
$$\mathcal{D}(B_{12})=\vec{J}_0(\Omega)\oplus\mathcal{D}(\gamma_n)\oplus\mathbb{R}^3,\tag{2.14}$$
$$\mathcal{D}(\gamma_n):=\Big\{\nabla\Phi\in\vec{G}_{h,S}(\Omega):\ \gamma_n\nabla\Phi:=\frac{\partial\Phi}{\partial n}\Big|_\Gamma\in L_{2,\Gamma}\Big\}.$$

2.3. We will further use the following well-known fact: if

$$z^0 \in \mathcal{D}(B), \quad f(t) \in C^1([0,T];\mathcal{H}), \tag{2.15}$$

then there exists a unique strong solution to problem (2.10) in the interval $[0,T]$. The proof of this assertion is based on the property that the operator $A^{-1}B$ with $\mathcal{D}(A^{-1}B) = \mathcal{D}(B)$ is skew-self-adjoint in the Pontryagin space with the scalar product $(A\cdot,\cdot)$ and therefore it is a generator of a group of the unitary operators $\exp\{-tA^{-1}B\}$.

Definition 2.1. We say that problem (2.1)–(2.5) has a strong solution in the interval $[0,T]$ if the following properties hold:

a) $\vec{u}(t,x) \in C^1([0,T];\vec{J}_{0,S}(\Omega))$, $\vec{J}_{0,S}(\Omega) := \vec{J}_0(\Omega) \oplus \vec{G}_{h,S}(\Omega)$,

$$\nabla p(t,x) \in C^1([0,T];\vec{G}(\Omega)),$$

$\vec{G}(\Omega) := \vec{G}_0(\Omega) \oplus \vec{G}_{h,S}(\Omega)$, and $\vec{\omega}(t) \in C^1([0,T];\mathbb{R}^3)$;

b) $\zeta(t,x_1,x_2) \in C^1([0,T];L_{2,\Gamma})$, and $\vec{\delta}(t) \in C^2([0,T];\mathbb{R}^3)$;

c) for any $t \in [0,T]$ the first equation (2.1) is valid and each term in it is a continuous function in t with values in $\vec{L}_2(\Omega)$; equations (2.3) are valid, and each term in it is a continuous function with values in $C^1([0,T];H_\Gamma^{1/2})$, $C^1([0,T];L_{2,\Gamma})$ and $C^1([0,T];\mathbb{R}^3)$, respectively;

d) for any $t \in [0,T]$ equation (2.4) is valid, and each term in t is a function in t with values in $C([0,T];\mathbb{R}^3)$;

e) initial conditions (2.5) hold.

On the base of the above formulated assertion, we prove the following main result (see [13]).

Theorem 2.1. *Let the conditions*

$$\begin{gathered}\vec{u}^0 \in \vec{J}_{0,S}(\Omega), \quad P_{h,S}\vec{u}^0 =: \nabla\Phi^0 \in \vec{G}_{h,S}(\Omega): \ \frac{\partial\Phi}{\partial n}\Big|_\Gamma \in L_{2,\Gamma}; \\ \zeta^0 \in H_\Gamma^{1/2}, \quad \vec{\omega}^0 \in \mathbb{R}^3, \quad \vec{\delta}^0 \in \mathbb{R}^3, \\ \vec{f}(t) \in C^1([0,T];\vec{L}_2(\Omega)), \quad \vec{M}(t) \in C^1([0,T];\mathbb{R}^3)\end{gathered} \tag{2.16}$$

hold.

Then the initial boundary-value problem (2.1)–(2.5) *on small motions of a pendulum with a cavity partially filled with a heavy ideal incompressible fluid has a unique strong solution in the interval* $[0,T]$. *The law of full energy balance* (2.6) *holds for the strong solution and each term in t is a function from* $C^1([0,T])$.

If, instead of (2.16), *the conditions*

$$\begin{gathered}\vec{u}^0 \in \vec{J}_{0,S}(\Omega), \quad \zeta^0 \in L_{2,\Gamma}, \quad \vec{\omega}^0 \in \mathbb{R}^3, \quad \vec{\delta}^0 \in \mathbb{R}^3, \\ \vec{f}(t,x) \in C([0,T];\vec{L}_2(\Omega)), \quad \vec{M}(t) \in C([0,T];\mathbb{R}^3)\end{gathered} \tag{2.17}$$

hold, then problem (2.1)–(2.5) *has a unique generalized solution, and the same law of full energy balance is valid for this solution.*

2.4. Consider solutions to the homogeneous problem (2.10) in the form $z(t) = e^{i\lambda t}z$, $z \in \mathcal{H}$, where λ is a frequency of an oscillation of the system and z is an amplitude element. Then we have the following spectral problem:

$$(iB)z = \lambda Az, \quad iB = (iB)^*, \quad z \in \mathcal{H}. \tag{2.18}$$

Lemma 2.3. *Problem* (2.18) *has an infinitely-multiple eigenvalue $\lambda = 0$ with eigenelements of the form*

$$z = (z_1;z_2)^t, \quad z_1 = (\vec{w};\vec{0};\vec{0}), \quad \forall\vec{w} \in \vec{J}_0(\Omega), \quad z_2 = (\zeta;P_2\vec{\delta}) = (0;\vec{0}). \tag{2.19}$$

Consider eigenoscillations for the case $\lambda \neq 0$ and for the situation when conditions (2.13) of static stability in linear approximation are fulfilled.

Theorem 2.2. *Under the above formulated assumptions, problem* (2.18) *has a discrete spectrum* $\{\mu_k\}_{k=1}^{\infty}$, $\mu_k = \lambda_k^2/g$, *consisting of finite-multiple positive eigenvalues* μ_k *with the limit point* $\mu = +\infty$. *The corresponding system of eigenelements* $(\nabla\Phi_k; \vec{\omega}_k)^t$ *form an orthogonal basis in the space* $\widetilde{H}_1 := \vec{G}_{h,S}(\Omega) \oplus \mathbb{R}^3$. *Eigenvalues* μ_k *are consequent minima of the functional (variation ratio)*

$$\frac{\rho\int_\Gamma \left|\frac{\partial\Phi}{\partial n} + (\vec{\omega}\times\vec{r})\cdot\vec{e}_3\right|^2 d\Gamma - \rho\int_\Gamma |(\vec{\omega}\times\vec{r})\cdot\vec{e}_3|^2\, d\Gamma + ml(|\omega_1|^2 + |\omega_2|^2)}{\rho\int_\Omega \left|\nabla\Phi + \sum_{k=1}^3 \omega_k \nabla\psi_k\right|^2 d\Omega}, \tag{2.20}$$

where

$$\Delta\Phi = 0 \quad (\text{in } \Omega), \qquad \frac{\partial\Phi}{\partial n} = 0 \quad (\text{on } S), \qquad \int_\Gamma \Phi\, d\Gamma = 0,$$

and

$$\Delta\psi_k = 0 \quad (\text{in } \Omega), \qquad \frac{\partial\psi}{\partial n} = (\vec{e}_k\times\vec{r})\cdot\vec{n} \quad (\text{on } \partial\Omega), \qquad k = 1,2,3,$$

(ψ_k are the so-called Zhukovsky potentials). The asymptotic behavior of eigenvalues μ_k is the following:

$$\mu_k = \left(\frac{|\Gamma|}{4\pi}\right)^{-1/2} k^{1/2}[1 + o(1)] \quad (k \longrightarrow \infty). \tag{2.21}$$

Theorem 2.3. (**inversion of the Lagrange theorem on stability**). *Let the condition* $\Delta_2 \neq 0$ *hold but not the conditions* (2.13), *i.e., the system be not statically stable. Then the investigated hydrodynamical system is not dynamically stable, i.e., there exist solutions to problem* (2.1)–(2.5) *that increase in* t *according to the law* $\exp(\lambda_0 t)$ *with* $\lambda_0 > 0$.

3. A capillary ideal fluid partially filled a vessel

3.1. If we consider oscillations of a fluid in conditions close to low-gravity then capillary forces should be taken into account.

Assume that an ideal incompressible fluid partially fills an arbitrary vessel and has an equilibrium free surface Γ. We suppose that this surface is stable under action of gravity and surface tension.

Considering small oscillations of a fluid in the vessel, we have the following initial boundary value problem (see, for instance, [7, p. 158–160] and [8, p. 200–209]):

$$\frac{\partial\vec{u}}{\partial t} + \frac{1}{\rho}\nabla p = \vec{f}(t,x), \quad \operatorname{div}\vec{u} = 0 \quad (\text{in } \Omega), \quad u_n = \vec{u}\cdot\vec{n} = 0 \quad (\text{on } S), \tag{3.1}$$

$$\begin{gathered}\frac{\partial\zeta}{\partial t} = u_n, \quad p = \sigma\mathcal{L}\zeta := \sigma[-\Delta_\Gamma\zeta + a(x)\zeta] \quad (\text{on } \Gamma), \\ a(x) := -(k_1^2 + k_2^2) + \rho\sigma^{-1} g\cos(\widehat{\vec{n}, x_3}), \quad x \in \Gamma,\end{gathered} \tag{3.2}$$

$$\frac{\partial\zeta}{\partial\nu} + \chi\zeta = 0 \quad (\text{on } \partial\Gamma), \quad \chi = \frac{k_\Gamma\cos\delta - k_S}{\sin\delta}, \quad 0 < \delta < \pi, \quad \int_\Gamma \zeta\, d\Gamma = 0, \tag{3.3}$$

$$\vec{u}(0,x) = \vec{u}^0(x) \quad (\text{in } \Omega), \quad \zeta(0,x) = \zeta^0(x) \quad (\text{on } \Gamma). \tag{3.4}$$

Here $\sigma > 0$ is the surface tension coefficient, Δ_Γ is the Laplace-Beltrami operator, $a(x)$ is a known function, k_1 and k_2 are the principal curvatures of Γ, $\partial/\partial\nu$ is the conormal derivative, δ is a wetting angle, k_Γ and k_S are the corresponding curvatures calculated on $\partial\Gamma$ of the cross-sections by a plane orthogonal to $\partial\Gamma$. Other functions are the same as in (2.1)–(2.5).

Introduce the operator $B:\ L_{2,\Gamma} \longrightarrow L_{2,\Gamma}$ by

$$B\zeta := \theta\mathcal{L}\zeta, \quad \zeta = \theta\zeta \in \mathcal{D}(B) := \Big\{\zeta \in L_{2,\Gamma}:\ \mathcal{L}\zeta \in L_{2,\Gamma},\ \frac{\partial\zeta}{\partial\nu} + \chi\zeta = 0 \text{ (on } \partial\Gamma)\Big\}. \tag{3.5}$$

Lemma 3.1. *If the functions $a(x)$ and $\chi(x)$ are continuous then the operator B is self-adjoint and bounded from below. It has a discrete spectrum with a limit point at $+\infty$. Its quadratic form*

$$(B\zeta, \zeta) = \int_\Gamma [|\nabla_\Gamma \zeta|^2 + a|\zeta|^2]\, d\Gamma + \oint_{\partial\Gamma} \chi|\zeta|^2\, ds \tag{3.6}$$

is proportional to the potential energy of the system.

3.2. With using the method of orthogonal projecting on subspaces (2.7), problem (3.1)–(3.4) can be transformed to the following Cauchy problem in the Hilbert space $L_{2,\Gamma}$:

$$\rho\frac{d^2}{dt^2}(A\zeta) + \sigma B\zeta = F(t), \quad \zeta(0) = \zeta^0, \quad \zeta'(0) = \zeta^1, \tag{3.7}$$

where $0 < A < \mathfrak{S}_\infty$, $\nabla F(t) = P_{h,S}\vec{f}$ (in Ω).

Theorem 3.1. *Let an equilibrium state of the fluid in the vessel be statically stable in linear approximation, i.e., $\lambda_{\min}(B)$ of the operator B is positive. Then, under the assumptions*

$$\begin{gathered} \zeta^0 \in H^{5/2}(\Gamma) \cap \mathcal{D}(B), \quad \vec{u}^0 \in \vec{J}_{0,S}(\Omega), \quad \big[(P_{h,S}\vec{u}^0)\cdot\vec{n}\big]_\Gamma \in H^1(\Gamma) \cap L_{2,\Gamma}, \\ \vec{f}(t) \in C([0,T]; \vec{L}_2(\Omega)), \quad (P_{h,S}\vec{f})(t) = \nabla F(t) \in C^1([0,T]; \vec{G}_{h,S}(\Omega)), \end{gathered} \tag{3.8}$$

problem (3.1)–(3.4) *has a unique strong solution in the segment* $[0,T]$*, i.e., the following assertions are valid:*

a) $\vec{u}(t,x) \in C^1([0,T]; \vec{J}_{0,S}(\Omega))$, $\nabla p(t,x) \in C([0,T]; \vec{G}(\Omega))$ *and the first equation* (3.1) *holds for any* $t \in [0,T]$;

b) $\dfrac{\partial\zeta}{\partial t} = (\vec{u}\cdot\vec{n})_\Gamma \in C([0,T]; L_{2,\Gamma})$;

c) $p = \sigma\mathcal{L}\zeta \in C([0,T]; H_\Gamma^{1/2})$;

d) conditions (3.4) *hold.*

The proof of Theorem 3.1 is based on the assertion that Cauchy problem (3.7) has a unique strong solution $\zeta(t)$ with values in $\mathcal{D}(A^{-1/2}) \subset L_{2,\Gamma}$ in the interval $[0,T]$ if the conditions

$$\zeta^0 \in \mathcal{D}(A^{-1/2}B), \quad \zeta^1 \in \mathcal{D}(B^{1/2}), \quad F(t) \in C^1([0,T]; \mathcal{D}(A^{-1/2})) \tag{3.9}$$

are fulfilled.

3.3. For eigenoscillations, i.e., for solutions of the homogeneous problem (3.7) of the form $\zeta(t) = \zeta\exp(i\omega t)$, we have the spectral problem

$$B\zeta = \lambda A\zeta, \quad \lambda = \rho\omega^2\sigma^{-1}, \quad \zeta \in \mathcal{D}(B) \subset L_{2,\Gamma}. \tag{3.10}$$

Theorem 3.2. *If $\lambda_{\min}(B) > 0$, then problem* (3.10) *has a discrete spectrum, consisting of positive eigenvalues λ_k with limit point $\lambda = +\infty$. The numbers λ_k are successive minima of the variation ratio*

$$\begin{gathered} \frac{(\zeta,\zeta)_B}{(\zeta,\zeta)_A} = \Big\{\int_\Gamma [|\nabla_\Gamma\zeta|^2 + a|\zeta|^2]\, d\Gamma + \oint_{\partial\Gamma} \chi|\zeta|^2\, ds\Big\} \Big/ \int_\Omega |\nabla\varphi|^2\, d\Omega, \\ \Delta\varphi = 0 \quad (\text{in } \Omega), \quad \frac{\partial\varphi}{\partial n} = 0 \quad (\text{on } S), \quad \frac{\partial\varphi}{\partial n} = \zeta \quad (\text{on } \Gamma), \quad \int_\Gamma \varphi\, d\Gamma = 0. \end{gathered} \tag{3.11}$$

The asymptotic behavior of λ_k is the following

$$\lambda_k = \left(\frac{|\Gamma|}{4\pi}\right)^{-3/2} k^{3/2}[1 + o(1)] \quad (k \longrightarrow \infty). \tag{3.12}$$

Eigenfunctions $\{\nabla\varphi_k\}_{k=1}^{\infty}$ *form an orthogonal basis in the space* $\vec{G}_{h,S}(\Omega)$.

Theorem 3.3. (**inversion of the Lagrange theorem on stability**). *Suppose that the operator* B *is not positive definite and has exactly* $\varkappa$ *(with regard to multiplicities) negative eigenvalues and the q-multiple eigenvalue* $\lambda = \lambda_0 = 0$. *Then problem* (3.10) *has also* $\varkappa$ *negative eigenvalues and the q-multiple eigenvalue* $\lambda = 0$. *Consequently, in the case, the hydrodynamical system is unstable.*

The proof of the theorem is based on properties of linear operators that are self-adjoint in a Pontryagin space.

4. A pendulum with a capillary ideal fluid

The problem was studied in works of Kopachevsky N. D. and Vadiaa Ali (see [15, 16]).

4.1. Consider the problem of Section 2 but now we will assume that this problem is two-dimensional and a fluid is a capillary one. In the case a pendulum performs small oscillations relatively the axis Oy_1.

Then the statement of the problem is the following (see Sections 2 and 3):

$$\rho\frac{\partial^2\vec{w}}{\partial t^2} + \rho\left(\frac{\partial^2\vec{\delta}}{\partial t^2}\times\vec{r}\right) + \nabla p = \rho\vec{f}, \quad \operatorname{div}\vec{w} = 0 \quad (\text{in } \Omega), \tag{4.1}$$

$$w_n := \vec{w}\cdot\vec{n} = 0 \quad (\text{on } S), \quad \int_\Gamma \zeta\, d\Gamma \quad (\zeta := w_n|_\Gamma), \tag{4.2}$$

$$p = \sigma\mathcal{L}\zeta + \rho g\left(\vec{\delta}\times\vec{r}\right)\cdot\vec{e}_3 \quad (\text{on } \Gamma), \tag{4.3}$$

$$\sigma\mathcal{L}\zeta = -\sigma\Delta_\Gamma\zeta - \sigma k_1^2\zeta + \rho g\cos(\widehat{\vec{n},\vec{e}_3})\zeta, \quad \frac{\partial\zeta}{\partial\nu} + \chi\zeta = 0 \quad (\text{on } \partial\Gamma), \tag{4.4}$$

$$J\frac{d^2\vec{\delta}}{dt^2} + \rho\int_\Omega\left(\vec{r}\times\frac{\partial^2\vec{w}}{\partial t^2}\right)d\Omega + mgl\vec{\delta} - \rho g\int_\Gamma(\vec{e}_3\times\vec{r})\,\zeta\, d\Gamma = \vec{M}(t), \tag{4.5}$$

$$\vec{w}(0,x) = \vec{w}^0(x), \quad \frac{\partial\vec{w}}{\partial t}(0,x) = \vec{u}^0(x), \quad x\in\Omega; \quad \vec{\delta}(0) = \vec{\delta}^0, \quad \vec{\delta}'(0) = \vec{\omega}^0. \tag{4.6}$$

Here $\vec{w} = \vec{w}(t,x)$ is the displacement field of relative motions of the fluid in the vessel, $\vec{\delta}(t) = \delta_1(t)\vec{e}_1$ is the angular displacement of the body, $J > 0$ is the inertia tensor (only one positive constant in the case). Other notations are the same as in Sections 2 and 3.

4.2. As in Section 2, we use the orthogonal decomposition (2.7) and the representations

$$\begin{gathered}\vec{w} = \vec{u} + \nabla\Phi, \quad \vec{u}\in\vec{J}_0(\Omega), \quad \nabla\Phi\in\vec{G}_{h,S}(\Omega),\\ \nabla p = \nabla\varphi + \nabla p_0, \quad \nabla\varphi\in\vec{G}_{h,S}(\Omega), \quad \nabla p_0\in\vec{G}_{0,\Gamma}(\Omega).\end{gathered} \tag{4.7}$$

After projecting (4.1) on subspaces (2.7), using the operator C of the auxiliary boundary value problem, we have

$$\Delta\Phi = 0 \quad (\text{in } \Omega), \quad \frac{\partial\Phi}{\partial n} = 0 \quad (\text{on } S), \quad \frac{\partial\Phi}{\partial n} = \zeta \quad (\text{on } \Gamma), \quad \int_\Gamma \Phi\, d\Gamma = 0, \tag{4.8}$$

acting by the law $C\zeta := \Phi|_\Gamma = C\left(\frac{\partial\Phi}{\partial n}\right)_\Gamma$, and taking into account the so-called Cauchy-Lagrange's integral, we obtain the following Cauchy problem from (4.1)–(4.6):

$$\frac{d^2}{dt^2}\left[\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}\zeta\\ \vec{\delta}\end{pmatrix}\right] + \begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}\begin{pmatrix}\zeta\\ \vec{\delta}\end{pmatrix} = \begin{pmatrix}\rho F\\ \vec{M} - \rho\int_\Omega\left(\vec{r}\times\vec{f}_0\right)d\Omega\end{pmatrix}. \tag{4.9}$$

Here

$$A_{11}\zeta := \rho C\zeta, \quad A_{12}\vec{\delta} := \rho\psi^0\delta_1, \quad A_{21}\zeta := \rho\vec{e}_1 \int_\Gamma \varphi^0\zeta\, d\Gamma, \quad A_{22}\vec{\delta} = (J_b + J_f)\,\vec{\delta}, \tag{4.10}$$

$\nabla\psi^0 = P_{0,S}(\vec{e}_1 \times \vec{r})$, J_b is the inertia tensor of the body and
$J_f := \rho \int_\Omega |(I - P_0)(\vec{e}_1 \times \vec{r})|^2\, d\Omega > 0$ is a modified inertia tensor for the fluid,
$\nabla F = P_{0,S}\vec{f}$, $\vec{f_0} = P_0\vec{f}$, and φ^0 is the Zhukovsky potential,

$$\Delta\varphi^0 = 0 \quad (\text{in } \Omega\,), \quad \frac{\partial\varphi^0}{\partial n} = (\vec{e}_1 \times \vec{r}) \cdot \vec{n} \quad (\text{on } \partial\Omega\,), \quad \int_\Gamma \varphi^0\, d\Gamma = 0. \tag{4.11}$$

Further,

$$\begin{gathered} B_{11}\zeta := B_0\zeta, \quad B_0\zeta := \theta(\sigma\mathcal{L}_\sigma)\theta\zeta, \quad B_{12}\vec{\delta} := \rho g\theta\,((\vec{e}_1 \times \vec{r}) \cdot \vec{e}_3)\,\delta_1, \\ B_{21}\zeta := -\rho g \int_\Gamma (\vec{e}_3 \times \vec{r})\,\zeta\, d\Gamma, \quad B_{22}\vec{\delta} := mgl\vec{\delta}, \end{gathered} \tag{4.12}$$

where $\mathcal{D}(B_0)$ is defined as in (3.5).

Thus, problem (4.1)–(4.6) is transformed to a Cauchy problem for the equation,

$$\frac{d^2}{dt^2}(A\xi) + B\xi = f(t), \quad \xi(0) = \xi^0, \quad \xi'(0) = \xi^1, \tag{4.13}$$

in the Hilbert space $\mathcal{H} = L_{2,\Gamma} \oplus \mathbb{R}$ with

$$\begin{gathered} A = (A_{kl})_{k,l=1}^2, \quad B = (B_{kl})_{k,l=1}^2, \quad \xi = (\zeta; \vec{\delta})^t, \quad f = (\rho F; M - \rho \int_\Omega (\vec{r} \times \vec{f_0})\, d\Omega)^t, \\ \xi^0 = (\zeta^0; \vec{\delta}^0)^t, \quad \zeta^0 = (\vec{w}^0 \cdot \vec{n})_\Gamma; \quad \xi^1 = (\zeta^1; \vec{\omega}^0)^t, \quad \zeta^1 := (\vec{u}^0 \cdot \vec{n})_\Gamma. \end{gathered} \tag{4.14}$$

4.3. Consider properties of the operator matrices A and B in (4.13).

Lemma 4.1. *The operator $A : \mathcal{H} \longrightarrow \mathcal{H}$ is a self-adjoint positive compact operator, acting in (real) Hilbert space $\mathcal{H} = L_{2,\Gamma} \oplus \mathbb{R}$.*

Lemma 4.2. *The operator $B_0 : \mathcal{D}(B_0) \subset L_{2,\Gamma} \longrightarrow L_{2,\Gamma}$ is a bounded from below self-adjoint operator with discrete spectrum. If $B_0 \gg 0$, then $\mathcal{D}(B_0^{1/2}) = H^1(\Gamma) \cap L_{2,\Gamma} =: H^1_\Gamma$.*

Lemma 4.3. *The operator $B : \mathcal{D}(B) = \mathcal{D}(B_0) \oplus \mathbb{R} \subset \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded from below self-adjoint operator with discrete spectrum. If*

$$\lambda_{\min}(B_0) - g\rho^2(\sigma ml)^{-1} > 0, \tag{4.15}$$

where l is the distance between the origin and the mass center, then $B \gg 0$.

If $\lambda_{\min}(B_0) < 0$, then the quadratic form $(B\xi, \xi)_\mathcal{H}$ can take negative values. If $\lambda_{\min}(B_0) = 0$ and $\int_\Gamma \zeta_1(B_0)x_2\, d\Gamma \neq 0$ (where $\zeta_1(B_0)$ is the first eigenelement of B_0), then the form $(B\xi, \xi)_\mathcal{H}$ can also take negative values.

On the base of these properties, we formulate, as in Section 2 and Section 3, the main result (see Theorem 3.1).

Theorem 4.1. *Under the assumptions*

$$\begin{gathered} \vec{w}^0 \in \vec{J}_{0,S}(\Omega), \quad \zeta^0 := (\vec{w}^0 \cdot \vec{n})_\Gamma \in H^{5/2}(\Gamma) \cap \mathcal{D}(B_0), \quad \left[(P_{h,S}\vec{u}^0) \cdot \vec{n}\right]_\Gamma \in H^1_\Gamma, \\ \vec{f} \in C([0,T]; \vec{L}_2(\Omega)), \quad P_{h,S}\vec{f} = \nabla F \in C^1([0,T]; \vec{G}_{h,S}(\Omega)), \\ \vec{\delta}^0 \in \mathbb{R}, \quad \vec{\omega}^0 \in \mathbb{R}, \quad \vec{M}(t) \in C^1([0,T]), \end{gathered} \tag{4.16}$$

problem (4.13) has a unique strong solution with values in $H^{1/2}_\Gamma = H^{1/2}(\Gamma) \cap L_{2,\Gamma}$, and problem (4.1)–(4.6) has a unique strong solution on the segment $[0,T]$, i.e., *a)* $\vec{w}(t,x) \in C^2([0,T];\vec{J}_{0,S}(\Omega))$, $\nabla p(t,x) \in C([0,T];\vec{G}(\Omega))$ *and the first equation* (4.1) *holds for any* $t \in [0,T]$;

b) $\zeta = w_n \in C([0,T]; H^{5/2}(\Gamma) \cap \mathcal{D}(B_0))$ *and equation* (4.3) *holds for any* $t \in [0,T]$;

c) $\vec{\delta}(t) = \delta_1(t)\vec{e}_1 \in C^2([0,T];\mathbb{R})$ *and equation* (4.5) *is valid;*

d) *the initial conditions hold.*

4.4. Consider the cases where condition (4.15) is fulfilled or not, i.e., the hydromechanical system is statically stable or unstable in linear approximation. In problem on eigenoscillations, we have once more problem (3.10):

$$B\xi = \lambda A\xi, \quad \lambda = \omega^2, \quad \xi \in \mathcal{D}(B) \subset \mathcal{H}. \tag{4.17}$$

Theorem 4.2. *Problem* (4.17) *has a real discrete spectrum with limit point at* $+\infty$. *Eigenvalues of* (4.17) *are successive minima of variation ratio,*

$$(B\xi,\xi)_{\mathcal{H}}/(A\xi,\xi)_{\mathcal{H}}, \quad \xi = (\zeta;\vec{\delta})^t \in \mathcal{D}(B), \tag{4.18}$$

$$\begin{aligned}
(A\xi,\xi)_{\mathcal{H}} &= \rho\int_\Omega |\nabla\Phi + \delta_1\nabla\varphi^0|^2\,d\Omega + J_b|\delta_1|^2,\\
(B\xi,\xi)_{\mathcal{H}} &= \rho g\int_\Gamma |\zeta + \delta_1(\theta x_2)|^2 ds + g\Big(ml - \rho\int_\Gamma |\theta x_2|^2\,d\Gamma\Big)|\delta_1|^2\\
&\quad + \int_\Gamma \Big\{\sigma|\zeta'(s)|^2 + \Big[\rho g(\cos(\widehat{\vec{n},\vec{e}_3}) - 1) - \sigma k_1^2(s)\Big]|\zeta(s)|^2\Big\}\,ds\\
&\quad + \sigma\sum_{i=1}^{2}\chi(s_i)|\zeta(s_i)|^2,
\end{aligned} \tag{4.19}$$

where s_i $(i=1,2)$ *are the ends of* Γ *and* Φ *is a solution to problem* (4.8).

If the conditions

$$\lambda_1(B) \leqslant \cdots \leqslant \lambda_\varkappa(B) < 0 = \lambda_{\varkappa+1}(B) = \cdots = \lambda_{\varkappa+q}(B) < \lambda_{\varkappa+q+1}(B) \leqslant \cdots \tag{4.20}$$

hold, then problem (4.17) *has eigenvalues* $\{\lambda_k(A;B)\}_{k=1}^\infty$ *with the same properties as in* (4.20). *In particular, if* $\varkappa \geqslant 1$, *then we have the assertion that solutions to problem* (4.1)–(4.16) *are unstable.*

5. A capillary viscous fluid uniformly rotating in a vessel

In the section, we follow [17, 18] and [9, pp. 217–228].

5.1. Consider small oscillations of a viscous incompressible capillary fluid partially filling an arbitrary uniformly rotating (relatively the axis Ox_3) vessel. We assume that an equilibrium state of the fluid is stable. Then we have the following initial boundary value problem (see [7, p. 356], [9, p. 188]):

$$\frac{\partial\vec{u}}{\partial t} - 2\omega_0\vec{u}\times\vec{e}_3 + \frac{1}{\rho}\nabla p = \nu\Delta\vec{u} + \vec{f}, \quad \operatorname{div}\vec{u} = 0 \quad (\text{in } \Omega); \tag{5.1}$$

$$\vec{u} = \vec{0} \quad (\text{on } S), \quad \frac{\partial\zeta}{\partial t} = \gamma_n\vec{u} := \vec{u}\cdot\vec{n} \quad (\text{on } \Gamma), \quad \int_\Gamma \zeta\,d\Gamma = 0; \tag{5.2}$$

$$\rho\nu(u_{i,3} + u_{3,i}) = 0, \quad i = 1,2; \quad -p + 2\rho\nu u_{3,3} = -\mathcal{L}_\sigma\zeta := \sigma\Delta_\Gamma\zeta - a_\Gamma\zeta \quad (\text{on } \Gamma), \tag{5.3}$$

$$a_\Gamma := -\sigma(k_1^2 + k_2^2) + \rho g\cos(\widehat{\vec{n},\vec{e}_3}) - \rho\omega_0^2 r\cos(\widehat{\vec{n},\vec{e}_r}), \quad \zeta = 0 \quad (\text{on } \partial\Gamma), \tag{5.4}$$

$$\vec{u}(0,x) = \vec{u}^0(x), \quad x \in \Omega; \quad \zeta(0,x) = \zeta^0(x), \quad x \in \Gamma. \tag{5.5}$$

Here $\vec{\omega}_0 = \omega_0\vec{e}_3$ is the angular velocity of the vessel, $\nu > 0$ is the kinematic viscosity, $u_{i,j}$ is the covariant derivative of u_i with respect to ξ^j (in a local coordinate system in some neighborhood of Γ, with the Lame coefficient $h^3 \equiv 1$ on Γ), $\vec{u}(t,x)$ is the relative velocity of the fluid, $p(t,x)$ is the dynamical pressure, $\zeta(t,x)$ is the displacement field along the external normal $\vec{n}$ of the moving free surface $\Gamma(t)$ from Γ.

5.2. Let $\vec{J}^1_{0,S}(\Omega)$ be the Hilbert space of velocity fields with a finite dissipation of energy, i.e., the vector space with the squared norm

$$\|\vec{u}\|^2_{1,\Omega} := \frac{1}{2}\int_\Omega \sum_{k,j=1}^{3} \left|\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}\right|^2 d\Omega, \quad \operatorname{div}\vec{u} = 0 \quad (\text{in } \Omega), \quad \vec{u} = \vec{0} \quad (\text{on } S). \tag{5.6}$$

The space $\vec{J}^1_{0,S}(\Omega)$ is dense in $\vec{J}_{0,S}(\Omega)$ ([7, p. 114]).

Using two so-called auxiliary boundary value S. Krein's problems ([8, p. 277–280]) we transform (5.1)–(5.5) to the problem

$$\frac{d\vec{v}}{dt} + \frac{d\vec{w}}{dt} + \nu A\vec{v} - 2i\omega_0 S_0(\vec{v} + \vec{w}) = P_{0,S}\vec{f}, \quad \vec{v}(0) = \vec{v}^0, \tag{5.7}$$

$$\frac{d\vec{w}}{dt} + \nu^{-1} V B_\sigma \gamma_n(\vec{v} + \vec{w}) = \vec{0}, \quad \vec{w}(0) = \vec{w}^0, \tag{5.8}$$

$$\vec{v}^0 = \vec{u}^0 - \vec{w}^0, \quad \vec{w}^0 = -\nu V B_\sigma \zeta^0, \tag{5.9}$$

where $A \gg 0$ is the operator of the first auxiliary problem, $\mathcal{D}(A^{1/2}) = \vec{J}^1_{0,S}(\Omega)$, $0 < A^{-1} \in \mathfrak{S}_\infty$, $S_0 = S_0^*$, $\sigma(S_0) = \sigma_{\text{ess}}(S_0) = [-1, 1]$, V is the operator of the second auxiliary problem, $B_\sigma = \theta\mathcal{L}_\sigma\theta$, $\mathcal{D}(B_\sigma) = \{\zeta \in L_{2,\Gamma} : \ B_\sigma\zeta \in L_{2,\Gamma}, \ \zeta = 0 \ (\text{on } \partial\Gamma)\}$, B_σ is a self-adjoint bounded from below operator of potential energy of the system.

Further,

$$\vec{J}^1_{0,S}(\Omega) = \vec{N}_1(\Omega) \oplus \vec{M}_1(\Omega), \quad \vec{N}_1(\Omega) := \{\vec{v} \in \vec{J}^1_{0,S}(\Omega) : \ \gamma_n\vec{v} = 0 \quad (\text{on } \Gamma)\} \tag{5.10}$$

and $\vec{M}_1(\Omega)$ is a subspace of all weak solutions to the second auxiliary problem.

After substitutions $\vec{v} = A^{-1/2}\vec{\xi}$, $\vec{w} = A^{-1/2}\vec{\eta}$, we have from (5.7)–(5.9) the problem

$$\frac{d\vec{\xi}}{dt} + \nu A\vec{\xi} - 2i\omega_0 A^{1/2} S_0 A^{-1/2}(\vec{\xi} + \vec{\eta}) - \nu^{-1} B(\vec{\xi} + \vec{\eta}) = A^{1/2} P_{0,S}\vec{f}, \tag{5.11}$$

$$\frac{d\vec{\eta}}{dt} + \nu^{-1} B(\vec{\xi} + \vec{\eta}) = \vec{0}, \quad \vec{\eta}(0) = -\nu^{-1} Q^* B_\sigma \zeta^0, \quad \vec{\xi}(0) = A^{1/2}\vec{u}^0 - \vec{\eta}(0), \tag{5.12}$$

$$B = Q^* B_\sigma Q, \quad Q = \gamma_n A^{-1/2}, \quad Q^* = A^{1/2} V. \tag{5.13}$$

Lemma 5.1. *The operator $B : \ \vec{M}_0(\Omega) \longrightarrow \vec{M}_0(\Omega)$, $\vec{M}_0(\Omega) := A^{1/2}\vec{M}_1(\Omega)$, is a self-adjoint operator with discrete spectrum with limit point at $+\infty$. Eigenvalues of the operators B and A have the asymptotic behavior, respectively,*

$$\lambda_j(B) = \sigma\left(\frac{|\Gamma|}{\pi}\right)^{-1/2} j^{1/2}[1 + o(1)], \quad \lambda_j(A) = \left(\frac{|\Omega|}{3\pi^2}\right)^{-2/3} j^{2/3}[1 + o(1)] \quad (j \longrightarrow \infty). \tag{5.14}$$

5.3. Suppose $\lambda_{\min}(B) > 0$. Introduce the operators

$$\begin{gathered} R := B^{1/2} P A^{-1/2} : \ \vec{J}_{0,S}(\Omega) \longrightarrow \vec{M}_0(\Omega), \\ R^+ := A^{-1/2} P B^{1/2}, \quad \mathcal{D}(R^+) := \mathcal{D}(B^{1/2}) \subset \vec{M}_0(\Omega), \end{gathered} \tag{5.15}$$

where $P : \ \vec{J}_{0,S}(\Omega) \longrightarrow \vec{M}_0(\Omega)$ is the orthoprojection.

Lemma 5.2. *For the operators R and R^+ the following properties hold:*

$$R \in \mathfrak{S}_\infty, \quad R^+ = R^* | \mathcal{D}(B^{1/2}), \quad \overline{R^+} = R^* \in \mathfrak{S}_\infty. \tag{5.16}$$

After substitution $\vec{w} = \nu^{-1}R^{+}\vec{z}$, problem (5.7)–(5.9) can be rewritten in the form

$$(\mathcal{I} + \frac{1}{\nu}\mathcal{R}^*)\frac{dy}{dt} + (\mathcal{I} + \mathcal{F})\mathcal{A}_0 y = f_0(t), \quad y(0) = y^0,$$

$$\mathcal{F} := \frac{1}{\nu}\mathcal{R} - 2i\omega_0 S = \frac{1}{\nu}\begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix} - 2i\omega_0 \begin{pmatrix} \frac{1}{\nu}S_0A^{-1} & S_0A^{-1/2}B^{-1/2} \\ 0 & 0 \end{pmatrix}, \tag{5.17}$$

$$\mathcal{A}_0 := \begin{pmatrix} \nu A & 0 \\ 0 & \nu^{-1}B \end{pmatrix}, \quad y = \begin{pmatrix} \vec{v} \\ \vec{z} \end{pmatrix}, \quad f_0(t) = \begin{pmatrix} P_{0,S}\vec{f} \\ \vec{0} \end{pmatrix}, \quad y^0 = \begin{pmatrix} \vec{v}(0) \\ \vec{z}(0) \end{pmatrix}.$$

Here we assumed that $B \gg 0$, i.e., the hydrodynamical system is statically stable in linear approximation.

Theorem 5.1. *Let for problem* (5.1)–(5.5) *the following assumptions be valid:*

$$\vec{f}(t,x) \in C^{\alpha}([0,T]; \vec{L}_2(\Omega)), \quad 0 < \alpha \leqslant 1, \quad \vec{u}^0 \in \vec{J}_{0,S}(\Omega), \quad \vec{u}^0 = \vec{v}^0 + \vec{w}^0, \tag{5.18}$$

$$\vec{v}^0 \in \mathcal{D}(A) \subset \vec{J}^1_{0,S}(\Omega), \quad \vec{w}^0 \in \vec{M}_1(\Omega) \subset \vec{J}^1_{0,S}(\Omega), \quad \gamma_n \vec{w}^0 \in \mathcal{D}(B_\sigma^{1/2}). \tag{5.19}$$

Then problem (5.17) *has a unique strong solution in the interval* $[0,T]$.

The proof of the theorem is based on the assertion that problem (5.17) is equivalent to an abstract parabolic Cauchy problem with operator coefficient that is a generator of an analytic semigroup.

Theorem 5.1 allows us, under some additional assumptions of smoothness for ζ^0 and $\vec{u}^0$, functions $B\vec{z}(t)$ and $PA\vec{v}(t)$, to prove that the initial boundary value problem (5.1)–(5.5) has a unique strong solution (see [18]).

5.4. Consider solutions to homogeneous problem (5.17) in the form $y(t) = ye^{-\lambda t}$, where λ is an eigenvalue (complex frequency of oscillations) and y is an amplitude element. Then for unknown y and λ we have the spectral problem

$$(\mathcal{I} + \mathcal{F})\mathcal{A}_0 y = \lambda(\mathcal{I} + \nu^{-1}\mathcal{R}^*)y, \quad y = (\vec{v}; \vec{z})^t \in \mathcal{D}(\mathcal{A}_0). \tag{5.20}$$

Further we will use the following definition (see [19, p. 248–249]). The sequence $\{f_j\}_{j=1}^{\infty}$ of elements from a Hilbert space $\mathcal{H}$ is said to be an Abel–Lidsky basis of the order $\alpha > 0$ if there exists an increasing index sequence m_k, $0 = m_0 < m_1 < \ldots < m_l < \ldots$, such that the series

$$\sum_{l=0}^{\infty} P_l f, \quad P_l f := \sum_{j=m_l+1}^{m_{l+1}} (f, g_j) e_j(t) f_j, \quad e_j(t) := \exp(-\mu_j^{\alpha} t), \quad \forall f \in \mathcal{H},$$

$$\mu_j \in \Lambda_\theta := \{\lambda \in \mathbb{C} : \ |\arg \lambda| < \theta\}, \quad e_j(t) \equiv 1 \ (\mu_j \notin \Lambda_\theta),$$

converges for $t > 0$ and the sum $f(t)$ of the series tends to f as $t \longrightarrow +0$. In the definition, $\{g_j\}$ is a system biorthogonal to $\{f_j\}$, $\{f_j\}$ is a sequence of eigenelements of an operator with discrete spectrum and all eigenvalues μ_j of the operator (except for a finite number of them) are located in Λ_θ.

Theorem 5.2. *The spectrum of problem* (5.20) *is discrete with limit point at ∞. All eigenvalues λ are finite-multiple and are located (except for a finite numbers of them) in the sector Λ_ε for any $\varepsilon > 0$. Root (eigen- and associated-)elements to problem* (5.20) *form an Abel–Lidsky basis of order $\alpha > 2$. The asymptotic behavior of eigenvalues to problem* (5.20) *is the following:*

$$\lambda_j = \sigma\nu^{-1}\left(\frac{1}{\pi}|\Gamma|\right)^{-1/2} j^{1/2}[1 + o(1)] \quad (j \longrightarrow \infty). \tag{5.21}$$

The proof of the theorem is based on assertions 1^0–2^0 from [19, p. 292].

As a corollary of Theorem 5.2 we have the following statement: if the hydrodynamical system is statically stable in linear approximation then all normal motions, i.e., solutions of the form $y\exp(-\lambda t)$, are asymptotically stable, because $\operatorname{Re}\lambda_j > 0$ for any $j \geqslant 1$.

5.5. Assume now that $\lambda_{\min}(B) < 0$ and therefore the hydrodynamical system is not statically stable (in linear approximation). Consider the spectral problem

$$A\vec{\xi} - \alpha B(\vec{\xi} + \vec{\eta}) = \widetilde{\lambda}\vec{\xi}, \quad \vec{\xi} \in \mathcal{D}(A), \quad \alpha = \nu^{-2}, \quad \widetilde{\lambda} := \lambda/\nu, \tag{5.22}$$

$$\alpha B(\vec{\xi} + \vec{\eta}) = \widetilde{\lambda}\vec{\eta}, \quad P\vec{\xi} + \vec{\eta} \in \mathcal{D}(B), \tag{5.23}$$

generated by the evolution problem (5.11)–(5.13) for the case $\omega_0 = 0$.

Lemma 5.3. *Let the conditions*

$$\lambda_1(B_\sigma) \leqslant \cdots \leqslant \lambda_\varkappa(B_\sigma) < 0 = \lambda_{\varkappa+1}(B_\sigma) = \cdots = \lambda_{\varkappa+q}(B_\sigma) < \lambda_{\varkappa+q+1}(B_\sigma) \leqslant \cdots \tag{5.24}$$

be valid for the operator B_σ. Then the properties

$$\lambda_1(B) \leqslant \cdots \leqslant \lambda_\varkappa(B) < 0 = \lambda_{\varkappa+1}(B) = \cdots = \lambda_{\varkappa+q}(B) < \lambda_{\varkappa+q+1}(B) \leqslant \cdots \tag{5.25}$$

hold for the operator $B = Q^* B_\sigma Q : \mathcal{D}(B) \subset \vec{M}_0(\Omega) \longrightarrow \vec{M}_0(\Omega)$.

Lemma 5.4. *If* $\operatorname{Ker} B \neq \{0\}$ *and $q > 0$ in* (5.25) *then problem* (5.22)–(5.23) *has a transient solution (from the right half-plane to the left) of the form*

$$\widetilde{\lambda} = \widetilde{\lambda}_0 = 0, \quad \vec{\eta} = \vec{\eta}_0 = \vec{\psi}, \quad \forall \psi \in \operatorname{Ker} B, \quad \vec{\xi} = \vec{0}. \tag{5.26}$$

Theorem 5.3. (**the principle of changing stability**). *The transition of the eigenvalues from the right complex half-plane to the left one occurs along the real axis, where the eigenelements corresponding to such λ_s' have no associated elements.*

Lemma 5.5. *For $\widetilde{\lambda} \neq 0$ problem* (5.22), (5.23) *is equivalent to the problem*

$$\alpha A^{-1}\vec{\xi} + \alpha A^{-1}P_1\vec{\eta} = \mu\vec{\xi}, \quad \mu := \alpha\widetilde{\lambda}^{-1}, \tag{5.27}$$

$$-\alpha P_1 A^{-1}\vec{\xi} + (B_1^{-1} - \alpha P_1 A^{-1} P_1)\vec{\eta} = \mu\vec{\eta}, \tag{5.28}$$

where $P_1 : \vec{M}_0(\Omega) \longrightarrow \vec{M}_0(\Omega) \ominus \operatorname{Ker} B$, $B_1 := P_1 B P_1$.

Lemma 5.6. *If the condition*

$$4\alpha < \lambda_1(A)/|\lambda_1(B_1)| \tag{5.29}$$

holds then problem (5.27)–(5.28) *has exactly $\varkappa$ eigenvalues μ (counting the multiplicities) in the left half-plane.*

As a corollary of Lemmas 5.5 and 5.6 we get the following main result.

Theorem 5.4. *Let for the operator B_σ conditions* (5.24) *be valid. Then the spectral problem*

$$A\{\vec{\xi} - 2i\omega_0\nu^{-1}S(\vec{\xi} + \vec{\eta})\} - \alpha B(\vec{\xi} + \vec{\eta}) = \widetilde{\lambda}\vec{\xi}, \tag{5.30}$$

$$\alpha B(P\vec{\xi} + \vec{\eta}) = \widetilde{\lambda}\vec{\eta}, \quad P\vec{\xi} + \vec{\eta} \in \mathcal{D}(B), \tag{5.31}$$

$$\vec{\xi} - 2i\omega_0\nu^{-1}S(\vec{\xi} + \vec{\eta}) \in \mathcal{D}(A), \quad S := A^{-1/2}S_0A^{-1/2}, \tag{5.32}$$

has exactly $\varkappa$ eigenvalues located in the left half-plane and the q-multiple eigenvalue $\lambda = \lambda_0 = 0$.

In particular, if $\varkappa \geqslant 1$, $q \geqslant 0$, then the hydrodynamical system under consideration is dynamical unstable.

The proof of Theorem 5.4 is based on perturbation theory and on the assertion that eigenvalues are continuous functions of the parameters of the problem (see [9, p. 217–228], [18]).

Hypothesis 5.1. *For arbitrary $\nu > 0$ and $\omega_0 = 0$, the spectral problem* (5.20) *has no more than a finite number of pairs of non-real eigenvalues (located symmetrically relatively to the real axis). If ν is sufficiently large then all the eigenvalues are real and positive.*

The hypothesis was expressed by N. D. Kopachevsky, S. G. Krein and A. D. Mishkis in 1966. There are some hydrodynamical problem (see [10], Section 7.3, 7.4) verifying it.

6. A pendulum with a capillary viscous fluid

This problem, on the one hand, is a generalization of the problem of Section 4 in the case of a viscous capillary fluid and, on the other hand, is a partial case of the problem in Section 5 for $\omega_0 = 0$.

6.1. As in Section 4, we consider a two-dimensional problem on small oscillations of a pendulum with a cavity partially filled with a capillary fluid, but now we assume that the fluid is viscous. Then we have (see Section 4 and 5) the following initial boundary value problem:

$$\rho\frac{\partial \vec{u}}{\partial t} + \rho\left(\frac{d\vec{\omega}}{dt}\times\vec{r}\right) + \nabla p = \rho\nu\Delta\vec{u} + \rho\vec{f}, \quad \operatorname{div}\vec{u} = 0 \quad (\text{in } \Omega); \tag{6.1}$$

$$\rho\int_\Omega \left(\vec{r}\times\frac{\partial\vec{u}}{\partial t}\right) d\Omega + J\frac{d\vec{\omega}}{dt} + \alpha\vec{\omega} + mgl\vec{\delta} - \rho g\int_\Gamma (\vec{e}_3\times\vec{r})\zeta\, d\Gamma = \vec{M}(t), \tag{6.2}$$

$$\vec{u} = \vec{0} \quad (\text{on } S), \qquad \int_\Gamma \zeta\, d\Gamma = 0, \qquad \frac{\partial\zeta}{\partial t} = u_n \quad (\text{on } \Gamma), \qquad \frac{d\vec{\delta}}{dt} - \vec{\omega} = 0, \tag{6.3}$$

$$\rho\nu(u_{2,3} + u_{3,2}) = 0, \quad -p + 2\rho\nu u_{3,3} = -\mathcal{L}_\sigma\zeta + \rho g(\vec{\delta}\times\vec{r})\cdot\vec{e}_3 \quad (\text{on } \Gamma), \tag{6.4}$$

$$\mathcal{L}_\sigma\zeta := -\sigma\Delta_\Gamma\zeta - \sigma k_1^2\zeta + \rho g\cos(\widehat{\vec{n},\vec{e}_3})\zeta, \quad \zeta = 0 \ (\text{on } \Gamma), \tag{6.5}$$

$$\vec{u}(0,x) = \vec{u}^0(x), \quad x\in\Omega; \quad \zeta(0,x) = \zeta^0(x), \quad x\in\Gamma; \quad \vec{\delta}(0) = \vec{\delta}^0, \quad \vec{\omega}(0) = \vec{\omega}^0. \tag{6.6}$$

Here the notations are the same as in Section 4 and 5 and $\alpha > 0$ is the friction coefficient on the axis Ox_1. Besides, on $\partial\Gamma$ (i.e., at the ends of the equilibrium line Γ) we have, as in Section 5, a Dirichlet condition for ζ.

For classical solutions to problem (6.1)–(6.6), the law of full energy balance holds in the following form (compare with (2.6)):

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{\left[\rho\int_\Omega |\vec{u}|^2\, d\Omega + 2\rho\int_\Omega \vec{u}\cdot(\vec{\omega}\times\vec{r})\, d\Omega + J|\vec{\omega}|^2\right]\right.\\ &\qquad\left. + \left[(\zeta,\zeta)_{B_0} + 2\rho\int_\Gamma ((\vec{\delta}\times\vec{r})\cdot\vec{e}_3)\zeta\, d\Gamma + mgl|\vec{\delta}|^2\right]\right\}\\ &\qquad = -\rho\nu\|\vec{u}\|^2_{1,\Omega} - \alpha|\vec{\omega}|^2 + \rho\int_\Omega \vec{f}\cdot\vec{u}\, d\Omega + \vec{M}\cdot\vec{\omega}, \end{aligned} \tag{6.7}$$

$$(\zeta,\zeta)_{B_0} := \int_\Gamma \left[\sigma|\nabla_\Gamma\zeta|^2 + a|\zeta|^2\right] d\Gamma, \quad a := -\sigma k_1^2 + \rho g\cos(\widehat{\vec{n},\vec{e}_3}), \tag{6.8}$$

see the definition of the norm $\|\cdot\|_{1,\Omega}$ in (5.6) for the two-dimensional case.

6.2. For problem (6.1)–(6.6) we use the same operator approach that was applied to problem (5.1)–(5.5). We represent $\vec{u}$ in the form

$$\vec{u} = \vec{v} + \nu^{-1} R^{+} \vec{z}, \quad \vec{v} \in \mathcal{D}(A) \subset \vec{J}_{0,S}(\Omega), \quad \vec{z} \in \mathcal{D}(B) \subset \vec{M}_0(\Omega), \tag{6.9}$$

and transform problem (6.1)–(6.6) to a Cauchy problem of the form (5.17) where now

$$y(t) = (\vec{v}(t); \vec{z}(t); \vec{\omega}(t); \zeta(t); \vec{\delta}(t))^t \in \mathcal{D}(A) \oplus \mathcal{D}(B) \oplus \mathbb{R} \oplus L_{2,\Gamma} \oplus \mathbb{R} =: \mathcal{H}. \tag{6.10}$$

More precisely, we have the problem

$$(\mathcal{A}_0 + \mathcal{R}_1)\frac{dy}{dt} + (\mathcal{I} + \mathcal{R}_2)\mathcal{B}_0 y + \mathcal{R}_3 y = f(t), \quad y(0) = y^0, \tag{6.11}$$

$$\mathcal{A}_0 = \operatorname{diag}(\rho I; \rho I; J; I; 1) \gg 0, \quad \mathcal{R}_1 \in \mathfrak{S}_\infty(\mathcal{H}), \quad \operatorname{Ker}(\mathcal{A}_0 + \mathcal{R}_1) = \{0\}, \tag{6.12}$$

$$\mathcal{B}_0 := \operatorname{diag}(\rho\nu A; \nu^{-1} B; \alpha; I; 1) \gg 0, \quad \mathcal{R}_2 \in \mathfrak{S}_\infty(\mathcal{H}), \quad \mathcal{R}_3 \in \mathcal{L}(\mathcal{H}). \tag{6.13}$$

Problem (6.11)–(6.13) is transformed, as in Section 5, to an abstract parabolic equation in the Hilbert space $\mathcal{H}$, and an analytic semigroup corresponds to it. Therefore, under the assumptions $f(t) \in C^\gamma([0,T]; \mathcal{H})$, $\gamma \in (0,1]$, $y_0 \in \mathcal{D}(B_0)$, problem (6.11) has a unique strong solution in the interval $[0,T]$. (For comparison, see Theorem 5.1). This assertion is proved under the assumption that the investigated hydromechanical system is stable in linear approximation, i.e., the quadratic form of the potential energy of the system, namely, the form

$$(\zeta, \zeta)_{B_0} + 2\rho \int_\Gamma ((\vec{\delta} \times \vec{r}) \cdot \vec{e}_3)\zeta \, d\Gamma + mgl|\vec{\delta}|^2, \tag{6.14}$$

takes positive values for $(\zeta; \vec{\delta})^t \neq 0$ (see (6.8)).

Theorem 6.1. (**inversion of the Lagrange theorem on stability**). *If the form* (6.14) *takes negative values for some pairs* $(\zeta; \vec{\delta})^t$*, i.e., the potential energy of the system has no minimum at an equilibrium state, then the spectral problem corresponding to solutions of homogeneous problem* (6.11) *of the form* $y(t) = y\exp(-\lambda t)$ *has eigenvalues with* $\operatorname{Re}\lambda < 0$*, and the investigated hydromechanical system is unstable.*

Results of this section are obtained in a joint work with O. A. Dudik.

7. Convective motions of a viscous fluid in a vessel

Here we consider only the problem on instability of the mechanical equilibrium for a nonunformly heated fluid in an arbitrary partially filled container (see [20], [9, pp. 162–186]).

7.1. Let the equilibrium gradient of the temperature be $\nabla T_0 = -\alpha \vec{e}_3$, $\alpha \neq 0$. Then $T_0 = T_0(x_3) = -\alpha x_3 + \alpha_0$, and the pressure field is

$$P_0 = P_0(x_3) = p_a - \rho g x_3 + \rho g \beta(-\alpha \frac{x_3^2}{2} + \alpha_0 x_3), \tag{7.1}$$

where $\beta > 0$ is the coefficient of thermal extension and p_a is the atmospheric pressure.

Small convective motions of a fluid in a container are described by the following system of equations, boundary value and initial conditions:

$$\frac{\partial \vec{u}}{\partial t} = -\frac{1}{\rho}\nabla p + \nu \Delta \vec{u} + g\beta\theta\vec{e}_3, \quad \operatorname{div}\vec{u} = 0 \quad (\text{in } \Omega), \quad \vec{u} = \vec{0} \quad (\text{on } S), \tag{7.2}$$

$$\frac{\partial \theta}{\partial t} = \alpha \vec{u} \cdot \vec{e}_3 + \chi \Delta\theta \quad (\text{in } \Omega), \quad \theta = 0 \quad (\text{on } S), \tag{7.3}$$

$$\rho\nu\left(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}\right) = 0, \quad i = 1, 2, \quad \frac{\partial \zeta}{\partial t} = \vec{u} \cdot \vec{e}_3, \quad p - 2\rho\nu\frac{\partial u_3}{\partial x_3} = \rho g \zeta \quad (\text{on } \Gamma), \tag{7.4}$$

(7.5) $$\chi \frac{\partial \theta}{\partial n} + a\theta = \alpha a \zeta \quad (\text{on } \Gamma),$$

(7.6) $$\vec{u}(0,x) = \vec{u}^0(x), \quad \theta(0,x) = \theta^0(x), \quad x \in \Omega; \quad \zeta(0,x) = \zeta^0(x), \quad x \in \Gamma.$$

Here $\theta(t,x)$ is the deviation of the temperature from the equilibrium one, $\chi > 0$ is the coefficient of temperature conductivity, $a \geqslant 0$ is the interfase coefficient of head exchange. Another notations are the same as in Section 6.

7.2. Consider solutions to problem (7.2)–(7.6) that depend on t according to the law $\exp(-\lambda t)$ and use auxiliary S. Krein's problems and the following two ones:

(7.7) $$-\Delta\varphi = f \quad (\text{in } \Omega), \quad \varphi = 0 \quad (\text{on } S), \quad \frac{\partial \varphi}{\partial n} + a\chi^{-1}\varphi = 0 \quad (\text{on } \Gamma);$$

(7.8) $$-\Delta\psi = 0 \quad (\text{in } \Omega), \quad \psi = 0 \quad (\text{on } S), \quad \frac{\partial \psi}{\partial n} + a\chi^{-1}\psi = \eta \quad (\text{on } \Gamma).$$

As it is known, $\varphi = A_1^{-1} f$, where $A_1 \gg 0$, $0 < A_1^{-1} \in \mathfrak{S}_\infty(L_2(\Omega))$,

(7.9) $$\lambda_j(A_1) = \left(\frac{|\Omega|}{6\pi^2}\right)^{-2/3} j^{2/3}[1 + o(1)] \quad (j \longrightarrow \infty).$$

Respectively, $\psi = V_1\eta$, $V_1 : \ H^{-1/2}(\Gamma) \longrightarrow H^1_{0,S}(\Omega) = \{\psi \in H^1(\Omega) : \ \psi = 0 \text{ (on } S)\}$,

(7.10) $$\|\psi\|^2_{H^1_{0,S}}(\Omega) := \int_\Omega |\nabla\psi|^2 \, d\Omega + \int_\Gamma a\chi^{-1}|\psi|^2 \, d\Gamma \quad (a \geqslant 0, \ \chi > 0).$$

Using the operators of all these boundary value problems we transform (7.2)–(7.5) to the following spectral problem for corresponding amplitude elements:

(7.11) $$\vec{\xi} - g\beta\nu^{-1}Cv = \lambda\nu^{-1}A^{-1}\vec{\xi} + \nu^{-1}g\lambda^{-1}B\vec{\xi}, \quad B := Q^*Q,$$

(7.12) $$-\alpha\chi^{-1}C^*\vec{\xi} + v = \lambda\chi^{-1}A_1^{-1}v - a\chi^{-1}\alpha\lambda^{-1}Q_1^*Q\vec{\xi}, \quad Q_1^* := A_1^{1/2}V_1,$$

(7.13)
$$Cv := A^{-1/2}P_{0,S}(A_1^{-1/2}v\vec{e}_3), \quad C^*\vec{\xi} := A_1^{-1/2}(A^{-1/2}\vec{\xi} \cdot \vec{e}_3), \quad \vec{\xi} = A^{1/2}\vec{u}, \quad v = A_1^{1/2}\theta.$$

Lemma 7.1. *The operators $C : \ L_2(\Omega) \longrightarrow \vec{J}_{0,S}(\Omega)$ and $C^* : \ \vec{J}_{0,S}(\Omega) \longrightarrow L_2(\Omega)$ are mutually adjoint and compact.*

Lemma 7.2. *The operator $B = Q^*Q : \ \vec{J}_{0,S}(\Omega) \longrightarrow \vec{M}_0(\Omega) \subset \vec{J}_{0,S}(\Omega)$ is compact and non-negative, positive eigenvalues of B have the asymptotic behavior*

(7.14) $$\lambda_j(B) = \left(\frac{|\Gamma|}{16\pi}\right)^{1/2} j^{-1/2}[1 + o(1)] \quad (j \longrightarrow \infty).$$

To make the system (7.11)–(7.12) more symmetric, we make the substitution

(7.15) $$v = \left(|\alpha|\nu(\chi g\beta)^{-1}\right)^{1/2}\varphi.$$

Then (7.11)–(7.12) transforms to the problem

(7.16) $$\widehat{I}_\varepsilon y = \lambda\widehat{A}y + \lambda^{-1}\widehat{B}y, \quad y = (\vec{\xi};\varphi)^t \in \mathcal{H} := \vec{J}_{0,S}(\Omega) \oplus L_2(\Omega),$$

(7.17) $$\widehat{I}_\varepsilon := \begin{pmatrix} I & -\varepsilon C \\ -\varepsilon \operatorname{sign}\alpha\, C^* & I_1 \end{pmatrix}, \quad \widehat{B} := \begin{pmatrix} g\nu^{-1}B & 0 \\ -a\,\varepsilon \operatorname{sign}\alpha\, B_1 & 0 \end{pmatrix},$$
$$\widehat{A} := \operatorname{diag}(\nu^{-1}A^{-1}; \chi^{-1}A_1^{-1}), \quad \varepsilon = (|\alpha|g\beta(\nu\chi)^{-1})^{1/2} > 0,$$

where I_1 is the identity operator in $L_2(\Omega)$ and $B_1 := Q_1^*Q$, $B_1 \in \mathfrak{S}_\infty$.

In (7.16), the operator $\widehat{A}$ is compact and positive, the operator $\widehat{B}$ is compact. If $a = 0$, then $\widehat{B}$ is nonnegative. Thus, the problem on normal convective motions of the fluid in the vessel is a spectral problem for the following operator pencil of S. Krein's type:

(7.18) $$L(\lambda) := \widehat{I} - \varepsilon\widehat{K} - \lambda\widehat{A} - \lambda^{-1}\widehat{B}, \quad \widehat{K} = \begin{pmatrix} 0 & C \\ \operatorname{sign}\alpha\, C^* & 0 \end{pmatrix} \in \mathfrak{S}_\infty(\mathcal{H}).$$

7.3. In [9, pp. 169–176], the cases $\alpha > 0$ (heating from below) and $\alpha < 0$ (heating from above) were studied. If was proved that the spectrum $L(\lambda)$ is discrete with limit points $\lambda = 0$ and $\lambda = \infty$. If $|\alpha|$ is sufficiently small then all eigenvalues are located in the right complex half-plane, i.e., all normal oscillations are stable.

Here we consider only the case $\alpha > 0$ and $a = 0$, i.e., the case of heating from below for a given heat flow on the surface. Then

$$\widehat{K} = \begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix} = K^* \in \mathfrak{S}_\infty, \quad \widehat{B} = g\nu^{-1} \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}. \tag{7.19}$$

If $\varepsilon > 0$ is sufficiently small then we have the following result ([9, pp. 174–176]).

Theorem 7.1. *Let the intensity of heating, ε, satisfies the condition*

$$0 \leqslant \varepsilon < \varepsilon_1 := \left(\lambda_{\max}(\widehat{K})\right)^{-1}. \tag{7.20}$$

Then the spectrum of the pencil $L(\lambda)$ is located in the right complex half-plane and consist of two branches of eigenvalues $\{\lambda_k^+\}_{k=1}^\infty$ and $\{\lambda_k^-\}_{k=1}^\infty$ situated on the real axis and of no more then a finite number of non-real eigenvalues. For large values of viscosity there are no non-real eigenvalues.

For sufficiently large ε the following assertion holds.

Theorem 7.2. *Let the intensity of heating be satisfied the condition*

$$\varepsilon > \varepsilon_2 := (\mu_1^+)^{-1}, \tag{7.21}$$

where μ_1^+ is the maximal eigenvalue of the auxiliary spectral problem

$$\begin{gathered} -\Delta\vec{u} + \nabla p = \mu^{-1} v \vec{e}_3, \quad \operatorname{div}\vec{u} = 0 \quad (\text{in } \Omega), \quad \vec{u} = \vec{0} \quad (\text{on } S), \\ u_n (= u_3) = 0, \quad \frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i} = 0 \quad (\text{on } \Gamma), \quad i = 1, 2; \\ -\Delta v = \mu^{-1} u_3 \quad (\text{in } \Omega), \quad v = 0 \quad (\text{on } S), \quad \frac{\partial v}{\partial n} = 0 \quad (\text{on } \Gamma). \end{gathered} \tag{7.22}$$

Then, under the condition of a constant heat flow on the free surface Γ and in the case of heating from below, the spectral convection problem (7.16) *has at least one negative eigenvalue and, therefore, it has an aperiodically increasing with time mode of normal convective motions.*

The proof of the theorem has several steps and uses some assertions of operator theory in the Pontryagin space. In particular, we consider the two-parameter pencil

$$M_{\varepsilon,\eta}(\tau) := \tau\widehat{A} - \widehat{B} - \eta(\widehat{I} - \varepsilon\widehat{K}) \tag{7.23}$$

and max-min principles for it, use the property

$$\operatorname{Ker}\widehat{K} \subset \operatorname{Ker}\widehat{B}, \quad \dim(\operatorname{Ker}\widehat{B} \ominus \operatorname{Ker}\widehat{K}) = \infty, \tag{7.24}$$

double-sided estimates for eigenvalues of $M_{\varepsilon,\eta}(\tau)$ and solutions properties to problem (7.22) (see [9, pp. 176–186]).

Unsolved problem. Condition (7.20) is sufficient for stability of normal convective motions of a fluid, and condition (7.21) is sufficient for instability of these motions. The case $0 < \varepsilon_1 \leqslant \varepsilon \leqslant \varepsilon_2$ is an unsolved problem.

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