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ON NON-DENSELY DEFINED INVARIANT HERMITIAN CONTRACTIONS

M. BEKKER

Abstract. We consider a non-densely defined Hermitian contractive operator which is unitarily equivalent to its linear-fractional transformation. We show that such an operator always admits self-adjoint extensions which are also unitarily equivalent to their linear-fractional transformation.

1. Introduction and Preliminary

Let $\mathfrak{D} = \overline{\mathfrak{D}}$ be a closed proper subspace of a separable Hilbert space $\mathfrak{H}$ with inner product $(\cdot,\cdot)$. Let $A : \mathfrak{D} \mapsto \mathfrak{H}$ be an operator defined on $\mathfrak{D}$ which possesses the following properties:

(1) $(Ah_1, h_2) = (h_1, Ah_2)$, for $h_1, h_2 \in \mathfrak{D}$ (Hermitian property);
(2) $\|A\| = \sup\{\|Ah\| : h \in \mathfrak{D}, \|h\| \leq 1\} \leq 1$.

Such an operator A is called a non-densely defined Hermitian contractive operator, or just a non-densely defined Hermitian contraction. Non-densely defined Hermitian contractions were apparently at first time considered by M. G. Kreĭn [11] in connection with positive self-adjoint extensions of positive symmetric operators.

For a non-densely defined Hermitian contraction A we denote by $\Delta(A)$ the set of all self-adjoint operators $\hat{A}$ which are norm-preserving extensions of A, that is

$$\Delta(A) = \{\hat{A} : \hat{A}f = Af, f \in \mathfrak{D}, \hat{A}^* = \hat{A}, \|\hat{A}\| = \|A\|\}. \tag{1}$$

In [11] M. G. Kreĭn proved that set $\Delta(A) \neq \emptyset$. Moreover in [11] it was proved that the set $\Delta(A)$ contains the smallest element $\hat{A}_\mu$ and the largest element $\hat{A}_M$.

A description of the set $\Delta(A)$ was originally obtained by M. G. Kreĭn [11] and presented in [1]. The article [12] among other important and interesting results contains a description of the resolvents of operators $\hat{A} \in \Delta(A)$. Other proofs of such type of results as well as further generalizations can be found in [3], [4], [9], [13]. Last two articles also contain extensive lists of references.

In the form that we use in the present article the description of the set $\Delta(A)$ was obtained or can be easily extracted from results of articles [5], [6], [10], [15].

Let $\mathfrak{N}$ be the orthogonal complement of $\mathfrak{D}$ in $\mathfrak{H}$, $\mathfrak{N} = \mathfrak{H} \ominus \mathfrak{D}$, and let P be the orthogonal projection onto $\mathfrak{D}$. We denote by B and C operators defined by:

$$B = PA, \quad C = (I - P)A. \tag{2}$$

The operator C maps $\mathfrak{D}$ into $\mathfrak{N}$ while B is an operator on $\mathfrak{D}$ and satisfies the condition $B^* = B$.

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Using these notations operator $A : \mathfrak{D} \mapsto \mathfrak{H}$ can be represented as a block operator matrix

$$A = \begin{bmatrix} B \\ C \end{bmatrix},$$

with respect to the decomposition $\mathfrak{H} = \mathfrak{D} \oplus \mathfrak{N}$, and any operator $\hat{A} \in \Delta(A)$ can be represented as a block operator matrix

$$\hat{A} = \begin{bmatrix} B & C^* \\ C & \mathcal{E} \end{bmatrix}, \tag{3}$$

where $\mathcal{E} : \mathfrak{N} \mapsto \mathfrak{N}$ satisfies $\mathcal{E}^* = \mathcal{E}$.

The condition $\|A\| \leq 1$ implies that for any $f \in \mathfrak{D}$ the following is fulfilled:

$$((I_{\mathfrak{D}} - B^2)f, f) \geq (C^*Cf, f).$$

The last inequality means that there exists a unique operator $X : \mathfrak{D} \to \mathfrak{N}$, such that $\|X\| \leq 1$ and

$$C = X(I_{\mathfrak{D}} - B^2)^{1/2}. \tag{4}$$

Initially the operator X is defined on $\overline{\mathfrak{R}(I_{\mathfrak{D}} - B^2)}$ (the closure of the range of $I_{\mathfrak{D}} - B^2$) and then defined as zero operator on $\mathfrak{D} \ominus \mathfrak{R}(I_{\mathfrak{D}} - B^2)$. The operator $\mathcal{E}$ is given by the following formulas (see above mentioned references):

$$\mathcal{E} = O + R^{1/2}ZR^{1/2} \tag{5}$$

where

$$O = -XBX^*, \quad R = I_{\mathfrak{N}} - XX^*, \tag{6}$$

$I_{\mathfrak{D}}$ and $I_{\mathfrak{N}}$ are identity operators in $\mathfrak{D}$ and $\mathfrak{N}$ respectively, and Z is an arbitrary self-adjoint contraction ($Z = Z^*$, $\|Z\| \leq 1$) on $\mathfrak{N}$. The set of all such contractive operators Z we denote by $\mathcal{B}(\mathfrak{N})$. In particular, the set $\Delta(A)$ contains only one element if and only if $R = 0$, or $XX^* = I_{\mathfrak{N}}$, that is if and only if operator the X is a coisometry.

We denote by $\mathcal{B}(\mathfrak{N})$ the set of all self-adjoint contractive operators on $\mathfrak{N}$. The set $\Delta(A)$ can be treated as the image of the set $\mathcal{B}(\mathfrak{N})$ under the mapping $\hat{A} : \mathcal{B}(\mathfrak{N}) \mapsto \Delta(A)$ defined by formulas (3)–(6). From these formulas it follows that $\hat{A}_\mu = \hat{A}(-I_{\mathfrak{N}})$, while $\hat{A}_M = \hat{A}(I_{\mathfrak{N}})$. Later on we use the notations $\mathcal{E}_\mu$ and $\mathcal{E}_M$ for right bottom blocks of $\hat{A}_\mu$ and $\hat{A}_M$ respectively. From the formulas above it follows that

$$\mathcal{E}_\mu = -I_{\mathfrak{N}} + X(I_{\mathfrak{D}} - B)X^*, \tag{7}$$

and

$$\mathcal{E}_M = I_{\mathfrak{N}} - X(I_{\mathfrak{D}} + B)X^*. \tag{8}$$

In this article we consider non-densely defined Hermitian contractions which are unitarily equivalent to their linear fractional transformation (see Definition 1). Such operators we call invariant contractions. In Section 2 we show that such contractions always admit self-adjoint extensions which are also unitarily equivalent to their linear fractional transformation (invariant extensions) and give a necessary condition for the operator $\mathcal{E}$ to be a right bottom block in the block representation (3) of an invariant extension.

In Section 3 we show that the extreme extensions $\hat{A}_\mu$ and $\hat{A}_M$ of the invariant contraction A are always invariant. From this result we deduce that if $\dim\mathfrak{N} = 1$ then $\hat{A}_\mu$ and $\hat{A}_M$ are only invariant self-adjoint extensions of A.

In Section 4 we consider an example of a non-densely defined invariant Hermitian contraction. We use Theorem 3 to construct the extreme extensions. Finally, in Section 5 we briefly discuss relation between non-densely defined invariant Hermitian contractions and positive symmetric operators which are scale-invariant (see Definition 2).

2. Invariant Hermitian contractions

Let $g : \mathbb{D} \mapsto \mathbb{D}$ be a linear fractional transformation of the unit disk $\mathbb{D}$ onto itself defined by

$$g(z) = \frac{z-\kappa}{1-\kappa z}, \quad -1 < \kappa < 1, \tag{9}$$

and let $G = \{g^n, n = 0, \pm 1, \pm 2, \dots,\}$ be the group of linear fractional transformations generated by g. Each transformation g^n from G is of the form

$$g^n : z \mapsto \frac{z-\kappa_n}{1-\kappa_n z}$$

where

$$\kappa_n = \frac{s^n - 1}{s^n + 1}, \quad s = \frac{1-\kappa}{1+\kappa}, \quad n = 0, \pm 1, \pm 2, \dots \tag{10}$$

Without loss of generality we may assume that $s > 1$ and, therefore, $0 < \kappa < 1$.

Let U be a unitary operator on a Hilbert space $\mathfrak{H}$.

Definition 1. *Let A be a non-densely defined Hermitian contraction on a Hilbert space $\mathfrak{H}$. The operator A is said to be (g,U)-invariant (or just invariant) if*

$$U^n A U^{*n} = g^n(A) = (A - \kappa_n I_{\mathfrak{D}})(I_{\mathfrak{D}} - \kappa_n A)^{-1}, \quad n = 0, \pm 1, \pm 2, \dots \tag{11}$$

Definition 1 is understood in the following sense: for any $h \in \mathfrak{D}$ there exists $h' \in \mathfrak{D}$ such that

$$U^n h = h' - \kappa_n A h' \tag{12}$$

and

$$U^n A h = A h' - \kappa_n h'. \tag{13}$$

Denote by $\mathfrak{M}_z$, $z \in \mathbb{C}$, the range of the operator $A - zI_{\mathfrak{D}}$. Then Definition 1 means the unitary operator U^n maps the subspace $\mathfrak{D}$ onto $\mathfrak{M}_{1/\kappa_n}$ and $\mathfrak{M}_0$ onto $\mathfrak{M}_{\kappa_n}$.

Remark 1. *From* (11) *one can easily deduce that for any $z \in \mathbb{C}$ the following is fulfilled:*

$$U^n \mathfrak{M}_z = \mathfrak{M}_{g^{-n}(z)}.$$

Theorem 1. *Let A be a non-densely defined (g,U)-invariant Hermitian contraction defined on a proper closed subspace $\mathfrak{D}$ of a Hilbert space $\mathfrak{H}$. Then it admits a (g,U)-invariant contractive self-adjoint extension.*

Proof. Denote by $\mathfrak{L}(\mathfrak{H})$ the algebra of all bounded operators on $\mathfrak{H}$. Recall that by $\mathcal{B}(\mathfrak{N})$ we denote the set of all self-adjoint contractions Z, $Z = Z^*$ on $\mathfrak{N}$. Observe that the set $\Delta(A)$ is convex and compact in the weak operator topology of $\mathfrak{L}(\mathfrak{H})$. Convexity of $\Delta(A)$ is obvious from formulas (3)–(6).

To prove compactness observe that the set $\mathcal{B}(\mathfrak{N})$ is a closed in the weak operator topology subset of the unit ball of $\mathfrak{L}(\mathfrak{N})$. Because the closed unit ball of $\mathfrak{L}(\mathfrak{N})$ is compact in the weak operator topology [7], so is $\mathcal{B}(\mathfrak{N})$.

The mapping $\hat{A} : \mathcal{B}(\mathfrak{N}) \to \Delta(A)$ is a continuous mapping from $\mathcal{B}(\mathfrak{N})$ with the weak operator topology into $\mathfrak{L}(\mathfrak{H})$ with the weak operator topology. Indeed if $\hat{A}(Z_0) \in \Delta(A)$ let

$$V = \{T \in \mathfrak{L}(\mathfrak{H}) : |((T - \hat{A}(Z_0))f_i, g_i)| < \epsilon, \epsilon > 0, f_i, g_i \in \mathfrak{H}, i = 1, 2, \dots, N\}$$

be a neighborhood of $\hat{A}(Z_0)$ in the weak operator topology of $\mathfrak{L}(\mathfrak{H})$. From formulas (3)–(6) it follows that

$$(\hat{A}(Z) - \hat{A}(Z_0))f_i, g_i) = \begin{bmatrix} 0 & 0 \\ 0 & ((Z - Z_0)R^{1/2}(I-P)f_i, R^{1/2}(I-P)g_i) \end{bmatrix}.$$

Therefore the neighborhood W of Z_0 defined by

$$W = \{S \in \mathfrak{L}(\mathfrak{N}) : |((S - Z_0)R^{1/2}(I - P)f_i, R^{1/2}(I - P)g_i)| < \epsilon, i = 1, 2, \ldots, N\}$$

satisfies the condition $\hat{A}(W) \subset V$. From continuity of the mapping $\hat{A}$ we deduce that $\Delta(A) = \hat{A}(\mathcal{B}(\mathfrak{N}))$ is a compact subset of $\mathfrak{L}(\mathfrak{H})$ with the weak operator topology.

For $T \in \mathfrak{L}(\mathfrak{H})$, $\|T\| \leq 1$, we denote by $\Psi_n(T)$ an operator from $\mathfrak{L}(\mathfrak{H})$ defined by

$$\Psi_n(T) = U^{n*} g^n(T) U^n. \tag{14}$$

Note that because $0 < \kappa < 1$, the operator

$$g(T) = (T - \kappa I)(I - \kappa T)^{-1}$$

is a bounded operator on $\mathfrak{H}$.

Observe also that

$$\Psi_{n_2}(\Psi_{n_1}(T)) = \Psi_{n_1+n_2}(T). \tag{15}$$

We claim that $\Psi_n(\hat{A})$ is in $\Delta(A)$ for $\hat{A} \in \Delta(A)$. It is clear that $\Psi_n(\hat{A}) = \Psi_n(\hat{A})^*$ and $\|\Psi_n(\hat{A})\| \leq 1$. We need to show that $\Psi_n(\hat{A})h = Ah$ for $h \in \mathfrak{D}$. But for $h \in \mathfrak{D}$ according to (12) and (13) we have

$$U^n h = (h' - \kappa_n A h') = (h' - \kappa_n \hat{A} h'), \quad h' \in \mathfrak{D}.$$

Therefore, for $h \in \mathfrak{D}$,

$$U^{n*} g^n(\hat{A}) U^n h = U^{n*}(\hat{A} - \kappa_n I)h' = U^{n*}(Ah' - \kappa_n h') = Ah$$

since $\hat{A}$ is an extension of A. Thus, for any n and for any $\hat{A} \in \Delta(A)$ the operator $\Psi_n(\hat{A})$ is in $\Delta(A)$.

Therefore, the mapping $\Psi_n : \hat{A} \mapsto \Psi_n(\hat{A})$ maps the compact convex subset $\Delta(A)$ of the locally convex space $\mathfrak{L}(\mathfrak{H})$ with the weak operator topology into itself (in fact, Ψ_n is a homeomorphism). Therefore, according to the Tychonoff fixed point theorem [16], $\Psi_n : \Delta(A) \mapsto \Delta(A)$ has a fixed point. It particular this is true for $n = 1$. That is, there exists an operator $\hat{A}_0 \in \Delta(A)$ such that $\hat{A}_0 = \Psi_1(\hat{A}_0)$, or

$$U\hat{A}_0 U^* = (\hat{A}_0 - \kappa I)(I - \kappa \hat{A}_0)^{-1}.$$

From (15) it follows that $\hat{A}_0$ is a fixed point for all Ψ_n, $n = 0, \pm 1, \pm 2, \ldots$ This means that $\hat{A}_0$ is an invariant extension of A and completes the proof. □

Theorem 1 along with the block representation (3) of an operator $\hat{A}$ from $\Delta(A)$ allow to give a necessary condition for the operator $\mathcal{E}$ on $\mathfrak{N}$ to be the right bottom block of the invariant extension.

With respect to the decomposition $\mathfrak{H} = \mathfrak{D} \oplus \mathfrak{N}$, operators U^n, $n = 0, \pm 1, \pm 2, \ldots$, are representable in a block form as follows

$$U^n = \begin{bmatrix} R_n & T_n \\ S_n & Q_n \end{bmatrix}. \tag{16}$$

The operators R_n, T_n, S_n, and Q_n satisfy the following relations which are a consequence of unitarity of U^n:

$$R_n^* = R_{-n}, \quad Q_n^* = Q_{-n}, \quad T_n^* = S_{-n}. \tag{17}$$

We rewrite (11) in the form

$$AU^{*n}(I_{\mathfrak{D}} - \kappa_n A) = U^{*n}(A - \kappa_n I_{\mathfrak{D}}),$$

from which we deduce that that $(g, U)-$invariance of A results in the following relations between blocks of A (see (2)) and blocks of U^n, $n = 0, \pm 1, \pm 2, \ldots$

$$T_n^*(I_{\mathfrak{D}} - \kappa_n B) - \kappa_n Q_n^* C = 0, \tag{18}$$

$$R_n^*(B-\kappa_n I_{\mathfrak{D}})+S_n^*C=BR_n^*(I_{\mathfrak{D}}-\kappa_n B)-\kappa_n BS_n^*C, \tag{19}$$

$$T_n^*(B-\kappa_n I_{\mathfrak{D}})+Q_n^*C=CR_n^*(I_{\mathfrak{D}}-\kappa_n B)-\kappa_n CS_n^*C. \tag{20}$$

Suppose now that $\hat{A}$ is an invariant contractive self-adjoint extension of A. Using (3) we obtain that in addition to formulas (18)–(20), the following relations are fulfilled for $n=0,\pm1,\pm2,\dots$

$$\begin{aligned}\kappa_n\mathcal{E}Q_n^*\mathcal{E}+(\kappa_n CS_n^*+Q_n^*)\mathcal{E}+\mathcal{E}(\kappa_n T_n^*C^*-Q_n^*)\\ +\kappa_n(CR_n^*C^*-Q_n^*)+T_n^*C^*-CS_n^*=0,\end{aligned} \tag{21}$$

that is $\mathcal{E}$ is a solution of a collection of Riccati equations. From this fact we immediately deduce that if $\dim\mathfrak{N}=1$, then the operator A has at most two (g,U)-invariant self-adjoint extensions. Later on it will be shown (Theorem 3 that the extreme extensions $\hat{A}_\mu$ and $\hat{A}_M$ are (g,U)-invariant).

In what follows we assume that the non-densely defined contraction A does not have numbers ±1 as eigenvalues. Then the self-adjoint operator $B=PA$ on $\mathfrak{D}$ (see (2)) also does not have ±1 in its point spectrum. Indeed, if $PAf=f$, $f\in\mathfrak{D}$, then

$$\begin{aligned}\|Af-f\|^2&=\|Af\|^2+\|f\|^2-(Af,f)-(f,Af)\\&=\|Af\|^2-\|f\|^2\le 0\end{aligned}$$

since A is a contraction. For $PAf=-f$ the proof is similar.

Our assumption does not cause a loss of generality. If $\lambda=1$ (or $\lambda=-1$ or both) is an eigenvalue of A then the corresponding eigenspace reduces the operator A and the restriction of A to it is the self-adjoint identity operator which is (g,U)-invariant.

Denote by D_n^+ and D_n^- the operators defined by

$$D_n^+=X(I_{\mathfrak{D}}-B^2)(I_{\mathfrak{D}}+\kappa_n B)^{-1}X^*, \tag{22}$$

and

$$D_n^-=X(I_{\mathfrak{D}}-B^2)(I_{\mathfrak{D}}-\kappa_n B)^{-1}X^*, \tag{23}$$

where equation (4) was used.

$D_n^\pm$ are bounded positive operators on the subspace $\mathfrak{N}$. Since $\kappa_{-n}=-\kappa_n$ (see (10)), we have $D_{-n}^+=D_n^-$.

Lemma 1. *Suppose that $\lambda=\pm1$ are not eigenvalues of B. Then the operators $D_n^\pm$ converge to the operators*

$$D^\pm=X(I_{\mathfrak{D}}\mp B)X^* \tag{24}$$

respectively as $n\to\infty$ in the weak operator topology of $\mathfrak{L}(\mathfrak{N})$.

Proof. It suffices to show that $(D_n^\pm h,h)\to(D^\pm h,h)$ as $n\to\infty$ for any $h\in\mathfrak{N}$. From the spectral representation of B it follows that

$$(D_n^+h,h)=\int_{-1}^{1}\frac{1-\lambda^2}{1+\kappa_n\lambda}\,d\sigma_h(\lambda)$$

where $\sigma_h(\lambda)=(E(\lambda)X^*h,X^*h)$, and $E(\lambda)$ is the resolution of the identity of B. Our assumption about the spectrum of B gives $\sigma_h(\{-1\})=\sigma_h(\{1\})=0$ and $\sigma_h(\lambda)$ is continuous at $\lambda=\pm1$.

Since $\kappa_n\to1$ as $n\to\infty$

$$\lim_{n\to\infty}\frac{1-\lambda^2}{1+\kappa_n\lambda}=\begin{cases}0 & \lambda=-1,\\ 1-\lambda & -1<\lambda\le 1\end{cases}.$$

Now the Lebesgue dominating convergence theorem gives

$$\lim_{n\to\infty}(D_n^+h,h)=\lim_{n\to\infty}\int_{-1}^{1}\frac{1-\lambda^2}{1+\kappa_n\lambda}\,d\sigma_h(\lambda)=\int_{-1}^{1}(1-\lambda)\,d\sigma_h(\lambda)=(D^+h,h).$$

For the operators D_n^- the proof is similar. Lemma is proved now. □

The same type of arguments show that in the weak operator topology of $\mathfrak{L}(\mathfrak{H})$

$$\lim_{n\to\infty}(1-\kappa_n^2)C(I_{\mathfrak{D}}-\kappa_nB)^{-2}C^*=\lim_{n\to\infty}(1-k_n^2)X(I_{\mathfrak{D}}-B^2)(I_{\mathfrak{D}}-\kappa_nB)^{-2}X^*=0. \tag{25}$$

Using (17), (18)–(20), and (22) and (23), we rewrite (21) in the form

$$\begin{aligned}\kappa_n\mathcal{E}Q_n^*\mathcal{E}+(I_{\mathfrak{N}}-\kappa_n^2D_n^+)Q_n^*\mathcal{E}-\mathcal{E}Q_n^*(I_{\mathfrak{N}}-\kappa_n^2D_n^-)&\\+\kappa_nD_n^+Q_n^*+\kappa_nQ_n^*D_n^--k_n^2D_n^+Q_n^*D_n^--\kappa_nQ_n^*&\\+\kappa_n(1-\kappa_n^2)Q_n^*C(I_{\mathfrak{D}}-\kappa_nB)^{-2}C^*=0.&\end{aligned} \tag{26}$$

Theorem 2. *Let A be a (g,U)-invariant Hermitian contraction with* $\dim\mathfrak{N}<\infty$, *and let $\hat{A}\in\Delta(A)$ be (g,U)-invariant. Then there exists a contraction Q on $\mathfrak{N}$ ($\|Q\|\le 1$) such that the parameter $\mathcal{E}$ in the block representation* (3) *of $\hat{A}$ satisfies the equation*

$$(\mathcal{E}-\mathcal{E}_\mu)Q^*(\mathcal{E}-\mathcal{E}_M)=0, \tag{27}$$

where $\mathcal{E}_\mu$ and $\mathcal{E}_M$ are defined by equations (7) *and* (8) *respectively.*

Proof. Since U^n, $n=0,\pm1,\pm2,\ldots$, are unitary operators, the operators $Q_n^*=(I-P)U^{n*}|\mathfrak{N}$ are in the unit ball of the space $\mathfrak{L}(\mathfrak{N})$. Since $\dim\mathfrak{N}<\infty$, this unit ball is compact. Let Q^* be a limit point (in norm topology of $\mathfrak{L}(\mathfrak{N})$) of the sequence $\{Q_n^*\}_{-\infty}^{\infty}$. It is clear that Q is a contraction. Suppose that $\lim_{l_j\to+\infty}Q_{l_j}^*=Q^*$ (if $l_j\to-\infty$, the proof is similar). Because the norm topology and the weak operator topology are equivalent on $\mathfrak{L}(\mathfrak{N})$, we can apply Lemma 1 and formula (25). Putting in (26) $n=l_j$ and taking the limit as $l_j\to\infty$ we obtain

$$\mathcal{E}Q^*\mathcal{E}+(I_{\mathfrak{N}}-D^+)Q^*\mathcal{E}-\mathcal{E}Q^*(I_{\mathfrak{N}}-D^-)+D^+Q^*+Q^*D^--D^+Q^*D^--Q^*=0.$$

The last expression can be factored as

$$[\mathcal{E}+(I_{\mathfrak{N}}-D^+)]Q^*[\mathcal{E}-(I_{\mathfrak{N}}-D^-)]=0,$$

which is (27) because of (7) and (8). This completes the proof. □

3. Invariance of $\hat{A}_\mu$ and $\hat{A}_M$

In this section we show directly without using Theorem 1 that for a (g,U)-invariant non-densely defined Hermitian contraction A the extreme extensions $\hat{A}_\mu$ and $\hat{A}_M$ are (g,U)-invariant. Theorem 3 also provides an alternative proof of the existence of invariant extensions.

Theorem 3. *Let A be a non-densely defined (g,U)-invariant Hermitian contraction. Then the self-adjoint operators $\hat{A}_\mu$ and $\hat{A}_M$ are (g,U)-invariant.*

We present a portion of the proof in the following lemmas.

Lemma 2. *Let A be a (g,U)-invariant non-densely defined Hermitian operator. Then the operator $\widehat{g(A)}$ belongs to the set $\Delta(g(A))$ if and only if it admits the following representation:*

$$\widehat{g(A)}=\begin{bmatrix}UBU^*|\mathfrak{M}_{1/\kappa} & UC^*U^*|\mathfrak{M}_{1/\kappa}\\ UCU^*|\mathfrak{M}_{1/\kappa} & \mathcal{E}'\end{bmatrix}, \tag{28}$$

where

$$\mathcal{E}' = UOU^*|\mathfrak{M}_{1/\kappa} + UR^{1/2}U^*Z'UR^{1/2}U^*|\mathfrak{M}_{1/\kappa}, \tag{29}$$

Z' is an arbitrary contractive self-adjoint operator on the subspace $\mathfrak{N}_{1/\kappa} = \mathfrak{M}_{1/\kappa}^{\perp}$, and the operators B, C, O, and R were defined by formulas (2), (5), *and* (6).

Proof. It is clear that the operator $g(A) = (A - \kappa I_{\mathfrak{D}})(I_{\mathfrak{D}} - \kappa A)^{-1}$ is also a non-densely defined Hermitian contraction. Its domain is the subspace $\mathfrak{M}_{1/\kappa}$ and its range is the subspace $\mathfrak{M}_{\kappa}$. We denote by $\Delta(g(A))$ the set of self-adjoint contractive extensions of the operator $g(A)$. The statement of the Lemma follows immediately from the Definition 1 and formulas (3)–(6). □

Remark 2. *The Lemma 2 states that*

$$\widehat{g(A)}(Z') = U\hat{A}(U^*Z'U)U^*.$$

In particular,

$$U\hat{A}_{\mu}U^* = [\widehat{g(A)}]_{\mu} \quad \textit{and} \quad U\hat{A}_M U^* = [\widehat{g(A)}]_M. \tag{30}$$

Lemma 3. *Let A be a non-densely defined Hermitian contraction. Then*

$$g(\Delta(A)) = \Delta(g(A)). \tag{31}$$

Proof. For an $\hat{A} \in \Delta(A)$ the operator $g(\hat{A}) = (\hat{A} - \kappa I)(I - \kappa\hat{A})^{-1}$ is self-adjoint and $\|g(\hat{A})\| \leq 1$. We only need to show that $g(\hat{A})$ is an extension of $g(A)$, i.e., $g(\hat{A})\varphi = g(A)\varphi$ for $\varphi \in \mathfrak{D}(g(A)) = \mathfrak{M}_{1/\kappa}$. But if $\varphi \in \mathfrak{M}_{1/\kappa}$, then $\varphi = h - \kappa Ah = h - \kappa\hat{A}h$, where $h \in \mathfrak{D}$, and $g(A)\varphi = Ah - \kappa h = \hat{A}h - \kappa h = g(\hat{A})\varphi$. Therefore, $g(\Delta(A)) \subset \Delta(g(A))$.

Conversely, suppose that $S \in \Delta(g(A))$, that is $S = S^*$, $\|S\| \leq 1$, and $S\varphi = g(A)\varphi = Ah - \kappa h$ for $\varphi = h - \kappa Ah$, $h \in \mathfrak{D}$. Therefore, any vector h from $\mathfrak{D}$ is representable in the form

$$h = \frac{1}{1-\kappa^2}(\varphi + \kappa S\varphi),$$

while

$$Ah = \frac{1}{1-\kappa^2}(S\varphi + \kappa\varphi).$$

These two equalities yield

$$Ah = (S + \kappa I')(I' + \kappa S)^{-1}h = g^{-1}(S)h.$$

Thus the self-adjoint operator $g^{-1}(S)$ is an extension of A, $g^{-1}(S) \in \Delta(A)$. Therefore, $S \in g(\Delta(A))$. This completes the proof. □

Lemma 4. *Let S_1 and S_2 be self-adjoint operators, $\|S_i\| \leq 1$ for $i = 1, 2$, and suppose that $S_1 \leq S_2$. Then for $g(x) = (x - \kappa)(1 - \kappa x)^{-1}$, $-1 < \kappa < 1$ the following inequality is fulfilled:*

$$g(S_1) \leq g(S_2).$$

Proof. We may assume that $\kappa > 0$. From the condition of the Lemma it follows that $(I - \kappa S_1) \geq (I - \kappa S_2)$. Because both operators $I - \kappa S_1$ and $I - \kappa S_2$ are boundedly invertible and positive, $(I - \kappa S_1)^{-1} \leq (I - \kappa S_2)^{-1}$. Now the conclusion of the lemma follows from the formula

$$g(S_1) - g(S_1) = \frac{1-\kappa^2}{k}[(I - \kappa S_1)^{-1} - (I - \kappa S_2)^{-1}].$$

For $\kappa < 0$ the proof is similar. □

Proof of the Theorem 3. From (30) it follows that it suffices to show that $[\widehat{g(A)}]_\mu = g(\hat{A}_\mu)$ and $[\widehat{g(A)}]_M = g(\hat{A}_M)$.

Suppose that $[\widehat{g(A)}]_\mu \neq g(\hat{A}_\mu)$. Then, according to Lemma 3, $[\widehat{g(A)}]_\mu = g(\hat{A})$, where $\hat{A} \in \Delta(A)$ and $\hat{A} \neq \hat{A}_\mu$. Then $\hat{A}_\mu \leq \hat{A}$ and from Lemma 4 it follows that $g(\hat{A}_\mu) \leq g(\hat{A}) = [\widehat{g(A)}]_\mu$. Since $g(\hat{A}_\mu) \in \Delta(g(A))$, it satisfies the inequality $g(\hat{A}_\mu) \geq [\widehat{g(A)}]_\mu$. Therefore $[\widehat{g(A)}]_\mu = g(\hat{A}_\mu)$.

The equality $[\widehat{g(A)}]_M = g(\hat{A}_M)$ can be proved using similar arguments. □

Corollary 1. *Let A be a non-densely defined (g, U)-invariant Hermitian contraction such that* $\dim\mathfrak{N} = 1$. *Suppose that the set $\Delta(A)$ contains more that one element. Then the extreme extensions $\hat{A}_\mu$ and $\hat{A}_M$ are the only invariant self-adjoint contractive extensions of A.*

Indeed, as it was pointed out in the previous section, if $\dim\mathfrak{N} = 1$, then operator A has no more than two invariant self-adjoint extensions. Assumptions of the lemma mean that $\hat{A}_\mu \neq \hat{A}_M$. Theorem 3 gives now desired conclusion.

4. Example

In this section we give an example of a non-densely defined Hermitian contraction with $\dim\mathfrak{N} = 1$. Using Theorem 3 it is possible in a simple way to construct extreme extensions $\hat{A}_\mu$ and $\hat{A}_M$ and, therefore, to describe the set $\Delta(A)$.

Recall at first that the Hardy space $H^2(\mathbb{C}_+)$ consists of functions $h(z)$ which are analytic in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ and satisfy the condition

$$\sup_{y>0} \int_{-\infty}^{\infty} |h(x+iy)|^2 dx < \infty.$$

Functions from $H^2(\mathbb{C}_+)$ can be identified with their boundary functions (as usual, we use the same notation for the analytic function $h(z)$, $z \in \mathbb{C}_+$, and its boundary function $h(\lambda)$, $\lambda \in \mathbb{R}$), which form a subspace of the space $L^2(\mathbb{R}, d\lambda/2\pi)$. This subspace is denoted by $H^2(\mathbb{R}, d\lambda/2\pi)$. We use notation H^2 if it clear whether we are speaking about analytic functions $h(z)$ of about their boundary values $h(\lambda)$. It is well known (see, for example [8]), that function $h(z) \in H^2(\mathbb{C}_+)$ can be recovered from its boundary function $h(\lambda)$ either by the Cauchy integral

$$h(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{h(\lambda)}{\lambda - z}\, d\lambda$$

or the Poisson integral

$$h(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-\lambda)^2 + y^2} h(\lambda)\, d\lambda, \quad z = x + iy.$$

The inner product (f, g) in $H^2(\mathbb{C}_+)$ is defined by

$$(f, g) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda)\overline{g(\lambda)}\, d\lambda.$$

Let $\mathfrak{D}$ be a subspace of $\mathfrak{H} = H^2(\mathbb{C}_+)$ of functions $h(\lambda)$, $\lambda \in \mathbb{R}$, which satisfy the following condition

$$\frac{1}{2\pi}\int_{\infty}^{\infty} \frac{h(\lambda)}{1+i\lambda}\, d\lambda = 0. \tag{32}$$

Condition (32) means that the analytic in $\mathbb{C}_+$ functions h from $\mathfrak{D}$ satisfy condition $h(i) = 0$. Clearly $\mathfrak{D}$ is a proper subspace of $\mathfrak{H}$, and $\dim\mathfrak{D}^\perp = 1$.

Define now an operator A on $\mathfrak{D}$ as follows:

$$(Ah)(\lambda) = \frac{1-\lambda^2}{1+\lambda^2} h(\lambda). \tag{33}$$

Since A is the operator of multiplication by a real-valued function of absolute value not greater than 1, A is a non-densely defined Hermitian contraction.

Let now a unitary operator U be defined by the formula

$$(Uh)(\lambda) = s^{1/4} h(s^{1/2}\lambda), \tag{34}$$

where $s > 1$, and let the linear-fractional transformation g be defined by (9) and (10). We are going to show that the operator A is (g, U)-invariant.

At first, according to the Definition 1 we need to show that for $h \in \mathfrak{D}$ function $Uh \in (I - \kappa A)\mathfrak{D}$, that is the equation

$$s^{1/4} h(s^{1/2}\lambda) = \left(1 - \kappa\frac{1-\lambda^2}{1+\lambda^2}\right) g(\lambda), \quad \kappa = \frac{s-1}{s+1}$$

has a solution $g \in \mathfrak{D}$. From the last relation we get

$$g(\lambda) = \frac{s^{1/4}(s+1)}{2}\frac{1+\lambda^2}{1+s\lambda^2} h(\sqrt{s}\lambda).$$

Now we get

$$\int_{-\infty}^{\infty} \frac{g(\lambda)}{1+i\lambda}\, d\lambda = \frac{s^{1/4}(s+1)}{2}\int_{-\infty}^{\infty}\left[\frac{1}{1+s\lambda^2} - i\frac{\lambda}{1+s\lambda^2}\right] h(\sqrt{s}\lambda)\, d\lambda = 0,$$

because for $h \in H^2(\mathbb{C}_+)$ the condition

$$\int_{-\infty}^{\infty} \frac{h(\lambda)}{1+i\lambda}\, d\lambda = 0$$

is equivalent to the conditions

$$\int_{-\infty}^{\infty} \frac{h(\lambda)}{1+\lambda^2}\, d\lambda = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\lambda h(\lambda)}{1+\lambda^2}\, d\lambda = 0.$$

Therefore, $g \in \mathfrak{D}$ and

$$\begin{aligned}(A - \kappa I)g &= \left(\frac{1-\lambda^2}{1+\lambda^2} - \frac{s-1}{s+1} I\right)\frac{s^{1/4}(s+1)}{2}\frac{1+\lambda^2}{1+s\lambda^2} h(\sqrt{s}\lambda) \\ &= s^{1/4}\frac{1-s\lambda^2}{1+s\lambda^2} h(\sqrt{s}\lambda) = (UAh)(\lambda).\end{aligned}$$

Thus A is a (g, U)-invariant operator.

It is easily seen that the orthogonal projection onto subspace $\mathfrak{N}$ is defined as

$$(I - P)f(\lambda) = \frac{2i}{\lambda+i} f(i), \quad f \in H^2.$$

Therefore, the operators B and C are defined as follows

$$(Bh)(\lambda) = \frac{1-\lambda^2}{1+\lambda^2}h(\lambda) - \frac{2}{\lambda+i}h'(i), \quad h \in \mathfrak{D}, \tag{35}$$

and

$$(Ch)(\lambda) = \frac{2}{\lambda+i}h'(i), \quad h \in \mathfrak{D}. \tag{36}$$

The operator C^* acts from $\mathfrak{N}$ into $\mathfrak{D}$. It suffices to calculate the action of C^* onto the function $(\lambda+i)^{-1}$. Direct verification shows that

$$C^*\varphi_0 = -\frac{1}{2}\frac{\lambda - i}{(\lambda+i)^2}, \quad \varphi_0 = \frac{1}{\lambda+i}. \tag{37}$$

Since $\dim\mathfrak{N} = 1$, the operator $\mathcal{E}$ in (3) is just the operator of multiplication by a real number. We represent a function g from $H^2(\mathbb{C}_+)$ in the form

$$g(\lambda) = \begin{bmatrix} g(\lambda) - \dfrac{2i}{\lambda+i}g(i) \\[2ex] \dfrac{2i}{\lambda+i}g(i) \end{bmatrix}$$

according to the decomposition $H^2(\mathbb{C}_+) = \mathfrak{D}\oplus\mathfrak{N}$. Now we obtain that for any $\hat{A} \in \Delta(A)$

$$\hat{A}g = \begin{bmatrix} \dfrac{1-\lambda^2}{1+\lambda^2}g(\lambda) + \dfrac{2i\lambda}{1+\lambda^2}g(i) - \dfrac{2}{\lambda+i}g'(i) \\[2ex] \dfrac{1}{\lambda+i}[2g'(i) + ig(i)(2\mathcal{E}-1)] \end{bmatrix}. \tag{38}$$

Theorem 4. *Let a unitary operator U be defined by* (34) *with $s > 1$, and let the linear-fractional transformation g be defined by* (9) *and* (10). *Then the operator $\hat{A}$ defined by* (38) *is (g,U)-invariant if and only if $\mathcal{E} = \pm\dfrac{1}{2}$.*

Proof. The proof of the theorem is based on direct calculations. An operator $\hat{A}$ is invariant if and only if for an arbitrary vector $h \in H^2(\mathbb{C}_+)$ and for the vector $g \in H^2(\mathbb{C}_+)$ defined by $(I - \kappa_n\hat{A})g = U^n h$ the following equality is fulfilled: $U^n\hat{A}h = (\hat{A} - \kappa_n I)g$. Note that since the operator A is invariant, the condition $PU^n\hat{A}h = P(\hat{A} - \kappa_n I)g$ is fulfilled automatically. The only nontrivial condition is

$$P_{\mathfrak{N}}U^n\hat{A}h = P_{\mathfrak{N}}(\hat{A} - \kappa_n I)g, \tag{39}$$

where $P_{\mathfrak{N}} = I - P$ is the orthogonal projection onto subspace $\mathfrak{N}$.

Pick a vector h from $H^2(\mathbb{C}_+)$ and rewrite the condition $U^n h = (I - \kappa_n\hat{A})g$, $n = 0, \pm1. \pm2, \ldots$ (of course, g depends on n) in the form

$$\begin{bmatrix} s^{n/4}[h(s^{n/2}\lambda) - \dfrac{2i}{\lambda+i}h(s^{n/2}i)] \\[2ex] \dfrac{2is^{n/4}}{\lambda+i}h(s^{n/2}i) \end{bmatrix} = \begin{bmatrix} g(\lambda) - \dfrac{2i}{\lambda+i}g(i) \\[2ex] \dfrac{2i}{\lambda+i}g(i) \end{bmatrix}$$

$$- \kappa_n \begin{bmatrix} \dfrac{1-\lambda^2}{1+\lambda^2}g(\lambda) + \dfrac{2i\lambda}{1+\lambda^2}g(i) - \dfrac{2}{\lambda+i}g'(i) \\[2ex] \dfrac{1}{\lambda+i}[2g'(i) + ig(i)(2\mathcal{E}-1)] \end{bmatrix},$$

from which we deduce that

$$h(\lambda) = s^{-n/4} g(s^{-n/2}\lambda)[1-\kappa_n \frac{s^n-\lambda^2}{s^n+\lambda^2}] - \frac{is^{n/4}\kappa_n}{s^n+\lambda^2} g(i)[(\lambda+is^{n/2})+2\mathcal{E}(\lambda-is^{n/2})]$$

(recall that $\kappa_n = (s^n-1)/(s^n+1)$). In particular,

$$2is^{n/4}h(s^{n/2}i) = 2ig(i) - \kappa_n[2g'(i)+ig(i)(2\mathcal{E}-1)] \tag{40}$$

and

$$s^{n/4}h(i) = \frac{s^{n/2}}{s^n+1} g(i)[(1+s^{n/2})+2\mathcal{E}(1-s^{n/2})]. \tag{41}$$

Using (33) we obtain that the vector $P_{\mathfrak{N}}U^n\hat{A}h$, the orthogonal projection of the vector $U^n\hat{A}h$ onto subspace $\mathfrak{N}$, is given by the formula

$$P_{\mathfrak{N}}U^n\hat{A}h = \frac{2is^{n/4}}{\lambda+i}\left\{\frac{1+s^n}{1-s^n}h(s^{n/2}i) + \frac{h(i)}{1-s^n}\left[2\mathcal{E}(1-s^{n/2})-(1+s^{n/2})\right]\right\}.$$

From (40) and (41) we obtain that in terms of the vector g last formula takes the following form:

$$\begin{aligned} P_{\mathfrak{N}}U^n\hat{A}h &= \frac{1}{\lambda+i}\left[2g'(i)+ig(i)(2\mathcal{E}-1)\right] \\ &+ \frac{2ig(i)}{\lambda+i}\left\{\frac{1+s^n}{1-s^n}+\frac{s^{n/2}}{1-s^{2n}}\left[4\mathcal{E}^2(1-s^{n/2})^2-(1+s^{n/2})^2\right]\right\}. \end{aligned} \tag{42}$$

Since $P_{\mathfrak{N}}(\hat{A}-\kappa_n I)g$ is given by

$$P_{\mathfrak{N}}(\hat{A}-\kappa_n I)g = \frac{1}{\lambda+i}\left\{[2g'(i)+ig(i)(2\mathcal{E}-1)]-2i\kappa_n g(i)\right\}, \tag{43}$$

from (42) and (43) we obtain that (39) results the equation

$$\frac{1+s^n}{1-s^n}+\frac{s^{n/2}}{1-s^{2n}}\left[4\mathcal{E}^2(1-s^{n/2})^2-(1+s^{n/2})^2\right]+\frac{s^n-1}{s^n+1}=0,$$

from which we obtain that $\mathcal{E} = \pm\frac{1}{2}$. This completes the proof. □

Now Theorem 1 gives the following statement.

Corollary 2. *For the operator A above the extreme extensions $\hat{A}_\mu$ and $\hat{A}_M$ are obtained according to the formula (38) with $\mathcal{E}_\mu = -1/2$ and $\mathcal{E}_M = 1/2$ respectively.*

Remark 3. *Note that the value $\mathcal{E} = 0$ corresponds to the operator of multiplication by $(1-\lambda^2)/(1+\lambda^2)$ followed by the projection from $L^2(\mathbb{R}, d\lambda/2\pi)$ onto $H^2(\mathbb{R}, d\lambda/2\pi)$, that is the Toeplitz operator on $H^2(\mathbb{C}_+)$ with the symbol $(1-\lambda^2)/(1+\lambda^2)$.*

Any operator $\hat{A}$ from $\Delta(A)$ satisfies the condition $\|\hat{A}g\| \le \|g\|$ for all $g \in H^2$. Using (38) it is not hard to calculate that $\hat{A}$ is a contraction if and only if

$$\begin{aligned} &2\mathcal{E}^2|g(i)|^2 + 2\mathcal{E}\left[2\mathrm{Re}\{ig(i)\overline{g'(i)}\} - |g(i)|^2\right] \\ &\qquad - \frac{1}{2}|g(i)|^2 - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4\lambda^2}{(1+\lambda^2)^2}|g(\lambda)|^2 d\lambda \le 0. \end{aligned} \tag{44}$$

From last inequality we see that without lost of generality it is possible to assume that $g(i) = 1$. Then from (44) it follows that the quantity $\mathcal{E}$ satisfies the following inequality:

$$\frac{1}{2}\left\{1 - 2\mathrm{Im}g'(i) - \sqrt{[1 - 2\mathrm{Im}g'(i)]^2 + 1 + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{4\lambda^2}{(1+\lambda^2)^2}|g(\lambda)|^2 d\lambda}\right\}$$

$$\le \mathcal{E} \le \frac{1}{2}\left\{1 - 2\mathrm{Im}g'(i) + \sqrt{[1 - 2\mathrm{Im}g'(i)]^2 + 1 + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{4\lambda^2}{(1+\lambda^2)^2}|g(\lambda)|^2 d\lambda}\right\}.$$

Therefore, $\mathcal{E}_\mu$ is equal to the supremum of the left-hand side to the last inequality over the set of functions g from H^2 which satisfies $g(i) = 1$, while $\mathcal{E}_M$ is equal to the infimum of the right-hand side over the same set.

5. Nondensely defined Hermitian contractions and positive scale-invariant symmetric operators

Let $\mathcal{H}$ be a densely defined positive closed unbounded symmetric operator on a Hilbert space $\mathfrak{H}$. We denote by $\mathfrak{D}(\mathcal{H})$ the domain of the operator $\mathcal{H}$.

In order to obtain positive self-adjoint extensions of operator $\mathcal{H}$, M. G. Kreĭn [11] considered a nondensely defined Hermitian contraction A defined as follows:
the domain $\mathfrak{D}$ of of A is the set of all vectors $h \in \mathfrak{H}$ representable in the form

$$h = f + \mathcal{H}f, \quad f \in \mathfrak{D}(\mathcal{H}),$$

and

$$Ah = f - \mathcal{H}f.$$

The last two equations can be written in the form

$$A = (I - \mathcal{H})(I + \mathcal{H})^{-1}. \tag{45}$$

The operator $\mathcal{H}$ can be recovered from A by the formula

$$\mathcal{H} = (I - A)(I + A)^{-1}.$$

Because $\mathcal{H}$ is not self-adjoint, set $\mathfrak{D} = \overline{\mathfrak{D}} \neq \mathfrak{H}$. The dimension of its orthogonal complement $\mathfrak{N} = \mathfrak{H} \ominus \mathfrak{D}$ is equal to the defect number of $\mathcal{H}$.

Any element $\hat{A} \in \Delta(A)$ defines a positive self-adjoint extension H of $\mathcal{H}$ according to the formula

$$H = (I - \hat{A})(I + \hat{A})^{-1}.$$

The extreme extensions $\hat{A}_\mu$ and $\hat{A}_M$ correspond to the Friedrichs extension H_F and the Kreĭn extension H_K respectively.

Definition 2. *Let $\mathcal{H}$ be a densely defined closed symmetric operator on a Hilbert space $\mathfrak{H}$ and let s be a positive number, $s \neq 1$. The operator $\mathcal{H}$ is said to be scale-invariant if there exists a unitary operator U on $\mathfrak{H}$ such that for all $n = 0, \pm 1, \pm 2, \ldots$*

$$U^n \mathfrak{D}(\mathcal{H}) = \mathcal{H},$$

and

$$U^n \mathcal{H} f = s^n \mathcal{H} U^n f, \quad f \in \mathfrak{D}(\mathcal{H}).$$

Without loss of generality we may assume that $s > 1$.

It is easy to check that a positive symmetric operator $\mathcal{H}$ is scale-invariant if and only if the nondensely defined Hermitian contraction A defined by formula(45) is (g, U)-invariant, where the transformation g is defined by (9). Therefore theorems that have been proved above for nondensely defined invariant Hermitian contractions can be reformulated in terms of unbounded symmetric operators in the following form:

Theorem 5. *Let $\mathcal{H}$ be a densely defined scale-invariant positive symmetric operator on a Hilbert space $\mathfrak{H}$. Then*

(1) *$\mathcal{H}$ always admits a positive scale-invariant self-adjoint extension H. In particular, the Friedrichs extension H_F and the Kreĭn extension H_K are scale invariant.*
(2) *If, in addition, the index of defect of $\mathcal{H}$ is $(1,1)$, then H_F and H_K are the only scale-invariant positive self-adjoint extensions of $\mathcal{H}$.*

Theorem 5 appeared for the first time in [14] and we presented an alternative proof of this theorem.

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Department of Mathematics and Statistics, University of Missouri-Rolla, Rolla, MO 65409, USA

E-mail address: bekkerm@umr.edu

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