Methods of Functional Analysis and Topology
Vol. 13 (2007), no. 3, pp. 236–261

GENERALIZED KREIN ALGEBRAS AND ASYMPTOTICS OF TOEPLITZ DETERMINANTS

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This paper is dedicated to the centenary of Mark Krein (1907–1989).

Abstract. We give a survey on generalized Krein algebras $K_{p,q}^{\alpha,\beta}$ and their applications to Toeplitz determinants. Our methods originated in a paper by Mark Krein of 1966, where he showed that $K_{2,2}^{1/2,1/2}$ is a Banach algebra. Subsequently, Widom proved the strong Szegő limit theorem for block Toeplitz determinants with symbols in $(K_{2,2}^{1/2,1/2})_{N\times N}$ and later two of the authors studied symbols in the generalized Krein algebras $(K_{p,q}^{\alpha,\beta})_{N\times N}$, where $\lambda := 1/p+1/q = \alpha+\beta$ and $\lambda = 1$. We here extend these results to $0 < \lambda < 1$. The entire paper is based on fundamental work by Mark Krein, ranging from operator ideals through Toeplitz operators up to Wiener-Hopf factorization.

1. Introduction and main results

1.1. **Krein algebras.** Suppose that $\mathcal{A}$ is a complex Banach space with norm $\|\cdot\|_{\mathcal{A}}$ which is also an algebra with unit e over the field of complex numbers and $\|e\|_{\mathcal{A}} \neq 0$. If the multiplication in $\mathcal{A}$ is continuous, then $\mathcal{A}$ is called a *unital Banach algebra*. Such an algebra can be equipped with a new norm $\|\cdot\|$ which is equivalent to $\|\cdot\|_{\mathcal{A}}$ and satisfies

$$\|e\| = 1, \qquad \|ab\| \le \|a\|\,\|b\| \quad \text{for all} \quad a, b \in \mathcal{A}. \tag{1.1}$$

Each norm satisfying (1.1) is called a *Banach algebra norm*.

Let $\mathbb{T}$ be the unit circle and, for $1 \le p \le \infty$, let $L^p := L^p(\mathbb{T})$ and $H^p := H^p(\mathbb{T})$ be the standard Lebesgue and Hardy spaces. Denote by $\{a_k\}_{k\in\mathbb{Z}}$ the sequence of the Fourier coefficients of a function $a \in L^1$,

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-ik\theta}d\theta \quad (k \in \mathbb{Z}).$$

It was Mark Krein [19] who first discovered that the set of all functions $a \in L^\infty$ satisfying $\sum_{k\in\mathbb{Z}} |a_k|^2|k| < \infty$ forms a Banach algebra. This algebra is called the *Krein algebra* and is denoted by $K_{2,2}^{1/2,1/2}$.

Now we give an equivalent definition of the Krein algebra. For $k \in \mathbb{Z}$, let $\chi_k(t) := t^k$, where $t \in \mathbb{T}$. Let $\sum_{k\in\mathbb{Z}} a_k\chi_k$ be the Fourier series of a function $a \in L^1$. The Riesz projections P and Q are defined formally by

$$P : \sum_{k\in\mathbb{Z}} a_k\chi_k \mapsto \sum_{k\ge 0} a_k\chi_k, \quad Q : \sum_{k\in\mathbb{Z}} a_k\chi_k \mapsto \sum_{k<0} a_k\chi_k.$$

2000 *Mathematics Subject Classification.* Primary 47B35; Secondary 15A15, 47B10.

Key words and phrases. Toeplitz matrix, generalized Krein algebra, Szegő-Widom limit theorem, Toeplitz operator, Hankel operator, Wiener-Hopf factorization, Besov space, Hölder space, Schatten-von Neumann class, regularized operator determinant.

The second author is partially supported by F.C.T. (Portugal) grants SFRH/BPD/11619/2002 and FCT/FEDER/POCTI/MAT/59972/2004.

These operators are bounded on L^2. Besides the projections P and Q, we now need the so-called flip operator J. This is the isometric operator acting on L^p, $1 < p < \infty$, by $(Jf)(t) = (1/t)\widetilde{f}(t)$, where $\widetilde{f}(t) := f(1/t)$. For $a \in L^\infty$, we define the Hankel operators $H(a)$ and $H(\widetilde{a})$ by

$$(1.2)\quad H(a) : H^2 \to H^2, \quad f \mapsto PM(a)QJf; \qquad H(\widetilde{a}) : H^2 \to H^2, \quad f \mapsto JQM(a)Pf,$$

where $M(a)f = af$ is the multiplication operator by a. It is well known and easy to see that if $a \in L^\infty$, then both $H(a)$ and $H(\widetilde{a})$ are Hilbert-Schmidt if and only if $a \in K_{2,2}^{1/2,1/2}$. Thus,

$$K_{2,2}^{1/2,1/2} = \{a \in L^\infty : H(a), H(\widetilde{a}) \text{ are Hilbert-Schmidt}\}.$$

The first natural generalization of the classical Krein algebra $K_{2,2}^{1/2,1/2}$ consists in replacing the ideal of Hilbert-Schmidt operators by the Schatten-von Neumann ideals $\mathcal{C}_p(H^2)$, $1 \le p \le \infty$ (see Section 2.3).

Let $1 \le p \le \infty$ and $1 \le q \le \infty$ (but not necessarily $1/p + 1/q = 1$). We put

$$(1.3)\quad \begin{aligned} K_{p,0}^{1/p,0} &:= \big\{a \in L^\infty : H(\widetilde{a}) \in \mathcal{C}_p(H^2)\big\}, \\ K_{0,q}^{0,1/q} &:= \big\{a \in L^\infty : H(a) \in \mathcal{C}_q(H^2)\big\}, \\ K_{p,q}^{1/p,1/q} &:= K_{p,0}^{1/p,0} \cap K_{0,q}^{0,1/q}. \end{aligned}$$

It is clear that the sets (1.3) are linear spaces. We define norms by

$$(1.4)\quad \begin{aligned} \|a\| &:= \|a\|_{L^\infty} + \|H(\widetilde{a})\|_{\mathcal{C}_p(H^2)} \quad (a \in K_{p,0}^{1/p,0}), \\ \|a\| &:= \|a\|_{L^\infty} + \|H(a)\|_{\mathcal{C}_q(H^2)} \quad (a \in K_{0,q}^{0,1/q}), \\ \|a\| &:= \|a\|_{L^\infty} + \|H(\widetilde{a})\|_{\mathcal{C}_p(H^2)} + \|H(a)\|_{\mathcal{C}_q(H^2)} \quad (a \in K_{p,q}^{1/p,1/q}). \end{aligned}$$

Let $A, B \subset L^\infty$ be two subsets. We put $A + B = \{\varphi + \psi : \varphi \in A, \psi \in B\}$. Let $\overline{H^p}$ be the set of functions f in L^p such that the complex conjugate $\overline{f}$ belongs to H^p and let $C := C(\mathbb{T})$ be the set of continuous functions. It is well known that $C + H^\infty$, and hence, $C + \overline{H^\infty}$ and $QC := (C + H^\infty) \cap (C + \overline{H^\infty})$ are closed subalgebras of L^∞ (see, e.g. [8, Section 6.31] or [23, Chap. 1, Theorem 5.1]).

Hartman's theorem (see, e.g. [5, Theorem 2.54] or [23, Chap. 1, Theorem 5.5]) shows that

$$K_{\infty,0}^{1/\infty,0} = C + H^\infty, \quad K_{0,\infty}^{0,1/\infty} = C + \overline{H^\infty}, \quad K_{\infty,\infty}^{1/\infty,1/\infty} = QC,$$

and since $\mathcal{C}_p(H^2) \subset \mathcal{C}_\infty(H^2)$, we have

$$(1.5)\quad K_{p,0}^{1/p,0} \subset C + H^\infty, \quad K_{0,q}^{0,1/q} \subset C + \overline{H^\infty}, \quad K_{p,q}^{1/p,1/q} \subset QC.$$

For a unital Banach algebra $\mathcal{A}$, we denote by $G\mathcal{A}$ its group of invertible elements.

Theorem 1.1. *Let $1 \le p, q \le \infty$.*

(a) *The sets* (1.3) *are Banach algebras under the Banach algebra norms* (1.4).

(b) *If $a \in K_{p,0}^{1/p,0}$, then $a \in GK_{p,0}^{1/p,0} \iff a \in G(C + H^\infty)$.*

(c) *If $a \in K_{0,q}^{0,1/q}$, then $a \in GK_{0,q}^{0,1/q} \iff a \in G(C + \overline{H^\infty})$.*

(d) *If $a \in K_{p,q}^{1/p,1/q}$, then*

$$a \in GK_{p,q}^{1/p,1/q} \iff a \in G(C + H^\infty) \iff a \in G(C + \overline{H^\infty}) \iff a \in GL^\infty.$$

This theorem was established by Krein [19] for the case $p = q = 2$. We therefore call (1.3) *Krein algebras*. Theorem 1.1 was proved in this form for the first time in [3, Sections 4.10–4.11], a complete proof is also given in [5, Theorem 10.9].

1.2. **Hankel operators in Schatten-von Neumann classes.** Of course, it is desirable to have equivalent definitions for $K_{p,q}^{1/p,1/q}$, $K_{p,0}^{1/p,0}$, and $K_{0,q}^{0,1/q}$ in terms of functions and not operators. As we already noted, for $K_{2,2}^{1/2,1/2}$ this is very easy. So, we need effective criteria guaranteeing that $H(a) \in \mathcal{C}_p(H^2)$ and $H(\widetilde{a}) \in \mathcal{C}_q(H^2)$ if $p \neq 2$ and $q \neq 2$.

Let X be one of the spaces L^p, $1 \le p < \infty$ or C. The moduli of continuity of $f \in X$ are defined for $s \ge 0$ by

$$\omega_X^1(f,s) := \sup_{|h|\le s} \|f(e^{i(\cdot+h)}) - f(e^{i\cdot})\|_X,$$

$$\omega_X^2(f,s) := \sup_{|h|\le s} \|f(e^{i(\cdot+h)}) - 2f(e^{i\cdot}) + f(e^{i(\cdot-h)})\|_X.$$

For $1 \le p < \infty$ and $0 < \alpha \le 1$, the *Besov space* B_p^α is defined as the set of all functions $f \in L^p$ such that

$$|f|_{B_p^\alpha} := \begin{cases} \left(\int_0^{2\pi} \left[s^{-\alpha}\omega_{L^p}^1(f,s)\right]^p \frac{ds}{s}\right)^{1/p} & (0<\alpha<1), \\ \left(\int_0^{2\pi} \left[s^{-1}\omega_{L^p}^2(f,s)\right]^p \frac{ds}{s}\right)^{1/p} & (\alpha = 1) \end{cases}$$

is finite. The Besov space is a Banach space under the norm

$$\|f\|_{B_p^\alpha} := \|f\|_{L^p} + |f|_{B_p^\alpha}.$$

These spaces are studied in detail (in a more general setting) in [25] and in many other monographs. The Riesz projections P and Q are bounded on the Besov spaces B_p^α for $1 \le p < \infty$ and $0 < \alpha \le 1$ (see [23, Appendix 2.6]).

Peller proved in the late 1970s that for $1 \le p, q < \infty$ and $a \in L^\infty$,

$$Pa \in B_q^{1/q} \iff H(a) \in \mathcal{C}_q(H^2), \qquad Qa \in B_p^{1/p} \iff H(\widetilde{a}) \in \mathcal{C}_p(H^2) \tag{1.6}$$

(see [23, Chap. 6, Theorems 1.1 and 2.1]). From those proofs one can see that there exist positive constants c_1 and c_2 depending only on p and q such that

$$c_1\|Pa\|_{B_q^{1/q}} \le \|H(a)\|_{\mathcal{C}_q(H^2)} \le c_2\|Pa\|_{B_q^{1/q}},$$

$$c_1\|Qa\|_{B_p^{1/p}} \le \|H(\widetilde{a})\|_{\mathcal{C}_p(H^2)} \le c_2\|Qa\|_{B_p^{1/p}}.$$

From this result and Theorem 1.1(a) we get the following.

Corollary 1.2. *If* $1 \le p, q < \infty$, *then*

$$K_{p,0}^{1/p,0} = \left\{a \in L^\infty : Qa \in B_p^{1/p}\right\} = L^\infty \cap (B_p^{1/p} + H^\infty),$$

$$K_{0,q}^{0,1/q} = \left\{a \in L^\infty : Pa \in B_q^{1/q}\right\} = L^\infty \cap (B_q^{1/q} + \overline{H^\infty}),$$

$$K_{p,q}^{1/p,1/q} = \left\{a \in L^\infty : Qa \in B_p^{1/p}, Pa \in B_q^{1/q}\right\} = L^\infty \cap (B_p^{1/p} + H^\infty) \cap (B_q^{1/q} + \overline{H^\infty}).$$

The norms

$$\|a\|_{L^\infty} + \|Qa\|_{B_p^{1/p}}, \quad \|a\|_{L^\infty} + \|Pa\|_{B_q^{1/q}}, \quad \|a\|_{L^\infty} + \|Qa\|_{B_p^{1/p}} + \|Pa\|_{B_q^{1/q}}$$

are equivalent norms in $K_{p,0}^{1/p,0}$, $K_{0,q}^{0,1/q}$, *and* $K_{p,q}^{1/p,1/q}$, *respectively.*

1.3. **Generalized Krein algebras.** We are going to extend the notion of Krein algebras $K_{p,0}^{1/p,0}$, $K_{0,q}^{0,1/q}$, and $K_{p,q}^{1/p,1/q}$. Now we take a generalization of the results of Corollary 1.2 as a definition.

Assume that $1 < p, q < \infty$ and $0 < \alpha, \beta < 1$. Define

$$K_{p,0}^{\alpha,0} := \left\{a \in L^\infty : Qa \in B_p^\alpha\right\} = L^\infty \cap (B_p^\alpha + H^\infty),$$

$$K_{0,q}^{0,\beta} := \left\{a \in L^\infty : Pa \in B_q^\beta\right\} = L^\infty \cap (B_q^\beta + \overline{H^\infty}),$$

$$K_{p,q}^{\alpha,\beta} := \left\{a \in L^\infty : Qa \in B_p^\alpha, Pa \in B_q^\beta\right\} = L^\infty \cap (B_p^\alpha + H^\infty) \cap (B_q^\beta + \overline{H^\infty}).$$

Theorem 1.3 (Main result 1)**.** *Let* $1 < p, q < \infty$ *and* $0 < \alpha, \beta < 1$.

(a) *If* $\alpha \ge 1/p$, *then* $K_{p,0}^{\alpha,0}$ *is a Banach algebra under the quasi-submultiplicative norm*

$$\|a\|_{K_{p,0}^{\alpha,0}} := \|a\|_{L^\infty} + \|Qa\|_{B_p^\alpha}. \tag{1.7}$$

(b) *If $\beta \ge 1/q$, then $K^{0,\beta}_{0,q}$ is a Banach algebra under the quasi-submultiplicative norm*

$$\|a\|_{K^{0,\beta}_{0,q}} := \|a\|_{L^\infty} + \|Pa\|_{B^\beta_q}. \tag{1.8}$$

(c) *If $\alpha > 1/p$, or $\beta > 1/q$, or $\alpha = 1/p$ and $\beta = 1/q$, then $K^{\alpha,\beta}_{p,q}$ is a Banach algebra under the quasi-submultiplicative norm*

$$\|a\|_{K^{\alpha,\beta}_{p,q}} := \|a\|_{L^\infty} + \|Qa\|_{B^\alpha_p} + \|Pa\|_{B^\beta_q}. \tag{1.9}$$

Theorem 1.4 (Main result 2)**.** *Let $1 < p, q < \infty$, $0 < \alpha, \beta < 1$, $1/p+1/q = \alpha+\beta \in (0, 1]$.*

(a) *Suppose $\alpha \ge 1/p$ and K is either $K^{\alpha,0}_{p,0}$ or $K^{\alpha,\beta}_{p,q}$. If $a \in K$, then*

$$a \in GK \iff a \in G(C + H^\infty).$$

(b) *Suppose $\beta \ge 1/q$ and K is either $K^{0,\beta}_{0,q}$ or $K^{\alpha,\beta}_{p,q}$. If $a \in K$, then*

$$a \in GK \iff a \in G(C + \overline{H^\infty}).$$

These statements were proved in [3, Chap. 4] for the particular case of $K^{\alpha,\beta}_{p,q}$ in which the parameters satisfy $1/p + 1/q = \alpha + \beta = 1$. Those proofs are based on Krein's ideas [19]. Since the book [3] is no longer available to a wide audience, we decided to present self-contained proofs of these results.

1.4. **Szegő-Widom type limit theorem.** Let N be a natural number. For a Banach space X, let X_N and $X_{N\times N}$ be the spaces of vectors and matrices with entries in X. The operators I, J, P, and Q are defined on vector spaces elementwise; the Hankel operators on H^2_N are defined for $a \in L^\infty_{N\times N}$ in the same way as in (1.2) and the Toeplitz operators are defined by

$$T(a) : H^2_N \to H^2_N, \quad f \mapsto PM(a)Pf, \quad T(\widetilde{a}) : H^2_N \to H^2_N, \quad f \mapsto JQM(a)QJf.$$

The matrix of the Toeplitz operator $T(a)$ in the standard basis of the space H^2_N is the infinite Toeplitz matrix

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

where $\{a_k\}_{k\in\mathbb{Z}}$ is the sequence of $N \times N$ matrices which are the Fourier coefficients of the generating function (symbol) $a \in L^\infty_{N\times N}$. Let $T_n(a) = (a_{j-k})^n_{j,k=0}$. This matrix of order $(n + 1)N$ is called a block Toeplitz matrix. We denote its determinant by $D_n(a)$.

In 1915, Szegő proved that $D_n(a)/D_{n-1}(a)$ tends to the geometric mean $G(a)$ of a if a is a scalar nonnegative function such that $a \in L^1$ and $\log a \in L^1$. This result, now called the first Szegő limit theorem, has been subsequently extended into different directions. We will not go into details, but notice that Krein [19] observed that his algebra $K^{1/2,1/2}_{2,2}$ can be useful in asymptotic analysis of Toeplitz determinants. The most general results for the case where positivity is replaced by some kind of sectoriality are those of Krein and Spitkovsky [20].

In 1952, Szegő proved his second (strong) limit theorem, which says that if a is a positive scalar function with Hölder continuous derivative, then $D_n(a) \sim G(a)^{n+1}E(a)$, where $E(a)$ is some completely determined nonzero constant. It was Widom [29] who proved the strong Szegő theorem in the block case $N > 1$ for the first time under the assumption that $a \in (K^{1/2,1/2}_{2,2})_{N\times N}$.

We suppose that the reader is familiar with basic facts on Fredholm operators and with properties of Schatten-von Neumann classes and (regularized) operator determinants (otherwise consult Section 2).

Functions in $K_{p,q}^{\alpha,\beta}$ may be discontinuous, hence some care is needed in the definition of the "geometric mean" $G(a)$. For $a \in L_{N\times N}^\infty$, we denote by $h_r a$ the harmonic extension,

$$h_r a(e^{i\theta}) := \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \quad (0 \le r < 1, \quad 0 \le \theta < 2\pi).$$

Lemma 1.5. *Let $a \in (C + H^\infty)_{N\times N}$ or $a \in (C + \overline{H^\infty})_{N\times N}$. If $T(a)$ is Fredholm of index zero on H_N^2, then the limit*

$$G(a) := \lim_{r\to 1-0} \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\det h_r a(e^{i\theta})d\theta\right) \tag{1.10}$$

exists, is finite and nonzero.

The proof of this lemma is given in [5, Proposition 10.6(a)] for $(C + H^\infty)_{N\times N}$ and it works equally also for $(C + \overline{H^\infty})_{N\times N}$.

Theorem 1.6. *Let K be one of the algebras $K_{1,0}^{1,0}$, $K_{0,1}^{0,1}$, or $K_{p,q}^{\alpha,\beta}$ with p, q, α, β satisfying*

$$1 < p, q < \infty, \quad 0 < \alpha, \beta < 1, \quad 1/p + 1/q = \alpha + \beta = 1, \quad -1/2 < \alpha - 1/p < 1/2.$$

If $a \in K_{N\times N}$ and $T(a)$ is Fredholm of index zero on H_N^2, then $a^{-1} \in K_{N\times N}$, the operator

$$H(a)H(\widetilde{a}^{-1}) = I - T(a)T(a^{-1})$$

is of trace class on H_N^2, and

$$\lim_{n\to\infty} \frac{D_n(a)}{G(a)^{n+1}} = \det T(a)T(a^{-1}),$$

where the last det *refers to the determinant defined for operators differing from the identity by an operator of trace class.*

For $1/p = 1/q = \alpha = \beta = 1/2$ this result goes back to Widom [29]. Two of the authors proved it in this form in [3, Theorem 6.14]. The proof is also reproduced in [5, Theorem 10.32].

Alternative proofs of the strong Szegő and Szegő-Widom limit theorems were found later. We refer to [1, 3, 4, 5, 6, 9, 15, 27, 29] and the references given there for more complete information.

1.5. **Higher order asymptotic formulas for block Toeplitz determinants.** For $a \in L_{N\times N}^\infty$ and $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$ define the operators P_n and Q_n on H_N^2 by

$$P_n : \sum_{k=0}^{\infty} a_k \chi_k \mapsto \sum_{k=0}^{n} a_k \chi_k, \quad Q_n := I - P_n.$$

The operator $P_n T(a) P_n : P_n H_N^2 \to P_n H_N^2$ may be identified with the finite block Toeplitz matrix $T_n(a) := (a_{j-k})_{j,k=0}^n$. Let W be the Wiener algebra of functions $a : \mathbb{T} \to \mathbb{C}$ with absolutely convergent Fourier series. For generalized Krein algebras, define the "conjugation number" by

$$\lambda := \begin{cases} 1/p & \text{for} \quad K_{p,0}^{\alpha,0} \quad \text{with} \quad \alpha \ge 1/p, \\ 1/q & \text{for} \quad K_{0,q}^{0,\beta} \quad \text{with} \quad \beta \ge 1/q, \\ 1/p + 1/q & \text{for} \quad K_{p,q}^{\alpha,\beta} \quad \text{with} \quad 1/p + 1/q = \alpha + \beta. \end{cases} \tag{1.11}$$

Theorem 1.7 (Main result 3)**.** *Let $1 < p, q < \infty$ and $0 < \alpha, \beta < 1$. Suppose K is one of the algebras*

$$W \cap K_{p,0}^{1/p,0}, \quad K_{p,0}^{\alpha,0} \text{ with } \alpha > 1/p, \quad W \cap K_{0,q}^{0,1/q}, \quad K_{0,q}^{0,\beta} \text{ with } \beta > 1/q, \quad W \cap K_{p,q}^{1/p,1/q},$$

or $K_{p,q}^{\alpha,\beta}$ with $\alpha \ne 1/p$, $1/p + 1/q = \alpha + \beta \in (0, 1)$, $-1/2 < \alpha - 1/p < 1/2$. If $a \in K_{N\times N}$ and both $T(a)$ and $T(\widetilde{a})$ are invertible on H_N^2, then the following statements hold.

(a) *The matrix function a is invertible in $K_{N\times N}$ and admits canonical right and left Wiener-Hopf factorizations in $K_{N\times N}$, that is, there exist*

$$u_-, v_- \in G(K\cap\overline{H^\infty})_{N\times N}, \quad u_+, v_+ \in G(K\cap H^\infty)_{N\times N}$$

such that $a = u_-u_+ = v_+v_-$.

(b) *Let λ be defined by* (1.11) *and m be the smallest integer such that $1 \le \lambda m$. If*

$$b := v_-u_+^{-1}, \quad c := u_-^{-1}v_+, \tag{1.12}$$

where $u_\pm, v_\pm$ are defined in part (a)*, then the operators $H(\widetilde{c})H(b)$ and $H(b)H(\widetilde{c})$ belong to the Schatten-von Neumann class $\mathcal{C}_m(H^2_N)$.*

(c) *Let $b, c,$ and m be defined as in part* (b)*. Then*

$$\lim_{n\to\infty}\frac{D_n(a)}{G(a)^{n+1}}\exp\left\{-\sum_{j=1}^{m-1}\frac{1}{j}\,\mathrm{trace}\left[\left(\sum_{k=0}^{m-1}F_{n,k}\right)^j\right]\right\} = \frac{1}{\det_m T(\widetilde{c})T(\widetilde{b})}, \tag{1.13}$$

where

$$F_{n,k} := P_nT(c)Q_n\big(Q_nH(b)H(\widetilde{c})Q_n\big)^kQ_nT(b)P_n \quad (k\in\mathbb{Z}_+) \tag{1.14}$$

and $\det_m$ denotes the m-regularized operator determinant defined for operators differing from the identity by an operator in $\mathcal{C}_m(H^2_N)$ (see Section 2.4*).*

Under the assumptions of Theorem 1.7 we cannot guarantee that $T(a)T(a^{-1}) - I$ is of trace class, which implies that the arguments of the proof of Theorem 1.6 do not work in this case. However, we can guarantee that $H(\widetilde{c})H(b)$ and $H(b)H(\widetilde{c})$ belong to $\mathcal{C}_m(H^2_N)$ where $m > 1$. In this case the asymptotic formulas for block Toeplitz determinants are subject to higher order corrections involving additional terms and regularized operator determinants.

Notice also that similar results were obtained in [2, Section 4], [3, Theorem 6.20], [5, Theorem 10.37], [17, Theorem 20] for weighted Wiener algebras and Hölder spaces C^γ. For Wiener algebras with power weights and for C^γ, the formula in (c) is a little bit simpler, because it does not contain $F_{n,m-1}$. In contrast to the case of weighted Wiener algebras and C^γ, which consist of continuous functions only, the generalized Krein algebras $K^{\alpha,\beta}_{p,q}$ may contain discontinuous functions.

1.6. **About this paper.** Section 2 contains operator-theoretic preliminaries. In Section 3, we slightly extend Krein's results [19] on Banach algebras generated by ideals. Section 4 contains auxiliary results from the theory of Besov spaces, algebras of multiplication operators on weighted ℓ_2-spaces, as well as Peller's results on the boundedness, compactness, and Schatten-von Neumann behavior of Hankel-type operators. Let c^γ be the closure of the set of all Laurent polynomials in the norm of the Hölder space C^γ, $0 < \gamma < 1$. In Section 5, we show that $C^\gamma + H^\infty$ and $c^\gamma + H^\infty$ are Banach algebras and that $c^\gamma + H^\infty$ is inverse closed in $C + H^\infty$. Section 6 contains the proofs of Theorems 1.3 and 1.4. The proof of Theorem 1.4 uses essentially the inverse closedness of $c^\gamma + H^\infty$ in $C + H^\infty$. In Section 7, we prove Theorem 1.7 by first applying the factorization theory developed in [16] to generalized Krein algebra and then using an abstract higher order asymptotic formula for Toeplitz determinants [17, Theorem 15], which is contained implicitly in [3, Theorem 6.20] and [5, Theorem 10.37].

2. Operator-theoretic preliminaries

2.1. **Commutative Banach algebras.** Let $\mathcal{A}$ be an algebra. A subalgebra $\mathcal{J}$ of $\mathcal{A}$ is called an algebraic *two-sided ideal* of $\mathcal{A}$ if $aj \in \mathcal{J}$ and $ja \in \mathcal{J}$ for all $a \in \mathcal{A}$ and $j \in \mathcal{J}$. Given two Banach algebras $\mathcal{A}$ and $\mathcal{B}$, a map $\varphi : \mathcal{A} \to \mathcal{B}$ is called a Banach algebra homomorphism if φ is a bounded linear operator and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{A}$. Now let $\mathcal{A}$ be a commutative Banach algebra with identity element e. The

Banach algebra homomorphisms of $\mathcal{A}$ into $\mathbb{C}$ which send e to 1 are called multiplicative linear functionals of $\mathcal{A}$. A proper closed two-sided ideal $\mathcal{J}$ of $\mathcal{A}$ is called a *maximal ideal* if it is not properly contained in any other proper closed two-sided ideal of $\mathcal{A}$. Let $\mathcal{M}_\mathcal{A}$ denote the set of all maximal ideals of $\mathcal{A}$ and let $M_\mathcal{A}$ stand for the set of all multiplicative linear functionals of $\mathcal{A}$. One can show that the map $M_\mathcal{A} \to \mathcal{M}_\mathcal{A}$, $\varphi \to \operatorname{Ker}\varphi$ is bijective. Therefore no distinction is usually made between multiplicative linear functionals and maximal ideals.

The formula $\widehat{a}(m) = m(a)$ $(m \in M_\mathcal{A})$ assigns a function $\widehat{a} : M_\mathcal{A} \to \mathbb{C}$ to each $a \in \mathcal{A}$. Let $\widehat{A}$ be the set $\{\widehat{a} : a \in \mathcal{A}\}$. The Gelfand topology on $M_\mathcal{A}$ is the coarsest topology on $M_\mathcal{A}$ which makes all functions $\widehat{a} \in \widehat{A}$ continuous. The set $M_\mathcal{A}$ equipped with the Gelfand topology is called the maximal ideal space of $\mathcal{A}$.

Theorem 2.1 (Gelfand)**.** *Let $\mathcal{A}$ be a commutative Banach algebra with identity element and let $M_\mathcal{A}$ be the maximal ideal space of $\mathcal{A}$. An element $a \in \mathcal{A}$ is invertible if and only if $\widehat{a}(m) \neq 0$ for all $m \in M_\mathcal{A}$.*

A proof of this result is in every textbook on Banach algebras.

2.2. **Fredholm operators.** The facts collected in this subsection can be found in most textbooks on operator theory (for instance, in [8, 10, 11]). Let X be a Banach space, $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on X, $\mathcal{C}_0(X)$ be the set of all finite-rank operators, and $\mathcal{C}_\infty(X)$ be the closed two-sided ideal of all compact operators on X. For $A \in \mathcal{B}(X)$, put $\operatorname{Ker} A = \{x \in X : Ax = 0\}$ and $\operatorname{Im} A = AX$. The operator A is said to be *Fredholm* (on X) if $\operatorname{Im} A$ is closed in X and both $\dim \operatorname{Ker} A$ and $\dim(X/\operatorname{Im} A)$ are finite. The integer

$$\operatorname{Ind} A := \dim \operatorname{Ker} A - \dim(X/\operatorname{Im} A)$$

is then referred to as the *index* of A.

If A and B are Fredholm, then AB is also Fredholm and $\operatorname{Ind}(AB) = \operatorname{Ind} A + \operatorname{Ind} B$. If A is Fredholm and $K \in \mathcal{C}_\infty(X)$, then $A + K$ is Fredholm and $\operatorname{Ind}(A + K) = \operatorname{Ind} A$. An operator $A \in \mathcal{B}(X)$ is Fredholm if and only if there exists an operator $R \in \mathcal{B}(X)$ such that $AR - I \in \mathcal{C}_\infty(X)$ and $RA - I \in \mathcal{C}_\infty(X)$. Such an operator R is called a *regularizer* of the operator A.

2.3. **Schatten-von Neumann ideals.** All facts stated in the rest of this section are proved in [14, Chap. 3–4]. Let H be a separable Hilbert space. Given an operator $A \in \mathcal{B}(H)$ define for $n \in \mathbb{Z}_+$,

$$s_n(A) := \inf\{\|A - F\|_{\mathcal{B}(H)} \ : \ F \in \mathcal{C}_0(H),\ \dim F(H) \le n\}.$$

For $1 \le p < \infty$, the collection of all operators $K \in \mathcal{B}(H)$ satisfying

$$\|K\|_{\mathcal{C}_p(H)} := \Big(\sum_{n \in \mathbb{Z}_+} s_n^p(K)\Big)^{1/p} < \infty \tag{2.1}$$

is denoted by $\mathcal{C}_p(H)$ and referred to as a *Schatten-von Neumann class*. This is a Banach space under the norm (2.1). Note that $\mathcal{C}_\infty(H) = \{K \in \mathcal{B}(H) : s_n(K) \to 0 \text{ as } n \to \infty\}$ and

$$\|K\|_{\mathcal{C}_\infty(H)} = \sup_{n \in \mathbb{Z}_+} s_n(K) = \|K\|_{\mathcal{B}(H)}.$$

The operators belonging to $\mathcal{C}_1(H)$ are called *trace class operators*.

Lemma 2.2. *Let $1 \le p, q, r \le \infty$.*

(a) *If $p < q$ and $K \in \mathcal{C}_p(H)$, then $K \in \mathcal{C}_q(H)$ and $\|K\|_{\mathcal{C}_q(H)} \le \|K\|_{\mathcal{C}_p(H)}$.*

(b) *If $A \in \mathcal{B}(H)$ and $K \in \mathcal{C}_p(H)$, then $AB, BA \in \mathcal{C}_p(H)$ and*

$$\max\{\|AK\|_{\mathcal{C}_p(H)}, \|KA\|_{\mathcal{C}_p(H)}\} \le \|K\|_{\mathcal{C}_p(H)} \|A\|_{\mathcal{B}(H)}.$$

(c) *If* $1/r = 1/p + 1/q$ *and* $K \in \mathcal{C}_p(H)$, $L \in \mathcal{C}_q(H)$, *then* $KL \in \mathcal{C}_r(H)$ *and*

$$\|KL\|_{\mathcal{C}_r(H)} \le \|K\|_{\mathcal{C}_p(H)} \|L\|_{\mathcal{C}_q(H)}.$$

Lemma 2.2(b) implies that $\mathcal{C}_p(H)$ is a two-sided ideal of $\mathcal{B}(H)$. This ideal is not closed in $\mathcal{B}(H)$ if $p < \infty$ and $\dim H = \infty$.

2.4. **Regularized operator determinants.** Let $A \in \mathcal{B}(H)$ be an operator of the form $I+K$ with $K \in \mathcal{C}_1(H)$. If $\{\lambda_j(K)\}_{j\ge 0}$ denotes the sequence of the nonzero eigenvalues of K counted up to algebraic multiplicity, then the product $\prod_{j\ge 0}(1+\lambda_j(K))$ is absolutely convergent. The *determinant* of A is defined by

$$\det A = \det(I+K) = \prod_{j\ge 0}(1+\lambda_j(K)).$$

If $K \in \mathcal{C}_m(H)$, where $m > 1$ is an integer, one can still define a determinant of $I+K$, but for ideals larger than $\mathcal{C}_1(H)$, the above definition requires a regularization. A simple computation (see [26, Lemma 6.1]) shows that then

$$R_m(K) := (I+K)\exp\left(\sum_{j=1}^{m-1} \frac{(-K)^j}{j}\right) - I \in \mathcal{C}_1(H).$$

Thus, it is natural to define $\det_1(I+K) := \det(I+K)$ and $\det_m(I+K) := \det(I+R_m(K))$ for $m > 1$. One calls $\det_m(I+K)$ the *m-regularized determinant* of $A = I+K$.

Regularized operator determinants have some useful properties. For instance, the operator $I+K$ is invertible on H if and only if $\det_m(I+K) \ne 0$.

3. Banach algebras generated by ideals

3.1. **Subalgebras generated by ideals.** The results of this section are essentially due to Krein [19]. They appeared in this form in [3, Chap. 4]. Since both sources may not be available to a wide audience, it seems reasonable to give here complete proofs.

Lemma 3.1. *Let $\mathcal{A}$ be an algebra and $e \in \mathcal{A}$ be the identity element. Suppose $\mathcal{J}_1 \subset \mathcal{A}$ and $\mathcal{J}_2 \subset \mathcal{A}$ are two-sided ideals in $\mathcal{A}$ and $\mathcal{L}$ is a subalgebra of $\mathcal{A}$. Let*

$$p \in \mathcal{A}, \quad p^2 = p, \quad q := e - p. \tag{3.1}$$

Then the sets

$$\mathcal{L}_1 := \{a \in \mathcal{L} \ : \ paq \in \mathcal{J}_1\}, \quad \mathcal{L}_2 := \{a \in \mathcal{L} \ : \ qap \in \mathcal{J}_2\}, \quad \mathcal{L}_* := \mathcal{L}_1 \cap \mathcal{L}_2 \tag{3.2}$$

are subalgebras of $\mathcal{L}$.

Proof. Because $peq = p(e-p) = p - p^2 = 0$ and $qep = (e-p)p = p - p^2 = 0$, we have $e \in \mathcal{L}_1$, $e \in \mathcal{L}_2$, and thus $e \in \mathcal{L}_1 \cap \mathcal{L}_2$. If $a, b \in \mathcal{L}_1$, then $paq \in \mathcal{J}_1$ and $pbq \in \mathcal{J}_1$. We have

$$pabq = pa(p+q)bq = pap \cdot pbq + paq \cdot qbp.$$

Since $\mathcal{J}_1$ is a two-sided ideal of $\mathcal{L}$ and $pap \in \mathcal{L}$, $qbp \in \mathcal{L}$, we conclude that $pabq \in \mathcal{J}_1$, that is, $ab \in \mathcal{L}_1$. Thus, $\mathcal{L}_1$ is an algebra with identity $e \in \mathcal{L}$. Analogously one can prove that $\mathcal{L}_2$ is a subalgebra of $\mathcal{L}$. The statement for $\mathcal{L}_* = \mathcal{L}_1 \cap \mathcal{L}_2$ is now trivial. □

3.2. **Banach algebras generated by complete normed ideals.** Throughout this section we assume that $\mathcal{A}$ is a Banach algebra with identity e and a Banach algebra norm $\|\cdot\|$.

Theorem 3.2. *Let $\mathcal{J}_1 \subset \mathcal{A}$, $\mathcal{J}_2 \subset \mathcal{A}$ be two-sided ideals of $\mathcal{A}$, which become Banach spaces under norms $\|\cdot\|_{\mathcal{J}_i}$, $i = 1, 2$. Assume that for every $i = 1, 2$ and every $x \in \mathcal{J}_i$,*

$$\|x\| \le \|x\|_{\mathcal{J}_i}. \tag{3.3}$$

If $\mathcal{L}$ is a closed subalgebra of $\mathcal{A}$, then $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_*$ defined by (3.1)–(3.2) *are Banach spaces under the norms*

$$\|a\|_1 := \left(\|pap\|^2 + \|qaq\|^2 + \|paq\|^2_{\mathcal{J}_1} + \|qap\|^2\right)^{1/2},$$
$$\|a\|_2 := \left(\|pap\|^2 + \|qaq\|^2 + \|paq\|^2 + \|qap\|^2_{\mathcal{J}_2}\right)^{1/2},$$
$$\|a\|_* := \left(\|pap\|^2 + \|qaq\|^2 + \|paq\|^2_{\mathcal{J}_1} + \|qap\|^2_{\mathcal{J}_2}\right)^{1/2},$$

respectively.

Proof. It is easy to see that $\|\cdot\|_i$ and $\|\cdot\|_*$ are norms. The triangle inequality for $\|\cdot\|_i$ as well as for $\|\cdot\|_*$ follows from the triangle inequality for $\|\cdot\|$ and $\|\cdot\|_{\mathcal{J}_i}$ and from the Minkowski inequality for sums.

If $a \in \mathcal{L}_1$, then $paq \in \mathcal{J}_1$. In view of (3.3), we have

$$\begin{aligned}\|a\| &\le \|pap\| + \|qaq\| + \|paq\| + \|qap\| \\ &\le \|pap\| + \|qaq\| + \|paq\|_{\mathcal{J}_1} + \|qap\| \\ &\le 4\left(\|pap\|^2 + \|qaq\|^2 + \|paq\|^2_{\mathcal{J}_1} + \|qap\|^2\right)^{1/2} \\ &= 4\|a\|_1.\end{aligned} \tag{3.4}$$

It is obvious that

$$\|paq\|_{\mathcal{J}_1} \le \|a\|_1. \tag{3.5}$$

Let $\{a_n\} \subset \mathcal{J}_1$ be a Cauchy sequence in $\mathcal{L}_1$, that is, $\|a_n - a_m\|_1 \to 0$ as $n, m \to \infty$. From (3.4) it follows that $\{a_n\}$ is a Cauchy sequence in $\mathcal{L}$. Since $\mathcal{L}$ is closed, there exists an $a \in \mathcal{L}$ such that $\|a - a_n\| \to 0$ as $n \to \infty$. This implies that as $n \to \infty$,

$$\|pap - pa_np\| \to 0, \tag{3.6}$$
$$\|paq - pa_nq\| \to 0, \tag{3.7}$$
$$\|qap - qa_np\| \to 0, \tag{3.8}$$
$$\|qaq - qa_nq\| \to 0. \tag{3.9}$$

On the other hand, from (3.5) we see that $\|pa_nq - pa_mq\|_{\mathcal{J}_1} \le \|a_n - a_m\|_1 \to 0$ as $n, m \to \infty$, that is, $\{pa_nq\}$ is a Cauchy sequence in $\mathcal{J}_1$. By the hypothesis, $(\mathcal{J}_1, \|\cdot\|_{\mathcal{J}_1})$ is a Banach space. Thus there is an element $c \in \mathcal{J}_1$ such that $\|c - pa_nq\|_{\mathcal{J}_1} \to 0$ as $n \to \infty$. In view of (3.3), this gives

$$\|c - pa_nq\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.10}$$

From (3.7) and (3.10) it follows that $c = paq$. Hence $a \in \mathcal{L}_1$ and

$$\|paq - pa_nq\|_{\mathcal{J}_1} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.11}$$

Combining (3.6), (3.8), (3.9), and (3.11), we get $\|a - a_n\|_1 \to 0$ as $n \to \infty$. Thus $a \in \mathcal{L}_1$ and $\mathcal{L}_1$ is closed.

It is clear that the same argument works also for $\mathcal{L}_2$ and $\mathcal{L}_*$. Thus $\mathcal{L}_2$ and $\mathcal{L}_*$ are closed, too. □

Theorem 3.3. *Under the assumptions of Theorem* 3.2, *the norms*

$$\|a\|'_1 := \|a\| + \|paq\|_{\mathcal{J}_1} \quad (a \in \mathcal{L}_1),$$
$$\|a\|'_2 := \|a\| + \|qap\|_{\mathcal{J}_2} \quad (a \in \mathcal{L}_2),$$
$$\|a\|'_* := \|a\| + \|paq\|_{\mathcal{J}_1} + \|qap\|_{\mathcal{J}_2} \quad (a \in \mathcal{L}_*)$$

are equivalent to the norms $\|a\|_1$, $\|a\|_2$, and $\|a\|_*$, respectively.

If, in addition,

$$\|p\| = \|q\| = 1$$

and for every $i = 1, 2$, for every $x \in \mathcal{J}_i$, and every $a \in \mathcal{A}$,

$$\max\{\|ax\|_{\mathcal{J}_i}, \|xa\|_{\mathcal{J}_i}\} \le \|x\|_{\mathcal{J}_i}\|a\|, \tag{3.12}$$

then $\|\cdot\|_1'$, $\|\cdot\|_2'$, *and* $\|\cdot\|_*'$ *are Banach algebra norms.*

Proof. Let us prove the statement for $\|\cdot\|_*'$. By analogy with (3.4), we get

$$\|a\| \le 4\|a\|_*. \tag{3.13}$$

It is obvious that

$$\|paq\|_{\mathcal{J}_1} \le \|a\|_*, \quad \|qap\|_{\mathcal{J}_2} \le \|a\|_*. \tag{3.14}$$

From (3.13) and (3.14) we see that $\|a\|_*' \le 6\|a\|_*$.

On the other hand,

$$\begin{aligned}\|a\|_* &\le \|pap\| + \|qaq\| + \|paq\|_{\mathcal{J}_1} + \|qap\|_{\mathcal{J}_2} \\ &\le (\|p\|^2 + \|q\|^2 + 1)(\|a\| + \|paq\|_{\mathcal{J}_1} + \|qap\|_{\mathcal{J}_2}) \\ &= (\|p\|^2 + \|q\|^2 + 1)\|a\|_*',\end{aligned}$$

that is, the norms $\|\cdot\|_*$ and $\|\cdot\|_*'$ are equivalent.

It is clear that $\|e\|_*' = \|e\| + \|pq\|_{\mathcal{J}_1} + \|qp\|_{\mathcal{J}_2} = \|e\| = 1$. If $\|p\| = \|q\| = 1$, then from (3.12) it follows that

$$\begin{aligned}\|pabq\|_{\mathcal{J}_1} &= \|pa(p+q)bq\|_{\mathcal{J}_1} \\ &\le \|pap\|\,\|pbq\|_{\mathcal{J}_1} + \|paq\|_{\mathcal{J}_1}\|qbq\| \\ &\le \|a\|\,\|pbq\|_{\mathcal{J}_1} + \|paq\|_{\mathcal{J}_1}\|b\|\end{aligned} \tag{3.15}$$

and

$$\|qabp\|_{\mathcal{J}_2} \le \|qap\|_{\mathcal{J}_2}\|b\| + \|a\|\,\|qbp\|_{\mathcal{J}_2}. \tag{3.16}$$

Combining $\|ab\| \le \|a\|\,\|b\|$ and (3.15)–(3.16), we get

$$\begin{aligned}\|ab\|_*' &\le \|a\|\,\|b\| + \|a\|\,\|pbq\|_{\mathcal{J}_1} + \|paq\|_{\mathcal{J}_1}\|b\| + \|qap\|_{\mathcal{J}_2}\|b\| + \|a\|\,\|qbp\|_{\mathcal{J}_2} \\ &= \|a\|\,\|b\|_*' + \|b\|(\|paq\|_{\mathcal{J}_1} + \|qap\|_{\mathcal{J}_2}) \\ &\le \|a\|\,\|b\|_*' + \|b\|_*'(\|paq\|_{\mathcal{J}_1} + \|qap\|_{\mathcal{J}_2}) \\ &= \|a\|_*'\|b\|_*'\end{aligned}$$

for all $a, b \in \mathcal{L}_*$. The proof for $\mathcal{L}_1$ and $\mathcal{L}_2$ is similar. □

Remark 3.4. One can show that under the assumptions of Theorem 3.2 and (3.12) one has $\|ab\|_i \le \|a\|_i\|b\|_i$ for all $a, b \in \mathcal{L}_i$ and $\|ab\|_* \le \|a\|_*\|b\|_*$ for all $a, b \in \mathcal{L}_*$. However, $\|e\|_i = \sqrt{2}$ and $\|e\|_* = \sqrt{2}$ if $\|p\| = \|q\| = 1$.

3.3. **Operator algebras generated by ideals of compact operators.** Let X be a Banach space and $\mathcal{P}, \mathcal{Q} \in \mathcal{B}(X)$ be two complementary projections, that is, $\mathcal{P}^2 = \mathcal{P}$ and $\mathcal{Q} = I - \mathcal{P}$.

Theorem 3.5. *Let $\mathcal{J}_1$, $\mathcal{J}_2$ be (not necessarily closed) two-sided ideals in $\mathcal{B}(X)$ such that $\mathcal{C}_0(X) \subset \mathcal{J}_1 \subset \mathcal{C}_\infty(X)$ and $\mathcal{C}_0(X) \subset \mathcal{J}_2 \subset \mathcal{C}_\infty(X)$, let $\mathcal{L}$ be a (not necessarily closed) subalgebra of $\mathcal{B}(X)$, and let*

$$\mathcal{L}_1 := \{A \in \mathcal{L} : \mathcal{P}A\mathcal{Q} \in \mathcal{J}_1\}, \quad \mathcal{L}_2 := \{A \in \mathcal{L} : \mathcal{Q}A\mathcal{P} \in \mathcal{J}_2\}, \quad \mathcal{L}_* := \mathcal{L}_1 \cap \mathcal{L}_2.$$

(a) *If $A \in \mathcal{L}_1$ is invertible and $\mathcal{P}A\mathcal{P}|\operatorname{Im}\mathcal{P}$ is Fredholm on $\mathcal{P}X$, then $A^{-1} \in \mathcal{L}_1$.*
(b) *If $A \in \mathcal{L}_2$ is invertible and $\mathcal{Q}A\mathcal{Q}|\operatorname{Im}\mathcal{Q}$ is Fredholm on $\mathcal{Q}X$, then $A^{-1} \in \mathcal{L}_2$.*
(c) *If $A \in \mathcal{L}_*$ is invertible, then $A^{-1} \in \mathcal{L}_*$.*

Proof. (a) Put $A_1 := \mathcal{P}A\mathcal{P} + \mathcal{Q}A\mathcal{Q}$. From the invertibility of A and

$$A - A_1 - \mathcal{Q}A\mathcal{P} = \mathcal{P}A\mathcal{Q} \in \mathcal{J}_1 \subset \mathcal{C}_\infty(X)$$

it follows that $A_1 + \mathcal{Q}A\mathcal{P}$ is Fredholm and has index zero.

Let $C \in \mathcal{B}(\mathcal{P}X)$ be a regularizer of the operator $D := \mathcal{P}A\mathcal{P}|\operatorname{Im}\mathcal{P}$, which is Fredholm on $\mathcal{P}X$. Then $CD = \mathcal{P} + \mathcal{P}T\mathcal{P}$, where $T \in \mathcal{C}_\infty(\mathcal{P}X)$. In that case

$$(I + \mathcal{Q}A\mathcal{P}C\mathcal{P})(\mathcal{P}A\mathcal{P} + \mathcal{Q}A\mathcal{Q}) = A_1 + \mathcal{Q}A\mathcal{P}CD = A_1 + \mathcal{Q}A\mathcal{P} + T_1, \tag{3.17}$$

where $T_1 \in \mathcal{C}_\infty(X)$. The operator $A_1 + \mathcal{Q}A\mathcal{P} + T_1$ is Fredholm and has index zero because $A_1 + \mathcal{Q}A\mathcal{P}$ is so. On the other hand, it is easy to check that

$$(I + \mathcal{Q}A\mathcal{P}C\mathcal{P})^{-1} = I - \mathcal{Q}A\mathcal{P}C\mathcal{P}. \tag{3.18}$$

Thus from (3.17) and (3.18) it follows that the operator A_1 is Fredholm and has index zero. Consequently, there exist closed subspaces $M, N \subset X$ such that

$$X = \operatorname{Ker} A_1 \oplus M, \quad X = \operatorname{Im} A_1 \oplus N.$$

Hence $\dim \operatorname{Ker} A_1 = \dim N$.

Denote by $\widehat{A_1}$ the restriction of A to M. Then $\widehat{A_1} : M \to \operatorname{Im} A_1$ is continuous, one-to-one, and hence its continuous inverse $(\widehat{A_1})^{-1}$ exists. Define $B_1 \in \mathcal{B}(X)$ by

$$B_1 x = \begin{cases} (\widehat{A_1})^{-1} x & \text{if} \quad x \in \operatorname{Im} A_1, \\ 0 & \text{if} \quad x \in N. \end{cases}$$

Denote the projection onto $\operatorname{Ker} A_1$ parallel to M by Q_1. We have for every $x \in X$,

$$B_1 A_1 x = B_1 A_1 (x - Q_1 x) = B_1 \widehat{A_1}(x - Q_1 x) = x - Q_1 x,$$

that is,

$$B_1 A_1 = I - Q_1. \tag{3.19}$$

Set

$$\begin{aligned} &A_{11} := \mathcal{P}A\mathcal{P}, \quad &&A_{12} := \mathcal{P}A\mathcal{Q}, \quad &&A_{21} := \mathcal{Q}A\mathcal{P}, \quad &&A_{22} := \mathcal{Q}A\mathcal{Q}, \\ &B_{11} := \mathcal{P}A^{-1}\mathcal{P}, \quad &&B_{12} := \mathcal{P}A^{-1}\mathcal{Q}, \quad &&B_{21} := \mathcal{Q}A^{-1}\mathcal{P}, \quad &&B_{22} := \mathcal{Q}A^{-1}\mathcal{Q}. \end{aligned} \tag{3.20}$$

From $AA^{-1} = I$ we get

$$0 = \mathcal{P}\mathcal{Q} = \mathcal{P}AA^{-1}\mathcal{Q} = \mathcal{P}A(\mathcal{P} + \mathcal{Q})A^{-1}\mathcal{Q} = A_{11}B_{12} + A_{12}B_{22}. \tag{3.21}$$

It is obvious that

$$A_1 = A_{11} + A_{12}, \quad A_{22}B_{12} = 0. \tag{3.22}$$

Combining (3.21) and (3.22), we get $A_1 B_{12} + A_{12} B_{22} = 0$. Multiplying this equality by B_1 from the left and taking into account (3.19), we get $(I - Q_1)B_{12} + B_1 A_{12} B_{22} = 0$. Hence

$$B_{12} = Q_1 B_{12} - B_1 A_{12} B_{22}. \tag{3.23}$$

By the hypothesis, $A \in \mathcal{L}_1$, whence $A_{12} = \mathcal{P}A\mathcal{Q} \in \mathcal{J}_1$. Thus,

$$B_1 A_{12} B_{22} \in \mathcal{J}_1. \tag{3.24}$$

Since A_1 is Fredholm, the projection Q_1 has finite rank. Hence $Q_1 \in \mathcal{C}_0(X) \subset \mathcal{J}_1$ and

$$Q_1 B_{12} \in \mathcal{J}_1. \tag{3.25}$$

From (3.23)–(3.25) we obtain that $B_{12} = \mathcal{P}A^{-1}\mathcal{Q} \in \mathcal{J}_1$. By the definition of $\mathcal{L}_1$, the operator A^{-1} belongs to $\mathcal{L}_1$. Part (a) is proved.

(b) This statement follows from (a) with $\mathcal{Q} = I - \mathcal{P}$ in place of $\mathcal{P}$.

(c) Since $\mathcal{P}A\mathcal{Q} \in \mathcal{J}_1 \subset \mathcal{C}_\infty(X)$, $\mathcal{Q}A\mathcal{P} \in \mathcal{J}_2 \subset \mathcal{C}_\infty(X)$, and the operator A is invertible, we conclude that

$$A_1 = A - \mathcal{P}A\mathcal{Q} - \mathcal{Q}A\mathcal{P} = \mathcal{P}A\mathcal{P} + \mathcal{Q}A\mathcal{Q}$$

is Fredholm. If R is its regularizer, then $\mathcal{P}R\mathcal{P}|\operatorname{Im}\mathcal{P}$ is a regularizer of $\mathcal{P}A\mathcal{P}|\operatorname{Im}\mathcal{P}$ on $\mathcal{P}X$ and $\mathcal{Q}R\mathcal{Q}|\operatorname{Im}\mathcal{Q}$ is a regularizer of $\mathcal{Q}A\mathcal{Q}|\operatorname{Im}\mathcal{Q}$ on $\mathcal{Q}X$. Thus $\mathcal{P}A\mathcal{P}|\operatorname{Im}\mathcal{P}$ and $\mathcal{Q}A\mathcal{Q}|\operatorname{Im}\mathcal{Q}$ are Fredholm. Now statement (c) follows from parts (a) and (b). ☐

Remark 3.6. A minor modification of the proof of part (a) shows that if A is invertible on X, $\mathcal{J}_1 = \{0\}$, and $\mathcal{P}A\mathcal{P}|\operatorname{Im}\mathcal{P}$ is invertible on $\mathcal{P}X$, then $A^{-1} \in \mathcal{L}_1$.

Remark 3.7. If H is a separable Hilbert space and $\mathcal{J}$ is any two-sided ideal of $\mathcal{B}(H)$ such that $\mathcal{J} \neq \{0\}$ and $\mathcal{J} \neq \mathcal{B}(H)$, then $\mathcal{C}_0(H) \subset \mathcal{J} \subset \mathcal{C}_\infty(H)$ by Calkin's theorem (see, e.g. [14, Chap. 3, Theorem 1.1]). Hence the statement of the above theorem can be simplified for separable Hilbert spaces.

4. Auxiliary results

4.1. **Some facts on Besov spaces.** We start with the following well known facts.

Lemma 4.1. *If $1 \le p < \infty$ and $0 < \alpha \le 1$, then $\mathcal{P}$ is dense in B_p^α.*

Lemma 4.2. *If $1 < p < \infty$ and $1/p < \alpha \le 1$, then $B_p^\alpha \subset C$.*

These lemmas are proved, for instance, in [25, Sections 3.5.1 and 3.5.5].

The following fact is certainly known to specialists (see, e.g. [23, p. 735]). We give its proof for the convenience of the reader.

Lemma 4.3. *Let $1 \le p < \infty$ and $0 < \alpha < 1$. If $a, b \in L^\infty \cap B_p^\alpha$, then*

$$\|ab\|_{B_p^\alpha} \le \|a\|_{L^\infty}\|b\|_{B_p^\alpha} + \|a\|_{B_p^\alpha}\|b\|_{L^\infty},$$

that is, $L^\infty \cap B_p^\alpha$ is a Banach algebra under the quasi-submultiplicative norm

$$\|a\|_{L^\infty \cap B_p^\alpha} := \|a\|_{L^\infty} + \|a\|_{B_p^\alpha}.$$

Proof. It is easy to see that

$$\omega^1_{L^p}(ab, s) \le \|a\|_{L^\infty}\omega^1_{L^p}(b, s) + \omega^1_{L^p}(a, s)\|b\|_{L^\infty} \quad (s \ge 0).$$

From this inequality and $\|ab\|_{L^p} \le \|a\|_{L^\infty}\|b\|_{L^p} + \|a\|_{L^p}\|b\|_{L^\infty}$ it follows that

$$\begin{aligned}\|ab\|_{B_p^\alpha} &\le \|a\|_{L^\infty}\|b\|_{L^p} + \|a\|_{L^p}\|b\|_{L^\infty}\\ &\quad + \|a\|_{L^\infty}\left(\int_0^{2\pi}\left[s^{-\alpha}\omega^1_{L^p}(b, s)\right]^p\frac{ds}{s}\right)^{1/p} + \|b\|_{L^\infty}\left(\int_0^{2\pi}\left[s^{-\alpha}\omega^1_{L^p}(a, s)\right]^p\frac{ds}{s}\right)^{1/p}\\ &= \|a\|_{L^\infty}\|b\|_{B_p^\alpha} + \|a\|_{B_p^\alpha}\|b\|_{L^\infty}.\end{aligned}$$

Now the fact that $\|\cdot\|_{L^\infty \cap B_p^\alpha}$ is a quasi-submultiplicative norm is obvious. ☐

4.2. **The algebra of multiplication operators.** For $\delta, \mu \in \mathbb{R}$, we denote by $\ell_2^{\delta,\mu}$ the set of all sequences $\varphi = \{\varphi_j\}_{j\in\mathbb{Z}}$ such that

$$\|\varphi\|^2_{\ell_2^{\delta,\mu}} := \sum_{j=-\infty}^{-1}|\varphi_j|^2(|j|+1)^{2\delta} + \sum_{j=0}^{\infty}|\varphi_j|^2(j+1)^{2\mu} < \infty.$$

It is clear that $\ell_2^{\delta,\mu}$ is a Hilbert space. Let ℓ^0 denote the collection of all sequences from $\ell_2^{\delta,\mu}$ with finite support. For a function $a \in L^1$, define $M(a)$ on ℓ^0 by

$$M(a) : \{\varphi_j\}_{j\in\mathbb{Z}} \mapsto \Big\{\sum_{k\in\mathbb{Z}} a_{j-k}\varphi_k\Big\}_j.$$

If

$$\sup\left\{\|M(a)\varphi\|_{\ell_2^{\delta,\mu}}/\|\varphi\|_{\ell_2^{\delta,\mu}} \;:\; \varphi \in \ell^0,\ \varphi \ne 0\right\} < \infty,$$

then $M(a)$ can be extended to a bounded operator on $\ell_2^{\delta,\mu}$. In this case we call $M(a)$ the *multiplication operator* with symbol a. The following basic properties of multiplication operators on $\ell_2^{\delta,\mu}$ can be proved in the same way as in [5, Sections 2.5 and 6.2] (see also [28]).

Theorem 4.4. *Let $\delta, \mu \in \mathbb{R}$ and $a \in L^1$.*

(a) *If $M(a) \in \mathcal{B}(\ell_2^{\delta,\mu})$, then the adjoint of $M(a)$ equals $M(\overline{a}) \in \mathcal{B}(\ell_2^{-\delta,-\mu})$ and*

$$\|M(a)\|_{\mathcal{B}(\ell_2^{\delta,\mu})} = \|M(\overline{a})\|_{\mathcal{B}(\ell_2^{\delta,\mu})} = \|M(a)\|_{\mathcal{B}(\ell_2^{-\delta,-\mu})} = \|M(\overline{a})\|_{\mathcal{B}(\ell_2^{-\delta,-\mu})}.$$

(b) *If* $M(a) \in \mathcal{B}(\ell_2^{\delta,\mu})$, *then* $a \in L^\infty$ *and*

$$\|a\|_{L^\infty} = \|M(a)\|_{\mathcal{B}(\ell_2^{0,0})} \le \|M(a)\|_{\mathcal{B}(\ell_2^{\delta,\mu})}.$$

(c) *The set* $\{a \in L^1 : M(a) \in \mathcal{B}(\ell_2^{\delta,\mu})\}$ *is a Banach algebra under the Banach algebra norm* $\|a\| := \|M(a)\|_{\mathcal{B}(\ell_2^{\delta,\mu})}$.

Lemma 4.5. *Let* $0 < \delta < \infty$. *For every* $k \in \mathbb{Z}$,

$$\lim_{m\to+\infty} \|M(\chi_{km})\|^{1/m}_{\mathcal{B}(\ell_2^{\delta,-\delta})} \le 1.$$

Proof. It is easy to see that $M(\chi_{km})$ is the shift operator $\{\varphi_j\}_{j\in\mathbb{Z}} \mapsto \{\varphi_{j-km}\}_{j\in\mathbb{Z}}$. Let $\varphi \in \ell_2^{\delta,-\delta}$ and suppose k is positive. Without loss of generality assume $km \ge 1$. Then

$$\begin{aligned}
\|M(\chi_{km})\varphi\|^2_{\ell_2^{\delta,-\delta}} &= \sum_{j=-\infty}^{-1} |\varphi_{j-km}|^2(|j|+1)^{2\delta} + \sum_{j=0}^{\infty} |\varphi_{j-km}|^2(|j|+1)^{-2\delta} \\
&= \sum_{i=-\infty}^{-1+km} |\varphi_i|^2(|i+km|+1)^{2\delta} + \sum_{i=km}^{\infty} |\varphi_i|^2(|i+km|+1)^{-2\delta} \\
&\le \sup_{i\le -1}\left(\frac{|i+km|+1}{|i|+1}\right)^{2\delta} \sum_{i=-\infty}^{-1} |\varphi_i|^2(|i|+1)^{2\delta} \\
&\quad + \max_{0\le i\le km-1}\left[(|i+km|+1)^{2\delta}(|i|+1)^{2\delta}\right] \sum_{i=0}^{km-1} |\varphi_i|^2(|i|+1)^{-2\delta} \\
&\quad + \sup_{i\ge km}\left(\frac{|i+km|+1}{|i|+1}\right)^{-2\delta} \sum_{i=km}^{\infty} |\varphi_i|^2(|i|+1)^{-2\delta} \\
&\le \big(S_1(k,m)+S_2(k,m)+S_3(k,m)\big)\|\varphi\|^2_{\ell_2^{\delta,-\delta}}
\end{aligned} \tag{4.1}$$

where

$$S_1(k,m) := \sup_{i\le -1}\left(\frac{|i+km|+1}{|i|+1}\right)^{2\delta},$$

$$S_2(k,m) := \max_{0\le i\le km-1}\left[(|i+km|+1)^{2\delta}(|i|+1)^{2\delta}\right],$$

$$S_3(k,m) := \sup_{i\ge km}\left(\frac{|i+km|+1}{|i|+1}\right)^{-2\delta}.$$

If $i \le -1$, then

$$\frac{|i+km|+1}{|i|+1} \le 1 + \frac{km}{|i|+1} \le 1 + \frac{km}{2} \le 2km.$$

Hence

$$S_1(k,m) \le (2km)^{2\delta}. \tag{4.2}$$

If $0 \le i \le km-1$, then

$$(|i+km|+1)(|i|+1) = (i+km+1)(i+1) \le 2(km)^2,$$

whence

$$S_2(k,m) \le 2^{2\delta}(km)^{4\delta}. \tag{4.3}$$

If $i \ge km$, then

$$\frac{|i+km|+1}{|i|+1} = \frac{i+km+1}{i+1} \ge 1.$$

Therefore,

$$S_3(k,m) \le 1. \tag{4.4}$$

Combining (4.1)–(4.4), we get for $\varphi \in \ell_2^{\delta,-\delta}$,

$$\|M(\chi_{km})\varphi\|^2_{\ell_2^{\delta,-\delta}} \le \big((2km)^{2\delta} + 2^{2\delta}(km)^{4\delta} + 1\big)\|\varphi\|^2_{\ell_2^{\delta,-\delta}}.$$

Thus,

$$\lim_{m\to+\infty} \|M(\chi_{km})\|^{1/m}_{\mathcal{B}(\ell_2^{\delta,-\delta})} \le \lim_{m\to+\infty} \big((2km)^{2\delta} + 2^{2\delta}(km)^{4\delta} + 1\big)^{1/(2m)} = 1.$$

If k is negative, then the proof is analogous. □

Denote by P the operator given on $\ell_2^{\delta,\mu}$ by $(P\varphi)_j = \varphi_j$ if $j \ge 0$ and $(P\varphi)_j = 0$ if $j < 0$. Let $Q := I - P$. It is easy to see that $P^2 = P$, $Q^2 = Q$, and $\|P\| = \|Q\| = 1$.

Lemma 4.6. *Suppose $\mu \in \mathbb{R}$ and $a \in L^1$. Then*

$$\|PM(a)P\|_{\mathcal{B}(\ell_2^{\mu,-\mu})} \le \|PM(a)P\|_{\mathcal{B}(\ell_2^{-\mu,-\mu})}, \quad \|QM(a)Q\|_{\mathcal{B}(\ell_2^{\mu,-\mu})} \le \|QM(a)Q\|_{\mathcal{B}(\ell_2^{\mu,\mu})}.$$

Proof. This statement follows from the definition of P and Q. □

The Besov space B^γ_∞, $0 < \gamma < 1$, is nothing else than the Hölder space C^γ, $0 < \gamma < 1$, defined as the set of all functions $f \in C$ such that

$$\|f\|_{C^\gamma} := \|f\|_C + \sup_{0<s\le 2\pi} \frac{\omega^1_C(f,s)}{s^\gamma} < \infty.$$

It is easy to see that $\|\cdot\|_{C^\gamma}$ is a Banach algebra norm.

Theorem 4.7 (Verbitsky [28]). *If $|\mu| < \gamma < 1$, then there exists a positive constant $L_{\gamma,\mu}$ depending only on γ and μ such that $\|M(a)\|_{\mathcal{B}(\ell_2^{\mu,\mu})} \le L_{\gamma,\mu}\|a\|_{C^\gamma}$ for all $a \in C^\gamma$.*

4.3. **Peller's theorems on Hankel-type operators.** For $\gamma \in (0,1)$, denote by c^γ the closure of the set of all Laurent polynomials in the norm of C^γ.

The following result is a corollary of Peller's theorems [23, Chap. 6, Theorem 8.1 and 8.2]. It deals with the boundedness and compactness of $QM(a)P$ and $PM(a)Q$ on the spaces $\ell_2^{\gamma/2,-\gamma/2}$ and $\ell_2^{-\gamma/2,\gamma/2}$, respectively.

Theorem 4.8. *If $0 < \gamma < 1$ and $a \in L^\infty$, then*

$$QM(a)P \in \mathcal{B}(\ell_2^{\gamma/2,-\gamma/2}) \iff Qa \in C^\gamma \iff a \in C^\gamma + H^\infty, \tag{4.5}$$

$$QM(a)P \in \mathcal{C}_\infty(\ell_2^{\gamma/2,-\gamma/2}) \iff Qa \in c^\gamma \iff a \in c^\gamma + H^\infty, \tag{4.6}$$

$$PM(a)Q \in \mathcal{B}(\ell_2^{-\gamma/2,\gamma/2}) \iff Pa \in C^\gamma \iff a \in C^\gamma + \overline{H^\infty},$$

$$PM(a)Q \in \mathcal{C}_\infty(\ell_2^{-\gamma/2,\gamma/2}) \iff Pa \in c^\gamma \iff a \in c^\gamma + \overline{H^\infty},$$

and there exist positive constants c_1 and c_2 depending only on γ such that

$$c_1\|Qa\|_{C^\gamma} \le \|QM(a)P\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})} \le c_2\|Qa\|_{C^\gamma}, \tag{4.7}$$

$$c_1\|Pa\|_{C^\gamma} \le \|PM(a)Q\|_{\mathcal{B}(\ell_2^{-\gamma/2,\gamma/2})} \le c_2\|Pa\|_{C^\gamma}.$$

The following theorem is a particular case of Peller's description of generalized Hankel matrices belonging to the Schatten-von Neumann classes (see [23, Chap. 6, Theorem 8.9]).

Theorem 4.9. *Let $1 \le p, q < \infty$ and $\delta, \mu \in \mathbb{R}$. Suppose $a \in L^\infty$.*

(a) *If $0 < 1/p + \delta + \mu \le 1$ and*

$$\min\{\delta,\mu\} > \max\{-1/2, -1/p\}, \tag{4.8}$$

then

$$QM(a)P \in \mathcal{C}_p(\ell_2^{\delta,-\mu}) \iff Qa \in B_p^{1/p+\delta+\mu} \iff a \in B_p^{1/p+\delta+\mu} + H^\infty,$$

and there exist positive constants c_1 and c_2 depending only on δ, μ, and p such that

$$c_1\|Qa\|_{B_p^{1/p+\delta+\mu}} \le \|QM(a)P\|_{\mathcal{C}_p(\ell_2^{\delta,-\mu})} \le c_2\|Qa\|_{B_p^{1/p+\delta+\mu}}.$$

(b) *If* $0 < 1/q - \delta - \mu \le 1$ *and*

$$\max\{\delta, \mu\} < \min\{1/2, 1/q\}, \tag{4.9}$$

then

$$PM(a)Q \in \mathcal{C}_q(\ell_2^{\delta,-\mu}) \Longleftrightarrow Pa \in B_q^{1/q-\delta-\mu} \Longleftrightarrow a \in B_q^{1/q-\delta-\mu} + \overline{H^\infty},$$

and there exist positive constants c_1 *and* c_2 *depending only on* δ, μ, *and* q *such that*

$$c_1 \|Pa\|_{B_q^{1/q-\delta-\mu}} \le \|PM(a)Q\|_{\mathcal{C}_q(\ell_2^{\delta,-\mu})} \le c_2 \|Pa\|_{B_q^{1/q-\delta-\mu}}.$$

Remark 4.10. The restrictions $1 \le p, q < \infty$ and $0 < 1/p + \delta + \mu \le 1$ (respectively, $0 < 1/q - \delta - \mu \le 1$) are not essential. The theorem is also true for $p, q \in (0,1)$ and large δ and μ (resp. large $-\delta$ and $-\mu$). We imposed these restrictions just to keep the presentation in the setting of Besov spaces B_p^α with $1 \le p < \infty$ and $0 < \alpha \le 1$, which is sufficient for our purposes.

On the other hand, hypotheses (4.8) and (4.9) are essential because without these hypotheses the theorem is not true (see the remark on p. 291 of [23]).

4.4. **Toeplitz operators with antianalytic symbols on analytic Besov spaces.** The following result is due to Peller and Khrushchev [24]. Its proof is contained in [5, Proposition 10.23].

Lemma 4.11. *Let* $1 < p < \infty$ *and* $0 < \alpha < 1$. *If* $g \in H^\infty$, *then the Toeplitz operator* $T(\overline{g}) : \varphi \mapsto P(\overline{g}\varphi)$ *is bounded on* PB_p^α *and there exists a positive constant* $L_{p,\alpha}$ *depending only on* p *and* α *such that*

$$\|T(\overline{g})\|_{\mathcal{B}(PB_p^\alpha)} \le L_{p,\alpha} \|g\|_{L^\infty}.$$

5. The Banach algebras $C^\gamma + H^\infty$ and $c^\gamma + H^\infty$

5.1. **The sets $C^\gamma + H^\infty$ and $c^\gamma + H^\infty$ are Banach algebras.** The results of this section may look curious at the first glance. However, they are important pieces of the proof of Theorem 1.4. The material of this section is taken from [3, Chap. 4].

Lemma 5.1. *If* $0 < \gamma < 1$ *and* $a \in L^1$, *then*

$$M(a) \in \mathcal{B}(\ell_2^{\gamma/2,-\gamma/2}) \Longleftrightarrow a \in C^\gamma + H^\infty.$$

Proof. If $M(a) \in \mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})$, then $a \in L^\infty$ in view of Theorem 4.4(b). It is clear that $QM(a)P \in \mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})$. By (4.5), $a \in C^\gamma + H^\infty$. The necessity portion is proved.

Let us prove the sufficiency part. If $a \in C^\gamma + H^\infty$, then there exist $c \in C^\gamma$ and $h \in H^\infty$ such that $a = c + h$. We represent $M(a)$ as

$$M(a) = PM(c)P + PM(h)P + QM(a)P + PM(a)Q + QM(c)Q + QM(h)Q$$

and show that all terms on the right-hand side are bounded on $\ell_2^{\gamma/2,-\gamma/2}$.

By Theorem 4.7, $PM(c)P \in \mathcal{B}(\ell_2^{-\gamma/2,-\gamma/2})$ and $QM(c)Q \in \mathcal{B}(\ell_2^{\gamma/2,\gamma/2})$. Thus, in view of Lemma 4.6, $PM(c)P$ and $QM(c)Q$ are bounded on $\ell_2^{\gamma/2,-\gamma/2}$.

Lemma 4.11 yields that the operators $T(\overline{h})$ and $T(\widetilde{h})$ are bounded on $PB_2^{\gamma/2}$. Hence the operators $PM(\overline{h})P$ and $QM(h)Q$ are bounded on the Besov space $B_2^{\gamma/2}$. It is well known that $f \in B_2^{\gamma/2}$ if and only if its sequence of the Fourier coefficients belongs to $\ell_2^{\gamma/2,\gamma/2}$ and that the corresponding norms are equivalent. Thus $PM(\overline{h})P$ and $QM(h)Q$ are bounded on $\ell_2^{\gamma/2,\gamma/2}$. By Theorem 4.4(a), the operator $PM(h)P$ is the adjoint of $PM(\overline{h})P$ and $PM(h)P \in \mathcal{B}(\ell_2^{-\gamma/2,-\gamma/2})$. From Lemma 4.6 it follows that the operators $PM(h)P$ and $QM(h)Q$ are bounded on $\ell_2^{\gamma/2,-\gamma/2}$.

Let $\varphi \in \ell_2^{\gamma/2,-\gamma/2}$. Then, taking into account that $\|M(a)\|_{\mathcal{B}(\ell_2^{0,0})} = \|a\|_{L^\infty}$, we have

$$\begin{aligned}\|PM(a)Q\varphi\|_{\ell_2^{\gamma/2,-\gamma/2}} &= \|PM(a)Q\varphi\|_{\ell_2^{0,-\gamma/2}} \le \|PM(a)Q\varphi\|_{\ell_2^{0,0}} \\ &\le \|a\|_{L^\infty}\|Q\varphi\|_{\ell_2^{0,0}} = \|a\|_{L^\infty}\|Q\varphi\|_{\ell_2^{0,-\gamma/2}} \le \|a\|_{L^\infty}\|\varphi\|_{\ell_2^{\gamma/2,-\gamma/2}}.\end{aligned}$$

Thus $PM(a)Q$ is bounded on $\ell_2^{\gamma/2,-\gamma/2}$. Finally, the operator $QM(a)P$ is bounded on $\ell_2^{\gamma/2,-\gamma/2}$ in view of (4.5). □

Theorem 5.2. *If $0 < \gamma < 1$, then the sets $C^\gamma + H^\infty$ and $c^\gamma + H^\infty$ are Banach algebras under the Banach algebra norm*

$$\|a\| := \|M(a)\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})}.$$

Proof. From the sufficiency portion of Lemma 5.1 it follows that

$$C^\gamma + H^\infty = \{a \in C^\gamma + H^\infty : M(a) \in \mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})\}.$$

It is obvious that $C^\gamma + H^\infty$ is an algebra. Let a_n be a Cauchy sequence in this algebra. By Theorem 4.4(c), there exists an $a \in L^1$ such that $\|M(a_n) - M(a)\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})} \to 0$ as $n \to \infty$ and $M(a) \in \mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})$. In view of the necessity part of Lemma 5.1, $a \in C^\gamma + H^\infty$. Hence $C^\gamma + H^\infty$ is closed. It is clear that $\|\cdot\|$ is a Banach algebra norm.

Let $\mathcal{A} := \mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})$ and $\mathcal{L} := \{M(a) \in \mathcal{A} : a \in C^\gamma + H^\infty\}$. We have proved that $\mathcal{L}$ is a closed subalgebra of $\mathcal{A}$. From Theorem 3.5(b) with $\mathcal{A}$, $\mathcal{L}$, and $\mathcal{J}_2 := \mathcal{C}_\infty(\ell_2^{\gamma/2,-\gamma/2})$ we obtain that $\mathcal{L}_2 := \{M(a) \in \mathcal{L} : QM(a)P \in \mathcal{J}_2\}$ is a Banach algebra under the Banach algebra norm

$$\|M(a)\|_1 := \|M(a)\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})} + \|QM(a)P\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})}.$$

It is clear that this norm is equivalent to $\|M(a)\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})}$. From (4.6) we get

$$\mathcal{L}_2 = \{M(a) \in \mathcal{B}(\ell_2^{\gamma/2,-\gamma/2}) \ : \ a \in c^\gamma + H^\infty\}.$$

Hence $c^\gamma + H^\infty$ is a Banach algebra under the norm $\|\cdot\|$. □

5.2. **Structure of the Banach algebra $c^\gamma + H^\infty$.** In this subsection we clarify the structure of the algebra $c^\gamma + H^\infty$.

Lemma 5.3. *The set H^∞ is a closed subalgebra of $c^\gamma + H^\infty$.*

Proof. It is obvious that H^∞ is a subalgebra of $c^\gamma + H^\infty$. Let $h_n \in H^\infty$ be a Cauchy sequence in the norm of $c^\gamma + H^\infty$. Since $c^\gamma + H^\infty$ is closed, there is a function $h \in c^\gamma + H^\infty$ such that $\|M(h_n) - M(h)\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})} \to 0$ as $n \to \infty$. From this fact and Theorem 4.4(b) we deduce that $\|h_n - h\|_{L^\infty} \to 0$ as $n \to \infty$. Because H^∞ is closed in L^∞, we conclude that $h \in H^\infty$, that is, H^∞ is closed in $c^\gamma + H^\infty$. □

Lemma 5.4. *We have $c^\gamma + H^\infty = \operatorname{clos}_{c^\gamma + H^\infty} \operatorname{span}\{\psi\overline{\chi_n} : \psi \in H^\infty, n \in \mathbb{Z}_+\}$.*

Proof. Put $A := \operatorname{clos}_{c^\gamma + H^\infty} \operatorname{span}\{\psi\overline{\chi_n} : \psi \in H^\infty, n \in \mathbb{Z}_+\}$. It is obvious that $A \subset c^\gamma + H^\infty$. Let us show the reverse inclusion. If $f \in c^\gamma + H^\infty$, then there exist $c \in c^\gamma$ and $h \in H^\infty$ such that $f = c + h$. By Privalov's theorem, the projections P and Q are bounded on C^γ, $0 < \gamma < 1$. Hence $Pc \in H^\infty$. It is clear that $Pc + h \in H^\infty \subset A$. By the definition of c^γ, there exists a sequence of Laurent polynomials p_m such that

$$\|c - p_m\|_{C^\gamma} \to 0 \quad \text{as} \quad m \to \infty. \tag{5.1}$$

Obviously, $PM(Qc - Qp_m)Q = 0$. Hence

$$M(Qc) - M(Qp_m) = PM(Qc - Qp_m)P + QM(Qc - Qp_m)P + QM(Qc - Qp_m)Q. \tag{5.2}$$

From Theorem 4.7 and Lemma 4.6 it follows that

$$\begin{aligned}\|PM(Qc-Qp_m)P\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})} &\le \|PM(Qc-Qp_m)P\|_{\mathcal{B}(\ell_2^{-\gamma/2,-\gamma/2})}\\ &\le \|M(Qc-Qp_m)\|_{\mathcal{B}(\ell_2^{-\gamma/2,-\gamma/2})} \qquad (5.3)\\ &\le L_{\gamma,-\gamma/2}\|Q\|_{\mathcal{B}(C^\gamma)}\|c-p_m\|_{C^\gamma}\end{aligned}$$

and similarly,

$$\|QM(Qc-Qp_m)Q\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})} \le L_{\gamma,\gamma/2}\|Q\|_{\mathcal{B}(C^\gamma)}\|c-p_m\|_{C^\gamma}. \qquad (5.4)$$

On the other hand, from (4.7) we see that

$$\|QM(Qc-Qp_m)P\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})} \le c_2\|Q(Qc-Qp_m)\|_{C^\gamma} \le c_2\|Q\|_{\mathcal{B}(C^\gamma)}\|c-p_m\|_{C^\gamma}. \qquad (5.5)$$

Combining (5.1)–(5.5), we get

$$\|M(Qc)-M(Qp_m)\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})} \le \text{const}\|c-p_m\|_{C^\gamma} \to 0 \quad \text{as} \quad m\to\infty.$$

It is clear that $Qp_m \in \operatorname{span}\{\psi\overline{\chi_n} : \psi \in H^\infty, n \in \mathbb{Z}_+\}$ and therefore $Qc \in A$. Thus $f \in A$ and $c^\gamma + H^\infty \subset A$. □

5.3. **Invertibility in the Banach algebra $c^\gamma + H^\infty$.** In this subsection we will show that $c^\gamma + H^\infty$ is inverse closed in $C + H^\infty$. From [8, Proposition 6.36, Corollary 6.38] we get the following description of the maximal ideal space of $C + H^\infty$.

Lemma 5.5. *We have* $M_{C+H^\infty} = \{m \in M_{H^\infty} : |\widehat{\chi_k}(m)| = 1$ *for* $k \in \mathbb{Z}_+\}$.

The next statement shows that the maximal ideal spaces of $c^\gamma + H^\infty$ and $C + H^\infty$ coincide.

Lemma 5.6. *We have* $M_{c^\gamma+H^\infty} = \{m \in M_{H^\infty} : |\widehat{\chi_k}(m)| = 1$ *for* $k \in \mathbb{Z}_+\}$.

Proof. The proof is developed by analogy with [8, Proposition 6.37]. By Lemma 5.3, H^∞ is a closed subalgebra of $c^\gamma + H^\infty$. If f is a multiplicative linear functional on $c^\gamma + H^\infty$, then $f|H^\infty$ is a multiplicative linear functional on H^∞. Hence $\Psi : f \mapsto f|H^\infty$ defines a continuous mapping from $M_{c^\gamma+H^\infty}$ into M_{H^∞}. If f_1 and f_2 are elements in $M_{c^\gamma+H^\infty}$ such that $\Psi(f_1) = \Psi(f_2)$, then for $k \in \mathbb{Z}_+$,

$$f_1(\overline{\chi_k}) = f_1(\chi_k^{-1}) = f_1(\chi_k)^{-1} = f_2(\chi_k)^{-1} = f_2(\chi_k^{-1}) = f_2(\overline{\chi_k}).$$

From this equality and Lemma 5.4 we conclude that $f_1 = f_2$. Therefore, Ψ is a homeomorphism of $M_{c^\gamma+H^\infty}$ into M_{H^∞}. Moreover, for every $f \in M_{c^\gamma+H^\infty}$ and $k \in \mathbb{Z}_+$, by multiplicativity of f and Lemma 4.5,

$$|f(\chi_k)| \le \lim_{m\to+\infty} \|\chi_k^m\|_{c^\gamma+H^\infty}^{1/m} = \lim_{m\to+\infty} \|M(\chi_{km})\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})}^{1/m} \le 1$$

and similarly

$$1/|f(\chi_k)| = |f(\chi_{-k})| \le 1.$$

That is, $|f(\chi_k)| = 1$ for every $k \in \mathbb{Z}_+$. Therefore, the range of Ψ is contained in

$$\big\{m \in M_{H^\infty} \ : \ |\widehat{\chi_k}(m)| = 1 \text{ for } k \in \mathbb{Z}_+\big\},$$

and only the reverse containment remains.

Let m be a point in M_{H^∞} such that $|\widehat{\chi_k}(m)| = 1$ for $k \in \mathbb{Z}_+$. If we define g on $\operatorname{span}\{\psi\overline{\chi_n} : \psi \in H^\infty, n \in \mathbb{Z}_+\}$ such that $g(\psi\overline{\chi_k}) = \widehat{\psi}(m)\widehat{\chi_k}(m)$, then g can be easily shown to be multiplicative. The inequality

$$\begin{aligned}|g(\psi\overline{\chi_k})| &= |\widehat{\psi}(m)|\,|\widehat{\chi_k}(m)| = |\widehat{\psi}(m)| \le \|\psi\|_{H^\infty}\\ &= \|\psi\overline{\chi_k}\|_{L^\infty} \le \|M(\psi\overline{\chi_k})\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})} = \|\psi\overline{\chi_k}\|_{c^\gamma+H^\infty}\end{aligned}$$

(recall Theorems 4.4(b) and 5.2) shows that g can be extended to a multiplicative linear functional on $\operatorname{clos}_{c^\gamma+H^\infty}\operatorname{span}\{\psi\overline{\chi_n} : \psi \in H^\infty, n \in \mathbb{Z}_+\}$. But the latter set coincides with

$c^\gamma + H^\infty$ by Lemma 5.4. Obviously, $\Psi(g) = m$ and thus $M_{c^\gamma+H^\infty}$ is homeomorphic to $\{m \in H^\infty : |\widehat{\chi_k}(m)| = 1 \text{ for } k \in \mathbb{Z}_+\}$. □

Combining Lemmas 5.5 and 5.6 with Gelfand's theorem, we get the following.

Theorem 5.7. *If $0 < \gamma < 1$ and $a \in c^\gamma + H^\infty$, then*

$$a \in G(c^\gamma + H^\infty) \Longleftrightarrow a \in G(C + H^\infty).$$

6. Generalized Krein algebras

6.1. **The sets $K^{\alpha,0}_{p,0}$, $K^{0,\beta}_{0,q}$, and $K^{\alpha,\beta}_{p,q}$ are Banach algebras.** Since the projections P and Q are bounded on the Besov spaces B^α_p and B^β_q, it is clear that $K^{\alpha,0}_{p,0}$, $K^{0,\beta}_{0,q}$, and $K^{\alpha,\beta}_{p,q}$ are Banach spaces under the norms (1.7), (1.8), and (1.9), respectively. In this subsection we show that these norms are quasi-submultiplicative.

Lemma 6.1. *Let $1 < p, q < \infty$ and $0 < \alpha, \beta < 1$.*

(a) *If $\alpha > 1/p$, then the projections P and Q are bounded on $K^{\alpha,0}_{p,0}$.*
(b) *If $\beta > 1/q$, then the projections P and Q are bounded on $K^{0,\beta}_{0,q}$.*
(c) *If $\alpha > 1/p$ or $\beta > 1/q$, then the projections P and Q are bounded on $K^{\alpha,\beta}_{p,q}$.*

Proof. (a) By Lemma 4.2, there exists a constant $C > 0$ such that $\|f\|_{L^\infty} \le C\|f\|_{B^\alpha_p}$ for all $f \in B^\alpha_p$. If $a \in K^{\alpha,0}_{p,0}$, then

$$\|Qa\|_{K^{\alpha,0}_{p,0}} = \|Qa\|_{L^\infty} + \|Qa\|_{B^\alpha_p} \le (C+1)\|Qa\|_{B^\alpha_p} \le (C+1)\|a\|_{K^{\alpha,0}_{p,0}}.$$

The boundedness of $P = I - Q$ is now obvious.

(b) The proof is analogous.

(c) The arguments of the proof of (a) or (b) apply. □

Now we are in a position to prove our first main result.

Proof of Theorem 1.3. Let us prove only part (c). The proofs of parts (a) and (b) are similar (and even a little simpler).

For $\alpha = 1/p$ and $\beta = 1/q$, the statement is contained in Corollary 1.2. Hence we can assume that $\alpha > 1/p$ or $\beta > 1/q$. Then the projections P and Q are bounded on $K^{\alpha,\beta}_{p,q}$ by Lemma 6.1(c). For brevity we will omit the subscript in $\|\cdot\|_{K^{\alpha,\beta}_{p,q}}$. If $a, b \in K^{\alpha,\beta}_{p,q}$, then we have

$$\begin{aligned}\|ab\| &\le \|PaPb\| + \|QaQb\| + \|PaQb\| + \|QaPb\| \\ &= \|PaPb\|_{L^\infty} + \|P(PaPb)\|_{B^\beta_q} + \|QaQb\|_{L^\infty} + \|Q(QaQb)\|_{B^\alpha_p} \\ &\quad + \|PaQb\|_{L^\infty} + \|P(PaQb)\|_{B^\beta_q} + \|Q(PaQb)\|_{B^\alpha_p} \\ &\quad + \|QaPb\|_{L^\infty} + \|P(QaPb)\|_{B^\beta_q} + \|Q(QaPb)\|_{B^\alpha_p}.\end{aligned} \tag{6.1}$$

It is clear that

$$\begin{aligned}&\|PaQb\|_{L^\infty} + \|QaQb\|_{L^\infty} + \|PaQb\|_{L^\infty} + \|QaPb\|_{L^\infty} \\ &\le \|Pa\|\,\|Pb\| + \|Qa\|\,\|Qb\| + \|Pa\|\,\|Qb\| + \|Qa\|\,\|Pb\| \\ &\le (\|P\| + \|Q\|)^2\|a\|\,\|b\|.\end{aligned} \tag{6.2}$$

In view of Lemma 4.3,

$$\begin{aligned}\|P(PaPb)\|_{B^\beta_q} &= \|PaPb\|_{B^\beta_q} \\ &\le \|Pa\|_{L^\infty}\|Pb\|_{B^\beta_q} + \|Pa\|_{B^\beta_q}\|Pb\|_{L^\infty} \\ &\le \|Pa\|\,\|b\| + \|a\|\,\|Pb\| \\ &\le 2\|P\|\,\|a\|\,\|b\|\end{aligned} \tag{6.3}$$

and analogously

$$\|Q(QaQb)\|_{B^\alpha_p} \le 2\|Q\|\,\|a\|\,\|b\|. \tag{6.4}$$

By Lemma 4.11,

$$\begin{aligned}
\|P(PaQb)\|_{B^\beta_q} &= \|PM(Qb)Pa\|_{B^\beta_q} \\
&= \|T(Qb)Pa\|_{PB^\beta_q} \\
&\le \|T(Qb)\|_{\mathcal{B}(PB^\beta_q)}\|Pa\|_{PB^\beta_q} \\
&\le L_{q,\beta}\|Qb\|_{L^\infty}\|Pa\|_{B^\beta_q} \\
&\le L_{q,\beta}\|Qb\|\,\|a\| \\
&\le L_{q,\beta}\|Q\|\,\|a\|\,\|b\|
\end{aligned} \tag{6.5}$$

and analogously

$$\|P(QaPb)\|_{B^\beta_q} \le L_{q,\beta}\|Q\|\,\|a\|\,\|b\|. \tag{6.6}$$

It is clear that the operator J is an isometry on B^α_p. Then applying Lemma 4.11 again, we get

$$\begin{aligned}
\|Q(PaQb)\|_{B^\alpha_p} &= \|QM(Pa)Qb\|_{B^\alpha_p} \\
&= \|JT(\widetilde{Pa})JQb\|_{QB^\alpha_p} \\
&\le \|T(\widetilde{Pa})\|_{\mathcal{B}(PB^\alpha_p)}\|JQb\|_{PB^\alpha_p} \\
&\le L_{p,\alpha}\|Pa\|_{L^\infty}\|Qb\|_{B^\alpha_p} \\
&\le L_{p,\alpha}\|Pa\|\,\|b\| \\
&\le L_{p,\alpha}\|P\|\,\|a\|\,\|b\|
\end{aligned} \tag{6.7}$$

and similarly

$$\|Q(QaPb)\|_{B^\alpha_p} \le L_{p,\alpha}\|P\|\,\|a\|\,\|b\|. \tag{6.8}$$

Combining (6.1)–(6.8), we get

$$\|ab\| \le C(p,q,\alpha,\beta)\|a\|\,\|b\|$$

with $C(p,q,\alpha,\beta) := (\|P\|+\|Q\|)^2 + 2\|P\| + 2\|Q\| + 2L_{q,\beta}\|Q\| + 2L_{p,\alpha}\|P\|$. □

6.2. **Invertibility in the Banach algebras $K^{\alpha,0}_{p,0}$, $K^{0,\beta}_{0,q}$, and $K^{\alpha,\beta}_{p,q}$.** In this subsection we show that generalized Krein algebras are inverse closed either in $C+H^\infty$ or in $C+\overline{H^\infty}$.

From Lemma 4.2 and (1.5) we immediately get the following.

Lemma 6.2. *Let $1 < p,q < \infty$ and $0 < \alpha,\beta < 1$.*

(a) *If $\alpha \ge 1/p$, then $K^{\alpha,0}_{p,0} \subset C+H^\infty$ and $K^{\alpha,\beta}_{p,q} \subset C+H^\infty$.*
(b) *If $\beta \ge 1/q$, then $K^{0,\beta}_{0,q} \subset C+\overline{H^\infty}$ and $K^{\alpha,\beta}_{p,q} \subset C+\overline{H^\infty}$.*

Now we prepare the proof of Theorem 1.4. We know from Theorem 5.7 that $c^\gamma + H^\infty$ is inverse closed in $C+H^\infty$. We show that the intersection of a generalized Krein algebra $K^{\alpha,\beta}_{p,q}$ (or $K^{\alpha,0}_{p,0}$) with $c^\gamma + H^\infty$ is inverse closed in $c^\gamma + H^\infty$ (and thus in $C+H^\infty$) if $\alpha > 1/p$.

Lemma 6.3. *Let $0 < \lambda \le 1$, $1 < p,q < \infty$, $1/p+1/q = \lambda$, and $0 < \gamma < \lambda - 1/p$. Suppose K is either $K^{1/p+\gamma,1/q-\gamma}_{p,q}$ or $K^{1/p+\gamma,0}_{p,0}$.*

(a) *The set $K \cap (c^\gamma + H^\infty)$ is a Banach algebra under the quasi-submultiplicative norm*

$$\|a\| := \|M(a)\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})} + \|a\|_K.$$

(b) *If $a \in K \cap (c^\gamma + H^\infty)$, then*

$$a \in G\big(K \cap (c^\gamma + H^\infty)\big) \Longleftrightarrow a \in G(c^\gamma + H^\infty).$$

Proof. (a) This statement follows from Theorem 5.2 and Theorem 1.3(a), (c).

(b) Let $\mathcal{A} := \mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})$ and $\mathcal{L} := \{M(a) \in \mathcal{A} : a \in c^\gamma + H^\infty\}$. From Theorem 5.2 it follows that the mapping

$$M : c^\gamma + H^\infty \to \mathcal{L}, \quad a \mapsto M(a)$$

is an isometry of Banach algebras. Hence

$$a \in G(c^\gamma + H^\infty) \Longleftrightarrow M(a) \in G\mathcal{L}. \tag{6.9}$$

By Lemma 2.2(b), (c) and Theorems 3.2 and 3.3,

$$\mathcal{L}_2 := \{M(a) \in \mathcal{L} \ : \ QM(a)P \in \mathcal{C}_p(\ell_2^{\gamma/2,-\gamma/2})\},$$
$$\mathcal{L}_* := \{M(a) \in \mathcal{L} \ : \ PM(a)Q \in \mathcal{C}_q(\ell_2^{\gamma/2,-\gamma/2}),\ QM(a)P \in \mathcal{C}_p(\ell_2^{\gamma/2,-\gamma/2})\}$$

are Banach algebras under the Banach algebra norms

$$\|M(a)\|_2 := \|M(a)\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})} + \|QM(a)P\|_{\mathcal{C}_p(\ell_2^{\gamma/2,-\gamma/2})},$$
$$\|M(a)\|_* := \|M(a)\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})} + \|PM(a)Q\|_{\mathcal{C}_q(\ell_2^{\gamma/2,-\gamma/2})} + \|QM(a)P\|_{\mathcal{C}_p(\ell_2^{\gamma/2,-\gamma/2})},$$

respectively.

If $M(a) \in G\mathcal{L}$, then from (6.9) and (4.6) it follows that $QM(a)P$ and $QM(a^{-1})P$ belong to $\mathcal{C}_\infty(\ell_2^{\gamma/2,-\gamma/2})$. Thus,

$$Q - QM(a)Q \cdot QM(a^{-1})Q = QM(a)P \cdot PM(a^{-1})Q \in \mathcal{C}_\infty(\ell_2^{\gamma/2,-\gamma/2}),$$
$$Q - QM(a^{-1})Q \cdot QM(a)Q = QM(a^{-1})P \cdot PM(a)Q \in \mathcal{C}_\infty(\ell_2^{\gamma/2,-\gamma/2}),$$

that is, $QM(a^{-1})Q|\operatorname{Im} Q$ is a regularizer of $QM(a)Q|\operatorname{Im} Q$ on $Q\ell_2^{\gamma/2,-\gamma/2}$. By Theorem 3.5(b), $M(a) \in G\mathcal{L}_2$. In view of Theorem 3.5(c), we conclude also that $M(a) \in G\mathcal{L}_*$. Thus,

$$M(a) \in G\mathcal{L} \Longleftrightarrow M(a) \in G\mathcal{L}_2, \quad M(a) \in G\mathcal{L} \Longleftrightarrow M(a) \in G\mathcal{L}_*. \tag{6.10}$$

Take $\mu = \delta = \gamma/2$. Then hypotheses (4.8) and (4.9) are simultaneously satisfied. Due to Theorem 4.9,

$$\mathcal{L}_2 = \{M(a) \in \mathcal{A} \ : a \in K_{p,0}^{1/p+\gamma,0} \cap (c^\gamma + H^\infty)\},$$
$$\mathcal{L}_* = \{M(a) \in \mathcal{A} \ : a \in K_{p,q}^{1/p+\gamma,1/q-\gamma} \cap (c^\gamma + H^\infty)\},$$

and the norms $\|PM(a)Q\|_{\mathcal{C}_q(\ell_2^{\gamma/2,-\gamma/2})}$ and $\|QM(a)P\|_{\mathcal{C}_p(\ell_2^{\gamma/2,-\gamma/2})}$ are equivalent to the norms $\|Pa\|_{B_q^{1/q-\gamma}}$ and $\|Qa\|_{B_p^{1/p+\gamma}}$, respectively. From Theorem 4.4(b) we conclude that $\|M(a)\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})}$ is equivalent to $\|a\|_{L^\infty} + \|M(a)\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})}$. Thus, the norms $\|M(a)\|_2$ and $\|M(a)\|_*$ are equivalent to the norms

$$\|M(a)\|_2' := \|M(a)\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})} + \|a\|_{K_{p,0}^{1/p+\gamma,0}},$$
$$\|M(a)\|_*' := \|M(a)\|_{\mathcal{B}(\ell_2^{\gamma/2,-\gamma/2})} + \|a\|_{K_{p,q}^{1/p+\gamma,1/q-\gamma}},$$

respectively. It is clear that the mappings

$$M : K_{p,0}^{1/p+\gamma,0} \cap (c^\gamma + H^\infty) \to (\mathcal{L}_2, \|\cdot\|_2'), \quad a \mapsto M(a),$$
$$M : K_{p,q}^{1/p+\gamma,1/q-\gamma} \cap (c^\gamma + H^\infty) \to (\mathcal{L}_*, \|\cdot\|_*'), \quad a \mapsto M(a)$$

are isometries. Thus,

$$\begin{aligned} M(a) \in G\mathcal{L}_2 &\Longleftrightarrow a \in G\big(K_{p,0}^{1/p+\gamma,0} \cap (c^\gamma + H^\infty)\big), \\ M(a) \in G\mathcal{L}_* &\Longleftrightarrow a \in G\big(K_{p,q}^{1/p+\gamma,1/q-\gamma} \cap (c^\gamma + H^\infty)\big). \end{aligned} \tag{6.11}$$

Combining (6.9)–(6.11), we get the assertion (b). □

Now we prove that the intersection of a generalized Krein algebra $K_{p,q}^{\alpha,\beta}$ (or $K_{0,q}^{0,\beta}$) with $c^\gamma + \overline{H^\infty}$ is inverse closed in $c^\gamma + \overline{H^\infty}$ (and thus in $C + \overline{H^\infty}$) if $\beta > 1/q$.

Lemma 6.4. *Let $0 < \lambda \le 1$, $1 < p, q < \infty$, $1/p + 1/q = \lambda$, and $0 < \gamma < \lambda - 1/q$. Suppose K is either $K_{p,q}^{1/p-\gamma,1/q+\gamma}$ or $K_{0,q}^{0,1/q+\gamma}$.*

(a) *The set $K \cap (c^\gamma + \overline{H^\infty})$ is a Banach algebra under the quasi-submultiplicative norm*

$$\|a\| := \|M(a)\|_{\mathcal{B}(\ell_2^{-\gamma/2,\gamma/2})} + \|a\|_K.$$

(b) *If* $a \in K \cap (c^\gamma + \overline{H^\infty})$, *then*

$$a \in G\big(K \cap (c^\gamma + \overline{H^\infty})\big) \Longleftrightarrow a \in G(c^\gamma + \overline{H^\infty}).$$

Proof. From Theorem 5.2 and Theorem 4.4(a) it follows that $c^\gamma + \overline{H^\infty}$ is a Banach algebra under the Banach algebra norm $\|a\| := \|M(a)\|_{\mathcal{B}(\ell_2^{-\gamma/2,\gamma/2})}$. Hence we can finish the proof by making obvious modifications in the proof of Lemma 6.3. □

We are in a position to prove our second main result. We will show that the intersection of a generalized Krein algebra K with $c^\gamma + H^\infty$ or with $c^\gamma + \overline{H^\infty}$ is dense in K. This allows us to prove that K is inverse closed in $C + H^\infty$ or in $C + \overline{H^\infty}$, respectively.

Proof of Theorem 1.4. (a) For $\alpha = 1/p$ the statement of Theorem 1.4(a) is nothing else than Theorem 1.1(b), (d).

Assume that $\alpha > 1/p$ and put $\gamma := \alpha - 1/p$. Let K be either $K_{p,0}^{1/p+\gamma,0}$ or $K_{p,q}^{1/p+\gamma,1/q-\gamma}$ and denote $B := K \cap (c^\gamma + H^\infty)$. Then, by Lemma 6.2(a), $K \subset C + H^\infty$. In view of Lemma 6.3(a), B is continuously embedded in K.

Due to Lemma 4.2, there exists a constant $c \in (0,\infty)$ such that $\|f\|_{L^\infty} \le c\|f\|_{B_p^{1/p+\gamma}}$ for all $f \in B_p^{1/p+\gamma}$. Let $a \in K$. By Lemma 4.1, for every $\varepsilon > 0$ there exists a Laurent polynomial p such that

$$\|Qa - p\|_{B_p^{1/p+\gamma}} < \frac{\varepsilon}{(c+1)\|Q\|_{\mathcal{B}(B_p^{1/p+\gamma})}}.$$

Hence $Qp + Pa \in B$ and

$$\begin{aligned}\|a - (Qp + Pa)\|_K = \|Q(Qa-p)\|_K &= \|Q(Qa-p)\|_{L^\infty} + \|Q(Qa-p)\|_{B_p^{1/p+\gamma}} \\ &\le (c+1)\|Q\|_{\mathcal{B}(B_p^{1/p+\gamma})}\|Qa - p\|_{B_p^{1/p+\gamma}} < \varepsilon.\end{aligned}$$

Thus, B is dense in K.

Combining Theorem 5.7 and Lemma 6.3(b), we obtain that if $b \in B$, then the spectrum of b in B coincides with the spectrum of b in $C + H^\infty$. Hence for each $\varphi \in M_B$ there is a $\psi \in M_{C+H^\infty}$ depending on φ and b such that

$$\psi(b) = \varphi(b). \tag{6.12}$$

Assume now that there exists an $a \in K$ being invertible in $C + H^\infty$ but not invertible in K. Thus, by Gelfand's theorem, we can find a $\varphi \in M_K$ with $\varphi(a) = 0$. Since B is dense in K, there exists a sequence $\{b_n\} \subset B$ such that

$$\|a - b_n\|_K \to 0 \quad \text{as} \quad n \to \infty. \tag{6.13}$$

Consequently, $\varphi(b_n) \to \varphi(a) = 0$ as $n \to \infty$. Since $\varphi \in M_K \subset M_B$, it follows from (6.12) that there are $\psi_n \in M_{C+H^\infty}$ such that

$$\psi_n(b_n) = \varphi(b_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{6.14}$$

On the other hand, from (6.13) and the definition of the norm in K we get $\|a - b_n\|_{L^\infty} \to 0$ as $n \to \infty$. Since $a \in G(C + H^\infty)$, it results that $b_n \in G(C + H^\infty)$ for all sufficiently large n. Hence $\|b_n^{-1} - a^{-1}\|_{L^\infty} \to 0$ as $n \to \infty$. Thus $\|b_n^{-1}\|_{L^\infty} \le 2\|a^{-1}\|_{L^\infty}$ for all n large enough. Since every multiplicative linear functional has norm 1, we obtain

$$|\psi_n(b_n^{-1})| \le \|\psi_n\|\,\|b_n^{-1}\|_{L^\infty} \le 2\|a^{-1}\|_{L^\infty}$$

for all sufficiently large n. From this inequality and $\psi_n(b_n)\psi_n(b_n^{-1}) = 1$ it follows that $|\psi_n(b_n)| \ge (2\|a^{-1}\|_{L^\infty})^{-1}$ for all n large enough, but this contradicts (6.14). Part (a) is proved.

(b) The proof is analogous to the proof of part (a). We only need to replace Lemma 6.3 by Lemma 6.4. □

7. Proof of higher order asymptotic formulas

7.1. **Higher order asymptotic formulas: an abstract version.** Our asymptotic analysis is based on the following fact [5, Section 10.34].

Lemma 7.1. *Suppose $a \in L^\infty_{N\times N}$ satisfies the following hypotheses:*

(i) *there are two factorizations $a = u_- u_+ = v_+ v_-$, where $u_-, v_- \in G(\overline{H^\infty})_{N\times N}$ and $u_+, v_+ \in G(H^\infty)_{N\times N}$;*

(ii) *$u_- \in G(C + H^\infty)_{N\times N}$ or $u_+ \in G(C + \overline{H^\infty})_{N\times N}$.*

Then $D_n(a) \neq 0$ and

$$\frac{G(a)^{n+1}}{D_n(a)} = \det\left(I - \sum_{k=0}^{\infty} F_{n,k}\right) \tag{7.1}$$

for sufficiently large n, where $G(a)$ is defined by (1.10) *and $F_{n,k}$ are defined by* (1.14).

From (i) and (ii) it follows that $a \in G(C+H^\infty)_{N\times N}$ or $a \in G(C+\overline{H^\infty})_{N\times N}$. Moreover, the operators $T(a)$ and $T(\widetilde{a})$ are invertible on H^2_N. Hence, in view of Lemma 1.5, the constant $G(a)$ is well defined.

An earlier version of Lemma 7.1 was obtained in [2] (see also [3, Section 6.15]) with

(iii) $u_- \in C_{N\times N}$ or $u_+ \in C_{N\times N}$

in place of (ii).

If the series $\sum_{k=0}^{\infty} F_{n,k}$ converges in the topology of the Schatten-von Neumann class $\mathcal{C}_m(H^2_N)$, $m \in \mathbb{N}$, then one can remove its remainder in (7.1) because the mapping

$$\det{}_m(I + \cdot) : \mathcal{C}_m(H^2_N) \to \mathbb{C}$$

is continuous (see, e.g. [26, Theorem 6.5]). More precisely, starting from (7.1) one can get the following.

Theorem 7.2. *Let $a \in L^\infty_{N\times N}$ satisfy the hypotheses* (i) *and* (ii) *of Lemma* 7.1. *Define the constant $G(a)$, the functions b, c, and the operators $F_{n,k}$ by* (1.10), (1.12), *and* (1.14), *respectively. If $m \in \mathbb{N}$ and $H(\widetilde{c})H(b)$, $H(b)H(\widetilde{c})$ belong to the Schatten-von Neumann class $\mathcal{C}_m(H^2_N)$, then* (1.13) *holds.*

This statement is proved in [17, Theorem 15] under the hypotheses (i) and (iii). Exactly the same proof works under the hypotheses (i) and (ii) because of Lemma 7.1. Theorem 7.2 is contained implicitly in [2, Section 5] (see also [3, Theorem 6.20], [5, Theorem 10.37]).

Notice that we need the statement in this (new) form because generalized Krein algebras contain discontinuous functions and therefore the (old) hypothesis (iii) is not satisfied for generalized Krein algebras.

7.2. **Wiener-Hopf factorization in decomposing algebras.** Mark Krein [18] was the first to understand the Banach algebraic background of Wiener-Hopf factorization and to present the method in a crystal-clear manner. Gohberg and Krein [12] proved that $a \in GW_{N\times N}$ admits a Wiener-Hopf factorization. Later Budyanu and Gohberg developed an abstract factorization theory in decomposing algebras of *continuous* functions. Their results are contained in [7, Chap. 2]. However, generalized Krein algebras contain discontinuous functions and the results of Budyanu and Gohberg are not applicable to them. One of the authors and Heinig [16] extended the theory of Budyanu and Gohberg to the case of decomposing algebras which may contain *discontinuous* functions. The following definitions and results are taken from [16] (see also [3, Chap. 5]).

Let A be a Banach algebra of complex-valued functions on the unit circle $\mathbb{T}$ under a Banach algebra norm $\|\cdot\|_A$. The algebra A is said to be *decomposing* if it possesses the following properties:

(a) A is continuously embedded in L^∞;
(b) A contains all Laurent polynomials;
(c) $PA \subset A$ and $QA \subset A$.

Using the closed graph theorem it is easy to deduce from (a)–(c) that P and Q are bounded on A and that PA and QA are closed subalgebras of A. Given a decomposing algebra A put

$$A_+ = PA, \quad \overset{\circ}{A}_- = QA, \quad \overset{\circ}{A}_+ = \chi_1 A_+, \quad A_- = \chi_1 \overset{\circ}{A}_- .$$

Let A be a decomposing algebra. A matrix function $a \in A_{N\times N}$ is said to *admit a right Wiener-Hopf factorization in* $A_{N\times N}$ if it can be represented in the form $a = a_- d a_+$, where $a_\pm \in G(A_\pm)_{N\times N}$ and

$$d = \operatorname{diag}(\chi_{\kappa_1}, \dots, \chi_{\kappa_N}), \quad \kappa_i \in \mathbb{Z}, \quad \kappa_1 \le \kappa_2 \le \dots \le \kappa_N.$$

The integers κ_i are usually called the *right partial indices* of a; they can be shown to be uniquely determined by a. If $\kappa_1 = \dots = \kappa_N = 0$, then the Wiener-Hopf factorization is said to be *canonical*. A decomposing algebra A is said to have the *factorization property* if every matrix function in $GA_{N\times N}$ admits a right Wiener-Hopf factorization in $A_{N\times N}$.

Let $\mathcal{R}$ be the restriction to the unit circle $\mathbb{T}$ of the set of all rational functions defined on the whole plane $\mathbb{C}$ and having no poles on $\mathbb{T}$.

Theorem 7.3. *Let A be a decomposing algebra. If at least one of the sets $(\mathcal{R} \cap \overset{\circ}{A}_-) + A_+$ or $\overset{\circ}{A}_- + (\mathcal{R} \cap A_+)$ is dense in A, then A has the factorization property.*

7.3. **Fredholmness and invertibility of Toeplitz operators.** In this subsection we collect some well known facts about the Fredholmness and invertibility of Toeplitz operators on H^2_N.

Theorem 7.4. *Let $a \in L^\infty_{N\times N}$.*

(a) *If $a \in W_{N\times N}$, then $T(a)$ is Fredholm on H^2_N if and only if $\det a \in GW$.*
(b) *If $a \in (C + \overline{H^\infty})_{N\times N}$, then $T(a)$ is Fredholm on H^2_N if and only if $\det a$ belongs to $G(C + \overline{H^\infty})$.*

Part (a) is essentially due to Gohberg and Krein [12], part (b) is due to Douglas, its proof is given in [5, Theorem 2.94(a)].

The following result was proved by Widom and in a slightly different setting by Simonenko.

Theorem 7.5. *If $a \in GL^\infty_{N\times N}$, then $T(a)$ is invertible on H^2_N if and only if the following two conditions hold:*

(a) *a admits a canonical right generalized factorization in L^2_N, that is, there exist a_-, a_+ such that*

$$a = a_- a_+, \quad a_-^{\pm 1} \in (\overline{H^2})_{N\times N}, \quad a_+^{\pm 1} \in (H^2)_{N\times N};$$

(b) *the operator $M(a_-)PM(a_-^{-1})$ is bounded on L^2_N.*

For a proof, see, e.g. [7, Chap. 7, Theorem 3.2] or [22, Theorem 3.14].

Theorem 7.6. *If $a \in GL^\infty_{N\times N}$, then $T(\widetilde{a})$ is invertible on H^2_N if and only if $T(a^{-1})$ is invertible on H^2_N.*

For a proof, see [5, Proposition 7.19(b)].

7.4. **Wiener-Hopf factorization in generalized Krein algebras.** Let us show that all algebras mentioned in Theorem 1.7 have the factorization property.

Lemma 7.7. *Let $1 < p, q < \infty$ and $0 < \alpha, \beta < 1$. Each of the algebras*

$W \cap K_{p,0}^{1/p,0}$, $K_{p,0}^{\alpha,0}$ *with* $\alpha > 1/p$, $W \cap K_{0,q}^{0,1/q}$, $K_{0,q}^{0,\beta}$ *with* $\beta > 1/q$, $W \cap K_{p,q}^{1/p,1/q}$,

and $K_{p,q}^{\alpha,\beta}$ with $\alpha \neq 1/p$, $1/p + 1/q = \alpha + \beta \in (0, 1]$ is a decomposing algebra with the factorization property.

Proof. This statement is proved for $W \cap K_{p,q}^{1/p,1/q}$ in [5, Section 10.24], the same argument applies also to $W \cap K_{p,0}^{1/p,0}$ and $W \cap K_{0,q}^{0,1/q}$.

Under the imposed conditions on the parameters p, q, α, β, by Lemma 6.1, the projections P and Q are bounded on each $K_{p,0}^{\alpha,0}$, $K_{0,q}^{0,\beta}$, and $K_{p,q}^{\alpha,\beta}$. Hence each of these algebras is decomposing.

If $\beta > 1/q$, then, by Lemma 4.2, B_q^β is continuously embedded in C and, by Lemma 4.1, the Laurent polynomials are dense in B_q^β. It follows that $\mathcal{R} \cap PB_q^\beta$ is dense in $PK_{p,q}^{\alpha,\beta}$ and in $PK_{0,q}^{0,\beta}$. Theorem 7.3 gives the factorization property of the algebras $K_{p,q}^{\alpha,\beta}$ and $K_{0,q}^{0,\beta}$ if $\beta > 1/q$. Analogously one can show that $K_{p,q}^{\alpha,\beta}$ and $K_{p,0}^{\alpha,0}$ have the factorization property if $\alpha > 1/p$. □

We are ready to start the proof of our last main result.

Proof of Theorem 1.7(a). From Lemma 6.2 we conclude that $K_{N\times N} \subset (C + H^\infty)_{N\times N}$ or $K_{N\times N} \subset (C + \overline{H^\infty})_{N\times N}$. In the first case $\widetilde{a} \in (C + \overline{H^\infty})_{N\times N}$. Since $T(\widetilde{a})$ is invertible, from Theorem 7.5(b) we get $\det \widetilde{a} \in G(C + \overline{H^\infty})$ (and $\det \widetilde{a} \in GW$ if, in addition, $a \in W_{N\times N}$ due to Theorem 7.5(a)). Hence $\det a \in G(C + H^\infty)$. By Theorem 1.4, $\det a \in GK$. Then, in view of [21, Chap. 1, Theorem 1.1], $a \in GK_{N\times N}$. If $K_{N\times N} \subset (C + H^\infty)_{N\times N}$, then, as before, the Fredholmness of $T(a)$ implies $a \in GK_{N\times N}$.

By Theorem 7.5, there exist a_-, a_+ such that $a = a_- a_+$, $a_-^{\pm 1} \in (\overline{H^2})_{N\times N}$, and $a_+^{\pm 1} \in (H^2)_{N\times N}$. On the other hand, in view of Lemma 7.7, a admits a right Wiener-Hopf factorization in $K_{N\times N}$, that is, there exist $u_\pm \in G(K_\pm)_{N\times N}$ such that $a = u_- d u_+$. It is clear that

$$u_-^{\pm 1} \in (K_-)_{N\times N} \subset (\overline{H^2})_{N\times N}, \quad u_+^{\pm 1} \in (K_+)_{N\times N} \subset (H^2)_{N\times N}.$$

Hence $a = u_- d u_+$ is a right generalized factorization of a in L_N^2. It is well known that the set of partial indices of such a factorization is unique (see, e.g. [22, Corollary 2.1]). Thus $d = 1$ and $a = u_- u_+$.

In view of Theorem 7.6, $T(a^{-1})$ is invertible on H_N^2. By what has just been proved, there exist $f_\pm \in G(K_\pm)_{N\times N}$ such that $a^{-1} = f_- f_+$. Put $v_\pm := f_\pm^{-1}$. In that case $v_\pm \in G(K_\pm)_{N\times N}$ and $a = v_+ v_-$. □

7.5. **Products of Hankel operators in Schatten-von Neumann classes.** Now we prove simple sufficient conditions guaranteeing the membership of products of two Hankel operators in Schatten-von Neumann classes.

Lemma 7.8. *Let $1 < p, q < \infty$ and $0 < \alpha, \beta < 1$.*

- (a) *If $\alpha \geq 1/p$ and $b, c \in K_{p,0}^{\alpha,0}$, then the operators $H(\widetilde{c})H(b)$ and $H(b)H(\widetilde{c})$ belong to the Schatten-von Neumann class $\mathcal{C}_p(H^2)$.*
- (b) *If $\beta \geq 1/q$ and $b, c \in K_{0,q}^{0,\beta}$, then the operators $H(\widetilde{c})H(b)$ and $H(b)H(\widetilde{c})$ belong to the Schatten-von Neumann class $\mathcal{C}_q(H^2)$.*

Proof. (a) If $c \in K_{p,0}^{\alpha,0}$, then $Qa \in B_p^\alpha$. Since $\alpha \geq 1/p$, we have $B_p^\alpha \subset B_p^{1/p}$. From (1.6) we conclude that $H(\widetilde{c}) \in \mathcal{C}_p(H^2)$. Thus $H(\widetilde{c})H(b), H(b)H(\widetilde{c}) \in \mathcal{C}_p(H^2)$.

(b) The proof is analogous. □

Lemma 7.9. *Suppose* $1 \le r < \infty$, $1 < p, q < \infty$, $0 < \alpha, \beta < 1$, *and*

$$1/p + 1/q = \alpha + \beta \in (0, 1], \quad -1/2 < \alpha - 1/p < 1/2. \tag{7.2}$$

If $1/r = 1/p + 1/q$ *and* $b, c \in K_{p,q}^{\alpha,\beta}$, *then the operators* $H(\widetilde{c})H(b)$ *and* $H(b)H(\widetilde{c})$ *belong to the Schatten-von Neumann class* $\mathcal{C}_r(H^2)$.

Proof. Put $\gamma := \alpha - 1/p = 1/q - \beta$. If $b, c \in K_{p,q}^{\alpha,\beta}$, then Qb, Qc belong to $B_p^{1/p+\gamma}$ and Pb, Pc are in $B_q^{1/q-\gamma}$. From (7.2) we easily get

$$\min\{\gamma, 0\} > \max\{-1/2, -1/p\}, \quad \max\{\gamma, 0\} < \min\{1/2, 1/q\}.$$

Then, by Theorem 4.9,

$$QM(c)P \in \mathcal{C}_p(\ell_2^{\gamma,0}), \ QM(c)P \in \mathcal{C}_p(\ell_2^{0,\gamma}), \ PM(b)Q \in \mathcal{C}_q(\ell_2^{\gamma,0}), \ PM(b)Q \in \mathcal{C}_q(\ell_2^{0,\gamma}).$$

Since $1/r = 1/p + 1/q$, from Lemma 2.2(c) we deduce that

$$PM(b)QM(c)P \in \mathcal{C}_r(\ell_2^{\gamma,0}), \quad QM(c)PM(b)Q \in \mathcal{C}_r(\ell_2^{0,\gamma}).$$

Applying the flip operator to the second operator, we obtain

$$JQM(c)PM(b)QJ \in \mathcal{C}_r(\ell_2^{\gamma,0}).$$

Considering the compressions of the above operators to $\ell_2(\mathbb{Z}_+) \sim H^2$, we finally get $H(b)H(\widetilde{c}), H(\widetilde{c})H(b) \in \mathcal{C}_r(H^2)$. □

Now we are ready to finish the proof of our last main result.

Proof of Theorem 1.7(b). From Theorem 1.7(a) we know that $b, c \in K_{N\times N}$. Let $b = (b_{ij})_{i,j=1}^N$ and $c = (c_{kl})_{k,l=1}^N$. Since $b_{ij}, c_{kl} \in K$, from Lemmas 7.8 and 7.9 we conclude that $H(\widetilde{c_{kl}})H(b_{ij})$ and $H(b_{ij})H(\widetilde{c_{kl}})$ belong to $\mathcal{C}_{1/\lambda}(H^2)$ for all $i, j, k, l \in \{1, \dots, N\}$, where λ is defined by (1.11). Hence $H(\widetilde{c})H(b), H(b)H(\widetilde{c}) \in \mathcal{C}_{1/\lambda}(H^2_N)$. Since $m \ge 1/\lambda$, we finally get $H(\widetilde{c})H(b), H(b)H(\widetilde{c}) \in \mathcal{C}_m(H^2_N)$ by Lemma 2.2(a). □

Proof of Theorem 1.7(c). It is only necessary to verify the hypotheses (i) and (ii) of Lemma 7.1. By Theorem 1.7(a), $u_-, v_- \in G(K \cap \overline{H^\infty})_{N\times N}$, $u_+, v_+ \in G(K \cap H^\infty)_{N\times N}$. Hence $u_-, v_- \in G(\overline{H^\infty})_{N\times N}$, $u_+, v_+ \in G(H^\infty)_{N\times N}$, and $u_\pm \in GK_{N\times N}$. From this fact and Theorem 1.4 we see that $u_- \in G(C + H^\infty)_{N\times N}$ or $u_+ \in G(C + \overline{H^\infty})_{N\times N}$. Thus, the hypotheses (i) and (ii) are met. By Theorem 1.7(b), $H(\widetilde{c})H(b)$ and $H(b)H(\widetilde{c})$ belong to $\mathcal{C}_m(H^2_N)$. Applying Theorem 7.2, we get (1.13). □

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Received 27/12/2006