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DIRECT THEOREMS IN THE THEORY OF APPROXIMATION OF VECTORS IN A BANACH SPACE WITH EXPONENTIAL TYPE ENTIRE VECTORS

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Abstract. For an arbitrary operator A on a Banach space $\mathfrak{X}$ which is the generator of a C_0–group with certain growth condition at infinity, direct theorems on connection between the degree of smoothness of a vector $x \in \mathfrak{X}$ with respect to the operator A, the rate of convergence to zero of the best approximation of x by exponential type entire vectors for the operator A, and the k-module of continuity are established. The results allow to obtain Jackson-type inequalities in a number of classic spaces of periodic functions and weighted L_p spaces.

1. Introduction

Direct and inverse theorems establishing a relationship between the degree of smoothness of a function with respect to a differentiation operator and the rate of convergence to zero of its best approximation by trigonometric polynomials are well known in the theory of approximation of periodic functions. Jackson's inequality is one among such results.

N. P. Kupcov proposed a generalized notion of the module of continuity, expanded onto C_0-groups in a Banach space [1]. Using this notion, N. P. Kupcov [1] and A. P. Terehin [2] proved the generalized Jackson's inequalities for the cases of a bounded group and s-regular group. Recall that a group $\{U(t)\}_{t\in\mathbb{R}}$ is called s-regular if the resolvent of its generator A satisfies the condition $\exists \theta \in \mathbb{R}: \quad \|R_\lambda(e^{i\theta}A^s)\| \le \frac{C}{\mathrm{Im}\lambda}$.

G. V. Radzievskii studied the direct and inverse theorems [3, 4], using the notion of a K-functional instead of module of continuity, but it should be noted that the K-functional has two-sided estimates with regard to the module of continuity at least for bounded C_0-groups.

In the papers [5, 6] and [7] the authors investigated the case of a group of unitary operators in a Hilbert space and established Jackson-type inequalities in Hilbert spaces and their rigs. These inequalities are used to estimate the rate of convergence to zero of the best approximation of both finite and infinite smoothness vectors for the operator A by exponential type entire vectors.

We consider the C_0-groups, generated by the so-called *non-quasianalytic operators* [8], i.e. the groups satisfying

$$\int_{-\infty}^{\infty} \frac{\ln\|U(t)\|}{1+t^2}\,dt < \infty. \tag{1.1}$$

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As was shown in [5], the set of exponential type entire vectors for the non-quasianalytic operator A is dense in $\mathfrak{X}$, so the problem of approximation by exponential type entire vectors is correct. On the other hand, it was shown in [9] that condition (1.1) is close to the necessary one, so in the case when (1.1) doesn't hold, the class of entire vectors isn't necessary dense in $\mathfrak{X}$, and the corresponding approximation problem loses its meaning.

The purpose of this work is to obtain Jackson-type inequalities in the case where a vector of a Banach space is approximated by exponential type entire vectors for a non-quasianalytic operator, and, in particular, Jackson-type inequalities in various classical function spaces.

2. Preliminaries

Let A be a closed linear operator with dense domain of definition $\mathcal{D}(A)$ in the Banach space $(\mathfrak{X}, \|\cdot\|)$ over the field of complex numbers.

Let $C^\infty(A)$ denotes the set of all infinitely differentiable vectors of the operator A, i.e.

$$C^\infty(A) = \bigcap_{n\in\mathbb{N}_0} \mathcal{D}(A^n), \quad \mathbb{N}_0 = \mathbb{N}\cup\{0\}.$$

For a number $\alpha > 0$ we set

$$\mathfrak{E}^\alpha(A) = \left\{x \in C^\infty(A) \,|\, \exists c = c(x) > 0 \ \forall k \in \mathbb{N}_0 \ \left\|A^k x\right\| \le c\alpha^k\right\}.$$

The set $\mathfrak{E}^\alpha(A)$ is a Banach space with respect to the norm

$$\|x\|_{\mathfrak{E}^\alpha(A)} = \sup_{n\in\mathbb{N}_0} \frac{\|A^n x\|}{\alpha^n}.$$

Then $\mathfrak{E}(A) = \bigcup_{\alpha>0} \mathfrak{E}^\alpha(A)$ is a linear locally convex space with respect to the topology of the inductive limit of the Banach spaces $\mathfrak{E}^\alpha(A)$:

$$\mathfrak{E}(A) = \liminf_{\alpha\to\infty} \mathfrak{E}^\alpha(A).$$

Elements of the space $\mathfrak{E}(A)$ are called exponential type entire vectors of the operator A. The type $\sigma(x, A)$ of a vector $x \in \mathfrak{E}(A)$ is defined as the number

$$\sigma(x, A) = \inf\{\alpha > 0 : x \in \mathfrak{E}^\alpha(A)\} = \limsup_{n\to\infty} \|A^n x\|^{\frac{1}{n}}.$$

Example 2.1. Let $\mathfrak{X}$ is one of the $L_p(2\pi)$ $(1 \le p < \infty)$ spaces of integrable in p-th degree over $[0, 2\pi]$, 2π-periodical functions or the space $C(2\pi)$ of continuous 2π-periodical functions (the norm in $\mathfrak{X}$ is defined in a standard way), and let A is the differentiation operator in the space $\mathfrak{X}$ ($\mathcal{D}(A) = \{x \in \mathfrak{X}\cap AC(\mathbb{R}) : x' \in \mathfrak{X}\}$; $(Ax)(t) = \frac{dx}{dt}$, where $AC(\mathbb{R})$ denotes the space of absolutely continuous functions over $\mathbb{R}$). It can be proved that in such case the space $\mathfrak{E}(A)$ coincides with the space of all trigonometric polynomials, and for $y \in \mathfrak{E}(A)$ $\sigma(y, A) = \deg(y)$, where $\deg(y)$ is the degree of the trigonometric polynomial y.

In what follows, we always assume that the operator A is the generator of the group of linear continuous operators $\{U(t) : t \in \mathbb{R}\}$ of class C_0 on $\mathfrak{X}$. We recall that belonging of the group to the C_0 class means that for every $x \in \mathfrak{X}$ the vector-function $U(t)x$ is continuous on $\mathbb{R}$ with respect to the norm of the space $\mathfrak{X}$.

For $t \in \mathbb{R}_+$, we set

$$M_U(t) := \sup_{\tau\in\mathbb{R},\,|\tau|\le t} \|U(\tau)\|.$$

The estimation $\|U(t)\| \le Me^{\omega t}$ for some $M, \omega \in \mathbb{R}$ implies $M_U(t) < \infty$ $(\forall t \in \mathbb{R}_+)$. It is easy to see that the function $M_U(\cdot)$ has the following properties:

1) $M_U(t) \ge 1$, $t \in \mathbb{R}_+$;
2) $M_U(\cdot)$ is monotonically non-decreasing on $\mathbb{R}_+$;
3) $M_U(t_1 + t_2) \le M_U(t_1)M_U(t_2)$, $t_1, t_2 \in \mathbb{R}_+$.

According to [1], for $x \in \mathfrak{X}$, $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$ we set

$$\omega_k(t, x, A) = \sup_{0 \le \tau \le t} \left\| \Delta_\tau^k x \right\|, \tag{2.1}$$

where

$$\Delta_h^k = (U(h) - \mathbb{I})^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{j}{k} U(jh), \quad k \in \mathbb{N}_0, \quad h \in \mathbb{R} \quad (\Delta_h^0 \equiv 1). \tag{2.2}$$

Moreover, let

$$\widetilde{\omega}_k(t, x, A) = \sup_{|\tau| \le t} \left\| \Delta_\tau^k x \right\|. \tag{2.3}$$

Remark 2.1. It is easy to see that in the case of the bounded group $\{U(t)\}$ ($\|U(t)\| \le M$, $t \in \mathbb{R}$) the quantities $\omega_k(t, x, A)$ and $\widetilde{\omega}_k(t, x, A)$ are equivalent within constant factor ($\omega_k(t, x, A) \le \widetilde{\omega}_k(t, x, A) \le M\, \omega_k(t, x, A)$), and in the case of isometric group ($\|U(t)\| \equiv 1$, $t \in \mathbb{R}$) these quantities coincide.

It is immediate from the definition of $\widetilde{\omega}_k(t, x, A)$ that for $k \in \mathbb{N}$

1) $\widetilde{\omega}_k(0, x, A) = 0$;
2) for fixed x the function $\widetilde{\omega}_k(t, x, A)$ is non-decreasing and is continuous by the variable t on $\mathbb{R}_+$;
3) $\widetilde{\omega}_k(nt, x, A) \le \big(1 + (n-1)M_U((n-1)t)\big)^k \widetilde{\omega}_k(t, x, A)$ ($n \in \mathbb{N}$, $t > 0$);
4) $\widetilde{\omega}_k(\mu t, x, A) \le \big(1 + \mu M_U(\mu t)\big)^k \widetilde{\omega}_k(t, x, A)$ ($\mu, t > 0$);
5) for fixed $t \in \mathbb{R}_+$ the function $\widetilde{\omega}_k(t, x, A)$ is continuous in x.

For arbitrary $x \in \mathfrak{X}$ we set, according to [7, 6],

$$\mathcal{E}_r(x, A) = \inf_{y \in \mathfrak{E}(A)\,:\, \sigma(y, A) \le r} \|x - y\|, \quad r > 0,$$

i.e. $\mathcal{E}_r(x, A)$ is the best approximation of the element x by exponential type entire vectors y of the operator A for which $\sigma(y, A) \le r$. For fixed x $\mathcal{E}_r(x, A)$ does not increase and $\mathcal{E}_r(x, A) \to 0$, $r \to \infty$ for every $x \in \mathfrak{X}$ if and only if the set $\mathfrak{E}(A)$ of exponential type entire vectors is dense in $\mathfrak{X}$. Particularly, as indicated above, the set $\mathfrak{E}(A)$ is dense in $\mathfrak{X}$ if the group $\{U(t) : t \in \mathbb{R}\}$ belongs to non-quasianalytic class.

3. Abstract Jackson's inequality in a Banach space

Theorem 3.1. *Suppose that $\{U(t) : t \in \mathbb{R}\}$ satisfies condition (1.1). Then $\forall k \in \mathbb{N}$ there exists a constant $\mathbf{m}_k = \mathbf{m}_k(A) > 0$, such that $\forall x \in \mathfrak{X}$ the following inequality holds:*

$$\mathcal{E}_r(x, A) \le \mathbf{m}_k \cdot \tilde{\omega}_k\left(\frac{1}{r}, x, A\right), \quad r \ge 1. \tag{3.1}$$

Remark 3.1. If, additionally, the group $\{U(t)\}$ is bounded ($M_U(t) \le \widetilde{M} < \infty$, $t \in \mathbb{R}$), then the assumption $r \ge 1$ can be changed to $r > 0$.

Integral kernels, constructed in [10], will be used in the proving of the theorem. Moreover, we need additional properties of these kernels, lacking in [10]. The following lemma shows how these kernels are constructed and continues the investigation of their properties.

In what follows we denote as $\mathfrak{Q}$ the class of functions $\alpha : \mathbb{R} \mapsto \mathbb{R}$, satisfying the following conditions:

I) $\alpha(\cdot)$ *is measurable and bounded on any segment* $[-T, T] \subset \mathbb{R}$.
II) $\alpha(t) > 0$, $t \in \mathbb{R}$.
III) $\alpha(t_1 + t_2) \le \alpha(t_1)\alpha(t_2)$, $\quad t_1, t_2 \in \mathbb{R}$.
IV) $\int_{-\infty}^{\infty} |\ln(\alpha(t))| / (1 + t^2)\, dt < \infty$.

Lemma 3.1. *Let $\alpha \in \mathfrak{Q}$. Then there exists such entire function $\mathcal{K}_\alpha : \mathbb{C} \mapsto \mathbb{C}$ that*

1) $\mathcal{K}_\alpha(t) \geq 0,\ t \in \mathbb{R}$;
2) $\int_{-\infty}^{\infty} \mathcal{K}_\alpha(t)\, dt = 1$;
3) $\forall r > 0\ \exists c_r = c_r(\alpha) > 0\ \ \forall z \in \mathbb{C}\quad |\mathcal{K}_\alpha(rz)| \leq c_r \frac{e^{r|\operatorname{Im} z|}}{\alpha(|z|)}$.

Proof. Without lost of generality we may assume that the function $\alpha(t)$ satisfies additional conditions

V) $\alpha(t) \geq 1$, $t \in \mathbb{R}$; [1]
VI) $\alpha(t)$ *is even on* $\mathbb{R}$ *and is monotonically increasing on* $\mathbb{R}_+$;
VII) $\left\|\alpha^{-1}\right\|_{L_1(\mathbb{R})} = \int_{-\infty}^{\infty} |\alpha^{-1}(t)| dt < \infty$.

It is easy to verify that assumptions V),VII) and condition that the function $\alpha(t)$ is even in VI) don't confine the general case if one examined the function $\alpha_1(t) = \widetilde{\alpha}(t)\widetilde{\alpha}(-t)$, where $\widetilde{\alpha}(t) = (1+\alpha(t))(1+t^2)$. In [11, Theorems 1 and 2] it has been proved that the monotony condition on $\alpha(t)$ in VI) doesn't confine the general case too.

It follows from VII) that

$$\alpha(t) \to \infty, \quad t \to \infty. \tag{3.2}$$

Let $\beta(t) = \ln \alpha(t),\ t \in \mathbb{R}$. Conditions III)–VII) and (3.2) lead to conclusion that

$$\beta(t) > 0, \quad \beta(-t) = \beta(t), \quad \beta(t) \to \infty, \quad t \to \infty;$$

$$\beta(t_1 + t_2) \leq \beta(t_1) + \beta(t_2), \quad t_1, t_2 \in \mathbb{R} \tag{3.3}$$

$$\int_1^\infty \frac{\beta(t)}{t^2}\, dt < \infty. \tag{3.4}$$

Because of (3.3) there exists limit $\lim_{t\to\infty} \frac{\beta(t)}{t}$. And, by virtue of (3.4)

$$\lim_{t\to\infty} \frac{\beta(t)}{t} = 0. \tag{3.5}$$

Also, using (3.4) it is easy to check that

$$\sum_{k=1}^{\infty} \frac{\beta(k)}{k^2} < \infty, \tag{3.6}$$

moreover, all terms of the series (3.6) are positive. From the convergence of series (3.6) follows the existence of such sequence $\{Q_n\}_{n=1}^\infty \subset \mathbb{R}$ that $Q_n > 1$, $Q_n \to \infty$, $n \to \infty$ and

$$\sum_{k=1}^{\infty} \frac{\beta(k)}{k^2} Q_k = S < \infty. \tag{3.7}$$

We set

$$a_k := \frac{\beta(k) Q_k}{S\, k^2}, \quad k \in \mathbb{N}.$$

The definition of a_k and (3.7) result in equality

$$\sum_{k=1}^{\infty} a_k = 1. \tag{3.8}$$

We construct the sequence of functions, which, obviously, are entire for every $n \in \mathbb{N}$

$$f_n(z) := \prod_{k=1}^{n} P_k(z), \quad \text{where} \quad P_k(z) = \left(\frac{\sin \frac{a_k z}{2}}{\frac{a_k z}{2}}\right)^2, \quad z \in \mathbb{C}, \quad n \in \mathbb{N}.$$

Similarly to the proof of the Denjoy-Carleman theorem [12, p.378] it can be concluded that the sequence of (entire) functions $f_n(z)$ converges uniformly to the function

$$f(z) = \prod_{k=1}^{\infty} \left(\frac{\sin \frac{a_k z}{2}}{\frac{a_k z}{2}}\right)^2, \quad z \in \mathbb{C}$$

[1] As shown in [8], for non-quasianalytic groups the condition $\|U(t)\| \geq 1$ always holds, therefore in this paper the condition V) automatically takes place.

in every disk $\{z \in \mathbb{C} \,|\, |z| \le R\}$. Thus, by Weierstrass theorem, the function $f(z)$ is entire.

Using the inequality $|\sin z| \le \min(1,|z|)e^{|\operatorname{Im} z|}$, $z \in \mathbb{C}$ and taking (3.8) into account, when $z \in \mathbb{C}$ and $r > 0$, we receive

$$|f(rz)| = \prod_{k=1}^{\infty} \left| \frac{\sin \frac{a_k r z}{2}}{\frac{a_k r z}{2}} \right|^2 \le \prod_{k=1}^{\infty} \left(\frac{2}{a_k r|z|} \min\left(1, \frac{a_k r|z|}{2}\right) e^{\frac{1}{2} a_k r |\operatorname{Im} z|} \right)^2$$

$$= e^{r|\operatorname{Im} z|} \prod_{k=1}^{\infty} \min^2\left(1, \frac{2}{a_k r|z|}\right) \le e^{r|\operatorname{Im} |} \prod_{k=1}^{N} \min^2\left(1, \frac{2}{a_k r|z|}\right)$$

for every $N \in \mathbb{N}$. Using the inequality $\min(1,a) \cdot \min(1,b) \le \min(1,ab)$, we get

(3.9)

$$|f(rz)| \le e^{r|\operatorname{Im} z|} \min^2\left(1, \prod_{k=1}^{N} \frac{2}{a_k r|z|}\right) = e^{r|\operatorname{Im} z|} \min^2\left(1, \frac{2^N}{\left(\prod_{k=1}^{N} \frac{\beta(k) Q_k}{S\, k^2}\right) (r|z|)^N}\right)$$

$$= e^{r|\operatorname{Im} z|} \min^2\left(1, \frac{2^N N!}{\frac{\beta(1)}{1} \cdots \frac{\beta(N)}{N} \left(\frac{r}{S}\right)^N |z|^N Q_1 \cdots Q_N}\right).$$

Because of the condition $Q_n \to \infty$, $n \to \infty$ there exists such number $n(r) \in \mathbb{N}$ that

$$\forall n > n(r) \quad Q_n \ge \frac{4\sqrt{e} S}{r}. \tag{3.10}$$

It follows from (3.5) that there is $T_0 \in (0, \infty)$ such that

$$\forall\, t > T_0 \quad \frac{\beta(t)}{t} \le 1. \tag{3.11}$$

In [10] the following statement was proved:

$$\forall\, t_1, t_2 \in \mathbb{R}_+ \quad t_1 \le t_2 \;\Rightarrow\; \frac{\beta(t_1)}{t_1} \ge \frac{1}{2} \frac{\beta(t_2)}{t_2}. \tag{3.12}$$

Let $z \in \mathbb{C}$ and $|z| \ge \max\left(\beta^{[-1]}(n(r)), T_0\right)$, where $\beta^{[-1]}$ is the inverse function of the function β on $[0,\infty)$ (the inverse function $\beta^{[-1]}$ exists due to monotony of β on $[0,\infty)$). We substitute as N in (3.9) $N := [\beta(|z|)]$, where $[\cdot]$ denotes the integer part of a number. Then for $k \in \{1, \ldots, N\}$, in accordance with (3.11) and (3.12), we obtain $k \le N \le \beta(|z|) \le |z|$ and

$$\frac{\beta(k)}{k} \ge \frac{1}{2} \frac{\beta(|z|)}{|z|}. \tag{3.13}$$

Using (3.9), (3.10), (3.13), we find

$$|f(rz)| \le e^{r|\operatorname{Im} z|} \left(\frac{2^N N!}{\left(\frac{1}{2} \frac{\beta(|z|)}{|z|}\right)^N \left(\frac{r}{S}\right)^N |z|^N Q_1 \cdots Q_N} \right)^2$$

$$\le e^{r|\operatorname{Im} z|} \left(\frac{2^N N!}{\left(\frac{1}{2} \frac{N}{|z|}\right)^N \left(\frac{r}{S}\right)^N |z|^N Q_1 \cdots Q_N} \right)^2 = e^{r|\operatorname{Im} z|} \left(\frac{2^{2N} N!}{N^N \left(\frac{r}{S}\right)^N Q_1 \cdots Q_N} \right)^2$$

$$\le e^{r|\operatorname{Im} z|} \left(\frac{2^{2N}}{\left(\frac{r}{S}\right)^N Q_1 \cdots Q_N} \right)^2 = e^{r|\operatorname{Im} z|} \left(\frac{\left(\frac{4S}{r}\right)^N}{Q_1 \cdots Q_N} \right)^2.$$

Since $Q_n \ge 1$, the last inequality leads to

$$\begin{aligned} |f(rz)| &\le e^{r|\operatorname{Im} z|} \left(\frac{\left(\frac{4S}{r}\right)^N}{\left(\frac{4\sqrt{e}S}{r}\right)^{N-n(r)}} \right)^2 = e^{r|\operatorname{Im} z|} \left(\frac{4\sqrt{e}S}{r} \right)^{2n(r)} e^{-[\beta(|z|)]} \\ &\le e^{r|\operatorname{Im} z|} \left(\frac{4\sqrt{e}S}{r} \right)^{2n(r)} e^{-(\beta(|z|)-1)} = C_r^{(1)} \frac{e^{r|\operatorname{Im} z|}}{\alpha(|z|)}, \end{aligned} \tag{3.14}$$

where $C_r^{(1)} = e\left(\frac{4\sqrt{e}S}{r}\right)^{2n(r)}$. When $z \in \mathbb{C}$ and $|z| < \max\left(\beta^{[-1]}(n(r)), T_0\right)$, using (3.9), we get

$$|f(rz)| \le e^{r|\operatorname{Im} z|} = e^{r|\operatorname{Im} z|}\frac{\alpha(|z|)}{\alpha(|z|)} \le e^{r|\operatorname{Im} z|}\frac{C_r^{(2)}}{\alpha(|z|)}, \tag{3.15}$$

where $C_r^{(2)} = \alpha(\max(\beta^{[-1]}(n(r)), T_0))$. It follows from (3.14), (3.15) that

$$|f(rz)| \le e^{r|\operatorname{Im} z|}\frac{C_r^{(0)}}{\alpha(|z|)}, \quad z \in \mathbb{C}, \quad \text{where} \quad C_r^{(0)} = \max(C_r^{(1)}, C_r^{(2)}). \tag{3.16}$$

Inequality (3.16) and Condition VII) imply that $\|f\|_{L_1(\mathbb{R})} < \infty$. Thus it is enough to set $\mathcal{K}_\alpha(z) := \frac{1}{\|f\|_{L_1(\mathbb{R})}} f(z)$, $z \in \mathbb{C}$ and use (3.16) to finish the proof. □

Let $\alpha \in \mathfrak{Q}$, and $\mathcal{K}_\alpha : \mathbb{C} \mapsto \mathbb{C}$ is the function constructed by the function α in Lemma 3.1. We set

$$\mathcal{K}_{\alpha,r}(z) := r\mathcal{K}_\alpha(rz), \quad z \in \mathbb{C}, \quad r \in (0, \infty).$$

The Lemma 3.1 ensures us that the function $\mathcal{K}_{\alpha,r}$ has the following properties:

1) $\mathcal{K}_{\alpha,r}(t) \ge 0, \quad t \in \mathbb{R}$;
2) $\int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t)\,dt = 1$;
3) $\forall z \in \mathbb{C}\ |\mathcal{K}_{\alpha,r}(z)| \le rc_r\frac{e^{r|\operatorname{Im} z|}}{\alpha(|z|)}; \quad r > 0.$

Lemma 3.2. *$\forall r \in (0, \infty)$ there exists constant $\widetilde{c}_r = \widetilde{c}_r(\alpha) > 0$, such that $\forall n \in \mathbb{N}$ the following inequality holds:*

$$|\mathcal{K}_{\alpha,r}^{(n)}(t)| \le \widetilde{c}_r\frac{\sqrt{2\pi n}\,\alpha\left(\frac{n}{r}\right)}{\alpha(|t|)}r^n, \quad t \in \mathbb{R}.$$

Proof. In what follows in this proof we assume $t \in \mathbb{R}$, $r \in (0, \infty)$, $n \in \mathbb{N}$. Let

$$\gamma_{n,r}(t) := \left\{\zeta \in \mathbb{C} : |\zeta - t| = \frac{n}{r}\right\}.$$

Using Cauchy's integral theorem and Stirling's approximation for $n!$, we get

$$\begin{aligned}|\mathcal{K}_{\alpha,r}^{(n)}(t)| &\le \frac{n!}{2\pi}\oint_{\gamma_{n,r}(t)}\frac{|\mathcal{K}_{\alpha,r}(\xi)|}{|\xi - t|^{n+1}}|d\xi| = \frac{n!}{2\pi}\frac{r^{n+1}}{n^{n+1}}\oint_{\gamma_{n,r}(t)}|\mathcal{K}_{\alpha,r}(\xi)||d\xi| \\ &\le \frac{c^{(!)}r^{n+1}}{\sqrt{2\pi n}}e^{-n}\oint_{\gamma_{n,r}(t)}|\mathcal{K}_{\alpha,r}(\xi)||d\xi|,\end{aligned}$$

where

$$c^{(!)} = \sup_{k\in\mathbb{N}}\frac{k!}{\sqrt{2\pi k}}\left(\frac{e}{k}\right)^k < e^{1/12}.$$

Using property 3) of the function $K_{\alpha,r}$, the condition $t \in \mathbb{R}$ and conditions III), VI) of the function α, one can find from the last inequality

$$\begin{aligned}|\mathcal{K}_{\alpha,r}^{(n)}(t)| &\le \frac{c^{(!)}r^{n+1}}{\sqrt{2\pi n}}e^{-n}rc_r\oint_{\gamma_{n,r}(t)}\frac{e^{r|\operatorname{Im}\xi|}}{\alpha(|\xi|)}|d\xi| \\ &= \frac{c^{(!)}r^{n+1}}{\sqrt{2\pi n}}e^{-n}\frac{rc_r}{\alpha(|t|)}\oint_{\gamma_{n,r}(t)}\frac{e^{r|\operatorname{Im}(\xi-t)|}\alpha(|(t-\xi)+\xi|)}{\alpha(|\xi|)}|d\xi| \\ &\le \frac{c^{(!)}r^{n+1}}{\sqrt{2\pi n}}e^{-n}\frac{rc_r}{\alpha(|t|)}\oint_{\gamma_{n,r}(t)}e^{r|\operatorname{Im}(\xi-t)|}\alpha(|t-\xi|)|d\xi| \\ &\le \frac{c^{(!)}r^{n+1}}{\sqrt{2\pi n}}e^{-n}\frac{rc_r}{\alpha(|t|)}\oint_{\gamma_{n,r}(t)}e^n\alpha\left(\frac{n}{r}\right)|d\xi| = \widetilde{c}_r\frac{\sqrt{2\pi n}\,\alpha\left(\frac{n}{r}\right)}{\alpha(|t|)}r^n,\end{aligned}$$

where $\widetilde{c}_r = c^{(!)}rc_r$. □

Remark 3.2. If the function $\alpha(t)$ satisfies the conditions of Lemma 3.1, but, moreover, has the polynomial order of growth at infinity, i.e. $\exists m \in \mathbb{N}_0$, $\exists M > 0$:

$$\alpha(t) \le M(1+|t|)^{2m}, \quad t \in \mathbb{R}, \tag{3.17}$$

another integral kernel may be used:

$$\tilde{K}_\alpha(z) = \frac{1}{\mathrm{K}_m}\left(\frac{\sin\frac{z}{2m}}{\frac{z}{2m}}\right)^{2m}, \quad \mathrm{K}_m = \int_{-\infty}^{\infty}\left(\frac{\sin\frac{x}{2m}}{\frac{x}{2m}}\right)^{2m} dx.$$

In much the same way to the proving of the Lemmas 3.1 and 3.2 one can show that

$$\left|\tilde{K}_\alpha(rz)\right| \le \tilde{C}_r \frac{e^{r|\operatorname{Im} z|}}{\alpha(|z|)}, \quad \text{where} \quad \tilde{C}_r = \frac{M}{\mathrm{K}_m}\left(1+\frac{2m}{r}\right)^{2m},$$

and

$$\left|\tilde{K}^{(n)}_{\alpha,r}(t)\right| \le \tilde{c}_r \frac{\sqrt{2\pi n}\,\alpha(\frac{n}{r})}{\alpha(|t|)} r^n, \quad \text{where} \quad \tilde{c}_r = c^{(!)} r \tilde{C}_r,$$

that is to say, defined in such a way integral kernel satisfies Lemmas 3.1 and 3.2.

Proof of Theorem 3.1. Let the group $\{U(t) : t \in \mathbb{R}\}$ satisfies (1.1). Then it follows from [11, Theorems 1 and 2] that

$$\int_{-\infty}^{\infty} \frac{\ln\left(M_U(|t|)\right)}{1+t^2} dt < \infty. \tag{3.18}$$

We fix arbitrary $k \in \mathbb{N}$ and set

$$\alpha(t) := \left(M_U(|t|)\right)^k (1+|t|)^{k+2}, \quad t \in \mathbb{R}.$$

The function α is, obviously, even on $\mathbb{R}$. Condition (3.18) and the properties of the function $M_U(\cdot)$ imply $\alpha \in \mathfrak{Q}$, and, moreover,

$$\int_{-\infty}^{\infty} \frac{\left((1+|t|)M_U(|t|)\right)^k}{\alpha(t)} dt = \int_{-\infty}^{\infty} \frac{dt}{(1+|t|)^2} = 2. \tag{3.19}$$

Using Lemma 3.1 (or Remark 3.2 if $\alpha(t) \le M(1+|t|)^m$) for the function $\alpha(t)$, we construct the family of kernels $K_{\alpha,r}$.

In what follows, we assume $x \in \mathfrak{X}$, $r \in (0,\infty)$ and $n \in \{1,\dots,k\}$. We define

$$x_{r,n} := \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) U(nt) x \, dt.$$

Let $\nu \in \mathbb{N}_0$. Let's prove that $x_{r,n} \in C^\infty(A) = \bigcap_{\nu \in \mathbb{N}_0} \mathcal{D}(A^\nu)$ and

$$A^\nu x_{r,n} = \frac{(-1)^\nu}{n^\nu} \int_{-\infty}^{\infty} \mathcal{K}^{(\nu)}_{\alpha,r}(t) U(nt) x \, dt. \tag{3.20}$$

It follows from the property 3) of the function $\mathcal{K}_{\alpha,r}$ and from Lemma 3.2 that there exists such constant $\widetilde{C}(\nu,r) > 0$ that $\mathcal{K}^{(\nu)}_{\alpha,r}(t) \le \frac{\widetilde{C}(\nu,r)}{\alpha(t)}$, $t \in \mathbb{R}$. Thus, using (3.19), we get

$$\begin{aligned} \int_{-\infty}^{\infty} \left\|\mathcal{K}^{(\nu)}_{\alpha,r}(t) U(nt) x\right\| dt &\le \int_{-\infty}^{\infty} \frac{\widetilde{C}(\nu,r)}{\alpha(t)} \|U(t)\|^n \|x\| \, dt \\ &\le \widetilde{C}(\nu,r)\|x\| \int_{-\infty}^{\infty} \frac{M_U(|t|)^k}{\alpha(t)} dt \le 2\widetilde{C}(\nu,r)\|x\| < \infty. \end{aligned} \tag{3.21}$$

Therefore the integral $\int_{-\infty}^{\infty} \mathcal{K}^{(\nu)}_{\alpha,r}(t) U(nt) x \, dt$ converges. We define

$$x^{(\nu)}_{r,n} = \frac{(-1)^\nu}{n^\nu} \int_{-\infty}^{\infty} \mathcal{K}^{(\nu)}_{\alpha,r}(t) U(nt) x \, dt.$$

Then, using closedness of the operator A and integration by parts, one can find for $x \in \mathcal{D}(A)$ that $x_{r,n}^{(\nu)} \in \mathcal{D}(A)$ and

$$
\begin{aligned}
Ax_{r,n}^{(\nu)} &= \frac{(-1)^\nu}{n^\nu} \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}^{(\nu)}(t) U(nt) Ax\, dt = \frac{(-1)^\nu}{n^\nu} \frac{1}{n} \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}^{(\nu)}(t) (U(nt)x)'\, dt \\
&= -\frac{(-1)^\nu}{n^\nu} \frac{1}{n} \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}^{(\nu+1)}(t) U(nt)x\, dt = x_{r,n}^{(\nu+1)}.
\end{aligned} \tag{3.22}
$$

Let x is an arbitrary element of the space $\mathfrak{X}$. Then there exists the sequence $\{x_m\}_{m=1}^{\infty} \subset \mathcal{D}(A)$ such that $\|x_m - x\| \to 0, \quad m \to \infty$. Consequently, using inequality (3.21) and relation (3.22), one can get

$$
\left\| (x_m)_{r,n}^{(\nu)} - x_{r,n}^{(\nu)} \right\| \le \frac{1}{n^\nu} \int_{-\infty}^{\infty} \left\| \mathcal{K}_{\alpha,r}^{(\nu)}(t) U(nt)(x_m - x) \right\| dt \le \frac{2\widetilde{C}(\nu, r)}{n^\nu} \|x_m - x\| \to 0;
$$

$$
\left\| A(x_m)_{r,n}^{(\nu)} - x_{r,n}^{(\nu+1)} \right\| = \left\| (x_m)_{r,n}^{(\nu+1)} - x_{r,n}^{(\nu+1)} \right\| \to 0, \quad m \to \infty.
$$

Hence, taking into account closedness of the operator A, we have

$$
x_{r,n}^{(\nu)} \in \mathcal{D}(A), \quad Ax_{r,n}^{(\nu)} = x_{r,n}^{(\nu+1)}. \tag{3.23}
$$

One can get (3.20) from (3.23) by induction.

Using relation (3.20) and Lemma 3.2, one can find

$$
\begin{aligned}
\|A^\nu x_{r,n}\| &\le \frac{\|x\|}{n^\nu} \int_{-\infty}^{\infty} \left| \mathcal{K}_{\alpha,r}^{(\nu)}(t) \right| \|U(nt)\|\, dt \\
&\le \frac{\|x\|}{n^\nu} \int_{-\infty}^{\infty} \widetilde{c}_r \frac{\sqrt{2\pi\nu}\, \alpha\left(\frac{\nu}{r}\right)}{\alpha(|t|)} r^\nu \|U(t)\|^n\, dt \\
&\le \widetilde{c}_r \|x\| \sqrt{2\pi\nu}\, \alpha\left(\frac{\nu}{r}\right) \left(\int_{-\infty}^{\infty} \frac{\|U(t)\|^n}{\alpha(t)} dt \right) \left(\frac{r}{n}\right)^\nu,
\end{aligned} \tag{3.24}
$$

where, accordingly to (3.19) and due to $n \le k$, $\int_{-\infty}^{\infty} \frac{\|U(t)\|^n}{\alpha(t)}\, dt \le \int_{-\infty}^{\infty} \frac{\|U(t)\|^k}{\alpha(t)}\, dt \le 2 < \infty$. Since $\beta(t) = \ln(\alpha(t))$, $t \in \mathbb{R}$, as was mentioned in the proof of Lemma 3.1, $\lim_{\tau \to \infty} \frac{\beta(\tau)}{\tau} = 0$ (cf. (3.5)). Thus

$$
\lim_{\nu \to \infty} \left(\alpha\left(\frac{\nu}{r}\right) \right)^{1/\nu} = \lim_{\nu \to \infty} e^{\frac{1}{r}\left(\frac{r}{\nu}\beta\left(\frac{\nu}{r}\right)\right)} = e^{\frac{1}{r}\cdot 0} = 1.
$$

Therefore from relation (3.24) one can get:

$$
\limsup_{\nu \to \infty} \left(\|A^\nu x_{r,n}\| \right)^{1/\nu} \le \frac{r}{n}.
$$

The last inequality brings us to the conclusion that

$$
x_{r,n} \in \mathfrak{E}(A) \quad \text{and} \quad \sigma(x_{r,n}, A) \le \frac{r}{n}. \tag{3.25}
$$

For arbitrary $x \in \mathfrak{X}$ we define

$$
\begin{aligned}
\widetilde{x}_{r,k} &:= \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) (x + (-1)^{k-1} (U(t) - \mathbb{I})^k x)\, dt \\
&= \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) \sum_{n=1}^{k} (-1)^{n+1} \binom{k}{n} U(nt)x\, dt
\end{aligned} \tag{3.26}
$$

(the absolute convergence by the norm of $\mathfrak{X}$ of the integral in the right part of (3.26) follows from inequality (3.21), so the definition of the vector $\widetilde{x}_{r,k}$ is correct). Using definition (3.26) one can get

$$
\widetilde{x}_{r,k} = \sum_{n=1}^{k} (-1)^{n+1} \binom{k}{n} \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) U(nt)x\, dt = \sum_{n=1}^{k} (-1)^{n+1} \binom{k}{n} x_{r,n}.
$$

Therefore, accordingly to (3.25),

$$
\widetilde{x}_{r,k} \in \mathfrak{E}(A) \quad \text{and} \quad \sigma(\widetilde{x}_{r,k}, A) \le r.
$$

Hence for an arbitrary $x \in \mathfrak{X}$ we have

$$\mathcal{E}_r(x, A) = \inf_{y \in \mathfrak{E}(A) \,:\, \sigma(y,A) \le r} \|x - y\| \le \|x - \widetilde{x}_{r,k}\| .$$

Using (3.26), the property 2) of the kernel $\mathcal{K}_{\alpha,r}$ and (2.3), the last inequality implies

$$\begin{aligned}\mathcal{E}_r(x, A) &\le \left\| \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) x \, dt - \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) \big(x + (-1)^{k-1} (U(t) - \mathbb{I})^k x\big) \, dt \right\| \\ &\le \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) \left\| (U(t) - \mathbb{I})^k x \right\| dt \le \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) \widetilde{\omega}_k(|t|, x, A) \, dt.\end{aligned}$$

So, in accordance with the property 4) of the function $\widetilde{\omega}_k(|t|, x, A)$,

$$\begin{aligned}\mathcal{E}_r(x, A) &\le \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) \widetilde{\omega}_k \left(|rt| \frac{1}{r}, x, A \right) dt \\ &\le \widetilde{\omega}_k \left(\frac{1}{r}, x, A \right) \int_{-\infty}^{\infty} \big(1 + |rt| M_U(|t|)\big)^k \mathcal{K}_{\alpha,r}(t) \, dt.\end{aligned} \tag{3.27}$$

Taking into account properties of the function $M_U(\cdot)$, the definition of $\mathcal{K}_{\alpha,r}$, Lemma 3.1 and equality (3.19), one can find for $r \ge 1$

$$\begin{aligned}&\int_{-\infty}^{\infty} \big(1 + |rt| M_U(|t|)\big)^k \mathcal{K}_{\alpha,r}(t) \, dt \le \int_{-\infty}^{\infty} \big(1 + |rt| M_U(rt)\big)^k r \mathcal{K}_{\alpha}(rt) \, dt \\ &\quad \le \int_{-\infty}^{\infty} \big((1 + \tau) M_U(\tau)\big)^k \mathcal{K}_{\alpha}(\tau) \, d\tau \le c_1 \int_{-\infty}^{\infty} \frac{\big((1 + |\tau|) M_U(|\tau|)\big)^k}{\alpha(\tau)} \, d\tau = 2c_1 < \infty.\end{aligned}$$

In accordance with (3.27), inequality (3.1) holds for all $r \in [1, \infty)$ with a constant $\mathbf{m}_k = 2c_1$. It should be noted that constant $\mathbf{m}_k$, indeed, depends on k, because due to 3.1, the constant $c_1 = c_1(\alpha)$ depends on the function $\alpha(t) = (M_U(|t|))^k (1 + |t|)^{k+2}$.

Moreover, let the group $\{U(t)\}$ is bounded $(M_U(t) \le \widetilde{M},\ t \in \mathbb{R},\ \widetilde{M} \ge 1)$. Taking into account properties of the function $M_U(\cdot)$, the definition of $\mathcal{K}_{\alpha,r}$, Lemma 3.1 and equality (3.19), one can find for $r \in (0, \infty)$

$$\begin{aligned}&\int_{-\infty}^{\infty} (1 + |rt| M_U(|t|))^k \mathcal{K}_{\alpha,r}(t) \, dt \le \int_{-\infty}^{\infty} (1 + |rt| \widetilde{M} M_U(rt))^k r \mathcal{K}_{\alpha}(rt) \, dt \\ &\qquad \le \widetilde{M}^k \int_{-\infty}^{\infty} ((1 + \tau) M_U(\tau))^k \mathcal{K}_{\alpha}(\tau) \, d\tau \le 2\widetilde{M}^k c_1 < \infty,\end{aligned}$$

which proves Remark 3.1 with the constant $\mathbf{m}_k = 2\widetilde{M}^k c_1$. □

Theorem 3.1 allows us to prove the analogue of the classic Jackson's inequality for m times differentiable functions:

Corollary 3.1. *Let $x \in \mathcal{D}(A^m)$, $m \in \mathbb{N}_0$. Then $\forall k \in \mathbb{N}_0$*

$$\mathcal{E}_r(x, A) \le \mathbf{m}_{k+m} \frac{M_U\left(\frac{m}{r}\right)}{r^m} \widetilde{\omega}_k \left(\frac{1}{r}, A^m x, A \right), \quad r \ge 1, \tag{3.28}$$

where the constants $\mathbf{m}_n$ $(n \in \mathbb{N})$ are the same as in Theorem 3.1.

Proof. Let $x \in \mathcal{D}(A^m)$ and $r \ge 1$. By Theorem 3.1,

$$\mathcal{E}_r(x, A) \le \mathbf{m}_{k+m} \cdot \widetilde{\omega}_{k+m} \left(\frac{1}{r}, x, A \right).$$

Let $t \in \mathbb{R}$, $0 \le |t| \le \frac{1}{r}$. Then, using properties of the groups of the C_0 class and properties of the function $M_U(t)$, one can get

$$\begin{aligned}\left\|(U(t)-\mathbb{I})^{k+m}x\right\| &= \left\|(U(t)-\mathbb{I})^{m}(U(t)-\mathbb{I})^{k}x\right\| \\ &\le \int_0^t \cdots \int_0^t \|U(\xi_1+\cdots+\xi_m)\| \left\|(U(t)-\mathbb{I})^{k}A^m x\right\| \, d\xi_1 \dots d\xi_m \\ &\le M_U(m|t|) \left\|(U(t)-\mathbb{I})^{k}A^m x\right\| t^m \le \frac{M_U(\frac{m}{r})}{r^m}\widetilde{\omega}_k\left(\frac{1}{r}, A^m x, A\right).\end{aligned}$$

This implies $\widetilde{\omega}_{k+m}\left(\frac{1}{r}, x, A\right) = \sup_{|t|\le\frac{1}{r}} \left\|(U(t)-\mathbb{I})^{k+m}x\right\| \le \frac{M_U(\frac{m}{r})}{r^m}\widetilde{\omega}_k\left(\frac{1}{r}, A^m x, A\right)$, which proves inequality (3.28). □

By setting in Corollary 3.1 $k=0$ and taking into account that $\tilde{\omega}_0(\cdot, A^m x, A) \equiv \|A^m x\|$, one can conclude the following inequality:

Corollary 3.2. *Let $x \in \mathcal{D}(A^m)$, $m \in \mathbb{N}_0$. Then*

$$\mathcal{E}_r(x,A) \le \frac{\mathbf{m}_m}{r^m}\big(M_U(1/r)\big)^m \|A^m x\| \qquad r \ge 1, \tag{3.29}$$

where the constants $\mathbf{m}_n$ $(n \in \mathbb{N})$ are the same as in Theorem 3.1.

4. The examples of application of the abstract Jackson's inequality in particular spaces

Lets consider several examples of application of Theorem 3.1 in particular spaces.

4.1. Jackson's inequalities in $L_p(2\pi)$ and $C(2\pi)$.

Example 4.1. Let the space $\mathfrak{X}$ and the operator A are the same as in the Example 2.1. Then for $x \in \mathfrak{X}$ the quantity $\mathcal{E}_r(x,A)$ is the value of the best approximation of function x by trigonometric polynomials whose degree does not exceed r with respect to the norm in $\mathfrak{X}$. It is generally known that differential operator A is a generator of (isometric) group of shifts in the space $\mathfrak{X}$

$$\begin{aligned}&(U(t)x)(\xi) = x(t+\xi), && x \in \mathfrak{X}; \quad t,\xi \in \mathbb{R}, \\ &\|U(t)\| \equiv 1, && t \in \mathbb{R},\end{aligned} \tag{4.1}$$

where $\|U(\cdot)\| = \|U(\cdot)\|_{\mathcal{L}(\mathfrak{X})}$ is the norm of the operator $U(t)$ in the space $\mathcal{L}(\mathfrak{X})$ of linear continuous operators over $\mathfrak{X}$. It follows from (4.1) that

$$\widetilde{\omega}_k(t,x,A) = \omega_k(t,x,A) = \sup_{0\le h\le t}\left\|\sum_{j=0}^{k}(-1)^{k-j}\binom{j}{k}x(\cdot+jh)\right\|_{\mathfrak{X}}, \quad t \in \mathbb{R}_+, \quad x \in \mathfrak{X}.$$

I.e., in that case, $\widetilde{\omega}_k(t,x,A)$ coincides with classic modulus of continuity of k-th degree in the space $\mathfrak{X}$.

Thus, from Theorem 3.1 and Corollary 3.1 one can conclude all classic Jackson-type inequalities in the spaces $C(2\pi)$ and $L_p(2\pi)$, $1 \le p < \infty$.

4.2. Jackson's inequalities of the approximation by exponential type entire functions in the space $L_p(\mathbb{R}, \mu^p)$.

We consider the real-valued function $\mu(t)$ satisfying the following conditions:

1) $\mu(t) \ge 1, \quad t \in \mathbb{R}$,
2) $\mu(t)$ is even, monotonically non-decreasing when $t > 0$,
3) $\mu(t)$ satisfies naturally occurring in many applications condition $\mu(t+s) \le \mu(t)\cdot\mu(s)$, $s,t \in \mathbb{R}$,
4) $\int_{-\infty}^{\infty} \frac{\ln\mu(t)}{1+t^2}\,dt < \infty$,

or alternatively, instead of 4), the equivalent condition holds

4') $\sum_{k=1}^{\infty} \frac{\ln\mu(k)}{k^2} < \infty$.

Lets consider several important classes of functions satisfying conditions 1)–4).

1. Constant function $\mu(t) \equiv 1, \quad t \in \mathbb{R}$.

2. Functions with polynomial order of growth at infinity. It is easy to check that for such functions following estimate holds: $\exists k \in \mathbb{N},\ \exists M \geq 1$

$$\mu(t) \leq M(1+|t|)^k, \quad t \in \mathbb{R}.$$

3. Functions of the form

$$\mu(t) = e^{|t|^\beta}, \quad 0 < \beta < 1, \quad t \in \mathbb{R}.$$

4. $\mu(t)$ represented as a power series for $t > 0$. I.e.,

$$\mu(t) = \sum_{n=0}^{\infty} \frac{|t|^n}{m_n},$$

where $\{m_n\}_{n\in\mathbb{N}}$ is the sequence of positive real numbers satisfying two conditions

- $m_0 = 1,\ m_n^2 \leq m_{n-1} \cdot m_{n+1},\ n \in \mathbb{N}$;
- $\forall k, l \in \mathbb{N}\ \frac{(k+l)!}{m_{k+l}} \leq \frac{k!}{m_k} \frac{l!}{m_l}$.

The function $\mu(t)$, defined above, obviously satisfies conditions 1) and 2). The condition $\forall k, l \in \mathbb{N}\ \frac{(k+l)!}{m_{k+l}} \leq \frac{k!}{m_k} \frac{l!}{m_l}$ implies

$$\sum_{k=0}^{n} \frac{t^k s^{n-k} n!}{k!(n-k)!\, m_n} \leq \sum_{k=0}^{n} \frac{t^k s^{n-k}}{m_k m_{n-k}}, \tag{4.2}$$

and it is easy to see that condition 3) follows from inequality (4.2). The Denjoy-Carleman theorem [12, p. 376] asserts that the following conditions are equivalent:

a) $\mu(t)$ satisfies condition 4);

b) $\sum_{n=1}^{\infty} \left(\frac{1}{m_n}\right)^{1/n} < \infty$;

c) $\sum_{n=1}^{\infty} \frac{m_{n-1}}{m_n} < \infty$.

5. $\mu(t)$ as a module of an entire function with zeroes on the imaginary axis. We consider

$$\omega(t) = C \prod_{k=1}^{\infty} \left(1 - \frac{t}{it_k}\right), \quad t \in \mathbb{R},$$

where $C \geq 1,\ 0 < t_1 \leq t_2 \leq \dots,\ \sum_{k=1}^{\infty} \frac{1}{t_k} < \infty$. We set $\mu(t) := |\omega(t)|$. Then $\mu(t)$ satisfies conditions 1) – 3), and, as shown in [8], $\mu(t)$ satisfies condition 4) also.

Lets proceed to the description of the spaces $L_p(\mathbb{R}, \mu^p)$. Let the function $\mu(t)$ satisfies conditions 1) – 4). One can consider the space $L_p(\mathbb{R}, \mu^p)$ of the functions $x(s),\ s \in \mathbb{R}$, integrable in p-th degree with the weight μ^p

$$\|x\|^p_{L_p(\mathbb{R},\mu^p)} = \int_{-\infty}^{\infty} |x(s)|^p \mu^p(s)\, ds.$$

$L_p(\mathbb{R}, \mu^p)$ is the Banach space. We consider the differential operator A ($\mathcal{D}(A) = \{x \in L_p(\mathbb{R}, \mu^p) \cap AC(\mathbb{R}) :\ x' \in L_p(\mathbb{R}, \mu^p)\},\ (Ax)(t) = \frac{dx}{dt}$). As in Example 4.1, the operator A generates the group of shifts $\{U(t)\}_{t\in\mathbb{R}}$ in the space $L_p(\mathbb{R}, \mu^p)$. But in contrast to Example 4.1, this group isn't bounded. Indeed, lets consider

$$x(s) = \begin{cases} 1, & s \in [0,1], \\ 0, & s \notin [0,1]. \end{cases}$$

Obviously, $x(s) \in L_p(\mathbb{R}, \mu^p)$, but for $t > 1$

$$\|U(t)x\|^p = \int_{-\infty}^{\infty} |x(t+s)|^p \mu^p(s)\, ds = \int_{t-1}^{t} \mu^p(s)\, ds \geq \mu^p(t-1) \to \infty, \quad t \to \infty.$$

On the other hand, because of the property 3),

$$\|U(t)x\|^p = \int_{-\infty}^{\infty} |x(t+s)|^p \mu^p(s)\, ds \le \mu^p(-t) \int_{-\infty}^{\infty} |x(t+s)|^p \mu^p(t+s)\, ds = \big(\mu(-t)\big)^p \|x\|^p,$$

so $\|U(t)\|_{L_p(\mathbb{R},\mu^p)} \le \mu(-t) = \mu(|t|), \quad t \in \mathbb{R}.$ [2]

By the same way as in the Example 4.1, modules of continuity ω_k and $\tilde{\omega}_k$ coincides with classic ones, but in contrast to the Example 4.1, they don't equal mutually. The space $\mathfrak{E}(A)$ consists of fast decreasing at the infinity entire functions. The examples of such functions have been given in [8]. By applying Theorem 3.1 one can get

Corollary 4.1. *$\forall k \in \mathbb{N}$ there exists constant $\mathbf{m}_k(p,\mu) > 0$ such that $\forall f \in L_p(\mathbb{R},\mu^p)$*

$$\mathcal{E}_r(f) \le \mathbf{m}_k \cdot \tilde{\omega}_k \left(\frac{1}{r}, x, A\right), \quad r \ge 1.$$

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[2]If $\mu(t)$ is continuous and $\mu(0) = 1$, it is possible to show in a similar manner that $\|U(t)\|_{L_p(\mathbb{R},\mu^p)} = \mu(|t|)$.