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THE ε_∞-PRODUCT OF A b-SPACE BY A QUOTIENT BORNOLOGICAL SPACE

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ABSTRACT. We define the ε_∞-product of a Banach space G by a quotient bornological space $E \mid F$ that we denote by $G\varepsilon_\infty(E \mid F)$, and we prove that G is an $\mathcal{L}_\infty$-space if and only if the quotient bornological spaces $G\varepsilon_\infty(E \mid F)$ and $(G\varepsilon E) \mid (G\varepsilon F)$ are isomorphic. Also, we show that the functor $.\varepsilon_\infty. : \mathbf{Ban} \times \mathbf{qBan} \longrightarrow \mathbf{qBan}$ is left exact. Finally, we define the ε_∞-product of a b-space by a quotient bornological space and we prove that if G is an εb-space and $E \mid F$ is a quotient bornological space, then $(G\varepsilon E) \mid (G\varepsilon F)$ is isomorphic to $G\varepsilon_\infty(E \mid F)$.

1. Introduction and basic notions

The ε-product of two locally convex spaces was introduced by L. Schwartz in his famous article on vector-valued distributions [13], where he also looked at the ε-product of two continuous linear mappings. Many spaces of vector-valued functions or distributions turn out to be the ε-product of the corresponding space of scalar functions and the range space. Also, ε-products allow to reduce the treatment of many spaces of functions or distributions on product sets to the one dimensional case.

L. Waelbroeck [14], rediscovered the ε-product of two Banach spaces much later, without giving any explicit reference to the ε-product of Schwartz (we guess that Waelbroeck simply forgot to quote Schwartz). But his objective was to give a different approach to the ε-product of Schwartz in his special case.

It is well known that the ε-product by a Banach space is always a left exact functor but in general is not right exact. To study this problem for space of vector-valued functions that can be interpreted as an ε-product, Kaballo [8] introduced ε-spaces as locally convex spaces G for which the ε-product of the identity map of G with any surjective continuous linear mapping between Banach spaces is surjective and showed that a Banach space is an ε-space if and only if it is an $\mathcal{L}_\infty$-space. As a consequence, if G is an $\mathcal{L}_\infty$**-**space, the left exact functor $G\varepsilon. : \mathbf{Ban} \longrightarrow \mathbf{Ban},\ E \longrightarrow G\varepsilon E$ is exact, and then by Theorem 4.1 of [17], it admits an exact extension $G\varepsilon. : \mathbf{qBan} \longrightarrow \mathbf{qBan}$, $E \mid F \longrightarrow G\varepsilon(E \mid F) = (G\varepsilon E) \mid (G\varepsilon F)$, where **qBan** is the category of quotient Banach spaces and **Ban** the category of Banach spaces. But there exist many important Banach spaces which are not $\mathcal{L}_\infty$-spaces. For example, Khenkin [9], showed that if U is an open subset of $\mathbb{R}^n$, $n \geq 2$ and $r \in \mathbb{N}^*$, the Banach space $C^r(\overline{U})$ is not an $\mathcal{L}_\infty$-space and Pelsczynski [11], proved that $A(\mathbf{D})$, the Banach space of continuous functions on the closed unit disc of $\mathbb{C}$ and holomorphic on the open unit disc of $\mathbb{C}$, is not an $\mathcal{L}_\infty$-space.

Now our interest in this paper is to discuss the following question:

Let G be a b-space and $E \mid F$ be a quotient bornological space, such that $G\varepsilon(E \mid F)$ is not isomorphic to $(G\varepsilon E) \mid (G\varepsilon F)$, is $G\varepsilon(E \mid F)$ a quotient of a b-space by a b-subspace? What is the relation between $(G\varepsilon E) \mid (G\varepsilon F)$ and $G\varepsilon(E \mid F)$?

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Clearly, our question arises from the problem of lifting in the category of quotient bornological spaces of Waelbroeck [17], and the present paper is aimed to give a positive answer to this problem.

Recall that in [2], we defined the ε-product of an $\mathcal{L}_\infty$-space by a quotient Banach space and we established a necessary and sufficient condition under which the ε-product is monic. Also, the ε_c-product of a Schwartz b-space by a quotient Banach space had been defined and some examples of applications were given. However, it is not clear how to define the ε_c-product of an arbitrary b-space by a quotient bornological space.

To do this, we shall define and study a new ε-product in the category of quotient bornological spaces of Waelbroeck [17] that we call the ε_∞-product and which coincides with the ε-product of Waelbroeck [14] for the class of $\mathcal{L}_\infty$-spaces and the class of εb-spaces. It is also isomorphic to the ε_c-product of the class of Schwartz b-spaces defined in [2]. This ε_∞-product is useful to describe some spaces $\Im(X)\varepsilon(E \mid F)$ as a quotient of a b-space by a b-subspace.

To prove our results, we need to recall some definitions and notations. Let **EV** be the category of vector spaces and linear mappings over the scalar field $\mathbb{R}$ or $\mathbb{C}$.

1. Let $(E, \| \|_E)$ be a Banach space. A Banach subspace F of E is a vector subspace endowed with a Banach norm $\| \|_F$ such that the inclusion map $(F, \| \|_F) \longrightarrow (E, \| \|_E)$ is bounded. A quotient Banach space $E \mid F$ is a vector space E/F, where E is a Banach space and F a Banach subspace. If $E \mid F$ and $E_1 \mid F_1$ are quotient Banach spaces, a strict morphism $u : E \mid F \longrightarrow E_1 \mid F_1$ is a linear mapping $u : x + F \longmapsto u_1(x) + F_1$, where $u_1 : E \longrightarrow E_1$ is a bounded linear mapping such that $u_1(F) \subseteq F_1$. We shall say that u_1 induces u. Two bounded linear mappings u_1, $u_2 : E \longrightarrow E_1$ both inducing a strict morphism, induce the same strict morphism iff the linear mapping $u_1 - u_2 : E \longrightarrow F_1$ is bounded. A pseudo-isomorphism $u : E \mid F \longrightarrow E_1 \mid F_1$ is a strict morphism induced by a surjective bounded linear mapping $u_1 : E \longrightarrow E_1$ such that $u_1^{-1}(F_1) = F$.

We call $\tilde{\mathbf{q}}\mathbf{Ban}$ the category of quotient Banach spaces and strict morphisms, it is a subcategory of **EV** and contains **Ban**, which is not abelian, in fact, if E is a Banach space and F a closed subspace of E, the quotient Banach space $E \mid F$ is not necessarily isomorphic to $(E/F) \mid \{0\}$.

Waelbroeck introduced in [16] an abelian category **qBan** generated by $\tilde{\mathbf{q}}\mathbf{Ban}$ and inverses of pseudo-isomorphisms. For more information about quotient Banach spaces we refer the reader to [16].

2. A b-space (E, β) is a vector space E with a bounded structure β such that

$$E = \bigcup_{B \in \beta} B,$$

with $B \in \beta$ if $B \subset B_1 \cup B_2$ whenever $B_1, B_2 \in \beta$, without any non-null vector subspace of E belonging to β, and in which for every $B \in \beta$ there exists a $B_1 \in \beta$ with $B \subset B_1$, B_1 absolutely convex, and E_{B_1}, the subspace absorbed by B_1 with the norm-gauge associated to B_1, being a Banach space.

A subspace F of a b-space E is bornologically closed if $F \cap E_B$ is closed in E_B for every completant bounded B of E.

Given two b-spaces (E, β_E) and (F, β_F), a linear mapping $u : E \longrightarrow F$ is bounded, if it maps boundeds of E into boundeds of F. The mapping u is bornologically surjective if for every $B' \in \beta_F$, there exists $B \in \beta_E$ such that $u(B) = B'$.

We denote by **b** the category of b-spaces and bounded linear mappings. For more information about b-spaces we refer the reader to [5], [6] and [15].

Let (E, β_E) be a b-space. A b-subspace of E is a subspace F with a boundedness β_F such that (F, β_F) is a b-space and $\beta_F \subseteq \beta_E$. A quotient bornological space $E \mid F$ is a vector space E/F, where E is a b-space and F a b-subspace of E. If $E \mid F$ and $E_1 \mid F_1$ are quotient bornological spaces, a strict morphism $u : E \mid F \longrightarrow E_1 \mid F_1$ is induced by a bounded linear mapping $u_1 : E \longrightarrow E_1$ whose restriction to F is a bounded linear mapping $F \longrightarrow F_1$. Two bounded linear mappings $u_1,v_1 : E \longrightarrow E_1$, both inducing a strict morphism, induce the same strict morphism $E \mid F \longrightarrow E_1 \mid F_1$ iff the linear mapping $u_1 - v_1 : E \longrightarrow F_1$ is bounded.

The class of quotient bornological spaces and strict morphisms is a category, that we call $\widetilde{\mathbf{q}}$. A pseudo-isomorphism $u : E \mid F \longrightarrow E_1 \mid F_1$ is a strict morphism induced by a bounded linear mapping $u_1 : E \longrightarrow E_1$ which is bornologically surjective and such that $u_1^{-1}(F_1) = F$ i.e. $B \in \beta_F$ if $B \in \beta_E$ and $u_1(B) \in \beta_{F_1}$. As for the category $\tilde{\mathbf{q}}\mathbf{Ban},$ there are pseudo-isomorphisms which do not have strict inverses, Waelbroeck constructed in [17] an abelian category $\mathbf{q}$ that contains $\widetilde{\mathbf{q}}$ and in which all pseudo-isomorphisms of $\widetilde{\mathbf{q}}$ are isomorphisms.

3. The ε-product of two Banach spaces E and F is the Banach space $E\varepsilon F$ of linear mappings $E_1 \longrightarrow F$ whose restrictions to the unit ball of E_1 are $\sigma(E_1, E)$-continuous, where E_1 is the topological dual of E. It follows from Proposition 2 of [14], that the ε-product is symmetric. If E_i end F_i are Banach spaces and $u_i : E_i \longrightarrow F_i$ are bounded linear mappings, $i = 1, 2$, the ε-product of u_1 and u_2 is the bounded linear mapping $u_1\varepsilon u_2 : E_1\varepsilon E_2 \longrightarrow F_1\varepsilon F_2$, $f \longmapsto u_2 \circ f \circ u_1'$, where u_1' is the dual mapping of u_1. It is clear that if G is a Banach space and F is a Banach subspace of another Banach space E, then $G\varepsilon F$ is a Banach subspace of $G\varepsilon E$. For more detail about the ε-product we refer the reader to [7] and [14].

4. A Banach space E is an $\mathcal{L}_{\infty,\lambda}$-space, $\lambda \geq 1$, if and only if every finite-dimensional subspace F of E is contained in a finite-dimensional subspace F_1 of E such that $d(F_1, l_n^\infty) \leq \lambda$, where $n = \dim F_1$, l_n^∞ is $\mathbf{K}^n$ ($\mathbf{K} = \mathbb{R}$ or $\mathbb{C}$) with the norm $\sup_{1\leq i\leq n} |x_i|$, and $d\,(X, Y) = \inf\{||T||\,||T^{-1}||\,,\, T : X \longrightarrow Y \text{ isomorphism}\}$ is the Banach-Mazur distance of the Banach spaces X and Y. A Banach space E is an $\mathcal{L}_\infty$-space if it is an $\mathcal{L}_{\infty,\lambda}$-space for some $\lambda \geq 1$. For more information about $\mathcal{L}_\infty$-spaces we refer to see [10].

2. The ε_∞-product of a Banach space

A Banach space G is called injective if the restriction mapping $\mathbf{Ban}(., G) : \mathbf{Ban}(E, G) \longrightarrow \mathbf{Ban}(F, G)$ is surjective, as soon as E is a Banach space and F is a closed subspace of E, where $\mathbf{Ban}(H, G)$ is the Banach space of all bounded linear mappings from H into G, $H = E$, F. Well known examples of injective Banach spaces are $l^\infty(I)$, I being any set. By [10], every injective Banach space is an $\mathcal{L}_\infty$-space.

As the ε-product is a left exact functor on the category $\mathbf{Ban}$, we shall consider strongly left exact sequences. A complex $0 \longrightarrow E \xrightarrow{u} F \xrightarrow{v} G$ is left exact in $\mathbf{Ban}$ if $\mathrm{Ker}(v) = \mathrm{Im}(u)$. The complex $0 \longrightarrow E \xrightarrow{u} F \xrightarrow{v} G$ is strongly left exact in $\mathbf{Ban}$ if it is left exact and the image of v is closed in G.

Definition 2.1. Let G be a Banach space and I, J be sets. Then the strongly left exact complex $0 \longrightarrow G \xrightarrow{u} l^\infty(I) \xrightarrow{v} l^\infty(J)$ will be called a l^∞-resolution of G.

Proposition 2.2. *Every Banach space G has l^∞-resolutions.*

Proof. Let I be a dense subset in the closed unit ball $B_{G'}$ of the topological dual space G' of G. It is obvious that the linear mapping $u : G \longrightarrow l^\infty(I)$, $x \longmapsto u(x)$ such that $u(x)(g) = g(x)$ for all $g \in I$, is an isometry. Since $u(G)$ is a closed subspace of $l^\infty(I)$, we identify G with $u(G)$. Then there exists a dense subset J in $B_{(l^\infty(I)/G)'}$ and an isometric mapping $l^\infty(I)/G \longrightarrow l^\infty(J)$ where $(l^\infty(I)/G)'$ is the topological dual of $l^\infty(I)/G$. The mapping $v : l^\infty(I) \longrightarrow l^\infty(J)$ is the composition of the quotient mapping $l^\infty(I) \longrightarrow l^\infty(I)/G$ and the isometry $l^\infty(I)/G \longrightarrow l^\infty(J)$. Its image is closed in $l^\infty(J)$. It follows that $0 \longrightarrow G \xrightarrow{u} l^\infty(I) \xrightarrow{v} l^\infty(J)$ is a l^∞-resolution of G. $\square$

Below, we define the ε-product of a Banach space by a quotient bornological space. For this we let $0 \longrightarrow G \xrightarrow{u} l^\infty(I) \xrightarrow{v} l^\infty(J)$ be a l^∞-resolution of G. Since $l^\infty(I)$ and $l^\infty(J)$ are $\mathcal{L}_\infty$-spaces, it follows from [7] that the functor $l^\infty(K)\varepsilon. : \mathbf{Ban} \longrightarrow \mathbf{Ban}$ is exact for $K = I, J$. On the other word, the inductive limit functor is exact on the category of b-spaces [6], hence the functor $l^\infty(K)\varepsilon. : \mathbf{b} \longrightarrow \mathbf{b}$ is exact for $K = I, J$. Now, by Theorem 4.1 of [17], this functor admits an exact extension $l^\infty(K)\varepsilon. : \mathbf{q} \longrightarrow \mathbf{q}$. As a consequence, if $E \mid F$ is a quotient bornological space we have $l^\infty(K)\varepsilon(E \mid F) = (l^\infty(K)\varepsilon E) \mid (l^\infty(K)\varepsilon F)$ for $K = I, J$.

On the other hand, the bounded linear mapping $v\varepsilon \mathrm{Id}_E : l^\infty(I)\varepsilon E \longrightarrow l^\infty(J)\varepsilon E$ induces a strict morphism $v\varepsilon\mathrm{Id}_{E|F} : (l^\infty(I)\varepsilon E) \mid (l^\infty(I)\varepsilon F) \longrightarrow (l^\infty(J)\varepsilon E) \mid (l^\infty(J)\varepsilon F)$, and as the category $\mathbf{q}$ is abelian, the object $\mathrm{Ker}(v\varepsilon\mathrm{Id}_{E|F})$ exists, and then we obtain the following left exact sequence:

$$0 \longrightarrow \mathrm{Ker}(v\varepsilon\mathrm{Id}_{E|F}) \overset{u\varepsilon\mathrm{Id}_{E|F}}{\longrightarrow} (l^\infty(I)\varepsilon E) \mid (l^\infty(I)\varepsilon F) \overset{v\varepsilon\mathrm{Id}_{E|F}}{\longrightarrow} (l^\infty(J)\varepsilon E) \mid (l^\infty(J)\varepsilon F)$$

where

$$\mathrm{Ker}(v\varepsilon\mathrm{Id}_{E|F}) = (v\varepsilon\mathrm{Id}_E)^{-1}(l^\infty(J)\varepsilon F) \mid (l^\infty(I)\varepsilon F)$$

and $(v\varepsilon\mathrm{Id}_E)^{-1}(l^\infty(J)\varepsilon F)$ is a b-subspace of the b-space $l^\infty(I)\varepsilon E$ for the following boundedness: a subset B of $(v\varepsilon\mathrm{Id}_E)^{-1}(l^\infty(J)\varepsilon F)$ is bounded if it is bounded in $l^\infty(I)\varepsilon E$ and its image $(v\varepsilon\mathrm{Id}_E)(B)$ is bounded in $l^\infty(J)\varepsilon F$.

We let $G\varepsilon_{Res}(E \mid F) = \mathrm{Ker}(v\varepsilon\mathrm{Id}_{E|F})$. This defines a functor $G\varepsilon_{Res}. : \mathbf{q} \longrightarrow \mathbf{q},$ $E \mid F \longrightarrow G\varepsilon_{Res}(E \mid F)$.

While the Banach space G has several l^∞-resolutions, we will prove that the object $G\varepsilon_{Res}(E \mid F)$ does not depend on l^∞-resolutions of G.

Proposition 2.3. *Let G_1, G_2 be two Banach spaces and $0 \longrightarrow G_i \xrightarrow{u_i} l^\infty(I_i) \xrightarrow{v_i} l^\infty(J_i)$ be a l^∞-resolution of G_i, $i = 1, 2$. Let $u : G_1 \longrightarrow G_2$ be a bounded linear mapping. Then there exist bounded linear mappings $v : l^\infty(I_1) \longrightarrow l^\infty(I_2)$ and $w : l^\infty(J_1) \longrightarrow l^\infty(J_2)$ making the following diagram commutative:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & G_1 & \xrightarrow{u_1} & l^\infty(I_1) & \xrightarrow{v_1} & l^\infty(J_1) \\ & & \downarrow^{u} & & \downarrow^{v} & & \downarrow^{w} \\ 0 & \longrightarrow & G_2 & \xrightarrow{u_2} & l^\infty(I_2) & \xrightarrow{v_2} & l^\infty(J_2) \end{array}$$

Proof. By the construction of l^∞-resolutions of the Banach spaces G_1 and G_2 (proof of Proposition 2.2), we have the following sequences:

$$0 \longrightarrow G_i \xrightarrow{u_i} l^\infty(I_i) \longrightarrow l^\infty(I_i)/u_i(G_i) \longrightarrow l^\infty(J_i), \quad i = 1, 2.$$

Since G_1 is a closed subspace of $l^\infty(I_1)$ and $l^\infty(J_2)$ is an injective Banach space, the mapping $u_2 \circ u : G_1 \longrightarrow l^\infty(I_2)$ can be extended to a bounded linear mapping $v : l^\infty(I_1) \longrightarrow l^\infty(I_2)$ such that the left square of the above diagram is commutative. The mapping v induces a bounded linear mapping $\overline{v} : l^\infty(I_1)/u_1(G_1) \longrightarrow l^\infty(I_2)/u_2(G_2)$. As $l^\infty(J_2)$ is injective, we can extend the composition $l^\infty(I_1)/u_1(G_1) \longrightarrow l^\infty(I_2)/u_2(G_2) \longrightarrow l^\infty(J_2)$ to a bounded linear mapping $w : l^\infty(J_1) \longrightarrow l^\infty(J_2)$ such that the right square of the above diagram is commutative. □

Now, we prove that the strict morphism defined by u is independent of v and w occuring in Proposition 2.3. In fact, let $E \mid F$ be a quotient bornological space. By applying the functor $.\varepsilon(E \mid F) : \mathbf{Ban} \longrightarrow \mathbf{q}$ to the right square of the diagram of Proposition 2.3 and taking the kernels of the horizontal arrows $v_i\varepsilon\mathrm{Id}_{E|F}$, $i = 1, 2$, we obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ker}(v_1\varepsilon \mathrm{Id}_{E|F}) & \longrightarrow & l^\infty(I_1)\varepsilon(E \mid F) & \overset{v_1\varepsilon\mathrm{Id}_{E|F}}{\longrightarrow} & l^\infty(J_1)\varepsilon(E \mid F) \\ & & \downarrow & & \downarrow^{v\varepsilon\mathrm{Id}_{E|F}} & & \downarrow^{w\varepsilon\mathrm{Id}_{E|F}} \\ 0 & \longrightarrow & \mathrm{Ker}(v_2\varepsilon \mathrm{Id}_{E|F}) & \longrightarrow & l^\infty(I_2)\varepsilon(E \mid F) & \overset{v_2\varepsilon\mathrm{Id}_{E|F}}{\longrightarrow} & l^\infty(J_2)\varepsilon(E \mid F) \end{array}$$

where the strict morphism $\mathrm{Ker}(v_1\varepsilon\mathrm{Id}_{E|F}) \longrightarrow \mathrm{Ker}(v_2\varepsilon\mathrm{Id}_{E|F})$ in the above diagram is induced by the restriction of $v\varepsilon\mathrm{Id}_E$ to the b-space $(v_1\varepsilon\mathrm{Id}_E)^{-1}(l^\infty(J_1)\varepsilon F)$. We call it $u\varepsilon_{Res}\mathrm{Id}_{E|F} : G_1\varepsilon_{Res}(E \mid F) \longrightarrow G_2\varepsilon_{Res}(E \mid F)$.

We must show that $u\varepsilon_{Res}\mathrm{Id}_{E|F}$ does not depend on the choice of the mappings v and w.

Indeed, if $u = 0$, then there exists a bounded linear mapping $\beta : l^\infty(J_1) \longrightarrow l^\infty(I_2)$ such that the following square is commutative:

$$\begin{array}{ccc} l^\infty(I_1) & \overset{v_1}{\longrightarrow} & l^\infty(J_1) \\ \downarrow^{v} & \swarrow_{\beta} & \downarrow^{w} \\ l^\infty(I_2) & \overset{v_2}{\longrightarrow} & l^\infty(J_2) \end{array}$$

Now, by applying the functor $.\varepsilon(E \mid F) : \mathbf{Ban} \longrightarrow \mathbf{q}$ to the above square, we obtain a strict morphism $\beta\varepsilon\mathrm{Id}_{E|F} : l^\infty(J_1)\varepsilon(E \mid F) \longrightarrow l^\infty(I_2)\varepsilon(E \mid F)$. Finally, it is easy to prove that the strict morphism $u\varepsilon_{Res}\mathrm{Id}_{E|F} = 0$. Hence the morphism $u\varepsilon_{Res}\mathrm{Id}_{E|F}$ is well defined.

Now, we are in position to prove that the object $G\varepsilon_{Res}(E \mid F)$ is independent of l^∞-resolutions of G. Namely, we have the following result:

Theorem 2.4. *Let G be a Banach space and* $0 \longrightarrow G \overset{u_i}{\longrightarrow} l^\infty(I_i) \overset{v_i}{\longrightarrow} l^\infty(J_i)$, $i = 1, 2$, *be two l^∞-resolutions of G. Then, for every quotient bornological space $E \mid F$, the objects $G\varepsilon_{Res_1}(E \mid F)$ and $G\varepsilon_{Res_2}(E \mid F)$ are naturally isomorphic.*

Proof. Let us consider the following commutative diagrams:

$$\begin{array}{ccccccc} 0 & \longrightarrow & G & \overset{u_1}{\longrightarrow} & l^\infty(I_1) & \overset{v_1}{\longrightarrow} & l^\infty(J_1) \\ & & \downarrow^{\mathrm{Id}_G} & & \downarrow^{u} & & \downarrow^{v} \\ 0 & \longrightarrow & G & \overset{u_2}{\longrightarrow} & l^\infty(I_2) & \overset{v_2}{\longrightarrow} & l^\infty(J_2) \\ & & \downarrow^{\mathrm{Id}_G} & & \downarrow^{u'} & & \downarrow^{v'} \\ 0 & \longrightarrow & G & \overset{u_1}{\longrightarrow} & l^\infty(I_1) & \overset{v_1}{\longrightarrow} & l^\infty(J_1) \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & G & \overset{u_1}{\longrightarrow} & l^\infty(I_1) & \overset{v_1}{\longrightarrow} & l^\infty(J_1) \\ & & \downarrow^{\mathrm{Id}_G} & & \downarrow^{u'\circ u} & & \downarrow^{v'\circ v} \\ 0 & \longrightarrow & G & \overset{u_1}{\longrightarrow} & l^\infty(I_1) & \overset{v_1}{\longrightarrow} & l^\infty(J_1) \end{array}$$

By applying the functor $.\varepsilon(E \mid F) : \mathbf{Ban}\longrightarrow\mathbf{q}$ to the above diagrams and by using the identities $(f'\varepsilon\mathrm{Id}_H) \circ (f\varepsilon\mathrm{Id}_H) = (f' \circ f)\varepsilon\mathrm{Id}_H$ for $f = u, v$ and $H = E, F$, we obtain

$$(\mathrm{Id}_G\varepsilon_{Res_{2,1}}\mathrm{Id}_{E|F}) \circ (\mathrm{Id}_G\varepsilon_{Res_{1,2}}\mathrm{Id}_{E|F}) = \mathrm{Id}_G\varepsilon_{Res_{1,1}}\mathrm{Id}_{E|F} = \mathrm{Id}_{G\varepsilon_{Res_1}(E|F)}.$$

Also, using a similar argument, we have the following identity:

$$(\mathrm{Id}_G\varepsilon_{Res_{1,2}}\mathrm{Id}_{E|F}) \circ (\mathrm{Id}_G\varepsilon_{Res_{2,1}}\mathrm{Id}_{E|F}) = \mathrm{Id}_G\varepsilon_{Res_{2,2}}\mathrm{Id}_{E|F} = \mathrm{Id}_{G\varepsilon_{Res_2}(E|F)}.$$

And this finishes the proof of Theorem 2.4. □

Definition 2.5. The ε_∞-product of a Banach space G and a quotient bornological space $E \mid F$ is the object $G\varepsilon_\infty(E \mid F)$ that we define as $G\varepsilon_{Res}(E \mid F)$ for some l^∞-resolution of G.

In [2], we proved the following result.

Proposition. ([2], Proposition 2.2). *If G_1, G_2 are $\mathcal{L}_\infty$-spaces and $u : G_1 \longrightarrow G_2$ is a bounded linear mapping, then u is injective with a closed range if and only if for every quotient Banach space $E \mid F$, the strict morphism $u\varepsilon \mathrm{Id}_{E|F} : G_1\varepsilon(E \mid F) \longrightarrow G_2\varepsilon(E \mid F)$ is injective.*

This proposition is still valid in the category of quotient bornological spaces. In fact, if G is an $\mathcal{L}_\infty$-space and $(E_i \mid F_i)_{i\in I}$ is an inductive system of quotient Banach spaces, since the category **q** is stable under inductive limit, we can show that $G\varepsilon(\varinjlim_i (E_i \mid F_i)) \simeq \varinjlim_i (G\varepsilon (E_i \mid F_i))$.

On the other hand, a quotient bornological space $E \mid F$ can be considered as an inductive limit of quotient Banach spaces $E_i \mid F_i$. Indeed, let (B, C) be a couple of bounded completant sets, B bounded in E, C bounded in F and $C \subset B$. This set of couples is ordered by the relation $(B, C) \prec (B_1, C_1)$ if and only if $B \subset B_1$ and $C \subset C_1$. For such an order, the set of couples (B, C) is a net and the family $(E_B \mid F_C)_{(B,C)}$ is an inductive system in the category **q.** Then we can write $E \mid F \simeq \varinjlim_{(B,C)}(E_B \mid F_C)$. It follows that if G is an $\mathcal{L}_\infty$-space and $E \mid F$ is a quotient bornological space, then $G\varepsilon (E \mid F)) \simeq \varinjlim_{B,C} (G\varepsilon (E_B \mid F_C))$.

Thus Proposition 2.2 of [2] holds in the category of quotient bornological spaces.

An important characterization of $\mathcal{L}_\infty$-spaces by the ε_∞-product is giving by the following result.

Theorem 2.6. *A Banach space G is an $\mathcal{L}_\infty$-space if and only if whenever $E \mid F$ is a quotient bornological space, the objects $(G\varepsilon E) \mid (G\varepsilon F)$ and $G\varepsilon_\infty(E \mid F)$ are isomorphic.*

Proof. Let $0 \longrightarrow G \xrightarrow{u} l^\infty(I) \xrightarrow{v} l^\infty(J)$ be a l^∞-resolution of G. By a dual result of Proposition II.5.13 of [10], the Banach space $l^\infty(I)/G$ is an $\mathcal{L}_\infty$-space. We consider the following exact sequence:

$$0 \longrightarrow G \xrightarrow{u} l^\infty(I) \xrightarrow{\pi} l^\infty(I)/G \longrightarrow 0.$$

If $E \mid F$ is a quotient bornological space, by applying the ε-product functor, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G\varepsilon F & \longrightarrow & l^\infty(I)\varepsilon F & \longrightarrow & (l^\infty(I)/G)\varepsilon F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G\varepsilon E & \longrightarrow & l^\infty(I)\varepsilon E & \longrightarrow & (l^\infty(I)/G)\varepsilon E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G\varepsilon(E \mid F) & \longrightarrow & l^\infty(I)\varepsilon(E \mid F) & \longrightarrow & (l^\infty(I)/G)\varepsilon(E \mid F) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the three columns are exact (because G, $l^\infty(I)$ and $l^\infty(I)/G$ are $\mathcal{L}_\infty$-spaces). By Proposition 2.5 and the example 2.4(i) of [8], the sequence

$$0 \longrightarrow G\varepsilon K \longrightarrow l^\infty(I)\varepsilon K \longrightarrow l^\infty(J)\varepsilon K \tag{1}$$

is exact for $K = E_B$, F_C.

On the other hand, $E = \varinjlim_B E_B$ and $F = \varinjlim_C F_C$, and since the inductive limit functor is exact on the category of b-spaces [6], it follows that the first and the second lines are exact. Finally, we deduce from Theorem 4.3.6 of [12] that the third line is exact.

If we consider the isometry $v_1 : l^\infty(I)/G \longrightarrow l^\infty(J)$ such that $v = v_1 \circ \pi$, the strict morphism $v_1\varepsilon\mathrm{Id}_{E|F} : (l^\infty(I)/G)\varepsilon(E \mid F) \longrightarrow l^\infty(J)\varepsilon(E \mid F)$ is injective (Proposition 2.2 of [2]) and $v\varepsilon\mathrm{Id}_{E|F} = (v_1\varepsilon\mathrm{Id}_{E|F}) \circ (\pi\varepsilon\mathrm{Id}_{E|F})$. Then $\mathrm{Ker}(v\varepsilon\mathrm{Id}_{E|F}) = \mathrm{Ker}(\pi\varepsilon\mathrm{Id}_{E|F})$, and this shows the result.

Conversely, if $a_1 : X \longrightarrow Y$ is a surjective bounded linear mapping between Banach spaces, it induces an isomorphism $a : X \mid a_1^{-1}(0) \longrightarrow Y \mid \{0\}$. Let $0 \longrightarrow G \xrightarrow{u} l^\infty(I) \xrightarrow{v} l^\infty(J)$ be a l^∞-resolution of G. By applying the left exact functors $.\varepsilon_\infty(X \mid a_1^{-1}(0))$, $.\varepsilon_\infty(Y \mid \{0\}) : \mathbf{Ban} \longrightarrow \mathbf{q}$ (Theorem 3.1 of this paper) to the above left exact l^∞-resolution (1), we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & G\varepsilon_\infty(X \mid a_1^{-1}(0)) & \longrightarrow & l^\infty(I)\varepsilon(X \mid a_1^{-1}(0)) & \longrightarrow & l^\infty(J)\varepsilon(X \mid a_1^{-1}(0)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & G\varepsilon_\infty(Y \mid \{0\}) & \longrightarrow & l^\infty(I)\varepsilon(Y \mid \{0\}) & \longrightarrow & l^\infty(J)\varepsilon(Y \mid \{0\})
\end{array}$$

As $l^\infty(I)$ and $l^\infty(J)$ are $\mathcal{L}_\infty$-spaces, the strict morphisms $\mathrm{Id}_{l^\infty(I)}\varepsilon a$ and $\mathrm{Id}_{l^\infty(J)}\varepsilon a$ are isomorphism. Now, by using Lemma 4.3.3 of [12], we deduce that the morphism $\mathrm{Id}_G\varepsilon_\infty a$ is an isomorphism.

On the other hand, it follows from the left square of the above diagram that

$$(2) \qquad ((\mathrm{Id}_{l^\infty(I)})\varepsilon a) \circ (u\varepsilon\mathrm{Id}_{(X|a_1^{-1}(0))}) = (u\varepsilon\mathrm{Id}_{Y|\{0\}}) \circ (\mathrm{Id}_G\varepsilon_\infty a)$$

and by the commutative square

$$\begin{array}{ccc}
G\varepsilon(X \mid a_1^{-1}(0)) & \longrightarrow & l^\infty(I)\varepsilon(X \mid a_1^{-1}(0)) \\
\downarrow & & \downarrow \\
G\varepsilon(Y \mid \{0\}) & \longrightarrow & l^\infty(I)\varepsilon(Y \mid \{0\})
\end{array}$$

we have

$$(3) \qquad (\mathrm{Id}_{l^\infty(I)}\varepsilon a) \circ (u\varepsilon\mathrm{Id}_{(X|a_1^{-1}(0))}) = (u\varepsilon\mathrm{Id}_{Y|\{0\}}) \circ (\mathrm{Id}_G\varepsilon a).$$

By using the equalities (2) and (3), we obtain

$$(u\varepsilon\mathrm{Id}_{Y|\{0\}}) \circ (\mathrm{Id}_G\varepsilon a) = (u\varepsilon\mathrm{Id}_{Y|\{0\}}) \circ (\mathrm{Id}_G\varepsilon_\infty a).$$

Finally, since the strict morphism $u\varepsilon\mathrm{Id}_{Y|\{0\}}$ is injective, we deduce $\mathrm{Id}_G\varepsilon a = \mathrm{Id}_G\varepsilon_\infty a$. This proves that $\mathrm{Id}_G\varepsilon a : G\varepsilon X \mid (G\varepsilon a_1^{-1}(0)) \longrightarrow G\varepsilon(Y \mid \{0\})$ is an isomorphism, and then the mapping $\mathrm{Id}_G\varepsilon a_1 : G\varepsilon X \longrightarrow G\varepsilon Y$ is surjective. This proves the result. □

3. The left exactness of the functor $.\varepsilon_\infty$.

To show that for every quotient Banach space $E \mid F$, the functor $.\varepsilon_\infty(E \mid F)$ changes a left exact complex of the category **Ban** into a left exact complex of the category **qBan,** we need the following lemma.

Lemma 3.1. *Let* $0 \longrightarrow G_1 \xrightarrow{u} G_2 \xrightarrow{v} G_3 \longrightarrow 0$ *be a short exact complex of the category* **Ban**. *Let* $b_1 : G_1 \longrightarrow l^\infty(X_1)$ *and* $b_3 : G_3 \longrightarrow l^\infty(X_3)$ *be isometric embeddings. Then there exists an isometric embedding* $b_2 : G_2 \longrightarrow l^\infty(X_1) \bigoplus l^\infty(X_3)$ *such that the diagram*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G_1 & \xrightarrow{u} & G_2 & \xrightarrow{v} & G_3 & \longrightarrow & 0 \\ & & \downarrow^{b_1} & & \downarrow^{b_2} & & \downarrow^{b_3} & & \\ 0 & \longrightarrow & l^{\infty}(X_1) & \xrightarrow{\pi} & l^{\infty}(X_1) \oplus l^{\infty}(X_3) & \xrightarrow{s} & l^{\infty}(X_3) & \longrightarrow & 0 \end{array}$$

is commutative, where $\pi : l^{\infty}(X_1) \longrightarrow l^{\infty}(X_1) \bigoplus l^{\infty}(X_3)$ *and* $s : l^{\infty}(X_1) \bigoplus l^{\infty}(X_3) \longrightarrow l^{\infty}(X_3)$ *are the classical bounded linear mappings into and from a direct sum.*

Proof. We assume that G_1 is a closed subspace of G_2 and G_3 is the Banach space G_2/G_1. Let $G_1 \subset l^{\infty}(X_1)$ and $G_3 \subset l^{\infty}(X_3)$ be isometric embeddings. Since $l^{\infty}(X_1)$ is injective, the bounded linear mapping $b_1 : G_1 \longrightarrow l^{\infty}(X_1)$ can be extended to a bounded linear mapping $b_2' : G_2 \longrightarrow l^{\infty}(X_1)$ such that $b_2' \circ u = b_1$. On the other hand, G_2 is mapped into G_3, and G_3 is mapped in $l^{\infty}(X_3)$. The composition of these mappings is a bounded linear mapping $b_2'' : G_2 \longrightarrow l^{\infty}(X_3)$. We let $b_2 = b_2' \bigoplus b_2''$. Let $x_2 = x_2' \bigoplus x_2'' \in b_2(G_2)$; $x_2'' \in b_3(G_3)$, we let $g_3 \in G_3$ be the element mapped onto x_2'', then we see that $\| g_3 \|_{G_3} = \| x_2'' \|_{l^{\infty}(X_3)}$. And g_3 can be lifted to $g_2'' \in G_2$ such that $v(g_2'') = g_3$ and $\| g_2'' \|_{G_2} < (1+\varepsilon) \| g_3 \|_{G_3} = (1+\varepsilon) \| x_2'' \|_{l^{\infty}(X_3)}$. The element x_2' belongs to $b_2'(G_2)$, then an element $g_1 \in G_1$ exists such that $b_1(g_1) = x_2'$. Of course, $\| g_1 \|_{G_1} = \| x_2' \|_{l^{\infty}(X_1)}$ and $u(g_1) = g_2'' \in G_2$ is such that $\| u(g_1) \|_{G_2} = \| x_2' \|_{l^{\infty}(X_3)}$.

We have lifted $x_2 \in b_2(G_2)$ to $g_2 = g_2' + g_2''$, $\| g_2 \|_{G_2} \leq (1+\varepsilon) \| x_2 \|$. The bounded linear mapping $b_2 : G_2 \longrightarrow l^{\infty}(X_1) \bigoplus l^{\infty}(X_3)$ is isometric and then has a closed range. □

Now, we are in position to prove the following result.

Theorem 3.2. *Let* $0 \longrightarrow G_1 \xrightarrow{u_1} G_2 \xrightarrow{u_2} G_3$ *be a left exact complex of the category* **Ban**. *Let* $E \mid F$ *be a quotient Banach space. Then* $0 \longrightarrow G_1 \varepsilon_{\infty}(E \mid F) \longrightarrow G_2 \varepsilon_{\infty}(E \mid F) \longrightarrow G_3 \varepsilon_{\infty}(E \mid F)$ *is a left exact complex of the category* **qBan**.

Proof. By Lemma 3.1, $b_1(G_1)$, $b_2(G_2)$ and $b_3(G_3)$ are closed subspaces of the Banach spaces $l^{\infty}(X_1)$, $l^{\infty}(X_1) \bigoplus l^{\infty}(X_3)$ and $l^{\infty}(X_3)$ respectively, and then the sequence $0 \longrightarrow b_1(G_1) \longrightarrow b_2(G_2) \longrightarrow b_3(G_3) \longrightarrow 0$ is a short exact complex. We obtain the following commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & b_1(G_1) & \longrightarrow & b_2(G_2) & \longrightarrow & b_3(G_3) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & l^{\infty}(X_1) & \longrightarrow & l^{\infty}(X_1) \oplus l^{\infty}(X_3) & \longrightarrow & l^{\infty}(X_3) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & l^{\infty}(X_1)/b_1(G_1) & \longrightarrow & (l^{\infty}(X_1) \oplus l^{\infty}(X_3))/b_2(G_2) & \longrightarrow & l^{\infty}(X_3)/b_3(G_3) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where the three columns are exact and the first and the second lines are exact. One of the several 3×3 Lemmas [12] shows that the sequence $0 \longrightarrow l^{\infty}(X_1)/b_1(G_1) \longrightarrow (l^{\infty}(X_1) \oplus l^{\infty}(X_3))/b_2(G_2) \longrightarrow l^{\infty}(X_3)/b_3(G_3) \longrightarrow 0$ is a short exact complex.

The Banach spaces $l^{\infty}(X_1)/b_1(G_1)$ and $(l^{\infty}(X_1) \oplus l^{\infty}(X_3))/b_2(G_2)$ are included in an isometric way in $l^{\infty}(Y_1)$ and $l^{\infty}(Y_3)$. By Lemma 3.1, there exists an isometric mapping $c_2' : (l^{\infty}(X_1) \oplus l^{\infty}(X_3))/b_2(G_2) \longrightarrow l^{\infty}(Y_1) \oplus l^{\infty}(Y_3)$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & l^{\infty}(X_1)/b_1(G_1) & \longrightarrow & (l^{\infty}(X_1) \oplus l^{\infty}(X_3))/b_2(G_2) & \longrightarrow & l^{\infty}(X_3)/b_3(G_3) & \longrightarrow & 0 \\ & & \downarrow^{c_1'} & & \downarrow^{c_2'} & & \downarrow^{c_3'} & & \\ 0 & \longrightarrow & l^{\infty}(Y_1) & \longrightarrow & l^{\infty}(Y_1) \oplus l^{\infty}(Y_3) & \longrightarrow & l^{\infty}(Y_3) & \longrightarrow & 0 \end{array}$$

We let c_2 be the composition of the projection $l^\infty(X_1)\oplus l^\infty(X_3)\longrightarrow (l^\infty(X_1)\oplus l^\infty(X_3))/b_2(G_2)$ with the embedding $(l^\infty(X_1)\oplus l^\infty(X_3))/b_2(G_2)\longrightarrow l^\infty(Y_1)\oplus l^\infty(Y_3)$. The sequence $(0,b_2,c_2)$ is a l^∞-resolution of G_2 such that the following diagram is commutative:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_1 & \xrightarrow{u} & G_2 & \xrightarrow{v} & G_3 & \longrightarrow & 0\\
 & & \downarrow^{b_1} & & \downarrow^{b_2} & & \downarrow^{b_3} & & \\
0 & \longrightarrow & l^\infty(X_1) & \longrightarrow & l^\infty(X_1)\oplus l^\infty(X_3) & \longrightarrow & l^\infty(X_3) & \longrightarrow & 0\\
 & & \downarrow^{c_1} & & \downarrow^{c_2} & & \downarrow^{c_3} & & \\
0 & \longrightarrow & l^\infty(Y_1) & \longrightarrow & l^\infty(Y_1)\oplus l^\infty(Y_3) & \longrightarrow & l^\infty(Y_3) & \longrightarrow & 0
\end{array}
$$

We have chosen one l^∞-resolution of G_2, but we have proved that the quotient Banach space $G\varepsilon_\infty(E\mid F)$ does not depend on the l^∞-resolution (modulo a natural isomorphism). Instead of using the chosen l^∞-resolution, we use the l^∞-resolution above. By applying the functor $.\varepsilon_\infty(E\mid F)$ to the above diagram, we obtain

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0\\
 & & \downarrow & & \downarrow & & \downarrow\\
0 & \longrightarrow & G_1\varepsilon_\infty(E\mid F) & \longrightarrow & G_2\varepsilon_\infty(E\mid F) & \longrightarrow & G_3\varepsilon_\infty(E\mid F)\\
 & & \downarrow & & \downarrow & & \downarrow\\
0 & \longrightarrow & l^\infty(X_1)\,\varepsilon(E\mid F) & \longrightarrow & (l^\infty(X_1)\oplus l^\infty(X_3))\varepsilon(E\mid F) & \longrightarrow & l^\infty(X_3)\,\varepsilon(E\mid F)\\
 & & \downarrow & & \downarrow & & \downarrow\\
0 & \longrightarrow & l^\infty(Y_1)\,\varepsilon(E\mid F) & \longrightarrow & (l^\infty(Y_1)\oplus l^\infty(Y_3))\varepsilon(E\mid F) & \longrightarrow & l^\infty(Y_3)\,\varepsilon(E\mid F)
\end{array}
$$

another 3×3 Lemma of [12], shows that the complex $0\longrightarrow G_1\varepsilon_\infty(E\mid F)\longrightarrow G_2\varepsilon_\infty(E\mid F)\longrightarrow G_3\varepsilon_\infty(E\mid F)$ is left exact.

Now, let $0\longrightarrow G_1\longrightarrow G_2\longrightarrow H$ be a left exact complex and let $G_3\simeq G_2/u(G_1)$, the complex $0\longrightarrow G_1\varepsilon_\infty(E\mid F)\longrightarrow G_2\varepsilon_\infty(E\mid F)\longrightarrow G_3\varepsilon_\infty(E\mid F)$ is left exact in **qBan**. Since G_3 is (isomorphic to) a closed subspace of H, we have a second short exact complex $0\longrightarrow G_3\longrightarrow H\longrightarrow H/G_3\longrightarrow 0$ in **Ban**, and the complex $0\longrightarrow G_3\varepsilon_\infty(E\mid F)\longrightarrow H\varepsilon_\infty(E\mid F)\longrightarrow (H/G_3)\varepsilon_\infty(E\mid F)$ is left exact. In particular, the strict morphism $G_3\varepsilon_\infty(E\mid F)\longrightarrow H\varepsilon_\infty(E\mid F)$ is injective. Hence, the complex $0\longrightarrow G_1\varepsilon_\infty(E\mid F)\longrightarrow G_2\varepsilon_\infty(E\mid F)\longrightarrow H\varepsilon_\infty(E\mid F)$ is left exact in **qBan**. □

Finally, if we consider the functor $G\varepsilon_\infty.:\mathbf{qBan}\longrightarrow\mathbf{qBan}$, where G is a Banach space, we have the following property:

Proposition 3.3. *Let G be a Banach space and $0\longrightarrow E_1\mid F_1\longrightarrow E_2\mid F_2\longrightarrow E_3\mid F_3\longrightarrow 0$ be a short exact complex of quotient Banach spaces. Then the complex $0\longrightarrow G\varepsilon_\infty(E_1\mid F_1)\longrightarrow G\varepsilon_\infty(E_2\mid F_2)\longrightarrow G\varepsilon_\infty(E_3\mid F_3)$ is left exact in* **qBan**.

Proof. Let $0\longrightarrow G\longrightarrow l^\infty(I)\longrightarrow l^\infty(J)$ be a l^∞-resolution of G. Since $l^\infty(I)$ and $l^\infty(J)$ are $\mathcal{L}_\infty$-spaces, the functors $l^\infty(I)\,\varepsilon.$ and $l^\infty(J)\,\varepsilon.$ are exact on **qBan,** and it follows that, in the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & G\varepsilon_\infty(E_1 \mid F_1) & \longrightarrow & l^\infty(I)\,\varepsilon(E_1 \mid F_1) & \longrightarrow & l^\infty(J)\,\varepsilon(E_1 \mid F_1) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & G\varepsilon_\infty(E_2 \mid F_2) & \longrightarrow & l^\infty(I)\,\varepsilon(E_2 \mid F_2) & \longrightarrow & l^\infty(J)\,\varepsilon(E_2 \mid F_2) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & G\varepsilon_\infty(E_3 \mid F_3) & \longrightarrow & l^\infty(I)\,\varepsilon(E_3 \mid F_3) & \longrightarrow & l^\infty(J)\,\varepsilon(E_3 \mid F_3) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

the last two columns are exact. On the other hand, by the definition of the ε_∞-product, the three lines are left exact. A 3×3 Lemma of [12] show that the first column is left exact. □

4. The ε_∞-product of a b-space

Let G be a b-space, then every completant bounded B of G is included in a completant bounded A of G such that the inclusion mapping $i_{AB} : G_B \longrightarrow G_A$ is bounded [5].

Let $0 \longrightarrow G_C \xrightarrow{\Phi_C} l^\infty(I_C) \xrightarrow{\Psi_C} l^\infty(J_C)$ be a l^∞-resolution of the Banach space G_C, $C = A, B$. By Proposition 2.3, there exist bounded linear mappings $v_{AB} : l^\infty(I_B) \longrightarrow l^\infty(I_A)$ and $w_{AB} : l^\infty(J_B) \longrightarrow l^\infty(J_A)$ making commutative the following diagram:

$$\begin{array}{lccccccc}
(0, \Phi_B, \Psi_B): & 0 & \longrightarrow & G_B & \longrightarrow & l^\infty(I_B) & \longrightarrow & l^\infty(J_B) \\
 & & & \downarrow^{i_{AB}} & & \downarrow^{v_{AB}} & & \downarrow^{w_{AB}} \\
(0, \Phi_A, \Psi_A): & 0 & \longrightarrow & G_A & \longrightarrow & l^\infty(I_A) & \longrightarrow & l^\infty(J_A)
\end{array}$$

If $E \mid F$ is a quotient bornological space, by applying the functor $.\varepsilon(E \mid F) : \mathbf{Ban} \longrightarrow \mathbf{q}$ to the right square of the above diagram and adding the kernels of the horizontal arrows $\Psi_C \varepsilon \mathrm{Id}_{E|F}$, $C = A, B$, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & G_B\varepsilon_\infty(E \mid F) & \xrightarrow{\Phi_B\varepsilon_\infty \mathrm{Id}_{E|F}} & l^\infty(I_B)\varepsilon(E \mid F) & \xrightarrow{\Psi_B\varepsilon \mathrm{Id}_{E|F}} & l^\infty(J_B)\varepsilon(E \mid F) \\
 & & \downarrow^{i_{AB}\varepsilon_\infty \mathrm{Id}_{E|F}} & & \downarrow^{v_{AB}\varepsilon \mathrm{Id}_{E|F}} & & \downarrow^{w_{AB}\varepsilon \mathrm{Id}_{E|F}} \\
0 & \longrightarrow & G_A\varepsilon_\infty(E \mid F) & \xrightarrow{\Phi_A\varepsilon_\infty \mathrm{Id}_{E|F}} & l^\infty(I_A)\varepsilon(E \mid F) & \xrightarrow{\Psi_A\varepsilon \mathrm{Id}_{E|F}} & l^\infty(J_A)\varepsilon(E \mid F)
\end{array}$$

where the strict morphism $i_{AB}\varepsilon_\infty \mathrm{Id}_{E|F} : G_B\varepsilon_\infty(E \mid F) \longrightarrow G_A\varepsilon_\infty(E \mid F)$ is induced by the restriction of $v_{AB}\varepsilon \mathrm{Id}_E$ to the b-space $(\Psi_B \varepsilon \mathrm{Id}_E)^{-1}(l^\infty(J_B)\varepsilon F)$. The system $(G_B\varepsilon_\infty(E \mid F), i_{AB}\varepsilon_\infty \mathrm{Id}_{E|F})_B$ is inductive in $\mathbf{q}$, and then has an inductive limit which is a quotient bornological space.

Definition 4.1. *The ε_∞-product of a b-space G and a quotient bornological space $E \mid F$ is the quotient bornological space $G\varepsilon_\infty(E \mid F) = \varinjlim_B G_B\varepsilon_\infty(E \mid F)$.*

Since the inductive limit functor is exact on the category of b-spaces [6], it follows from Theorem 4.1 of [17] that this functor admits an exact extension to the category of quotient bornological spaces.

Our aim now is to show some properties of this ε_∞-product.

Proposition 4.2. *Let G be a b-space and $E \mid F$ be a quotient bornological space. Then there exists an injective strict morphism $(G\varepsilon E) \mid (G\varepsilon F) \longrightarrow G\varepsilon_\infty(E \mid F)$.*

Proof. Let $G = \varinjlim_B G_B$. Then by the same proof as in Proposition 2.12 of [3], we show the existence of an injective strict morphism $(G_B\varepsilon E) \mid (G_B\varepsilon F) \longrightarrow G_B\varepsilon_\infty(E \mid F)$.

Now, by applying the functor $\varinjlim_B(.)$ which is exact on $\mathbf{q}$, we obtain the injective strict morphism $\varinjlim_B((G_B\varepsilon E) \mid (G_B\varepsilon F)) \longrightarrow \varinjlim_B(G_B\varepsilon_\infty(E \mid F))$. On the other hand, the quotient bornological space $(G_B\varepsilon E) \mid (G_B\varepsilon F)$ defines the following exact sequence:

$$0 \longrightarrow G_B\varepsilon F \longrightarrow G_B\varepsilon E \longrightarrow (G_B\varepsilon E) \mid (G_B\varepsilon F) \longrightarrow 0.$$

Its image by the functor $\varinjlim_B(.)$ is the following exact sequence:

$$0 \longrightarrow \varinjlim_B(G_B\varepsilon F) \longrightarrow \varinjlim_B(G_B\varepsilon E) \longrightarrow \varinjlim_B((G_B\varepsilon E) \mid (G_B\varepsilon F)) \longrightarrow 0.$$

This shows that

$$\varinjlim_B(G_B\varepsilon E) \mid \varinjlim_B(G_B\varepsilon F) = \varinjlim_B((G_B\varepsilon E) \mid (G_B\varepsilon F))$$

and hence, we obtain an injective strict morphism $\varinjlim_B(G_B\varepsilon E) \mid \varinjlim_B(G_B\varepsilon F) \longrightarrow \varinjlim_B(G_B\varepsilon_\infty(E \mid F))$. □

Now, we introduce the class of εb-spaces. A b-space G is an εb-space if the mapping $\mathrm{Id}_G\varepsilon u : G\varepsilon E \longrightarrow G\varepsilon F$ is bornologically surjective whenever $u : E \longrightarrow F$ is a surjective bounded linear mapping between Banach spaces.

For example, each $\mathcal{L}_\infty$-space is an εb-space, and if the b-space G is an inductive limit of $\mathcal{L}_\infty$-spaces in the category **b**, then G is an εb-space.

Theorem 4.3.

1. *If a b-space G is a bornological inductive limit of $\mathcal{L}_\infty$-spaces and $E \mid F$ is a quotient bornological space, then $(G\varepsilon E) \mid (G\varepsilon F) = G\varepsilon_\infty(E \mid F)$.*
2. *Let G be a b-space. If for each quotient bornological space $E \mid F$, we have $(G\varepsilon E) \mid (G\varepsilon F) = G\varepsilon_\infty(E \mid F)$, then G is an εb-space.*

Proof. 1. Since $G = \varinjlim_B G_B$, where each G_B is an $\mathcal{L}_\infty$-space, it follows from Theorem 2.6, that $(G_B\varepsilon E) \mid (G_B\varepsilon F) = G_B\varepsilon_\infty(E \mid F)$. As the functor $\varinjlim_B(.)$ is exact, we obtain $\varinjlim_B((G_B\varepsilon E) \mid (G_B\varepsilon F)) = \varinjlim_B(G_B\varepsilon_\infty(E \mid F))$, and hence $(G\varepsilon E) \mid (G\varepsilon F) = G\varepsilon_\infty(E \mid F)$.

2. Let $u_1 : X \longrightarrow Y$ be a surjective bounded linear mapping between Banach spaces, it induces a pseudo-isomorphism $u : X \mid u_1^{-1}(0) \longrightarrow Y \mid \{0\}$. As $G = \varinjlim_B G_B$, let

$$0 \longrightarrow G_B \xrightarrow{\Phi_B} l^\infty(I_B) \xrightarrow{\Psi_B} l^\infty(J_B) \longrightarrow 0$$

be a l^∞-resolution of the Banach space G_B. By applying the left exact functors $.\varepsilon_\infty(X \mid u_1^{-1}(0))$, $.\varepsilon_\infty(Y \mid \{0\}) : \mathbf{Ban} \longrightarrow \mathbf{q}$ to the above l^∞-resolution of G_B, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & G_B\varepsilon_\infty(X \mid u_1^{-1}(0)) & \longrightarrow & l^\infty(I_B)\varepsilon(X \mid u_1^{-1}(0)) & \longrightarrow & l^\infty(J_B)\varepsilon(X \mid u_1^{-1}(0)) \\
 & & \downarrow{\scriptstyle \mathrm{Id}_{G_B}\varepsilon_\infty u} & & \downarrow{\scriptstyle \mathrm{Id}_{l^\infty(I_B)}\varepsilon u} & & \downarrow{\scriptstyle \mathrm{Id}_{l^\infty(J_B)}\varepsilon u} \\
0 & \longrightarrow & G_B\varepsilon_\infty(Y \mid \{0\}) & \longrightarrow & l^\infty(I_B)\varepsilon(Y \mid \{0\}) & \longrightarrow & l^\infty(J_B)\varepsilon(Y \mid \{0\})
\end{array}$$

Since $l^\infty(I)$ and $l^\infty(J)$ are $\mathcal{L}_\infty$-spaces, the strict morphisms $\mathrm{Id}_{l^\infty(I_B)}\varepsilon u$ and $\mathrm{Id}_{l^\infty(J_B)}\varepsilon u$ are isomorphism. It follows from Lemma 4.3.3 of [12], that the strict morphism $\mathrm{Id}_{G_B}\varepsilon_\infty u : G_B\varepsilon_\infty(X \mid u_1^{-1}(0)) \longrightarrow G_B\varepsilon_\infty(Y \mid \{0\})$ is an isomorphism.

Now, by applying the exact functor $\varinjlim_B(.)$, we obtain the isomorphism

$$\varinjlim_B(\mathrm{Id}_{G_B}\varepsilon_\infty u) : \varinjlim_B(G_B\varepsilon_\infty(X \mid u_1^{-1}(0))) \longrightarrow \varinjlim_B(G_B\varepsilon_\infty(Y \mid \{0\}))$$

i.e. $\mathrm{Id}_G\varepsilon_\infty u : G\varepsilon_\infty(X \mid u_1^{-1}(0)) \longrightarrow G\varepsilon_\infty(Y \mid \{0\})$ is an isomorphism. As $(G\varepsilon X) \mid \big(G\varepsilon(u_1^{-1}(0))\big) = G\varepsilon_\infty(X \mid u_1^{-1}(0))$ and $G\varepsilon_\infty(Y \mid \{0\}) = (G\varepsilon Y) \mid \{0\}$, the bounded linear mapping $\mathrm{Id}_G\varepsilon u : G\varepsilon X \longrightarrow G\varepsilon Y$ is bornologically surjective, and hence G is an εb-space. This ends the proof. □

Remark 4.4. In [3], we defined the b-space $O_1(U,E)$ as the kernel of the following morphism $\overline{\partial} : \mathcal{E}(U,E) \longrightarrow \mathcal{E}(U,E) \otimes \mathbb{C}^{n*}$, where $\mathcal{E}(U,E) = \varinjlim_{V \in \mathcal{C}_U} (\mathcal{E}(V)\, \varepsilon E)$ and $\mathbb{C}^{n*}$ is the space of antilinear forms on $\mathbb{C}^n$. We proved that if U is an open pseudo-convex subset of $\mathbb{C}^n$, E a b-space and F a bornologically closed subspace of E, then the b-spaces $O_1(U,E/F)$ and $O(U,E)/O(U,F)$ are naturally isomorphic. (Proposition 2.14 of [3]). This result proves that the functor $O_1(U,.) : \mathbf{b} \longrightarrow \mathbf{b} \subset \mathbf{q}$ is exact. Then it admits an unique and exact extension $O_1(U,.) : \mathbf{q} \longrightarrow \mathbf{q}$ (Theorem 4.1 of [17]). As a consequence, for each quotient bornological space $E \mid F$, we obtain $O_1(U,E \mid F)$ and $O(U,E) \mid O(U,F)$ are isomorphic in the category $\mathbf{q}$.

On the other hand, the b-space $O(U)$ is nuclear (i.e. all bounded completant subset B of $O(U)$ is included in a bounded completant subset A of $O(U)$ such that the inclusion mapping $i_{AB} : O(U)_B \longrightarrow O(U)_A$ is nuclear), and then $O(U)$ is a bornological inductive limit of Banach spaces $O(U)_B$, where each $O(U)_B$ is isometrically isomorphic to the $\mathcal{L}_\infty$-space c_0 (i.e. the space of sequences which converge to 0) ([5]). Hence, by Theorem 4.3 (1), we have $O(U,E) \mid O(U,F) = O(U)\, \varepsilon_\infty (E \mid F)$.

Finally, for each quotient bornological space $E \mid F$, the spaces $O_1(U,E \mid F)$ and $O(U)\, \varepsilon_\infty (E \mid F)$ are isomorphic in the category $\mathbf{q}$.

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