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ON μ-SCALE INVARIANT OPERATORS

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Dedicated to the memory of M. Krein on the occasion of his one hundredth birthday anniversary.

Abstract. We introduce the concept of a μ-scale invariant operator with respect to a unitary transformation in a separable complex Hilbert space. We show that if a nonnegative densely defined symmetric operator is μ-scale invariant for some $\mu > 0$, then both the Friedrichs and the Krein-von Neumann extensions of this operator are also μ-scale invariant.

1. Introduction

Given a unitary operator U in a separable complex Hilbert space $\mathcal{H}$ and a (complex) number $\mu \in \mathbb{C} \setminus \{0\}$, we introduce the concept of a μ-scale invariant operator T (with respect to the transformation U) as a (bounded) "solution" of the following equation

$$UTU^* = \mu T. \tag{1.1}$$

Note that in this case U and T commute up to a factor, that is,

$$UT = \mu TU, \tag{1.2}$$

and then necessarily $|\mu| = 1$ (see [6]), provided that T is a bounded operator and

$$\operatorname{spec}(UT) \neq \{0\}.$$

The search for pairs of unitaries U and T satisfying the canonical (Heisenberg) commutation relations (1.2) with $|\mu| = 1$ leads to realizations of the rotation algebra, the C^*-algebra generated by the monomials $T^m U^n$, $m, n \in \mathbb{Z}$ (see, e.g., [15]). The irreducible representations of this algebra play a crucial role in the study of the Hofstadter type models. For instance, the Hofstadter Hamiltonian $H = T + T^* + U + U^*$ typically has fractal spectrum that is rather sensitive to the algebraic properties of the "magnetic flux" θ, $\mu = e^{i\theta}$, which is captured in the beauty of the famous Hofstadter butterfly (see [15] and references therein). We also note that self-adjoint realizations U and T of commutation relations (1.1) or (1.2) for $|\mu| = 1$ are obtained in [6] while the case of contractive (not necessarily self-adjoint) solutions T, and unitary U, has been discussed in [14].

To incorporate the case of $|\mu| \neq 1$, where unbounded solutions to (1.1) are of necessity considered, we extend the concept of the μ-scale invariance to the case of unbounded operators T by the requirement that $\operatorname{Dom}(T)$ is invariant, that is,

$$U^* \operatorname{Dom}(T) \subseteq \operatorname{Dom}(T), \tag{1.3}$$

and

$$UTU^* f = \mu T f \quad \text{for all} \quad f \in \operatorname{Dom}(T). \tag{1.4}$$

In this short Note we restrict ourselves to the case $\mu > 0$ and focus on the study of symmetric as well as self-adjoint unbounded solutions T of (1.3) and (1.4). Our

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main result (see Theorem 2.2) states that if a densely defined nonnegative (symmetric) operator T is μ-scale invariant with respect to a unitary transformation U, then the two classical extremal nonnegative self-adjoint extensions, the Friedrichs and the Krein-von Neumann extensions, are μ-scale invariant as well.

The paper is organized as follows: In Section 2, based on a result by Ando and Nishio [3], we provide the proof of Theorem 2.2. Section 3 is devoted to further generalizations and a discussion of the μ-scale invariance concept from the standpoint of group representation theory.

2. Main result

Recall that if $\dot{A}$ is a densely defined (closed) nonnegative operator, then the set of all nonnegative self-adjoint extensions of $\dot{A}$ has the minimal element A_K, the Krein-von Neumann extension (different authors refer to the minimal extension A_K by using different names, see, e.g., [2], [3], [4], [5]), and the maximal one A_F, the Friedrichs extension. This means, in particular, that for any nonnegative self-adjoint extension $\tilde{A}$ of $\dot{A}$ the following operator inequality holds [11]:

$$(A_F+\lambda I)^{-1} \le (\tilde{A}+\lambda I)^{-1} \le (A_K+\lambda I)^{-1}, \quad \text{for all } \lambda > 0.$$

The following result characterizes the Friedrichs and the Krein-von Neumann extensions a form convenient for our considerations.

Theorem 2.1. ([1], [3]). *Let $\dot{A}$ be a (closed) densely defined nonnegative symmetric operator. Denote by $\mathbf{a}$ the closure*[1] *of the quadratic form*

$$\dot{\mathbf{a}}[f] = (\dot{A}f, f), \quad \mathrm{Dom}[\dot{\mathbf{a}}] = \mathrm{Dom}(\dot{A}). \tag{2.1}$$

Then,

(i) *the Friedrichs extension A_F of $\dot{A}$ coincides with the restriction of the adjoint operator $\dot{A}^*$ on the domain*

$$\mathrm{Dom}(A_F) = \mathrm{Dom}(\dot{A}^*) \cap \mathrm{Dom}[\mathbf{a}];$$

(ii) *the Krein-von Neumann extension A_K of $\dot{A}$ coincides with the restriction of the adjoint operator $\dot{A}^*$ on the domain $\mathrm{Dom}(A_K)$ which consists of the set of elements f for which there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$, $f_n \in \mathrm{Dom}(\dot{A})$, such that*

$$\lim_{n,m\to\infty} \mathbf{a}[f_n - f_m] = 0 \quad \text{and} \quad \lim_{n\to\infty} \dot{A}f_n = \dot{A}^*f.$$

We now state the main result of this Note.

Theorem 2.2. *Assume that $\mu > 0$ and that a densely defined (closed) nonnegative symmetric operator $\dot{A}$ is μ-scale invariant with respect to a unitary transformation U; that is,*

$$U^*\mathrm{Dom}(\dot{A}) \subseteq \mathrm{Dom}(\dot{A}))$$

and that

$$U\dot{A}U^* = \mu\dot{A} \quad \text{on} \quad \mathrm{Dom}(\dot{A}).$$

Then

(i) *the adjoint operator $\dot{A}^*$,*
(ii) *the Friedrichs extension A_F of $\dot{A}$, and*
(iii) *the Krein-von Neumann extension A_K of $\dot{A}$*

are μ-scale invariant with respect to the unitary transformation U.

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[1]Recall that $f \in \mathrm{Dom}[\mathbf{a}]$ if and only if there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$, $f_n \in \mathrm{Dom}(\dot{A})$, such that $\lim_{n,m\to\infty} \dot{\mathbf{a}}[f_n - f_m] = 0$ and $\lim_{n\to\infty} f_n = f$.

Proof. Clearly, it is sufficient to prove (i) followed by the proof of the fact that the domains of both the Friedrichs and the Krein-von Neumann extensions are invariant with respect to the operator U^*.

(i). Given $f \in \mathrm{Dom}(\dot{A})$ and $h \in \mathrm{Dom}(\dot{A}^*)$, one obtains

$$(\dot{A}f, U^*h) = (U\dot{A}f, h) = (U\dot{A}U^*Uf, h)$$
$$= (\mu\dot{A}Uf, h) = (Uf, \mu\dot{A}^*h) = (f, U^*\mu\dot{A}^*h),$$

thereby proving the inclusion $U^*\mathrm{Dom}(\dot{A}^*) \subseteq \mathrm{Dom}(\dot{A}^*)$ as well as the equality

$$\dot{A}^*U^*h = \mu U^*\dot{A}^*h, \quad h \in \mathrm{Dom}(\dot{A}). \tag{2.2}$$

The proof of (i) is complete.

(ii). First we show that the domain of the closure of the quadratic form (2.1) is invariant with respect to operator the U^*.

Recall that $f \in \mathrm{Dom}[\mathbf{a}]$ if and only if there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$, $f_n \in \mathrm{Dom}(\dot{A})$, such that

$$\lim_{n,m\to\infty} \dot{\mathbf{a}}[f_n - f_m] = 0 \quad \text{and} \quad \lim_{n\to\infty} f_n = f.$$

Take an $f \in \mathrm{Dom}[\mathbf{a}]$ and a sequence $\{f_n\}_{n\in\mathbb{N}}$ satisfying the properties above. Clearly

$$\lim_{n\to\infty} U^*f_n = U^*f` \tag{2.3}$$

with $U^*f_n \in \mathrm{Dom}(\dot{A})$. Moreover,

$$\mathbf{a}[U^*f_n - U^*f_m] = (\dot{A}U^*(f_n - f_m), U^*(f_n - f_m)) = (\dot{U}AU^*(f_n - f_m), (f_n - f_m))$$
$$= \mu(\dot{A}(f_n - f_m), (f_n - f_m)) = \mu\mathbf{a}[f_n - f_m].$$

Since $\lim_{n,m\to\infty} \mathbf{a}[f_n - f_m] = 0$, one proves that

$$\lim_{n,m\to\infty} \mathbf{a}[U^*f_n - U^*f_m] = 0$$

which together with (2.3) implies that $U^*f \in \mathrm{Dom}[\mathbf{a}]$. Hence, we have proven the inclusion

$$U^*\mathrm{Dom}[\mathbf{a}] \subseteq \mathrm{Dom}[\mathbf{a}]. \tag{2.4}$$

Next, by (i) the domain $\mathrm{Dom}(\dot{A}^*)$ is invariant with respect to U^*. This combined with (2.4) and Theorem 2.1 (i) proves that the domain of the Friedrichs extension A_F of $\dot{A}$ is invariant with respect to the operator U^*. Therefore, A_F is μ-scale invariant as a restriction of the μ-scale invariant operator $\dot{A}^*$ onto a U^*-invariant domain.

(iii). Analogously, in order to show that the Krein-von Neumann extension A_K is μ-scale invariant with respect to the transformation U, it is sufficient to show that its domain is invariant with respect to U^*.

Take $f \in \mathrm{Dom}(A_K)$. By Theorem 2.1 (ii) there exists an $\mathbf{a}$-Cauchy sequence[2] $\{f_n\}_{n\in\mathbb{N}}$, $f_n \in \mathrm{Dom}(\dot{A})$, such that

$$\lim_{n\to\infty} \dot{A}f_n = \dot{A}^*f. \tag{2.5}$$

From (2.2) it follows that

$$\dot{A}U^*f_n = \dot{A}^*U^*f_n = \mu U^*\dot{A}^*f_n = \mu U^*\dot{A}f_n \quad \text{and} \quad \dot{A}^*U^*f = \mu U^*\dot{A}^*f. \tag{2.6}$$

Combining (2.5) and (2.6), for the $\mathbf{a}$-Cauchy sequence $\{U^*f_n\}_{n\in\mathbb{N}}$ one gets

$$\lim_{n\to\infty} \dot{A}U^*f_n = \mu U^*\dot{A}^*f = \dot{A}^*U^*f$$

[2]in the “metric” generated by the form $\mathbf{a}$

proving that $U^*f \in \mathrm{Dom}(A_K)$ by Theorem 2.1 (ii). Thus, $\mathrm{Dom}(A_K)$ is U^*- invariant and, therefore, the Krein-von Neumann extension A_K is μ-scale invariant as a restriction of the μ-scale invariant operator $\dot{A}^*$ onto a U^*-invariant domain. □

Remark 2.3. We remark that the concept of μ-scale invariant operators can immediately be extended to the case of liner relations: we say that a linear relation S is μ-scale invariant with respect to the unitary transformation U if its domain is U^*-invariant and $(f,g) \in S$ implies $(U^*f, \mu U^*g) \in S$.

Recall that the Friedrichs extension S_F of a semi-bounded from below relation S is defined as the restriction of S^* onto the domain of the closure of the quadratic form associated with the operator part of S [7] and the Krein-von Neumann extension S_K is defined by

$$S_K = \left((S^{-1})_F\right)^{-1}, \tag{2.7}$$

provided that S is, in addition, nonnegative [8] (no care should be taken about inverses, for they always exist).

Assume that a nonnegative linear relation S is μ-scale invariant. Almost literally repeating the arguments of the proof of Theorem 2.2 (i) one concludes that the adjoint relation S^* is also μ-scale invariant. Given the above characterization of the Friedrichs extension of a semi-bounded relation, applying Theorem 2.2 (ii) proves the μ-scale invariance of S_F. As it follows from (2.7), a simple observation that S is μ-scale invariant if and only if the inverse relation S^{-1} is μ^{-1}-scale invariant ensures that the Krein-von Neumann extension S_K of S is also μ-scale invariant. Thus, Part (iii) of Theorem 2.2 is a direct consequence of Parts (i) and (ii) up to the representation theorem that states that Krein-von Neumann extension A_K of a nonnegative densely defined symmetric operator $\dot{A}$ can be "evaluated" as

$$A_K = \left((\dot{A}^{-1})_F\right)^{-1}, \tag{2.8}$$

with $\dot{A}^{-1}$ being understood as a linear relation (for the proof of (2.8) we refer to [8], also see [3] and [4]).

Remark 2.4. Note without proof that if the symmetric nonnegative operator $\dot{A}$ referred to in Theorem 2.2 has deficiency indices $(1,1)$ the Friedrichs and the Krein-von Neumann extensions of $\dot{A}$ are the only ones μ-scale invariant self-adjoint extensions.

The following simple example illustrates the statement of Theorem 2.2.

Example 2.5. Assume that $\mu > 0$, $\mu \neq 1$, and that U is the unitary scaling transformation on the Hilbert space $\mathcal{H} = L^2(0,\infty)$ defined by

$$(Uf)(x) = \mu^{-\frac{1}{4}} f(\mu^{-\frac{1}{2}} x), \quad f \in L^2(0,\infty).$$

T is the maximal operator on the Sobolev space $H^{2,2}(0,\infty)$ defined by

$$T = -\frac{d^2}{dx^2}, \quad \mathrm{Dom}(T) = H^{2,2}(0,\infty).$$

Let A_F and A_K be the restrictions of T onto the domains

$$\mathrm{Dom}(A_F) = \{f \in \mathrm{Dom}(T) \mid f(0) = 0\}$$

and

$$\mathrm{Dom}(A_K) = \{f \in \mathrm{Dom}(T) \mid f'(0) = 0\}$$

respectively. Denote by $\dot{A}$ the restriction of T onto the domain

$$\mathrm{Dom}(\dot{A}) = \mathrm{Dom}(A_F) \cap \mathrm{Dom}(A_K).$$

It is well known that $\dot{A}$ is a closed nonnegative symmetric operator with deficiency indices $(1,1)$ and that A_F and A_K are the Friedrichs and the Krein-von Neumann extensions of $\dot{A}$ respectively and $T=\dot{A}^*$. A straightforward computation shows that all the operators $\dot{A}$, A_F, A_K and T are μ-scale invariant with respect to the transformation U. Moreover, note that any other nonnegative self-adjoint extensions of $\dot{A}$ different from the extremal ones, A_F and A_K, can be obtained by the restriction of T onto the domain (see, e.g., [13], also see [9] and [10])

$$\mathrm{Dom}(\tilde{A}_s)=\{f\in\mathrm{Dom}(T)\,|\,f'(0)=sf(0)\},\qquad \text{for some } s>0,$$

which is obviously not U^*-invariant. Thus, the operator $\dot{A}$ admits the only two μ-scale invariant extensions, the Friedrichs and the Krein-von Neumann extensions (cf. Remark 2.4).

3. Concluding remarks

We remark that any μ-scale invariant operator T with respect to a unitary transformation U is also μ^n-scale invariant with respect to the (unitary) transformations U^n, $n=0,1,\dots$. That is,

$$U^nTU^{-n}=\mu^nT,\qquad \text{for all } n\in\{0\}\cup\mathbb{N}. \tag{3.1}$$

If, in addition,

$$U^*\mathrm{Dom}(T)=\mathrm{Dom}(T),$$

then relation (3.1) holds for all $n\in\mathbb{Z}$. Thus, we naturally arrive at a slightly more general concept of scale invariance with respect to a one-parameter unitary representation of the additive group $\mathcal{G}$ ($\mathcal{G}=\mathbb{N}$ or $\mathcal{G}=\mathbb{R}$): *Given a character μ, $\mu:G\to\mathbb{C}$, of the group $\mathcal{G}$ and its one-parameter unitary representation $g\mapsto U_g$, a densely defined operator T is said to be μ-character-scale invariant with respect to the representation U_g if*

$$U_g\mathrm{Dom}(T)=\mathrm{Dom}(T),\quad g\in\mathcal{G},$$

and

$$U_gTU_{-g}=\mu(g)T,\quad \textit{on}\quad \mathrm{Dom}(T),\quad g\in\mathcal{G}. \tag{3.2}$$

Clearly, an appropriate version of Theorem 2.2 can almost literally be restated in this more general setting. It is also worth mentioning that upon introducing the representation $V_g=\mu^gU_g$, $g\in\mathcal{G}$, one can rewrite (3.2) in the form

$$U_gT=TV_g,\quad g\in\mathcal{G}, \tag{3.3}$$

and we refer the interested reader to the papers [12] and [14] where commutation relations (3.3) for general groups $\mathcal{G}$ with not necessarily unitary representations U_g and V_g, $g\in\mathcal{G}$, of the group $\mathcal{G}$ are discussed.

Note that an infinitesimal analog of the commutation relation in (3.2) is also available provided that $\mathcal{G}=\mathbb{R}$ and the unitary representation U_t, $t\in\mathbb{R}$, is strongly continuous. In this case infinitesimal version of (3.2) can heuristically be written down as the following commutation relation

$$[B,T]=i\hbar T, \tag{3.4}$$

with $[\cdot,\cdot]$ the usual commutator and

$$\hbar=-\log\mu, \tag{3.5}$$

the structure constant of the simplest noncommutative two-dimensional Lie algebra (3.4) and (3.5). Here B is the infinitesimal generator of the group U_t, so that $U_t=e^{iBt}$, $t\in\mathbb{R}$. And in conclusion, note that Theorem 2.2 paves the way for realizations of the Lie algebra by self-adjoint operators, provided that some "trial" symmetric realizations of the Lie algebra are available.

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