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ON SOLUTIONS TO "ALMOST EVERYWHERE"— EULER–LAGRANGE EQUATION IN SOBOLEV SPACE W_2^1

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Dedicated to the memory of Mark Krein.

Abstract. It is known, that if the Euler–Lagrange variational equation is fulfilled everywhere in classical case C^1 then it's solution is twice continuously differentiable. The present note is devoted to the study of a similar problem for the Euler–Lagrange equation in the Sobolev space W_2^1.

1. Introduction. Preliminaries

The well-known results of I. V. Skrypnik [1] show that in the "Hilbert" case W_2^1, the variational functionals have special differential properties. In this case the Euler–Lagrange functional is not, with the exception of degenerate case, twice Frechet differentiable.

A thorough analysis of the situation shows, that the well-definiteness conditions of the basic variational functional in a Sobolev space are already connected to an additional requirement of "pseudoquadraticity" of the integrand with respect to y'. In addition, the continuity of the functional in the classical "Banach" case corresponds to the K-continuity, differentiability to the K-differentiability, and so on. For the "Sobolev" case W_2^1, the conditions of well-definiteness, compact continuity, compact and twice compact differentiability of variational functionals were obtained in [2, 3].

The situation with extrema of variational functionals in W_2^1 is analogous. In this case the extrema are, as a rule, ([2, 4, 5]) not local, but compact (K-extrema). In the papers [6–10] various both necessary and sufficient conditions of K-extrema for variational functionals in W_2^1 are considered.

Let us give necessary definitions [2–5].

In what follows, E, Y, Z be real Banach spaces.

Definition 1.1. A functional $\Phi : E \to \mathbb{R}$ is called *compactly differentiable* at $y \in E$ if, for every absolutely convex compact set $C \subset E$, the restriction of Φ to $(y + \text{span } C)$ is Frechet differentiable with respect to the norm $\|\cdot\|_C$ induced by C.

We call a K-differentiable functional Φ *twice K-differentiable* at a point $y \in E$, if, for every absolutely convex compact sets C_1 and C_2, there exists a bilinear form $g_{C_1C_2}$ continuous on span $C_1 \times$ span C_2 such that

$$(\Phi'_K(y+h) - \Phi'_K(y)) \cdot k = g_{C_1C_2}(y) \cdot (h, k) + o(\|h\|_{C_1} \cdot \|k\|_{C_2}).$$

Here Φ'_K and Φ''_K are the first K-derivative and the second K-derivative, respectively.

Definition 1.2. We call a functional

$$\varphi : [a; b] \times Y \times Z \to \mathbb{R}$$

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Weierstrass K-quadratic with respect to z ($\varphi \in W_2^K(z)$) if for every convex compact set $C_Y \subset Y$ there exists an absolutely convex compact set $C_Z \subset Z$ such that

(i) φ is continuous on $[a;b] \times C_Y \times C_Z$;

(ii) $\varphi(x,y,z)/\|z\|^2$ is uniform continuous and bounded on $[a;b] \times C_Y \times (Z \backslash C_Z)$.

Let us formulate a condition of the twice K-differentiability of a variational functional [2, 5].

Theorem 1.3. *Let $f(x,y,z)$ be a twice continuously Frechet differentiable function on $[a;b] \times E \times E$, $\Gamma(f)$ the Hessian of f in the variables y and z. If $\Gamma(f) \in W_2^K(z)$, then the variational functional*

$$\Phi(y) = \int_a^b f(x, y(x), y'(x))\, dx, \quad y \in W_2^1([a;b], E),$$

is twice K-differentiable, and

$$\Phi''_K(y)(h,k) = \int_a^b \Big[\frac{\partial^2 f}{\partial y^2}(x,y,y')(h,k) + \frac{\partial^2 f}{\partial y \partial z}(x,y,y')\big((h',k) + (h,k')\big) + \frac{\partial^2 f}{\partial z^2}(x,y,y')(h',k') \Big] dx.$$

In particular in [2, 5], the classical Euler–Lagrange variational equation is obtained in the following form.

Theorem 1.4. *If, under the hypothesis of Theorem 1.3, $y(\cdot) \in \overset{\circ}{W}{}_2^2([a;b], E)$, then $\Phi'_K(y) = 0$ if and only if the Euler–Lagrange variational equation*

$$\frac{\partial f}{\partial y}(x,y,y') - \frac{d}{dx}\left(\frac{\partial f}{\partial z}(x,y,y') \right) = 0 \tag{1}$$

is fulfilled a.e. on $[a;b]$.

In the classical case $y(\cdot) \in C^1$, as is known [11], fulfillment of the variational equation (1) everywhere implies twice continuous differentiability of $y(\cdot)$. The present note is devoted to the solution of an analogous problem for "a.e."— Euler–Lagrange equation in the Sobolev space W_2^1.

2. Main results

At first, we need some generalization of the mean value theorem in locally convex spaces (LCS) to the case of mappings of several real variables.

Lemma 2.1. *Let E be a complete real LCS, $F : \prod_{i=1}^{n} [x_i; x_i + \Delta x_i] =: [\overline{x}; \overline{x} + \overline{\Delta x}] \to E$. If the mapping F is differentiable on $[\overline{x}; \overline{x} + \overline{\Delta x}]$, then*

$$F(\overline{x} + \overline{\Delta x}) - F(\overline{x}) \in \sum_{i=1}^{n} \Delta x_i \cdot \overline{\operatorname{conv}}\, \frac{\partial F}{\partial x_i}([\overline{x}; \overline{x} + \overline{\Delta x}]). \tag{2}$$

Proof. Following to a standard scheme, we apply the classical mean value theorem [12] to the auxiliary function $\widetilde{F}(t) = F(\overline{x} + t \cdot \overline{\Delta x})$, $0 \le t \le 1$,

$$\begin{aligned} F(\overline{x} + \overline{\Delta x}) - F(\overline{x}) &= \widetilde{F}(1) - \widetilde{F}(0) \in \overline{\operatorname{conv}}\, \widetilde{F}'([0;1]) \\ &= \overline{\operatorname{conv}} \left\{ \nabla F(\overline{x} + t \cdot \overline{\Delta x}) \cdot \overline{\Delta x} \,\middle|\, 0 \le t \le 1 \right\} \\ &= \left(\overline{\operatorname{conv}} \left\{ \nabla F(\overline{x} + t \cdot \overline{\Delta x}) \,\middle|\, 0 \le t \le 1 \right\} \right) \cdot \overline{\Delta x} \\ &\subset \left(\overline{\operatorname{conv}}\, \nabla F([\overline{x}; \overline{x} + \overline{\Delta x}]) \right) \cdot \overline{\Delta x} \subset \left\{ \overline{\operatorname{conv}}\, \frac{\partial F}{\partial x_i}([\overline{x}; \overline{x} + \overline{\Delta x}]) \right\}_{i=1}^{n} \cdot \overline{\Delta x} \\ &= \sum_{i=1}^{n} \Delta x_i \cdot \overline{\operatorname{conv}}\, \frac{\partial F}{\partial x_i}([\overline{x}; \overline{x} + \overline{\Delta x}]). \end{aligned}$$

□

The following basic result of the work is not directly connected to the Euler–Lagrange variational equation.

Theorem 2.2. *Let E be a complete real LCS, $f : [a;b] \times E \times E \to \mathbb{R}$, $u = f(x,y,z)$ be a C^2-mapping, a mapping $y(\cdot) : [a;b] \to E$ be everywhere continuous and almost everywhere differentiable on $[a;b]$. If*

(i) *the mapping $\frac{\partial f}{\partial z}(x, y(x), y'(x))$ is differentiable a.e. on $[a;b]$;*

(ii) *the operators $\frac{\partial^2 f}{\partial z^2}(x, y(x), y'(x))$ are continuously invertible for a.e. $x \in [a;b]$,*

then the function $y(\cdot)$ is twice approximately differentiable almost everywhere on $[a;b]$, and

$$\begin{aligned} y''_{ap}(x) = (y')'_{ap} &\overset{\text{a.e.}}{=} \left(\frac{\partial^2 f}{\partial z^2}(x,y,y') \right)^{-1} \\ &\times \left[\frac{d}{dx} \left(\frac{\partial f}{\partial z}(x,y,y') \right) - \frac{\partial^2 f}{\partial z \partial x}(x,y,y') - \frac{\partial^2 f}{\partial z \partial y}(x,y,y') \cdot y' \right]. \end{aligned} \tag{3}$$

Proof. Applying (2) to the function $F(x,y,z) = \frac{\partial f}{\partial z}(x,y,z)$ on $[x; x+\Delta x] \times [y; y+\Delta y] \times [z; z+\Delta z] =: [\overline{h}; \overline{h} + \overline{\Delta h}]$ we obtain

$$\begin{aligned} \frac{\partial f}{\partial z}(x+\Delta x, y+\Delta y, z+\Delta z) - \frac{\partial f}{\partial z}(x,y,z) &\in \overline{\operatorname{conv}} \frac{\partial^2 f}{\partial z \partial x}([\overline{h}; \overline{h} + \overline{\Delta h}]) \cdot \Delta x \\ &+ \overline{\operatorname{conv}} \frac{\partial^2 f}{\partial z \partial y}([\overline{h}; \overline{h} + \overline{\Delta h}]) \cdot \Delta y + \overline{\operatorname{conv}} \frac{\partial^2 f}{\partial z^2}([\overline{h}; \overline{h} + \overline{\Delta h}]) \cdot \Delta z. \end{aligned} \tag{4}$$

Fix a point $x \in [a;b]$ in which y' exists and is approximately continuous and the conditions (i)–(ii) of Theorem 2.2 are fulfilled. Choose a subset $A \subset [a;b]$, having x as a density point [13], such that $y'(x+\Delta x) \to y'(x)$ as $x + \Delta x \to x$ along A. Put in (3) $y = y(x)$, $\Delta y = y(x+\Delta x) - y(x)$, $\Delta z = y'(x+\Delta x) - y'(x)$; in addition, $\Delta y \to 0$ as $\Delta x \to 0$ by continuity of $y(\cdot)$, $\Delta z \to 0$ as $x + \Delta x \overset{A}{\to} x$ by approximate continuity of y' at the point x. We get

$$\begin{aligned} &\frac{\partial f}{\partial z}(x+\Delta x, y(x+\Delta x), y'(x+\Delta x)) - \frac{\partial f}{\partial z}(x, y(x), y'(x)) \\ &\quad \in \overline{\operatorname{conv}}\, \frac{\partial^2 f}{\partial z \partial x}([\overline{h}; \overline{h} + \overline{\Delta h}]) \cdot \Delta x + \overline{\operatorname{conv}}\, \frac{\partial^2 f}{\partial z \partial y}([\overline{h}; \overline{h} + \overline{\Delta h}]) \cdot \Delta y(x) \\ &\quad + \overline{\operatorname{conv}}\, \frac{\partial^2 f}{\partial z^2}([\overline{h}; \overline{h} + \overline{\Delta h}]) \cdot \Delta y'(x). \end{aligned} \tag{5}$$

By setting $F(x) = \frac{\partial f}{\partial z}(x, y(x), y'(x))$ in (5), dividing both sides of the inclusion (5) by Δx and passing to the limit as $\Delta x \to 0$, taking into account that $f \in C^2$ and condition

(ii) of Theorem 2.2, we come to the existence of the limit $\Delta y'(x)/\Delta x$ as $\Delta x \to 0$ along A and, hence, to the equality

$$F'(x) \overset{\text{a.e.}}{=} \frac{\partial^2 f}{\partial z \partial x}(x,y,y') + \frac{\partial^2 f}{\partial z \partial y}(x,y,y') \cdot y' + \frac{\partial^2 f}{\partial z^2}(x,y,y') \cdot (y')'_{ap},$$

whence (3) follows. □

Corollary 2.3. *If, under the hypothesis of Theorem 2.2, the function $y(\cdot)$ satisfies the Euler–Lagrange variational equation a.e. on $[a;b]$,*

$$\frac{\partial f}{\partial y}(x,y,y') - \frac{d}{dx}\left[\frac{\partial f}{\partial z}(x,y,y')\right] \overset{\text{a.e.}}{=} 0, \tag{6}$$

then the function $y''_{ap}(x)$ is approximately continuous in the same points where $y'(x)$ is approximately continuous and the operator $\frac{\partial^2 f}{\partial z^2}(x,y(x),y'(x))$ is continuously invertible.

Proof. It follows from (3) and (5) that

$$y''_{ap}(x) \overset{\text{a.e.}}{=} \left(\frac{\partial^2 f}{\partial z^2}(x,y,y')\right)^{-1}\left[\frac{\partial f}{\partial y}(x,y,y') - \frac{\partial^2 f}{\partial z \partial x}(x,y,y') - \frac{\partial^2 f}{\partial z \partial y}(x,y,y') \cdot y'\right]. \tag{7}$$

If x_0 is a density point of A and $y'(x)$ is continuous at a point y_0 along A, then the right hand-side of (7) is also, obviously, continuous at the point x_0 along A, whence the statement of Corollary follows. □

There arises a natural question on conditions of the existence of the usual (not approximate) derivative $y''(x)$.

Corollary 2.4. *If, under the hypothesis of Theorem 2.2, $y'(x)$ is continuous a.e. on $[a;b]$, then $y''(x)$ exists a.e. on $[a;b]$.*

Proof. Let $x \in [a;b]$ be a point of continuity of y', in which the function $\frac{\partial f}{\partial z}(x,y,y')$ is differentiable. Then it is possible to pass to the limit in (5) for Δx arbitrarily approaching zero. Whence we obtain

$$y''(x) \overset{\text{a.e.}}{=} \left(\frac{\partial^2 f}{\partial z^2}(x,y,y')\right)^{-1}\left[\frac{\partial f}{\partial y}(x,y,y') - \frac{\partial^2 f}{\partial z \partial x}(x,y,y') - \frac{\partial^2 f}{\partial z \partial y}(x,y,y') \cdot y'\right].$$

□

Consider now the case of the Sobolev space $y(\cdot) \in W_2^1$. In this situation, it seems to be natural to assume that the function $y(\cdot) \in W_2^1$ satisfying the "a.e."— Euler–Lagrange equation belongs to the class W_2^2. However this is not the case.

Example 2.5. Consider a simplest variational functional

$$\Phi(y) = \int_0^1 (y')^2 dx, \quad y(\cdot) \in W_2^1([0;1],\mathbb{R}).$$

Here $f(x,y,z) = z^2$ and the Euler–Lagrange equation (6) has the form

$$y''(x) \overset{\text{a.e.}}{=} 0. \tag{8}$$

Let $\chi(t)$ be the "Cantor ladder" [13] on $[0;1]$, $y_0(x) = \int_0^x \chi(t)dt$, $0 \le x \le 1$. Then $y_0''(x) = \chi'(x) = 0$ a.e. on $[0;1]$, i.e. $y_0(\cdot)$ satisfies the equation (8). However, under the hypothesis of Example, $y_0(\cdot) \notin W_2^2([0;1],\mathbb{R})$, as $y_0(\cdot)$ is not an absolutely continuous function.

Let us note in conclusion that the question about sufficient conditions for the extremal $y(\cdot)$ from W_2^1 to be in W_2^2 is actually solved in the work of I. V. Orlov “Compact ellipsoids and compact extrema” (this issue, Example 3.5). It is shown in the work that if a compact elliptic extremum is indeed realized on an extremal $y_0(\cdot)$ and the lengths of the semiaxis of the appropriate compact ellipsoid admit the estimate $\varepsilon_k = O(1/k)$, then the extremum realizes as a local one in the Sobolev space W_2^2. In particular in this case, $y_0(\cdot) \in W_2^2$.

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