Methods of Functional Analysis and Topology
Vol. 13 (2007), no. 3, pp. 201–210

PG-FRAMES IN BANACH SPACES

M. R. ABDOLLAHPOUR, M. H. FAROUGHI, AND A. RAHIMI

Abstract. For extending the concepts of p-frame, frame for Banach spaces and atomic decomposition, we will define the concept of pg-frame and g-frame for Banach spaces, by which each $f \in X$ (X is a Banach space) can be represented by an unconditionally convergent series $f = \sum g_i \Lambda_i$, where $\{\Lambda_i\}_{i \in J}$ is a pg-frame, $\{g_i\} \in (\sum \oplus Y_i^*)_{l_q}$ and $\frac{1}{p} + \frac{1}{q} = 1$. In fact, a pg-frame $\{\Lambda_i\}$ is a kind of an overcomplete basis for X^*. We also show that every separable Banach space X has a g-Banach frame with bounds equal to 1.

1. Introduction

Various generalization of frames for Hilbert spaces have been proposed recently. For example, frame of subspaces [3], pseudo-frames [14], bounded quasi-projectors [10], oblique frames [7], [8] and so on. The most recent of these belongs to Wenchang Sun. In this generalization, W. Sun chose a family of bounded operators on a sequence of Hilbert spaces and called this system a generalized frame or a g-frame. By his extension, if $\{\Lambda_i\}_{i \in J}$ is a g-frame then every element $f \in \mathcal{H}$ can be represented as $f = \sum_{i \in J} \Lambda_i^* \Lambda_i S^{-1} f$.

The concept of frames in Banach spaces have been introduced by Christensen and Stoeva [5], Casazza, Han and Larson [4] and Grochenig [11]. In the present paper, by using Sun's extension and some techniques in a frame for Banach spaces, we shall introduce pg-frames and g-frames for Banach spaces that allows every element $f \in X$ to be represented by an unconditionally convergent series $f = \sum_{i \in J} g_i \Lambda_i f$, where $\{\Lambda_i\}_{i \in J}$ is a pg-frame, $\{g_i\}_{i \in J} \in (\sum \oplus Y_i^*)_{l_q}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Throughout this paper, J is a subset of $\mathbb{N}$, $\mathcal{H}$ is a separable Hilbert space, $\{\mathcal{H}_i\}_{i \in J}$ is a sequence of separable Hilbert spaces, X is a Banach space with dual X^* and also $\{Y_i\}_{i \in J}$ is a sequence of Banach spaces.

Definition 1.1. We call a sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in J}$ if there exist two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i \in J} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

We call A and B the lower and upper g-frame bounds, respectively.

We call $\{\Lambda_i\}_{i \in J}$ a tight g-frame if $A = B$ and Parseval g-frame if $A = B = 1$.

The following proposition was proved in [18] and gives a representation for each $f \in \mathcal{H}$.

Proposition 1.2. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ be a g-frame for $\mathcal{H}$. The operator*

$$S : \mathcal{H} \to \mathcal{H},$$
$$Sf = \sum_{i \in J} \Lambda_i^* \Lambda_i f$$

2000 *Mathematics Subject Classification.* Primary 41A58; Secondary 42C15.

Key words and phrases. Atomic decomposition, Bessel sequence, frame, Banach frame, g-frame, g-Banach frame, pg-frame, qg-Riesz basis, perturbation.

is a positive invertible operator and every $f \in \mathcal{H}$ has an expansion

$$f = \sum_{i\in J} S^{-1}\Lambda_i^*\Lambda_i f = \sum_{i\in J} \Lambda_i^*\Lambda_i S^{-1} f.$$

The operator S is called the g-frame operator of $\{\Lambda_i\}_{i\in J}$.

Definition 1.3. Let $1 < p < \infty$. A countable family $\{g_i\}_{i\in J} \subseteq X^*$ is a p-frame for X, if there exist constants $A, B > 0$ such that

$$A\|f\|_X \leq (\sum |g_i(f)|^p)^{\frac{1}{p}} \leq B\|f\|_X, \quad f \in X.$$

We will use the following lemma; its proof can be found in [13].

Lemma 1.4. *If $U : X \to Y$ is a bounded operator from a Banach space X into a Banach space Y then its adjoint $U^* : Y^* \to X^*$ is surjective, if and only if, U has a bounded inverse on $\mathcal{R}_U$.*

2. Duals of g-frames

Definition 2.1. Let $\{\Lambda_i\}_{i\in J}$ and $\{\Theta_i\}_{i\in J}$ be two g-frames for $\mathcal{H}$ such that

$$f = \sum_{i\in J} \Theta_i^*\Lambda_i f, \quad f \in \mathcal{H},$$

then $\{\Theta_i\}_{i\in J}$ is called an alternate dual of $\{\Lambda_i\}_{i\in J}$.

We have the following situation which shows that if $\{\Theta_i\}_{i\in J}$ is an alternate dual of $\{\Lambda_i\}_{i\in J}$ then $\{\Lambda_i\}_{i\in J}$ is an alternate dual of $\{\Theta_i\}_{i\in J}$.

Proposition 2.2. *Let $\{\Lambda_i\}_{i\in J}$ and $\{\Theta_i\}_{i\in J}$ be g-frames for a Hilbert space $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in J}$ such that*

$$f = \sum_{i\in J} \Lambda_i^*\Theta_i f, \quad f \in \mathcal{H},$$

then for each $f \in \mathcal{H}$, $f = \sum_{i\in J} \Theta_i^*\Lambda_i f$.

Proof. Let us define $T : \mathcal{H} \to \mathcal{H}$ by $Tf = \sum_{i\in J} \Theta_i^*\Lambda_i f$. If the upper g-frame bounds of $\{\Lambda_i\}_{i\in J}$ and $\{\Theta_i\}_{i\in J}$ are B and B', respectively, then

$$\begin{aligned}\|T\| &= \sup_{\|f\|=1} |\langle Tf, f\rangle| \\ &\leq \sup_{\|f\|=1} \left(\sum_{i\in J} \|\Lambda_i f\|^2\right)^{\frac{1}{2}} \left(\sum_{i\in J} \|\Theta_i f\|^2\right)^{\frac{1}{2}} \leq \sqrt{BB'}.\end{aligned}$$

Hence $T \in B(\mathcal{H})$. For $f, g \in \mathcal{H}$, we have

$$\langle Tf, g\rangle = \langle \sum_{i\in J} \Theta_i^*\Lambda_i f, g\rangle = \sum_{i\in J} \langle \Lambda_i f, \Theta_i f\rangle.$$

Also,

$$\langle f, g\rangle = \langle f, \sum_{i\in J} \Lambda_i^*\Theta_i g\rangle = \sum_{i\in J} \langle \Lambda_i f, \Theta_i g\rangle.$$

So $\langle Tf, g\rangle = \langle f, g\rangle$ for all $f, g \in \mathcal{H}$, which implies that $T = I$. □

Let $\{f_i\}$ be a frame for a Hilbert space $\mathcal{H}$ and $V : \mathcal{H} \to \mathcal{H}$ be an invertible operator. Then $\{Vf_i\}$ is a frame for $\mathcal{H}$ and the same result holds for g-frames.

Proposition 2.3. *Let $\{\Lambda_i\}_{i\in J}$ be a g-frame for a Hilbert space $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in J}$ and $V \in B(\mathcal{H})$ be an invertible operator. Then $\{\Lambda_i V\}_{i\in J}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in J}$ and its g-frame operator is $S' = V^*SV$.*

Proof. Let $\{\Lambda_i\}_{i\in J}$ be a g-frame for $\mathcal{H}$. We have

$$A\|Vf\|^2 \leq \sum_{i\in J} \|\Lambda_i Vf\|^2 \leq B\|Vf\|^2, \quad f \in \mathcal{H}.$$

Since V is invertible,

$$A\|V^{-1}\|^{-2}\|f\|^2 \leq \sum_{i\in J} \|\Lambda_i V f\|^2 \leq B\|V\|^2\|f\|^2, \quad f \in \mathcal{H},$$

so $\{\Lambda_i V\}_{i\in J}$ is a g-frame for $\mathcal{H}$.

For each $f \in \mathcal{H}$, we have

$$SVf = \sum_{i\in J} \Lambda_i^* \Lambda_i V f,$$

therefore

$$V^*SVf = \sum_{i\in J} V^*\Lambda_i^* \Lambda_i V f.$$

Let S' be the g-frame operator of $\{\Lambda_i V\}_{i\in J}$, then for each $f \in \mathcal{H}$,

$$S'f = \sum_{i\in J} V^*\Lambda_i^* \Lambda_i V f,$$

hence $S^{'} = V^*SV$. □

Note that when $\{\Lambda_i\}_{i\in J}$ is a g-frame for a Hilbert space $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in J}$ and $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i\in J}$ is a family of bounded operators such that $f = \sum_{i\in J} \Theta_i^*\Lambda_i f$ for each $f \in \mathcal{H}$. Then $\{\Theta_i\}_{i\in J}$ is not necessarily a g-frame. For instance, let $\mathcal{H} = \mathbb{C}$ and $K_i = \mathbb{C}$, choose sequences $\{c_i\}$ and $\{d_i\}$ in $\mathbb{C}$ such that $\sum_{i\in J} |d_i|^2 = \infty$, $\sum_{i\in J} |c_i|^2 = 1$ and $\sum_{i\in J} c_i\bar{d}_i = 1$. If $\Lambda_i f = c_i f$ and $\Theta_i f = d_i f$ then $\{\Lambda_i\}_{i\in J}$ is a normalized tight g-frame for $\mathbb{C}$ and

$$\sum_{i\in J} \Theta_i^*\Lambda_i f = \sum_{i\in J} \Theta_i^*(c_i f) = \sum_{i\in J} c_i\bar{d}_i f = f, \quad f \in \mathbb{C}.$$

Also we have

$$\sum_{i\in J} \|\Theta_i f\|^2 = \sum_{i\in J} \|d_i f\|^2 = \sum_{i\in J} |d_i|^2\|f\|^2 = \infty.$$

Therefore $\{\Theta_i\}_{i\in J}$ is not a g-frame for $\{c_i\}$.

Let $\{\mathcal{H}_i\}_{i\in J}$ be a sequence of Hilbert spaces. Then, the orthogonal sum of $\{\mathcal{H}_i\}_{i\in J}$ is the Hilbert space

$$\oplus_{i\in J}\mathcal{H}_i = \left\{\{f_i\} : f_i \in \mathcal{H}_i, \sum_{i\in J} \|f_i\|^2 < \infty\right\}$$

with the inner product defined by

$$\langle\{f_i\}, \{g_i\}\rangle = \sum_i \langle f_i, g_i\rangle.$$

Let for all $i \in J$, $\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)$. Then, we define the operator $\oplus_{i\in J}\Lambda_i$ on $\oplus_{i\in J}\mathcal{H}_i$ by $\oplus_{i\in J}\Lambda_i(\{f_i\}) = \{\Lambda_i f_i\}_{i\in J}$.

Proposition 2.4. *Let $\{\Lambda_i\}_{i\in J}$ and $\{\Theta_i\}_{i\in J}$ be two g-frames for Hilbert spaces $\mathcal{H}$ and K with respect to $\{\mathcal{H}_i\}_{i\in J}$ and $\{K_i\}_{i\in J}$, respectively. Then $\{\Lambda_i \oplus \Theta_i\}_{i\in J}$ is a g-frame for $\mathcal{H} \oplus K$ with respect to $\{\mathcal{H}_i \oplus K_i\}_{i\in J}$ and*

$$S_{\Lambda\oplus\Theta} = S_\Lambda \oplus S_\Theta,$$

where $S_{\Lambda\oplus\Theta}$, S_Λ and S_Θ are the g-frame operators of $\{\Lambda_i \oplus \Theta_i\}_{i\in J}$, $\{\Lambda_i\}_{i\in J}$ and $\{\Theta_i\}_{i\in J}$, respectively.

Proof. Let $\{\Lambda_i\}_{i\in J}$ be a g-frame for $\mathcal{H}$ with bounds A_1 and B_1 with respect to $\{\mathcal{H}_i\}_{i\in J}$, then

$$A_1\|f\|^2 \leq \sum_{i\in J} \|\Lambda_i f\|^2 \leq B_1\|f\|^2 \tag{2.1}$$

for all $f \in \mathcal{H}$. Suppose that $\{\Theta_i\}_{i\in J}$ is a g-frame for K with bounds A_2 and B_2 with respect to $\{K_i\}_{i\in J}$, we have

$$A_2\|g\|^2 \leq \sum_{i\in J} \|\Theta_i g\|^2 \leq B_2\|g\|^2 \tag{2.2}$$

for each $g \in \mathcal{H}$. From (2.1) and (2.2) we conclude that for each $f \in H$ and $g \in K$,

$$A_1\|f\|^2 + A_2\|g\|^2 \leq \sum_{i\in J} \|\Lambda_i f\|^2 + \|\Theta_i g\|^2 \leq B_1\|f\|^2 + B_2\|g\|^2.$$

Let $A = \min\{A_1, A_2\}$, $B = \max\{B_1, B_2\}$ and $f \oplus g \in H \oplus K$. We have

$$A\|f \oplus g\|^2 \leq \sum_{i\in J} \|(\Lambda_i \oplus \Theta_i)(f \oplus g)\|^2 \leq B\|f \oplus g\|^2.$$

So,

$$\begin{aligned} S_{\Lambda\oplus\Theta}(f \oplus g) &= \sum_{i\in J} (\Lambda_i \oplus \Theta_i)^*(\Lambda_i \oplus \Theta_i)(f \oplus g) = \sum_{i\in J} (\Lambda_i^* \oplus \Theta_i^*)(\Lambda_i f \oplus \Theta_i g) \\ &= \sum_{i\in J} (\Lambda_i^* \oplus \Theta_i^*)(\Lambda_i f \oplus \Theta_i g) = \sum_{i\in J} (\Lambda_i^*\Lambda_i f \oplus \Theta_i^*\Theta_i g) \\ &= \left(\sum_{i\in J} (\Lambda_i^*\Lambda_i f\right) \oplus \left(\sum_{i\in J} \Theta_i^*\Theta_i g\right) = (S_\Lambda \oplus S_\Theta)(f \oplus g). \end{aligned}$$

Hence, $S_{\Lambda\oplus\Theta} = S_\Lambda \oplus S_\Theta$. □

Corollary 2.5. *If $\Lambda_i = \{\Lambda_{ij}\}_{j\in J}$ is a g-frame for a Hilbert space $\mathcal{H}_i$ with respect to $\{\mathcal{H}_{ij}\}_{j\in J}$, with bounds A_i and B_i such that $inf_{i\in J} A_i = A > 0$ and $\sup_{i\in J} B_i = B < \infty$. Then $\Lambda = \{\oplus_{i\in\mathbb{N}}\Lambda_i\}$ is a g-frame for the Hilbert space $\oplus_{i\in\mathbb{N}}\mathcal{H}_i$ with respect to $\{\oplus_{i\in\mathbb{N}}\mathcal{H}_{ij}\}_{i\in J}$ with bounds A and B.*

3. PG-frame

As mentioned earlier, a p-frame for Banach spaces was introduced by Christensen and Stoeva [5] and a p-frame of subspaces by Faroughi and Najati [15]. The following definition is a generalization of g-frames that helps for every $f \in X^*$ to be represented as an unconditionally convergent series.

Definition 3.1. We call a sequence $\{\Lambda_i \in B(X, Y_i) : i \in J\}$ a pg-frame for X with respect to $\{Y_i : i \in J\}$ if there exist $A, B > 0$ such that

$$A\|x\|_X \leq \left(\sum_{i\in J} \|\Lambda_i x\|^p\right)^{\frac{1}{p}} \leq B\|x\|_X, \quad x \in X. \tag{3.1}$$

A, B is called the pg-frame bounds of $\{\Lambda_i\}_{i\in J}$.

If only the second inequality in (3.1) is satisfied, $\{\Lambda_i\}_{i\in J}$ is called a pg-Bessel sequence for X with respect to $\{Y_i : i \in J\}$ with bound B.

Similar to frames and g-frames [16], the following propositions show that the image of a pg-frame under a bounded operator is also a pg-frame.

Proposition 3.2. *Let $\{\Lambda_i\}_{i\in J}$ be a pg-frame for X with respect to $\{Y_i\}_{i\in J}$. Let S be a bounded invertible operator on X and $\Gamma_i = \Lambda_i S$. Then $\{\Gamma_i\}_{i\in J}$ is a pg-frame for X with pg-frame bounds $A\|S^{-1}\|^{-1}$ and $B\|S\|$.*

Proof. Let $\{\Lambda_i\}_{i\in J}$ be a pg-frame for X. Then

$$A\|Sx\|_X \leq \left(\sum_{i\in J} \|\Lambda_i Sx\|^p\right)^{\frac{1}{p}} \leq B\|Sx\|_X, \quad x \in X.$$

Since S is invertible,

$$A\|S^{-1}\|^{-1}\|x\|_X \leq \left(\sum_{i\in J}\|\Gamma_i x\|^p\right)^{\frac{1}{p}} \leq B\|S\|\|x\|_X, \quad x\in X,$$

so $\{\Gamma_i\}_{i\in J}$ is a *pg*-frame for X. □

Corollary 3.3. *Let $\{\Lambda_i\}_{i\in J}$ be a pg-frame for X with respect to $\{Y_i\}_{i\in J}$ and $S: X\to X$ be an isometry. If $\Gamma_i = \Lambda_i S$ then $\{\Gamma_i\}_{i\in J}$ is a pg-frame for X with the same bounds.*

Proposition 3.4. *Let $\{\Lambda_i\}_{i\in J}$ be a pg-frame for X with respect to $\{Y_i\}_{i\in J}$ and $S: X\to X$ be a bounded operator. Then $\{\Lambda_i S\}_{i\in J}$ is a pg-frame for X if and only if S is bounded below.*

Proof. Let $\{\Lambda_i S\}_{i\in J}$ be a *pg*-frame for X with bounds m, n. We have

$$m\|x\|_X \leq \left(\sum_{i\in J}\|\Lambda_i Sx\|^p\right)^{\frac{1}{p}} \leq n\|x\|_X, \quad x\in X.$$

Let A, B be *pg*-frame bounds of $\{\Lambda_i\}_{i\in J}$. Since

$$A\|Sx\|_X \leq \left(\sum_{i\in J}\|\Lambda_i Sx\|^p\right)^{\frac{1}{p}} \leq B\|Sx\|_X, \quad x\in X,$$

$m\|x\|_X \leq B\|Sx\|_X$. Thus, for each $x\in X$, $\|Sx\|_X \geq \frac{\delta}{m}\|x\|_X$. Now, suppose there exists $\delta > 0$ such that for each $x\in X$, $\|Sx\|_X > \delta\|x\|_X$. Since

$$A\delta\|x\|_X \leq A\|Sx\|_X \leq \left(\sum_{i\in J}\|\Lambda_i Sx\|^p\right)^{\frac{1}{p}} \leq B\|Sx\|_X \leq B\|S\|\|x\|_X,$$

$\{\Lambda_i S\}$ is a *pg*-frame for X with bounds $A\delta$ and $B\|S\|$. □

Definition 3.5. Let $\{Y_i\}_{i\in J}$ be a sequence of Banach spaces. We define

$$\left(\sum_{i\in J}\oplus Y_i\right)_{l_p} = \left\{\{x_i\}_{i\in J} | x_i\in Y_i, \left(\sum\|x_i\|^p\right)^{\frac{1}{p}} < +\infty\right\}.$$

Then $\left(\sum_{i\in J}\oplus Y_i\right)_{l_p}$ is a Banach space with the norm

$$\|\{x_i\}_{i\in J}\|_p = \left(\sum_{i\in J}\|x_i\|^p\right)^{\frac{1}{p}}.$$

Let $1 < p, q < \infty$ be conjugate exponents, i.e. $\frac{1}{p}+\frac{1}{q} = 1$. If $x^* = \{x_i^*\}_{i\in J} \in (\sum_{i\in J}\oplus Y_i^*)_{l_q}$ then an easy computation shows that the formula

$$\langle x, x^*\rangle = \sum_{i\in J}\langle x_i, x_i^*\rangle, \quad x = \{x_i\} \in \left(\sum_{i\in J}\oplus Y_i\right)_{l_p}$$

defines a continuous functional on $(\sum_{i\in J}\oplus Y_i)_{l_p}$ whose norm is equal to $\|x^*\|_q$ and its dual can be characterized with the following lemma whose proof can be found in [1].

Lemma 3.6. *Let $1 < p, q < \infty$ be such that $\frac{1}{p}+\frac{1}{q} = 1$. Then*

$$\left(\sum_{i\in J}\oplus Y_i\right)^*_{l_p} = \left(\sum_{i\in J}\oplus Y_i^*\right)_{l_q},$$

where the equality holds under the duality

$$\langle x, x^*\rangle = \sum_{i\in J}\langle x_i, x_i^*\rangle.$$

Definition 3.7. If $\{\Lambda_i\}_{i\in J}$ is a *pg*-frame, we define the operators T and U, by

$$U: X\to \left(\sum_{i\in J}\oplus Y_i\right)_{l_p},$$

(3.2) $$Ux = \{\Lambda_i x\}_{i\in J},$$

$$T : \Big(\sum_{i\in J} \oplus Y_i^*\Big)_{l_q} \to X^*,$$

$$T\{g_i\}_{i\in J} = \sum_{i\in J} g_i\Lambda_i. \tag{3.3}$$

The operators U, T are called the analysis and synthesis operators of $\{\Lambda_i\}_{i\in J}$.

Now, we characterize *pg*-Bessel sequence and *pg*-frames by the operator T defined by (3.3).

Proposition 3.8. *$\{\Lambda_i \in B(X, Y_i) : i \in J\}$ is a pg-Bessel sequence for X with respect to $\{Y_i\}$ if and only if the operator T defined by* (3.3) *is a well defined and bounded operator.*

Proof. Suppose that $\{\Lambda_i\}_{i\in J}$ is a *pg*-Bessel sequence with bound B, then we show that for each $\{f_i\}_{i\in J} \in (\sum_{i\in J} \oplus Y_i^*)_{l_q}$ the series $\sum_{i\in J} f_i\Lambda_i$ is convergent unconditionally. For finite subsets $J_1, J_2 \subset J$ and $J_2 \subsetneqq J_1$, we have

$$\begin{aligned}\Big\|\sum_{i\in J_1} f_i\Lambda_i - \sum_{i\in J_2} f_i\Lambda_i\Big\| &= \Big\|\sum_{i\in J_1\setminus J_2}^{k} f_i\Lambda_i\Big\| = \sup_{\|x\|=1}\Big\|\sum_{i\in J_1\setminus J_2} f_i\Lambda_i x\Big\| \\ &\le \sup_{\|x\|=1}\sum_{i\in J_1\setminus J_2} \|f_i\|\|\Lambda_i x\| \\ &\le \Big(\sum_{i\in J_1\setminus J_2}\|f_i\|^q\Big)^{\frac{1}{q}} \sup_{\|x\|=1}\Big(\sum_{i\in J_1\setminus J_2}\|\Lambda_i x\|^p\Big)^{\frac{1}{p}} \\ &\le B\Big(\sum_{i\in J_1\setminus J_2}\|f_i\|^q\Big)^{\frac{1}{q}},\end{aligned}$$

so, $\sum_{i\in J} f_i\Lambda_i$ is unconditionally convergent. By the same argument,

$$\Big\|\sum_{i\in J} f_i\Lambda_i\Big\| \le B\Big(\sum_{i\in J}\|f_i\|^q\Big)^{\frac{1}{q}}.$$

Hence,

$$\|T\{f_i\}_{i\in J}\| \le B\Big(\sum_{i\in J}\|f_i\|^q\Big)^{\frac{1}{q}} = B\|\{f_i\}\|_q,$$

so, T is bounded and $\|T\| \le B$.

For the converse, assume that T is well define and bounded. For $x \in X$, consider

$$F_x : \Big(\sum_{i\in J} \oplus Y_i^*\Big)_{l_q} \to \mathbb{C},$$

$$F_x(\{g_i\}) = (T\{g_i\})(x) = \sum_{i\in J} g_i\Lambda_i x,$$

then F_x is in $(\sum_{i\in J} \oplus Y_i^*)^*_{l_q}$, so

$$\{\Lambda_i x\} \in \Big(\sum_{i\in J} \oplus Y_i\Big)_{l_p}$$

and

$$\|F_x(\{g_i\})\| \le \|T\|\|\{g_i\}\|_q\|x\|.$$

By the Hahn-Banach theorem, there is $\{g_i\} \in (\sum_{i\in J} \oplus Y_i^*)_{l_q}$ with $\|\{g_i\}\|_q \le 1$ such that

$$\|\{\Lambda_i x\}\|_p = \Big|\sum_{i\in J} g_i\Lambda_i x\Big|.$$

Therefore,

$$\Big(\sum_{i\in J}\|\Lambda_i x\|^p\Big)^{\frac{1}{p}} = \|\{\Lambda_i x\}\|_p \le \sup_{\|\{g_i\}\|_q\le 1}\Big|\sum_{i\in J} g_i\Lambda_i x\Big| = \|F_x\| \le \|T\|\|x\|. \qquad \square$$

Lemma 3.9. *If $\{\Lambda_i\}_{i\in J}$ is a pg-frame, then the operator U has closed range.*

Proof. Let $\{\Lambda_i\}_{i\in J}$ be a *pg*-frame. Then there exist $A, B > 0$ such that

$$A||x||_X \leq \left(\sum_{i\in J} \|\Lambda_i x\|^p\right)^{\frac{1}{p}} \leq B\|x\|_X, \quad x \in X.$$

So,

$$A\|x\| \leq \|Ux\| \leq B\|x\|.$$

If $Ux = 0$ then $x = 0$, hence U is one-to-one and so $X \simeq \mathcal{R}_U$, therefore U has closed range. □

Lemma 3.10. *If all of Y_i's are reflexive and $\{\Lambda_i\}_{i\in J}$ is a pg-frame for X with respect to $\{Y_i\}_{i\in J}$ then X is reflexive.*

Proof. By lemma (3.9), $\mathcal{R}_U$ is a closed subspace of $\left(\sum_{i\in J} \oplus Y_i\right)_{l_p}$ and $X \simeq \mathcal{R}_U$ so X is reflexive. □

Lemma 3.11. *Let $\{\Lambda_i\}_{i\in J}$ be a pg-Bessel sequence for X with respect to $\{Y_i\}_{i\in J}$. Then*

(i) $U^* = T$.

(ii) *If $\{\Lambda_i\}_{i\in J}$ has the lower pg-frame condition and all of Y_i's are reflexive, then $T^* = U$.*

Proof. (i) For any $x \in X$ and $\{g_i\}_{i\in J} \in \left(\sum_{i\in J} \oplus Y_i^*\right)_{l_q}$, we have

$$\langle Ux, \{g_i\}_{i\in J}\rangle = \langle \{\Lambda_i x\}_{i\in J}, \{g_i\}_{i\in J}\rangle = \sum_{i\in J}\langle \Lambda_i x, g_i\rangle = \sum_{i\in J} g_i\Lambda_i x$$

and

$$\langle x, T\{g_i\}_{i\in J}\rangle = \left\langle x, \sum_{i\in J} g_i\Lambda_i\right\rangle = \sum_{i\in J} g_i\Lambda_i x,$$

so $T^* = U$.

(ii) By Lemma (3.9) $\mathcal{R}_U$ is a closed subspace of $(\sum_{i\in J} \oplus Y_i)_{l_p}$ and so is reflexive, so $U^{**} = T^*$ hence $U = T^*$. □

Theorem 3.12. *$\{\Lambda_i\}_{i\in J}$ is a pg-frame for X with respect to $\{Y_i\}_{i\in J}$ if and only if the operator T defined by* (3.3) *is a surjective bounded operator.*

Proof. If $\{\Lambda_i\}_{i\in J}$ is a *pg*-frame, by Proposition (3.8), T is well-defined and bounded. The proof of Lemma (3.9) shows that U is injective, so by Lemma (1.4) and (3.11)(i) $U^* = T$ is onto.

Conversely, assume that T is bounded and onto. Then Proposition (3.8) implies that $\{\Lambda_i\}_{i\in J}$ is a *pg*-Bessel sequence. Since $T = U^*$ is onto, by Lemma (1.4) , U has a bounded inverse . So there exists $A > 0$ such that for all $x \in X$, $\|Ux\| \geq A\|x\|$. In other words, $\{\Lambda_i\}_{i\in J}$ satisfies the lower *pg*-frame condition. □

Corollary 3.13. *If $\{\Lambda_i \in B(X, Y_i) : i \in J\}$ is a pg-frame for X with respect to $\{Y_i\}_{i\in J}$ then for any $x^* \in X^*$ there exists a $\{g_i\}_{i\in J} \in (\sum \oplus Y_i^*)_{l_q}$ such that*

$$x^* = \sum_{i\in J} g_i\Lambda_i.$$

Definition 3.14. Let $1 < q < \infty$. A family $\{\Lambda_i \in B(X, Y_i) : i \in J\}$ is called a *qg*-Riesz basis for X^* with respect to $\{Y_i\}_{i\in J}$, if

(i) $\{f : \Lambda_i f = 0, i \in J\} = \{0\}$ (i.e. $\{\Lambda_i\}_{i\in J}$ is *g*-complete);

(ii) there are positive constants A, B such that for any finite subset $J_1 \subseteq J$ and $g_i \in Y_i^*$, $i \in J_1$,

$$A\left(\sum_{i\in J_1} \|g_i\|^q\right)^{\frac{1}{q}} \leq \Big\|\sum_{i\in J_1} g_i\Lambda_i\Big\| \leq B\left(\sum_{i\in J_1} \|g_i\|^q\right)^{\frac{1}{q}}.$$

The assumptions of definition (3.14) imply that $\sum_{i\in J} g_i\Lambda_i$ converges unconditionally for all $\{g_i\} \in (\sum_{i\in J} \oplus Y_i^*)_{l_q}$, and

$$A\left(\sum_{i\in J} \|g_i\|^q\right)^{\frac{1}{q}} \leq \|\sum_{i\in J} g_i\Lambda_i\| \leq B\left(\sum_{i\in J} \|g_i\|^q\right)^{\frac{1}{q}}.$$

Therefore $\{\Lambda_i \in B(X, Y_i) : i \in J\}$ is a qg-Riesz basis for X, if and only if, the operator T defined by (3.3) is an invertible operator from $\left(\sum_{i\in J} \oplus Y_i^*\right)_{l_q}$ onto X^*.

The following Proposition shows that a qg-Riesz basis for X^* is a special case of pg-frames for X.

Proposition 3.15. *Let $\{\Lambda_i \in B(X, Y_i) : i \in J\}$ be a qg-Riesz basis for X^* with respect to $\{Y_i\}_{i\in J}$ with the optimal upper qg-Riesz basis bound B. Then $\{\Lambda_i \in B(X, Y_i) : i \in J\}$ is a pg-frame for X with respect to $\{Y_i\}_{i\in J}$ with optimal upper pg-frame bound B.*

Proof. Assume that $\{\Lambda_i \in B(X, Y_i) : i \in J\}$ is a qg-Riesz basis for X^*, the operator T defined by (3.3) is a bounded and invertible operator. Theorem (3.12) implies that $\{\Lambda_i\}_{i\in J}$ is a pg-frame for X. By Proposition (3.8) the upper qg-Riesz basis bound coincides with the upper pg-frame bound. □

Theorem 3.16. *Let $\{Y_i\}_{i\in J}$ be a sequence of reflexive Banach spaces. Let $\{\Lambda_i \in B(X, Y_i) : i \in J\}$ be a pg-frame for X with respect to $\{Y_i\}_{i\in J}$. Then the following statements are equivalent:*

(i) *$\{\Lambda_i\}_{i\in J}$ is a qg-Riesz basis for X^*.*
(ii) *If $\{g_i\}_{i\in J} \in \left(\sum_{i\in J} \oplus Y_i^*\right)_{l_q}$ and $\sum_{i\in J} g_i\Lambda_i = 0$ then $g_i = 0,\ i \in J$.*
(iii) *$\mathcal{R}_U = (\sum_{i\in J} \oplus Y_i)_{l_p}$.*

Proof. It is clear that (i) ⇒(ii).

Suppose that (ii) holds. By Theorem (3.12), the operator T is bounded and onto, by (ii), T is also injective, therefore, T has a bounded inverse $T^{-1} : X^* \to \left(\sum_{i\in J} \oplus Y_i^*\right)_{l_q}$ and so $\{\Lambda_i\}_{i\in J}$ is a qg-Riesz basis for X.

(i) ⇒ (iii) Since $\{\Lambda_i\}_{i\in J}$ is a qg-Riesz basis for X^*, T has a bounded inverse on $\mathcal{R}_T$. By Lemma (1.4) the adjoint $T^* : X^{**} \to \left(\sum_{i\in J} \oplus Y_i\right)_{l_p}$ is surjective on $\mathcal{R}_T$. By Lemma (3.10) X is reflexive, and so Theorem (3.12) and Lemma (3.11) imply that $\mathcal{R}_U = (\sum_{i\in J} \oplus Y_i)_{l_p}$.

(iii) ⇒ (i) Since the operator U is bijective, by Theorem 4.12 in [17], $T = U^* : \left(\sum_{i\in J} \oplus Y_i^*\right)_{l_q} \to X^*$ is invertible. □

4. G-Banach frames

A Banach space of vector-valued sequences (or BV-space) is a linear space of sequences with a norm which makes it a Banach space. Let X be a Banach space and $1 < p < \infty$ then

$$Y = \left\{\{x_i\}_{i\in J} \,|\, x_i \in X,\ \left(\sum_{i\in J} \|x_i\|^p\right)^{\frac{1}{p}} < +\infty\right\}$$

and

$$l^\infty = \{\{x_i\} | \sup_{i\in J} \|x_i\| < \infty,\, x_i \in X\}$$

are BV-space for X.

In [11] Grochenig and in [4] Casazza, Han and Larson generalized frames to Banach spaces and defined Banach frames for Banach space X with respect to a BV-space, and in this paper we shall extend its definition to g-Banach frames for a Banach space X with respect to a BV-space.

Definition 4.1. Let X be a Banach space and $\mathcal{H}$ be a separable Hilbert space. Let X_d be an associated Banach space of vector-valued sequences indexed by $\mathbb{N}$. Let $\{\Lambda_i\}_{i\in\mathbb{N}} \subset B(X,\mathcal{H})$ and $S : X_d \to X$ are given. If

(i) $\{\Lambda_i x\}_{i\in\mathbb{N}} \in X_d$ for each $x \in X$,
(ii) the norms $\|x\|_X$ and $\|\{\Lambda_i x\}_{i\in\mathbb{N}}\|_{X_d}$ are equivalent, and
(iii) S is bounded and linear and $S\{\Lambda_i x\}_{i\in\mathbb{N}} = x$ for each $x \in X$,

then $(\{\Lambda_i\}_{i\in\mathbb{N}}, S)$ is a g-Banach frame for X with respect to $\mathcal{H}$ and X_d. The mapping S is the reconstruction operator. If the norm equivalence is given by

$$A\|x\|_X \le \|\{\Lambda_i x\}_{i\in\mathbb{N}}\|_{X_d} \le B\|x\|_X$$

for all $x \in X$, then A, B are called the frame bounds for $(\{\Lambda_i\}_{i\in\mathbb{N}}, S)$.

Theorem 4.2. *Let $\mathcal{H}$ be a separable Hilbert Space. Then every separable Banach space has a g-Banach frame with respect to $\mathcal{H}$ with frame bounds $A = B = 1$.*

Proof. If X is a separable Banach space, there exists $E \subset X$ such that $\overline{E} = X$ and E is a countable set. Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. We define the operators Λ_i from E into $\mathcal{H}$ by

$$\Lambda_i(x_j) = \delta_{ij}\|x_j\|e_j, \quad j \in \mathbb{N}.$$

Then

$$\sup_i \|\Lambda_i(x_j)\| = \|x_j\|.$$

Since $\overline{E} = X$, Λ_i can be extended to a bounded operator $\tilde{\Lambda}_i$ on X such that

$$\sup_i \|\tilde{\Lambda}_i(x)\| = \|x\|, \quad x \in X. \tag{4.1}$$

Let X_d be the subspace of $l^\infty(X)$ given by

$$X_d = \{\{\tilde{\Lambda}_i x\} : x \in X\}.$$

Let $S : X_d \to X$ be given by $S(\{\tilde{\Lambda}_i x\}) = x$. Now, by equality (4.1), S is an isometry of X onto X_d and $(\{\tilde{\Lambda}_i\}, S)$ is a g-Banach frame for X with respect to X_d. □

Perturbation of frames as a type of Paley-Winer theorem was proved by Casazza and Christensen [2], for Banach frames by Christensen and Heil [9] and for g-frames in Hilbert spaces by Faroughi, Najati and Rahimi [16]. In this section we present the perturbation of g-Banach frames.

Theorem 4.3. *Let $(\{\Lambda_i\}_{i\in\mathbb{N}}, S)$ be a g-Banach frame for X with respect to X_d. Let $\{\Gamma_i\}_{i\in\mathbb{N}} \subseteq B(X,\mathcal{H})$. If there exist $\lambda, \mu \ge 0$ such that*

(i) $\lambda\|U\| + \mu < \|S\|^{-1}$,
(ii) $\|\{\Lambda_i(x) - \Gamma_i(x)\}\|_{x_d} \le \lambda\|\{\Lambda_i(x)\}\|_{x_d} + \mu\|x\|_X, \quad x \in X,$

then there exists an operator T such that $(\{\Gamma_i\}_{i\in\mathbb{N}}, T)$ is a g-Banach frame for X with respect to X_d with frame bounds $\|S\| - (\lambda\|U\| + \mu)$ and $\|U\| + (\lambda\|U\| + \mu)$, where U is the operator $Ux = \{\Lambda_i(x)\}_{i\in\mathbb{N}}, x \in X$.

Proof. Let us define the operator $V : X \to X_d$ by $Vx = \{\Gamma_i(x)\}_{i\in\mathbb{N}}$. Since $(\{\Lambda_i\}_{i\in\mathbb{N}}, S)$ is a g-Banach frame for X hence there exist $A, B > 0$ such that

$$A\|x\|_X \le \|\{\Lambda_i(x)\}\|_{X_d} \le B\|x\|_X, \quad x \in X.$$

So U is bounded and by (ii) for every $x \in X$,

$$\|Ux - Vx\|_{X_d} \le \lambda\|Ux\|_{X_d} + \mu\|x\|_X.$$

Therefore,

$$\|Vx\|_{X_d} \le (\|U\| + \lambda\|U\| + \mu)\|x\|_X,$$

so the upper g-frame bound is $(\|U\| + \lambda\|U\| + \mu)$. For the lower bound, we have $SU = I$ so

$$\|I - SV\| \le \|S\|\|U - V\| \le \|S\|(\lambda\|U\| + \mu) < 1,$$

therefore, SV is invertible, and $\|(SV)^{-1}\| \leq (1 - \|U\| + \mu) < 1$. If we consider $T = (SV)^{-1}S$ then $TV = I$,

$$\|x\|_X \leq \|T\|\|Vx\|_{X_d} \leq \frac{\|S\|}{1 - (\lambda\|U\| + \mu)\|S\|}\|Vx\|_{X_d},$$

and so

$$(\|S\|^{-1} - (\lambda\|U\| + \mu))\|x\|_X \leq \|Vx\|_{X_d},$$

and this concludes the proof. □

Acknowledgments. The authors would like to express their sincere thanks to Ole Christensen and the referee for the useful comments.

References

1. C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis: A Hitchhikers Guide*, Springer-Verlag, New York—Berlin, 1999.
2. P. G. Cazassa and O. Christensen, *Perturbation of operators and applications to frame theory*, J. Fourier Anal. Appl. **3** (1997), no. 5, 543–557.
3. P. G. Cazassa and Gitta Kutyniok, *Frames of subspaces. Wavelets, frames and operator theory*, Contemp. Math., Vol. 345, Amer. Math. Soc., Providence, R. I., 2004, 87–113.
4. P. G. Cazassa, D. Han, and D. R. Larson, *Frames for Banach spaces*, Contemp. Math. Vol. 247, Amer. Math. Soc., Providence, R. I., 1999, 149–182.
5. O. Christensen and D. Stoeva, *p-frames in separable Banach spaces*, Adv. Comput. Math. **18** (2003), no. 2–4, 117–126.
6. O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhauser, Boston, 2002.
7. O. Christensen and Y. C. Eldar, *Oblique dual frames and shift-invariant spaces*, Appl. Comput. Harmon. Anal. **17** (2004), 48–68.
8. Y. Eldar, *Sampling with arbitrary sampling and reconstruction spaces and oblique dual frame vectors*, J. Fourier Anal. Appl. **9** (2003), 77–96.
9. O. Christensen and C. Heil, *Perturbations of Banach frames and atomic decompositions*, Math. Nachr. **185** (1997), 33–47.
10. M. Fornasier, *Decompositions of Hilbert space: local construction of global frames*, Proc. Int. Conf., Constructive Theory of Functions (ed. B. Bojanov), Varna 2002. DARBA, Sofia, 2003, pp. 275–281.
11. K. Grochenig, *Describing functions: atomic decomposition versus frames*, Monatsh. Math. **112** (1991), 1–41.
12. D. Han, D. R. Larson, *Frames, bases and group representations*, Memoirs Amer. Math. Soc. **147** (2000), no. 697, 1–91.
13. H. Heuser, *Functional Analysis*, John Wiley, New York, 1982.
14. S. Li and H. Ogawa, *Pseudoframes for subspaces with applications*, J. Fourier Anal. Appl. **10** (2004), 409–431.
15. A. Najati and M. H. Faroughi, *p-frames of subspaces in separable Hilbert spaces*, Southest Asian Bull. Math. (to appear).
16. A. Najati, A. Rahimi, and M. H. Faroughi, *G-frames in Hilbert spaces* (submitted for publication).
17. W. Rudin, *Functional Analysis*, MacGraw-Hill, New York, 1973.
18. W. Sun, *G-frames and G-Riesz bases*, J. Math. Anal. Appl. **322** (2006), 437–452.

Department of Mathematics, Tabriz University, Tabriz, Iran
E-mail address: mr_abdollahpour@yahoo.com

Department of Mathematics, Tabriz University, Tabriz, Iran
E-mail address: mhfaroughi@yahoo.com

Department of Mathematics, Tabriz University, Tabriz, Iran
E-mail address: asgharrahimi@yahoo.com

Received 25/01/2006; Revised 28/07/2006