

THE SQUARE-TRANSFORM OF HERMITE-BIEHLER FUNCTIONS. A GEOMETRIC APPROACH

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This paper is dedicated to the memory of M. G. Krein.

ABSTRACT. We investigate the subclass of symmetric indefinite Hermite-Biehler functions which is obtained from positive definite Hermite-Biehler functions by means of the square-transform. It is known that functions of this class can be characterized in terms of location of their zeros. We give another, more elementary and geometric, proof of this result. The present proof employs a ‘shifting-of-zeros’ perturbation method. We apply our results to obtain information on the eigenvalues of a concrete boundary value problems.

1. INTRODUCTION AND MAIN RESULT

The *Hermite-Biehler class* is the set of all entire functions E which have no zeros in the open upper half-plane \mathbb{C}^+ and satisfy

$$(1.1) \quad |E(\bar{z})| \leq |E(z)|, \quad z \in \mathbb{C}^+.$$

An indefinite generalization of this notion is obtained when (1.1) is replaced by the a kernel condition:

If $\Omega \subseteq \mathbb{C}$ is a domain and $K(w, z)$ is a function defined on $\Omega \times \Omega$, which is analytic in the variables z and \bar{w} and has the property that $K(w, z) = \overline{K(z, w)}$, then K is called an *analytic symmetric kernel* (shortly *a kernel*) on Ω . Let $\kappa \in \mathbb{N} \cup \{0\}$. We say that the kernel K has κ *negative squares*, if for each choice of $n \in \mathbb{N}$ and $z_1, \dots, z_n \in \Omega$ the quadratic form

$$Q_K(\xi_1, \dots, \xi_n) := \sum_{i,j=1}^n K(z_j, z_i) \xi_i \bar{\xi}_j$$

has at most κ negative squares, and if for some choice of n, z_1, \dots, z_n this upper bound is actually attained.

Definition 1.1. Let $\kappa \in \mathbb{N} \cup \{0\}$. The set \mathcal{HB}_κ of *Hermite-Biehler functions with κ negative squares* is defined to be the set of all entire functions E which satisfy

- (i) E and $E^\#$ have no common nonreal zeros;
- (ii) the kernel

$$(1.2) \quad K_E(w, z) := i \frac{E(z) \overline{E(w)} - E^\#(z) \overline{E^\#(w)}}{z - \bar{w}}, \quad z, w \in \mathbb{C},$$

has κ negative squares.

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Moreover, we define the set of *indefinite Hermite-Biehler functions* as

$$\mathcal{HB}_{<\infty} := \bigcup_{\kappa \in \mathbb{N} \cup \{0\}} \mathcal{HB}_{\kappa}.$$

If $E \in \mathcal{HB}_{<\infty}$, we will denote by $\text{ind}_- E$ the actual number of negative squares of the kernel (1.2).

The fact that positive definiteness of the kernel K_E coincides with the condition (1.1) is a classical result, see e.g. [10].

In this paper we are concerned with two subclasses of indefinite Hermite-Biehler functions and the relationship between them.

Definition 1.2.

- (i) The set $\mathcal{HB}_{\kappa}^{\text{sym}}$ of *symmetric Hermite-Biehler functions with κ negative squares* is defined as the set of all functions $E \in \mathcal{HB}_{\kappa}$ which satisfy the functional equation

$$(1.3) \quad \overline{E(\bar{z})} = E(-z).$$

- (ii) The set $\mathcal{HB}_{\kappa}^{\text{sb}}$ of *semibounded Hermite-Biehler functions with κ negative squares* is defined as the set of all functions $E \in \mathcal{HB}_{\kappa}$ which have the property that the meromorphic function $\frac{E(z)+\overline{E(\bar{z})}}{E(z)-\overline{E(\bar{z})}}$ has only finitely many poles in $\mathbb{C} \setminus [0, \infty)$.

The notations $\mathcal{HB}_{<\infty}^{\text{sym}}$ and $\mathcal{HB}_{<\infty}^{\text{sb}}$ are defined correspondingly.

To any entire function F , an entire function $\mathfrak{sq} F$, its *square-transform*, can be associated by setting $A(z) := \frac{1}{2}(E(z) + \overline{E(\bar{z})})$, $B(z) := \frac{1}{2i}(E(z) - \overline{E(\bar{z})})$, and defining

$$(\mathfrak{sq} F)(z) := A(z^2) + izB(z^2).$$

Clearly, $\mathfrak{sq} F$ satisfies the functional equation (1.3), i.e., $\overline{\mathfrak{sq} F(\bar{z})} = \mathfrak{sq} F(-z)$. It is a consequence of [4, Theorem 4.1], and was explicitly shown in [13], that \mathfrak{sq} induces a bijection of the set

$$\{F \in \mathcal{HB}_{<\infty}^{\text{sb}} : F \text{ has no zeros in } (-\infty, 0)\}$$

onto $\mathcal{HB}_{<\infty}^{\text{sym}}$. Hence we obtain a partition of $\mathcal{HB}_{<\infty}^{\text{sym}}$ into the classes

$$\mathfrak{sq}(\{F \in \mathcal{HB}_{\kappa}^{\text{sb}} : F \text{ has no zeros in } (-\infty, 0)\}), \quad \kappa \in \mathbb{N} \cup \{0\}.$$

In [13] the following result was proved, and, as a consequence, information on the distribution of eigenvalues of some concrete boundary value problems was obtained.

Theorem 1.3. *Let $E \in \mathcal{HB}_{<\infty}^{\text{sym}}$. Then*

$$(1.4) \quad E \in \mathfrak{sq}(\{F \in \mathcal{HB}_0^{\text{sb}} : F \text{ has no zeros in } (-\infty, 0)\})$$

if and only if its zeros are distributed according to the following two rules:

- (Z1) *All zeros of E in \mathbb{C}^+ are simple and located on the imaginary axis.*
 (Z2) *Denote the zeros of E which lie in \mathbb{C}^+ by iy_1, \dots, iy_{κ} with $0 < y_1 < \dots < y_{\kappa}$. Then, for every $k = 2, \dots, \kappa$, the number of zeros of E in $(-iy_{k-1}, -iy_k)$ is odd. The number of zeros of E in $[0, -iy_1)$ is even. Thereby all zeros are counted according to their multiplicities.*

The proof of necessity that the zeros of an element of (1.4) are distributed according to (Z1) and (Z2) is a bit elaborate but in essence elementary. In contrast to necessity, the proof of sufficiency of these conditions which is given in [13] is more involved and rather implicit. It relies on the theory of (symmetric and semibounded) de Branges Pontryagin spaces as developed in [6], [5]. The geometric meaning of the conditions (Z1) and (Z2) remains unrevealed.

It is our aim in the present note to give another proof of the sufficiency part of Theorem 1.3, which is more elementary and beautifully explains the geometry behind

(Z1) and (Z2). We use a perturbation method, which was already employed in [12] for the case of polynomials.

Moreover, we will, as another application of Theorem 1.3, determine the distribution of eigenvalues of the boundary value problem

$$(1.5) \quad -y'' + ip\lambda y + q(x)y = \lambda^2 y, \quad x \in (0, b) \cup (b, a),$$

$$(1.6) \quad y(0) = 0, \quad y(a) = 0, \quad y(b-0) = y(b+0),$$

$$(1.7) \quad y'(b-0) - y'(b+0) + (\beta + i\alpha\lambda - m\lambda^2)y(b-0) = 0,$$

where $b \in (0, a)$, $p > 0$, $\alpha > 0$, $m > 0$, $\beta \in \mathbb{R}$, $q(x)$ is real valued and its restrictions belong to $L_2(0, b)$ and $L_2(b, a)$, correspondingly. This problem arises in the study of small transversal vibrations of a smooth inhomogeneous string with damping. The conditions (1.6) mean that the ends of the string are fixed, the conditions (1.7) describe a ring of mass m which is located at $x = b$ and which moves with damping proportional to α in the direction orthogonal to the equilibrium position of the string. The parameter p is the coefficient of damping of the string. We will show:

Theorem 1.4. *Let σ be the spectrum of the problem (1.5)-(1.7). There exist two disjoint sets Σ_1, Σ_2 , where Σ_1 consists of a finite number of pairs of conjugate purely imaginary and nonzero points which lie in the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| \leq \frac{p}{2}\}$, and Σ_2 satisfies the conditions (Z1) and (Z2), such that σ is represented as follows:*

- (i) if $\alpha = mp$, then $\Sigma_2 \subseteq \mathbb{R}$, and $\sigma = (\Sigma_1 \cup \Sigma_2) + \frac{ip}{2}$;
- (ii) if $\alpha < mp$, then Σ_2 is contained in the half-plane $\{z \in \mathbb{C} : |\operatorname{Im} z| \geq -\frac{p}{2}\}$, and $\sigma = (\Sigma_1 \cup \Sigma_2) + \frac{ip}{2}$;
- (iii) if $\alpha > mp$, then Σ_2 is contained in the half-plane $\{z \in \mathbb{C} : |\operatorname{Im} z| \leq \frac{p}{2}\}$, and $\sigma = (\Sigma_1 \cup \overline{\Sigma_2}) + \frac{ip}{2}$, where $\overline{\Sigma_2}$ denotes the reflection of Σ_2 with respect to the real line.

The content of the present paper is divided into three sections. In Section 2 we set up our notation and provide some preliminary results. In Section 3 we give the proof of the sufficiency part of Theorem 1.3. Finally, in Section 4, we apply Theorem 1.3 to the boundary value problem (1.5)–(1.7), and establish Theorem 1.4.

2. PRELIMINARIES

Let $H(\mathbb{C})$ denote the set of all entire functions. Throughout this paper all topological terms concerning elements of $H(\mathbb{C})$ will refer to locally uniform convergence, that is the topology induced by the metric

$$\rho(F, G) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{|z| \leq n} |F(z) - G(z)|}{1 + \sup_{|z| \leq n} |F(z) - G(z)|}, \quad F, G \in H(\mathbb{C}).$$

For $F \in H(\mathbb{C})$ put $F^\#(z) := \overline{F(\bar{z})}$. The map $F \mapsto F^\#$ is a conjugate linear and isometric involution of $H(\mathbb{C})$. We can decompose any entire function into its ‘real-’ and ‘imaginary-’ part with respect to this involution: For $F \in H(\mathbb{C})$ put

$$\Re F := \frac{F + F^\#}{2}, \quad \Im F := \frac{F - F^\#}{2i}.$$

Then $(\Re F)^\# = \Re F$, $(\Im F)^\# = \Im F$, and $F = \Re F + i\Im F$. An entire function F is called *real*, if $\Im F = 0$, i.e., if $F = F^\#$.

Remark 2.1. The real- and imaginary- part of a complex number w is denoted by $\operatorname{Re} w$ and $\operatorname{Im} w$, respectively. One has to distinguish the entire functions $\Re F$ and $\Im F$ from the C^∞ -functions $\operatorname{Re} F$ and $\operatorname{Im} F$ which assign to a point $z \in \mathbb{C}$ the value $\operatorname{Re}(F(z))$ and $\operatorname{Im}(F(z))$, respectively.

a. Symmetry

We deal with entire functions which satisfy the functional equation (1.3).

Definition 2.2. Denote by $H(\mathbb{C})^{\text{sym}}$ the set

$$H(\mathbb{C})^{\text{sym}} := \{F \in H(\mathbb{C}) : F^\#(z) = F(-z)\}.$$

Note that $H(\mathbb{C})^{\text{sym}}$ is a closed subset of $H(\mathbb{C})$. Moreover, it contains the sum and the product of each two of its elements, and all real multiples of each of its elements.

Since $F^\#(z) = \Re F(z) - i\Im F(z)$ and $F(-z) = \Re F(-z) + i\Im F(-z)$, we have $F \in H(\mathbb{C})^{\text{sym}}$ if and only if $\Re F$ is an even function and $\Im F$ is odd. This enables us to make the following construction:

Definition 2.3. Let $F \in H(\mathbb{C})^{\text{sym}}$. Then real entire functions $A_\pm(z)$, $B_\pm(z)$ are well-defined by the equations

$$\begin{aligned} A_+(z^2) &= A_-(z^2) = \Re F(z), \\ B_+(z^2) &= z\Im F(z), \quad B_-(z^2) = \frac{\Im F(z)}{z}. \end{aligned}$$

Define mappings $\mathfrak{T}_\pm : H(\mathbb{C})^{\text{sym}} \rightarrow H(\mathbb{C})$ by

$$(\mathfrak{T}_+ F)(z) := A_+(z) + iB_+(z), \quad (\mathfrak{T}_- F)(z) := A_-(z) + iB_-(z).$$

The following properties of the transformations \mathfrak{T}_\pm are seen by elementary computation. We will, therefore, omit their proofs.

- (i) The map \mathfrak{T}_+ is a bijection of $H(\mathbb{C})^{\text{sym}}$ onto $\{F \in H(\mathbb{C}) : \Im F(0) = 0\}$.
- (ii) The map \mathfrak{T}_- is a bijection of $H(\mathbb{C})^{\text{sym}}$ onto $H(\mathbb{C})$, and $(\mathfrak{T}_-)^{-1} = \text{sq}$.
- (iii) \mathfrak{T}_+ and \mathfrak{T}_- are continuous.
- (iv) We have $\mathfrak{T}_\pm(F^\#) = (\mathfrak{T}_\pm F)^\#$.
- (v) Let $G \in H(\mathbb{C})$ be real and even, and let $g \in H(\mathbb{C})$ be defined by the equation $g(z^2) = G(z)$. Then $\mathfrak{T}_\pm(GF) = g\mathfrak{T}_\pm F$.

b. The indefinite Hermite-Biehler class

Let us discuss the class $\mathcal{HB}_{<\infty}$ in a bit more detail.

Remark 2.4.

- (i) The powerful condition in Definition 1.1 is the requirement that $K_E(w, z)$ has a finite number of negative squares. However, the innocent looking condition that E and $E^\#$ have no common nonreal zeros will, especially in the present context, play an important role. For example its presence allows us to evaluate the number of negative squares of $K_E(w, z)$ in terms of the zeros of E , cf. [13, Remark 2.3, (ii)], [8]:

Let $E \in \mathcal{HB}_{<\infty}$, then $\text{ind}_- E$ is equal to the number of zeros of E which are located in the open upper half plane counted according to their multiplicities.

- (ii) The condition that a kernel has a finite number of negative squares is stable with respect to convergent sequences of uniformly bounded negative index. If $E_n \rightarrow E$ and each of the kernels K_{E_n} has at most κ negative squares, then also the kernel K_E has at most κ negative squares. I.e., in the limit the negative index may decrease but cannot increase. This shows the following statement.

Assume that $E_n \in \bigcup_{\nu=0}^{\kappa} \mathcal{HB}_\nu$ converges to an entire function E , and assume that E has no pairs of conjugate nonreal zeros. Then $E \in \bigcup_{\nu=0}^{\kappa} \mathcal{HB}_\nu$.

The set \mathcal{N}_κ of *generalized Nevanlinna functions with κ negative squares* is defined as the set of all functions q which are meromorphic in $\mathbb{C} \setminus \mathbb{R}$, satisfy $q(\bar{z}) = \overline{q(z)}$, and have the property that the kernel

$$(2.1) \quad L_q(w, z) := \frac{q(z) - \overline{q(w)}}{z - \bar{w}}, \quad z, w \in \mathbb{C}^+$$

has κ of negative squares. The set of all *generalized Nevanlinna functions* is then $\mathcal{N}_{<\infty} := \bigcup_{\kappa \in \mathbb{N} \cup \{0\}} \mathcal{N}_\kappa$. For $q \in \mathcal{N}_{<\infty}$, the actual number of negative squares of the kernel (4.5) will be denoted by $\text{ind}_- q$.

This is a generalization of a classical class of functions also known as the Nevanlinna class (not to be mixed up with the set of all functions of bounded characteristic, which is sometimes also referred to as the ‘Nevanlinna class’). It follows e.g. from [10] that $q \in \mathcal{N}_0$ if and only if q is analytic in $\mathbb{C} \setminus \mathbb{R}$, symmetric with respect to the real axis, and satisfies $\text{Im } q(z) \geq 0$ for all $z \in \mathbb{C}^+$.

Indefinite Hermite-Biehler functions are closely related to generalized Nevanlinna functions: Let $E \in H(\mathbb{C})$ be such that E and $E^\#$ have no common nonreal zeros, and put $q := \frac{\Re E}{\Im E}$. Then $E \in \mathcal{HB}_{<\infty}$ if and only if $q \in \mathcal{N}_{<\infty}$, and, in this case, $\text{ind}_- E = \text{ind}_- q$.

A product representation of an indefinite Hermite-Biehler function can be given, cf. [7], [13].

Theorem 2.5. (Kreĭn’s Factorization Theorem). *Let $E \in \mathcal{HB}_{<\infty}$, then only finitely many zeros of E lie in \mathbb{C}^+ , E has no pairs of conjugate nonreal zeros, and the nonreal zeros a_1, a_2, \dots of E satisfy the Blaschke condition*

$$\sum_n \left| \text{Im} \frac{1}{a_n} \right| < \infty.$$

Whenever numbers $p_n \in \mathbb{N} \cup \{0\}$ are chosen such that $\sum_n \frac{1}{|a_n|^{p_n+1}} < \infty$, the function E can be factorized as a convergent product,

$$(2.2) \quad E(z) = \gamma D(z) e^{-iaz} \prod_n \left(1 - \frac{z}{a_n} \right) \exp \left(\sum_{l=1}^{p_n} \frac{z^l}{l} \text{Re} \frac{1}{a_n^l} \right),$$

where D is real and has only real zeros, $|\gamma| = 1$, and $a \geq 0$.

Conversely, if $\gamma, D, a, a_1, a_2, \dots, p_1, p_2, \dots$, are given and subject to the mentioned conditions, then (2.2) defines an entire function which belongs to $\mathcal{HB}_{<\infty}$.

As a consequence we obtain that $\mathcal{HB}_{<\infty}$ is stable with respect to dividing out zeros.

Corollary 2.6. *Let $E \in \mathcal{HB}_{<\infty}$ and let P be a polynomial such that $\frac{E}{P} \in H(\mathbb{C})$. Then $\frac{E}{P} \in \mathcal{HB}_{<\infty}$ and*

$$\text{ind}_- \frac{E}{P} = \text{ind}_- E - N,$$

where N is the number of zeros of P in \mathbb{C}^+ counted according to their multiplicities.

Proof. Write

$$E(z) = \gamma D(z) e^{-iaz} \prod_n \left(1 - \frac{z}{a_n} \right) \exp \left(\sum_{l=1}^{p_n} \frac{z^l}{l} \text{Re} \frac{1}{a_n^l} \right)$$

according to Kreĭn’s Factorization Theorem. Since every zero of P must also be a zero of E with at least the same multiplicity, we can enumerate the nonreal zeros a_1, a_2, \dots of E in such a way that $P(z) = \tilde{P}(z) \prod_{n=1}^k \left(1 - \frac{z}{a_n} \right)$ where $\tilde{P}(z)$ has only real zeros. It follows that

$$\frac{E(z)}{P(z)} = \gamma \cdot \left[\frac{D(z)}{\tilde{D}(z)} \exp \left(\sum_{n=1}^k \sum_{l=1}^{p_n} \frac{z^l}{l} \text{Re} \frac{1}{a_n^l} \right) \right] e^{-iaz} \prod_{n>k} \left(1 - \frac{z}{a_n} \right) \exp \left(\sum_{l=1}^{p_n} \frac{z^l}{l} \text{Re} \frac{1}{a_n^l} \right).$$

From the converse statement in Kreĭn’s Factorization Theorem we obtain that $\frac{E}{P} \in \mathcal{HB}_{<\infty}$. The assertion on negative indices follows from Remark 2.4, (i). \square

c. More on the relation between $\mathcal{HB}_{<\infty}^{\text{sym}}$ and $\mathcal{HB}_{<\infty}^{\text{sb}}$

Since $\mathfrak{T}_- = \mathfrak{sq}^{-1}$, we know from [13, Proposition 2.10] that, if $E \in \mathcal{HB}_{<\infty}^{\text{sym}}$, then the transformed function $\mathfrak{T}_- E$ is again an indefinite Hermite-Biehler function. It was proved

in [5] under an additional assumption that E has no real zeros, that also \mathfrak{T}_+ has this property. It is an important observation, which appears as a consequence of [4, Proposition 3.2], that thereby negative indices sum up. Since this fact is essential for our further arguments, we provide an explicit proof.

Lemma 2.7. *Let $E \in \mathcal{HB}_{<\infty}^{\text{sym}}$. Then $\mathfrak{T}_{\pm}E \in \mathcal{HB}_{<\infty}^{\text{sb}}$ and we have*

$$\text{ind}_- \mathfrak{T}_+E + \text{ind}_- \mathfrak{T}_-E = \text{ind}_- E.$$

Proof. Let A_{\pm}, B_{\pm} be as in Definition 2.3. Assume that $w \in \mathbb{C} \setminus [0, \infty)$ is a zero of A_+ or A_- . Let $z \in \mathbb{C}^+$ be such that $z^2 = w$. Then, by the definition of A_{\pm} , it follows that $\Re E(z) = 0$. Similarly, we see that if $B_+(w) = 0$ or $B_-(w) = 0$, then z is a zero of $\Im E$.

Hence, if $w \in \mathbb{C} \setminus [0, \infty)$ is a common zero of A_+ and B_+ , or of A_- and B_- , then z is a common zero of $\Re E$ and $\Im E$, a contradiction. In particular it follows that \mathfrak{T}_+E as well as \mathfrak{T}_-E has no pair of conjugate nonreal zeros.

Consider the functions

$$q(z) := \frac{\Re E(z)}{\Im E(z)}, \quad q_+(z) := \frac{A_+(z)}{B_+(z)}, \quad q_-(z) := \frac{A_-(z)}{B_-(z)}.$$

Then $q \in \mathcal{N}_{<\infty}$ and $\text{ind}_- q = \text{ind}_- E$. By the definition of A_{\pm} and B_{\pm} , we have

$$q(z) = zq_+(z^2) = \frac{q_-(z^2)}{z}.$$

Now we can apply [4, Proposition 3.2], and obtain that $q_+, q_- \in \mathcal{N}_{<\infty}$ and $\text{ind}_- q_+ + \text{ind}_- q_- = \text{ind}_- q$. This shows that $\mathfrak{T}_-E, \mathfrak{T}_+E \in \mathcal{HB}_{<\infty}$, and that $\text{ind}_- \mathfrak{T}_-E + \text{ind}_- \mathfrak{T}_+E = \text{ind}_- E$.

As we have seen above in the first paragraph of this proof, a zero $w \in \mathbb{C} \setminus [0, \infty)$ of B_+ or B_- gives rise to a zero $z \in \mathbb{C}^+$ of $\Re E$ or $\Im E$, respectively. However, since $\frac{\Re E}{\Im E} \in \mathcal{N}_{<\infty}$, this function can have only finitely many poles in \mathbb{C}^+ . Since $\Re E$ and $\Im E$ have no common nonreal zeros, the function $\Im E$, and with it also B_+ and B_- , can have only finitely many zeros in $\mathbb{C} \setminus [0, \infty)$. In particular, \mathfrak{T}_-E and \mathfrak{T}_+E satisfy the requirement of Definition 1.2, (ii). \square

3. PROOF OF SUFFICIENCY IN THEOREM 1.3

The present proof is based on a ‘shifting-of-zeros’ perturbation argument. We present its core in the form of three lemmas. The first one is an immediate consequence of Kreĩn’s Factorization Theorem.

Lemma 3.1. *Let $E \in \mathcal{HB}_{<\infty}$ and let $\eta_1, \dots, \eta_N : [0, 1] \rightarrow \mathbb{C}$ be continuous and such that for each $t \in [0, 1]$ the set*

$$M_t := \{z \in \mathbb{C} : E(z) = 0\} \cup \{\eta_1(t), \dots, \eta_N(t)\}$$

does not contain pairs of conjugate nonreal points, cf. Figure 1.

Define

$$e(t) := \prod_{j=1}^N (z - \eta_j(t)) \cdot E(z), \quad t \in [0, 1].$$

Then e maps $[0, 1]$ continuously into $\mathcal{HB}_{<\infty}$ and we have

$$\text{ind}_- e(t) = \text{ind}_- E + \#\{j : \text{Im } \eta_j(t) > 0\}.$$

Proof. The fact that $e(t)$ is a continuous map into $H(\mathbb{C})$ is obvious. Write

$$E(z) = \gamma D(z) e^{-iaz} \prod_n \left(1 - \frac{z}{a_n}\right) \exp\left(\sum_{l=1}^{p_n} \frac{z^l}{l} \text{Re} \frac{1}{a_n^l}\right)$$

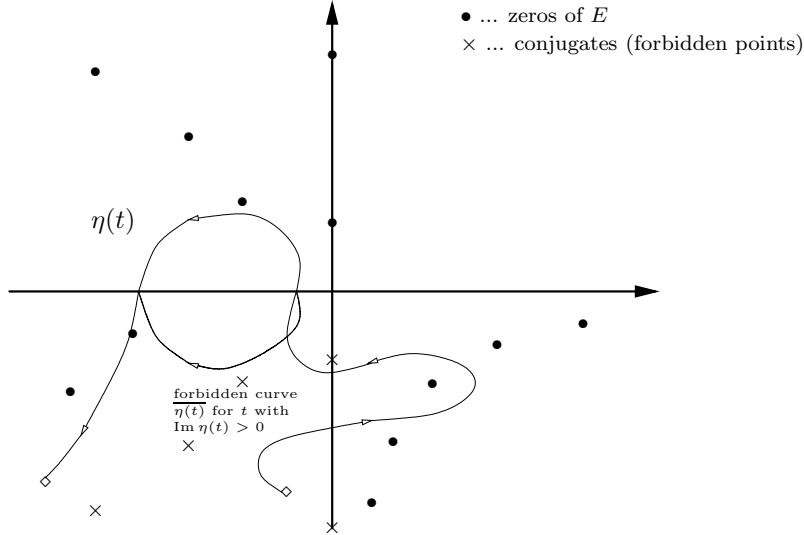


FIGURE 1. Path of $\eta(t)$ ($N = 1$)

according to Kreĭn’s Factorization Theorem. Put $c(t) := \prod_{j:\eta_j(t) \neq 0} \eta_j(t)$ and let $k(t) := \#\{j : \eta_j(t) = 0\}$. Then

$$e(t) = \left(\gamma \frac{c(t)}{|c(t)|} \right) \cdot \left[|c(t)| (-1)^{N-k(t)} z^{k(t)} D(z) \right] \cdot e^{-iaz}$$

$$\times \prod_{j:\eta_j(t) \neq 0} \left(1 - \frac{z}{\eta_j(t)} \right) \cdot \prod_n \left(1 - \frac{z}{a_n} \right) \exp \left(\sum_{l=1}^{p_n} \frac{z^l}{l} \operatorname{Re} \frac{1}{a_n^l} \right).$$

From the converse part of Kreĭn’s Factorization Theorem, we obtain $e(t) \in \mathcal{HB}_{<\infty}$. The assertion on negative indices follows from Remark 2.4, (i). \square

Lemma 3.2. *Let $\kappa \in \mathbb{N} \cup \{0\}$, let I be a nonempty and connected subset of \mathbb{R} , and let $e : I \rightarrow \mathcal{HB}_{\kappa}^{\text{sym}}$ be continuous. Then the functions $\operatorname{ind}_- \mathfrak{T}_+ e(t)$ and $\operatorname{ind}_- \mathfrak{T}_- e(t)$ are constant on I .*

Proof. Put $\kappa_0 := \min_{t \in I} \operatorname{ind}_- \mathfrak{T}_+ e(t)$. By Lemma 2.7 we have $\kappa_0 \leq \kappa$. Moreover, if $\kappa_0 = \kappa$, we must have $\operatorname{ind}_- \mathfrak{T}_+ e(t) = \kappa$ and $\operatorname{ind}_- \mathfrak{T}_- e(t) = 0$ for all $t \in I$, and we are done. Hence let us assume that $\kappa_0 < \kappa$.

Consider the set

$$J := \{t \in I : \operatorname{ind}_- \mathfrak{T}_+ e(t) = \kappa_0\}.$$

Then J is nonempty. Remark 2.4, (ii), and the continuity of \mathfrak{T}_+ imply that J is closed, for if $s \in \overline{J}$, then

$$\kappa_0 \leq \operatorname{ind}_- \mathfrak{T}_+ e(s) \leq \sup_{t \in J} \operatorname{ind}_- \mathfrak{T}_+ e(t) = \kappa_0.$$

Consider the set $I \setminus J$. By Lemma 2.7 we have

$$I \setminus J = \{t \in I : \operatorname{ind}_- \mathfrak{T}_+ e(t) > \kappa_0\} = \{t \in I : \operatorname{ind}_- \mathfrak{T}_- e(t) < \kappa - \kappa_0\}$$

$$= \{t \in I : \operatorname{ind}_- \mathfrak{T}_- e(t) \leq \kappa - \kappa_0 - 1\}.$$

From Remark 2.4, (ii), and the continuity of \mathfrak{T}_- , it now follows that also $I \setminus J$ is closed.

Since I is connected, we conclude that $I = J$. This means that $\operatorname{ind}_- \mathfrak{T}_+ e(t) = \kappa_0$ and $\operatorname{ind}_- \mathfrak{T}_- e(t) = \kappa - \kappa_0$ for all $t \in I$. \square

Lemma 3.3. *Let $E \in \mathcal{HB}_{<\infty}^{\text{sym}}$ and assume that E has a simple zero at the origin. Let $I = (\alpha, \beta)$, $\alpha \in [-\infty, 0)$, $\beta \in (0, \infty]$, be the largest interval which contains zero and is such that E has no zeros in $(-i\beta, -i\alpha) \setminus \{0\}$. Define a function $e : I \rightarrow H(\mathbb{C})$ by*

$$e(t) := \frac{z - it}{z} E(z), \quad t \in I.$$

Then e maps I continuously into $\mathcal{HB}_{<\infty}^{\text{sym}}$, and the function $\text{ind}_- \mathfrak{T}_- e(t)$ is constant on I .

Proof. We have $\frac{E(z)}{z} \in \mathcal{HB}_{<\infty}$ and $e(t) = (z - it) \frac{E(z)}{z}$. Hence, by the definition of I , we may apply Lemma 3.1 and obtain that $e(t) \in \mathcal{HB}_{<\infty}$ for all $t \in I$, and that

$$\text{ind}_- e(t) = \begin{cases} \text{ind}_- E + 1, & t > 0, \\ \text{ind}_- E, & t \leq 0. \end{cases}$$

Since

$$\left(\frac{z - it}{z}\right)^\# = \frac{z + it}{z} = \frac{(-z) - it}{(-z)},$$

we see that $e(t) \in H(\mathbb{C})^{\text{sym}}$.

By Lemma 3.2 both of the functions $\text{ind}_- \mathfrak{T}_+ e(t)$ and $\text{ind}_- \mathfrak{T}_- e(t)$ are constant on $(\alpha, 0]$ as well as on $(0, \beta)$. Put

$$\begin{aligned} \nu_\pm &:= \text{ind}_- \mathfrak{T}_\pm e(t), \quad t \in (\alpha, 0], \\ \pi_\pm &:= \text{ind}_- \mathfrak{T}_\pm e(t), \quad t \in (0, \beta). \end{aligned}$$

Then, by Remark 2.4 (ii), and the continuity of \mathfrak{T}_- ,

$$(3.1) \quad \nu_- = \text{ind}_- \mathfrak{T}_- e(0) \leq \pi_-.$$

We claim that

$$(3.2) \quad \pi_+ \geq \nu_+ + 1.$$

Assuming this claim, it follows that

$$\nu_+ + \nu_- = \text{ind}_- E = \pi_+ + \pi_- - 1 \geq (\nu_+ + 1) + \pi_- - 1 = \nu_+ + \pi_-$$

and hence $\nu_- \geq \pi_-$. Together with (3.1) we obtain $\nu_- = \pi_-$, and this is the assertion of the lemma.

It remains to establish the claim (3.2). To this end let us compute $\mathfrak{T}_+ e(t)$ more explicitly. We have

$$\begin{aligned} e(t) &= \left(1 - i\frac{t}{z}\right) E(z) = \left(1 - i\frac{t}{z}\right) \left(\Re E(z) + i\Im E(z)\right) \\ &= \underbrace{\left[\Re E(z) + t \frac{\Im E(z)}{z}\right]}_{=\Re e(t)(z)} + i \underbrace{\left[\Im E(z) - t \frac{\Re E(z)}{z}\right]}_{=\Im e(t)(z)}. \end{aligned}$$

To shorten the notation put $A := \Re \mathfrak{T}_- E$ and $B := \Im \mathfrak{T}_- E$. Then the above formula rewrites as

$$\Re e(t)(z) = A(z^2) + tB(z^2), \quad \Im e(t)(z) = \frac{1}{z}(z^2 B(z^2) - tA(z^2)).$$

The definition of \mathfrak{T}_+ now gives

$$(\Re \mathfrak{T}_+ e(t))(z) = A(z) + tB(z), \quad (\Im \mathfrak{T}_+ e(t))(z) = zB(z) - tA(z),$$

i.e.,

$$(3.3) \quad \mathfrak{T}_+ e(t)(z) = [A(z) + tB(z)] + i[zB(z) - tA(z)].$$

Since $E(0) = 0$, we have $\Re E(0) = \Im E(0) = 0$, and it follows from the definition of \mathfrak{T}_- that $A(0) = 0$. Since $\Re E$ is even, it has a zero of multiplicity at least 2 at 0. Due to our assumption that 0 is a simple zero of E , the function $\Im E$ must therefore have a simple zero at the origin. This implies that $B(0) = \Im \mathfrak{T}_- E(0) \neq 0$, and that $\Im \mathfrak{T}_+ E$ has a simple zero at 0. Hence $\mathfrak{T}_+ E = \mathfrak{T}_+ e(0)$ has a simple zero at the origin.

By Hurwitz's Theorem there exist $r_1 > 0$ and $\epsilon_1 > 0$ such that, for every $t \in (-\epsilon_1, \epsilon_1)$, the function $\mathfrak{T}_+ e(t)$ has exactly one zero $\zeta(t)$ in the disk $U_0 := \{z \in \mathbb{C} : |z| < r_1\}$ and

this zero is simple. By the generalized Theorem of Logarithmic Residues, $\zeta(t)$ depends real-analytically on t .

We differentiate the identity $\mathfrak{T}_+e(t)(\zeta(t)) = 0$ with respect to t using the representation (3.3) of $\mathfrak{T}_+e(t)$ (the prime denotes derivation with respect to the complex variable z , the dot the derivative with respect to the real variable t),

$$\begin{aligned} 0 &= [A'(\zeta(t))\dot{\zeta}(t) + B(\zeta(t)) + tB'(\zeta(t))\dot{\zeta}(t)] \\ &\quad + i[\dot{\zeta}(t)B(\zeta(t)) + \zeta(t)B'(\zeta(t))\dot{\zeta}(t) - A(\zeta(t)) - tA'(\zeta(t))\dot{\zeta}(t)] \\ &= \dot{\zeta}(t)[A'(\zeta(t)) + tB'(\zeta(t)) + iB(\zeta(t)) + i\zeta(t)B'(\zeta(t)) - itA'(\zeta(t))] \\ &\quad + [B(\zeta(t)) - iA(\zeta(t))]. \end{aligned}$$

Evaluating at $t = 0$, and keeping in mind that $\zeta(0) = 0$ and $A(0) = 0$, yields

$$0 = \dot{\zeta}(0)[A'(0) + iB(0)] + B(0).$$

Since $B(0) \neq 0$, we obtain

$$\dot{\zeta}(0) = \frac{-B(0)}{A'(0) + iB(0)} = \frac{-B(0)A'(0) + iB(0)^2}{|A'(0) + iB(0)|^2},$$

and thus

$$\operatorname{Im} \dot{\zeta}(0) > 0.$$

Since t is a real variable,

$$\frac{d}{dt}(\operatorname{Im} \zeta(t)) = \operatorname{Im} \left(\frac{d}{dt} \zeta(t) \right).$$

We conclude that, locally at 0, the function $\operatorname{Im} \zeta(t)$ is strictly increasing. In particular, there exists $\delta \in (0, \epsilon_1]$ such that

$$\operatorname{Im} \zeta(t) \begin{cases} > 0, & t \in (0, \delta), \\ < 0, & t \in (-\delta, 0). \end{cases}$$

Let a_1, \dots, a_n denote the zeros of \mathfrak{T}_+E in the open upper half plane, and let $\alpha_1, \dots, \alpha_n$ be their multiplicities, so that $\alpha_1 + \dots + \alpha_n = \operatorname{ind}_- \mathfrak{T}_+E = \nu_+$. By Hurwitz's Theorem, there exist $r_2 > 0$ and $\epsilon_2 > 0$ such that the disks $U_j := \{z \in \mathbb{C} : |z - a_j| < r_2\}$ are pairwise disjoint, are entirely contained in the open upper half plane, do not intersect the disk U_0 , and have the property that for every $t \in (-\epsilon_2, \epsilon_2)$ and $j = 1, \dots, n$ the function $\mathfrak{T}_+e(t)$ has zeros inside U_j whose total multiplicity is equal to α_j . It follows that, for every $t \in (0, \min\{\delta, \epsilon_2\})$, the total multiplicity of zeros of $\mathfrak{T}_+e(t)$ in the open upper half plane is at least equal to $\alpha_1 + \dots + \alpha_n + 1 = \nu_+ + 1$. \square

We are now ready for our new proof of the sufficiency part of Theorem 1.3.

Proof. Assume that $E \in \mathcal{HB}_{<\infty}^{\text{sym}}$ and that the zeros of E are distributed according to the rules (Z1) and (Z2). Since $\mathfrak{sq}^{-1} = \mathfrak{T}_-$, we have to show that $\operatorname{ind}_- \mathfrak{T}_-E = 0$. We will use induction on $\operatorname{ind}_- E$.

Step 1, $\operatorname{ind}_- E = 0$. In this case the fact that $\operatorname{ind}_- \mathfrak{T}_-E = 0$ is immediate from Lemma 2.7.

Step 2, reduction to the case where E has no zeros in $(-iy_1, iy_1)$. Assume that $\operatorname{ind}_- E > 0$, let y_1 be as in Theorem 1.3, and denote by ia_1, \dots, ia_{2k} the zeros of E in the interval $[0, -iy_1)$ listed according to their multiplicities and enumerated such that $0 \geq a_1 \geq \dots \geq a_{2k}$. We will move these zeros away from the imaginary axis. Since their total number is even, this can be done retaining the symmetry property of E .

If $k = 0$, there is nothing to do. Assume that $k > 0$. Put

$$\eta_j(t) := t \frac{-iy_1}{2} + (1-t)ia_j, \quad t \in [0, 1], \quad j = 1, \dots, 2k,$$

and consider the map

$$e(t) := \prod_{j=1}^{2k} (z - \eta_j(t)) \cdot \frac{E(z)}{\prod_{j=1}^{2k} (z - ia_j)}, \quad t \in [0, 1],$$

cf. Figure 2.

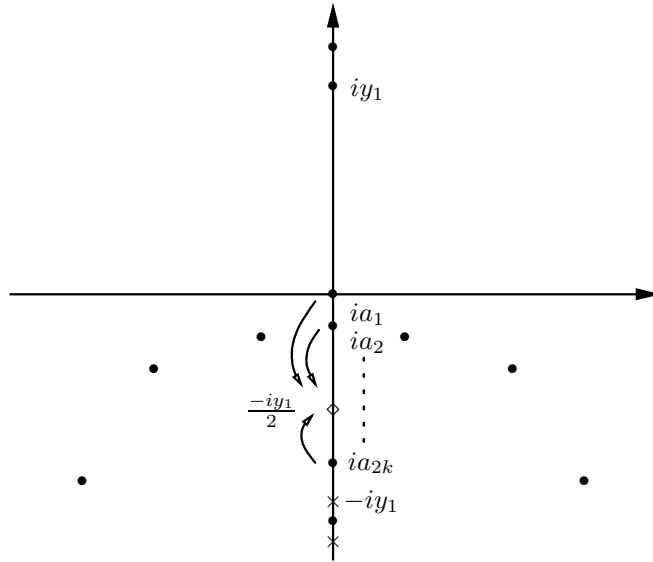


FIGURE 2. Producing one multiple zero with even multiplicity

For all values of t and j , the point $\eta_j(t)$ lies on the imaginary axis. Hence,

$$\left[\frac{z - \eta_j(t)}{z - ia_j} \right]^\# = \frac{z + \eta_j(t)}{z + ia_j} = \frac{(-z) - \eta_j(t)}{(-z) - ia_j},$$

and it follows that $e(t) \in H(\mathbb{C})^{\text{sym}}$. Since η_j depends continuously on t and $\text{Im } \eta_j(t) \leq 0$, Corollary 2.6 and Lemma 3.1 give $e(t) \in \mathcal{HB}_{<\infty}$ and $\text{ind}_- e(t) = \text{ind}_- E, t \in [0, 1]$. We therefore may apply Lemma 3.2 and conclude that

$$\text{ind}_- \mathfrak{T}_- e(1) = \text{ind}_- \mathfrak{T}_- E.$$

Choose $\epsilon > 0$ such that no zero of $e(1)$ lies in $[-\frac{iy_1}{2} - \epsilon, -\frac{iy_1}{2} + \epsilon] \setminus \{-\frac{iy_1}{2}\}$. Put

$$\lambda_l(t) := -\frac{iy_1}{2} - t\epsilon, \quad \lambda_r(t) := -\frac{iy_1}{2} + t\epsilon, \quad t \in [0, 1],$$

and consider the map

$$f(t) := \lambda_l(t)^k \lambda_r(t)^k \cdot \frac{e(1)(z)}{(z + \frac{iy_1}{2})^{2k}}, \quad t \in [0, 1],$$

cf. Figure 3.

The same argument as above will show that

$$\begin{aligned} \text{ind}_- f(t) &= \text{ind}_- f(0) & (= \text{ind}_- e(1) = \text{ind}_- E), \quad t \in [0, 1], \\ \text{ind}_- \mathfrak{T}_- f(1) &= \text{ind}_- \mathfrak{T}_- f(0) & (= \text{ind}_- \mathfrak{T}_- e(1) = \text{ind}_- \mathfrak{T}_- E). \end{aligned}$$

The function $f(1)$ belongs to $\mathcal{HB}_{<\infty}^{\text{sym}}$, its zeros are distributed according to the rules (Z1) and (Z2), and we have

$$\text{ind}_- f(1) = \text{ind}_- E, \quad \text{ind}_- \mathfrak{T}_- f(1) = \text{ind}_- \mathfrak{T}_- E.$$

Moreover, $f(1)$ has no zeros in $(-iy_1, iy_1)$.

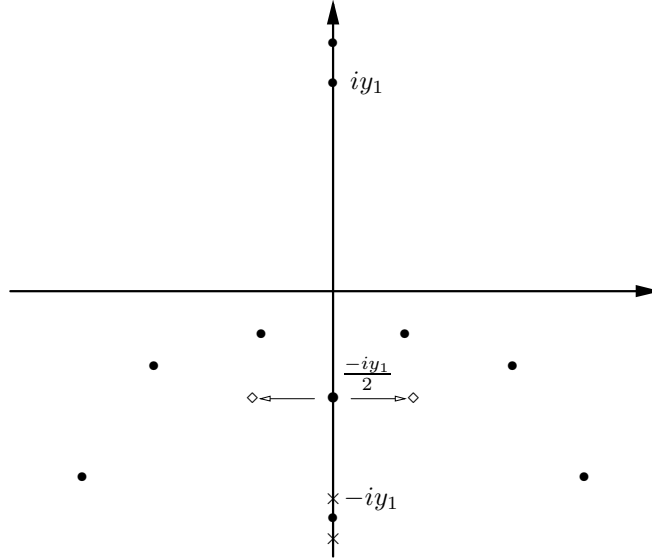


FIGURE 3. Moving the zeros symmetrically away from the imaginary axis

Step 3, the inductive step. Due to the considerations in Step 2, we may assume that E has no zeros in $(-iy_1, iy_1)$. Put

$$\hat{E}(z) := \frac{zE(z)}{z - iy_1},$$

then $\hat{E} \in H(\mathbb{C})^{\text{sym}}$. Moreover, by Corollary 2.6, $\hat{E} \in \mathcal{HB}_{<\infty}$ and $\text{ind}_- \hat{E} = \text{ind}_- E - 1$. Let $I = (\alpha, \beta)$ be the largest interval which contains 0 and is such that \hat{E} has no zeros in $(-i\beta, -i\alpha) \setminus \{0\}$. Then $\beta > y_1$, in particular $y_1 \in I$. An application of Lemma 3.3 to \hat{E} , cf. Figure 4, yields

$$\text{ind}_- \mathfrak{I}_- \hat{E} = \text{ind}_- \mathfrak{I}_- E.$$

However, the function \hat{E} belongs to $\mathcal{HB}_{<\infty}^{\text{sym}}$ and satisfies the conditions (Z1) and (Z2) since the total number of zeros in $(-iy_2, -iy_1)$ is odd and we have produced a simple zero at the origin. Moreover, $\text{ind}_- \hat{E} < \text{ind}_- E$. Hence, by the inductive hypothesis, $\text{ind}_- \mathfrak{I}_- \hat{E} = 0$. This completes the proof of Theorem 1.4, sufficiency. \square

4. AN APPLICATION

We come to the proof of Theorem 1.4. Here we will actually use the necessity part in Theorem 1.3, the proof of which can be found in [13]. First let us substitute $\lambda := z + \frac{ip}{2}$ for the spectral parameter λ . Then (1.5)–(1.7) become

$$(4.1) \quad y''(x) + z^2 y(x) + \left(\frac{p^2}{4} - q(x)\right) = 0, \quad x \in (0, b) \cup (b, a),$$

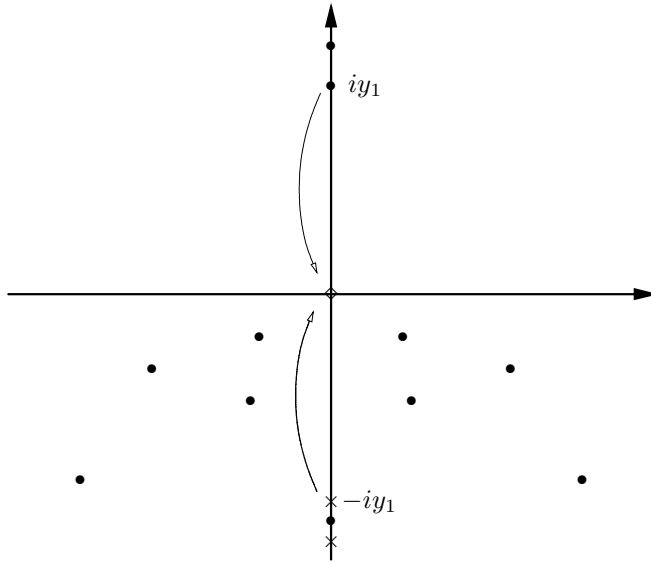
$$(4.2) \quad y(0) = y(a) = 0, \quad y(b-0) = y(b+0),$$

$$(4.3) \quad y'(b-0) - y'(b+0) + (\beta_1 + i(\alpha - mp)z - mz^2)y(b-0) = 0,$$

where $\beta_1 := \beta + \frac{mp^2}{4} - \frac{\alpha p}{2}$.

Let $s_1(\zeta, x), c_1(\zeta, x), x \in (0, b)$, and $s_2(\zeta, x), c_2(\zeta, x), x \in (b, a)$, be the solutions of the equation

$$y''(x) + \zeta y(x) + \left(\frac{p^2}{4} - q(x)\right) = 0$$

FIGURE 4. Moving one zero out of \mathbb{C}^+

on the interval $(0, b)$ and (b, a) , respectively, which satisfy

$$\begin{aligned} s_1(\zeta, 0) = 0, \quad s_1'(\zeta, 0) = 1, \quad s_2(\zeta, b) = 0, \quad s_2'(\zeta, b) = 1, \\ c_1(\zeta, 0) = 1, \quad c_1'(\zeta, 0) = 0, \quad c_2(\zeta, b) = 1, \quad c_2'(\zeta, b) = 0. \end{aligned}$$

We will use the following standard properties of these functions, see e.g. [3], [2], [9].

Remark 4.1.

- (i) All zeros of each of the functions $s_1(\zeta, b), c_1(\zeta, b), s_2(\zeta, a), c_2(\zeta, a)$, are real, simple, and lie in $[-\frac{p^2}{4}, \infty)$.
- (ii) The functions $s_1(\zeta, b)$ and $c_1(\zeta, b)$ have no common zeros. The same holds for each of the pairs $s_1(\zeta, b)$ and $s_1'(\zeta, b)$, $s_2(\zeta, b)$ and $c_2(\zeta, b)$, $s_2(\zeta, b)$ and $s_2'(\zeta, b)$.
- (iii) The functions

$$\frac{s_1(\zeta, b)}{c_1(\zeta, b)}, \quad \frac{s_1(\zeta, b)}{s_1'(\zeta, b)}, \quad \frac{s_2(\zeta, b)}{c_2(\zeta, b)}, \quad \frac{s_2(\zeta, b)}{s_2'(\zeta, b)}$$

belong to the Nevanlinna class \mathcal{N}_0 , i.e., have nonnegative imaginary part throughout the upper half-plane.

A standard argument will show that the spectrum of the problem (4.1)–(4.3) coincides with the set of zeros of the function

$$\begin{aligned} \phi(z) := s_1'(z^2, b)s_2(z^2, b) + s_1(z^2, b)c_2(z^2, b) \\ + (\beta_1 + i(\alpha - mp)z - mz^2)s_1(z^2, b)s_2(z^2, b). \end{aligned}$$

This function belongs to $H(\mathbb{C})^{\text{sym}}$ and, hence, can be written as a square-transform of some entire function. In fact,

$$\begin{aligned} \phi(z) = \left[s_1'(z^2, b)s_2(z^2, b) + s_1(z^2, b)c_2(z^2, b) + (\beta_1 - mz^2)s_1(z^2, b)s_2(z^2, b) \right] \\ + iz \left[(\alpha - mp)s_1(z^2, b)s_2(z^2, b) \right] = \mathfrak{sq} F(z), \end{aligned}$$

where

$$\begin{aligned} F(z) := \left[s_1'(z, b)s_2(z, b) + s_1(z, b)c_2(z, b) + (\beta_1 - mz)s_1(z, b)s_2(z, b) \right] \\ + i \left[(\alpha - mp)s_1(z, b)s_2(z, b) \right]. \end{aligned}$$

Case (I), $\alpha = mp$. In this case the term $i(\alpha - mp)z$ in (4.3) is not present. Hence, if we substitute $\zeta = z^2$, the problem (4.1)–(4.3) rewrites as

$$(4.4) \quad y''(x) + \zeta y(x) + \left(\frac{p^2}{4} - q(x)\right) = 0, \quad x \in (0, b) \cup (b, a),$$

$$(4.5) \quad y(0) = y(a) = 0, \quad y(b-0) = y(b+0),$$

$$(4.6) \quad y'(b-0) - y'(b+0) + (\beta_1 - m\zeta)y(b-0) = 0.$$

It is known, see e.g. [11], that the eigenvalues of this problem are all real, simple, and lie in $[-\frac{p^2}{4}, \infty)$. However, the eigenvalues of (4.4)–(4.6) are just the zeros of the function F , hence the zeros of F are all real, simple, and contained in $[-\frac{p^2}{4}, \infty)$.

Changing back the spectral parameter ζ to z , we see that the spectrum of (4.1)–(4.3) can be written as a disjoint union of the set Σ_1 which contains all square-roots of negative zeros of F (each with multiplicity 1), and of the set Σ_2 which contains all square-roots of positive zeros of F , each with multiplicity 1, and, in case $F(0) = 0$, the point 0 with multiplicity 2. Clearly, for all $z \in \Sigma_1$, we have $\|\operatorname{Im} z\| \leq \frac{p}{2}$. Changing back the spectral parameter z to λ , yields the assertion of Theorem 1.1 in the present case.

Cases (II), (III), $\alpha \neq mp$. By Remark 4.1 (iii), and since $m > 0$, we have

$$\begin{aligned} q(z) &:= -\frac{s'_1(z, b)s_2(z, b) + s_1(z, b)c_2(z, b) + (\beta_1 - mz^2)s_1(z, b)s_2(z, b)}{s_1(z, b)s_2(z, b)} \\ &= -\frac{s'_1(z, b)}{s_1(z, b)} - \frac{c_2(z, b)}{s_2(z, b)} - \beta_1 + mz \in \mathcal{N}_0. \end{aligned}$$

It follows that

$$\frac{\Re F}{\Im F} \in \begin{cases} \mathcal{N}_0, & \alpha < mp \\ -\mathcal{N}_0, & \alpha > mp \end{cases}.$$

By property Remark 4.1 (i), all zeros of $\Im F$ are real. In particular, $\Re F$ and $\Im F$ cannot have common nonreal zeros.

In order to apply Theorem 1.3, we have to take care about possible zeros of F located on the negative real line. Assume that $w \in (-\infty, 0)$ is such that $F(w) = 0$. Then $\Re F(w) = 0$ and $\Im F(w) = 0$. We see from the second relation that either $s_1(w, b) = 0$ or $s_2(w, a) = 0$. If, say $s_1(w, b) = 0$, then by Remark 4.1 (ii), we have $s'_1(w, b) \neq 0$, and hence the first relation implies that $s_2(w, a) = 0$. Similarly, if $s_2(w, a) = 0$, then $c_2(w, a) \neq 0$, and hence $s_1(w, b) = 0$. Thus $\Im F$ has a zero of multiplicity 2 at w . The multiplicity of w as a zero of $\Re F$ is, by what was said in Case (I) above, equal to 1. Hence $\frac{F(z)}{z-w}$ takes a nonzero real value at w .

All real negative zeros of $s_1(\zeta, b)$, and hence also all real negative zeros of F , lie in $[-\frac{p^2}{4}, \infty)$. Denote them by w_1, \dots, w_N , and put

$$\hat{F}(z) := \frac{F(z)}{\prod_{j=1}^N (z - w_j)}.$$

From now on we have to distinguish the cases $\alpha < mp$ and $\alpha > mp$.

Case (II), $\alpha < mp$. In this case we have $F \in \mathcal{HB}_0^{\text{sb}}$, and thus also $\hat{F} \in \mathcal{HB}_0^{\text{sb}}$. However, \hat{F} has no real negative zeros, and hence the zeros of $\mathfrak{sq} \hat{F}$ are distributed according to (Z1) and (Z2). We have

$$\mathfrak{sq} F(z) = \prod_{j=1}^N (z^2 - w_j) \cdot \mathfrak{sq} \hat{F}(z).$$

Let Σ_1 denote the set of all square-roots of the numbers w_1, \dots, w_N , each with multiplicity 1, and let Σ_2 be the collection of zeros of $\mathfrak{sq} \hat{F}$. Then Σ_1 is distributed as required in the assertion of Theorem 1.4, and Σ_2 satisfies (Z1) and (Z2). Since all eigenvalues of the problem (4.1)–(4.3) are geometrically simple, we must have $\Sigma_1 \cap \Sigma_2 = \emptyset$. Changing back the spectral parameter z to λ yields the asserted representation of the spectrum of (1.5)–(1.7) with these sets Σ_1 and Σ_2 . Now it also follows that $\operatorname{Im} z \geq \frac{\beta}{2}$ for all $z \in \Sigma_2$, since we know that, by our conditions on β and q , the spectrum of (1.5)–(1.7) lies in the closed upper half-plane, cf. [11].

Case (III), $\alpha > mp$. In this case we have $F^\# \in \mathcal{HB}_0^{\text{sb}}$, and thus Theorem 1.3 may be applied to $\hat{F}^\#$. Let Σ_1 be as above and let Σ_2 be the collection of zeros of $\hat{F}^\#$. The same arguments as above will yield the assertion of Theorem 1.4 also in this case. \square

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