Methods of Functional Analysis and Topology
Vol. 13 (2007), no. 3, pp. 279–283

ON CLOSED IDEALS OF ENTIRE FUNCTIONS OF FINITE GAMMA-GROWTH

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This note is dedicated to 100th anniversary of Mark Krein.

Abstract. We extend the result of Beurling on the closure in H^p of the linear manifold $F(z)\cdot\{$polynomials of $z\}$ to the classes of entire functions of finite gamma-growth.

Let H^p $(p > 0)$ be the space of analytic functions $F(z)$ in the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ such that

$$\sup_{0\le r<1} \int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta < \infty.$$

Let $F = I_F Q_F \in H^p$, $p > 0$, where I_F is the inner function and Q_F is the exterior function of F [1, Chap. IV]. The following theorem for $p = 2$ was proved by A. Beurling [2], the general case was considered by T. Srinivassan and J. K. Wang [3].

Theorem (Beurling). *Let $F = I_F Q_F \in H^p$, $p > 0$. Then the closure in H^p of the linear manifold $F(z)\cdot$ $\{$polynomials of $z\}$ is $I_F \cdot H^p$.*

This theorem was extended by K. G. Malyutin and Nazim Sadik [4] to the classes of entire functions of finite order. In this note, we generalize the Beurling theorem to the classes of entire functions of finite gamma-growth.

Definition 1. A growth function $\gamma(r)$ is a function defined for $0 < r < \infty$ that is positive, nondecreasing, continuous, and unbounded.

Let $\mathcal{E}$ be the set of entire functions on the plane $\mathbb{C}$. For some real constants $A, B > 0$, we denote the Banach space

$$\mathcal{E}_{A,B}(\gamma) := \{f : f \in \mathcal{E}, ||f||_{A,B} = \sup_{z\in\mathbb{C}} |f(z)| \exp(-A\gamma(B|z|)) < \infty\}$$

and set $\mathcal{E}(\gamma) = \bigcup\limits_{A,B>0} \mathcal{E}_{A,B}(\gamma)$. The set $\mathcal{E}(\gamma)$ is a linear locally convex space with the topology of inductive limit. Furthermore, the space $\mathcal{E}(\gamma)$ is a topological algebra with respect to the product and the sum of functions.

Definition 2. A sequence of entire functions $\{f_n(z)\}$ converges in the space $\mathcal{E}(\gamma)$ as $n \to \infty$ to a function $f(z)$ if and only if $\{f_n(z)\}$ converges to the function $f(z)$ uniformly on every compact subset of $\mathbb{C}$ and there exist constants $A, B > 0$ such that

$$|f_n(z)| \le \exp(A\gamma(B|z|)), \quad n \in \mathbb{N},$$

for all $z \in \mathbb{C}$.

If $\gamma(r) = r^\rho$ $(\rho > 0)$ then the space $\mathcal{E}(\gamma)$ is a space of entire functions of finite order ρ of mean type.

2000 *Mathematics Subject Classification.* Primary 30D20; Secondary 30D55.
Key words and phrases. Beurling theorem, entire function, linear manifold.

Let f be an entire function. Then the Fourier coefficients of f are the functions

$$c_k(r,f) = \frac{1}{2\pi}\int_0^{2\pi} (\log|f(re^{i\theta})|)e^{-ik\theta}\,d\theta, \quad k = 0, \pm 1, \pm 2, \dots$$

By $Z(f)$ we will denote the set of zeros of the function f.

Following Rubel [5], we consider the distribution of sequences $Z = \{z_n\}_{n=1}^{\infty}$, with multiplicity taken into account, of nonzero complex numbers z_n, $n = 1, 2, \dots$ Such sequences Z are studied in relation to the growth function γ.

Let $Z = \{z_n\}_{n=1}^{\infty}$ be a sequence of nonzero complex numbers such that $\lim z_n = \infty$ as $n \to \infty$. We define

$$N(r,Z) = \int_0^r \frac{n(t,Z)}{t}\,dt\,,$$

where $n(t,Z)$ is the counting function of Z.

We say that the sequence Z has finite γ-density if there exist constants A, B such that, for all $r > 0$,

$$N(r,Z) \le A\gamma(Br)\,.$$

We define, for $k = 1, 2, \dots$ and $r \ge 0$,

$$S(r;k,Z) = \frac{1}{k}\sum_{|z_n|\le r}\left(\frac{1}{z_n}\right)^k, \quad S'(r;k,Z) = \frac{1}{k}\sum_{|z_n|\le r}\left(\frac{z_n}{r}\right)^k.$$

We define, for $k = 1, 2, \dots$ and $r_2 \ge r_1 \ge 0$,

$$S(r_1,r_2;k,Z) = S(r_2;k,Z) - S(r_1;k,Z)\,.$$

When no confusion will result, we will drop the Z from the above notation and write $N(r)$, $S(r;k)$, etc.

We say that the sequence Z is γ-balanced if there exist constants A, B such that

$$S(r_1,r_2;k) \le \frac{A\gamma(Br_1)}{r_1^k} + \frac{A\gamma(Br_2)}{r_2^k}$$

for all $r_2 > r_1 > 0$ and $k = 1, 2, \dots$.

We say that the sequence Z is γ-admissible if Z has finite γ-density and is γ-balanced.

Let $f(z) \in \mathcal{E}(\gamma)$, and let $Z = \{z_n\}_{n=1}^{\infty}$ be the set of all nonzero roots of the function $f(z)$. Then the set Z is γ-admissible [5, Theorem 13.5.2].

Suppose now that a sequence Z is γ-balanced, with A, B being the corresponding constants. Let

$$p(\gamma) = \inf\left\{p = 1, 2, \dots : \liminf_{r\to\infty}\frac{\gamma(r)}{r^p} = 0\right\}.$$

Naturally, we let $p(\gamma) = \infty$ in the case $\liminf\limits_{r\to\infty}\gamma(r)r^{-p} > 0$ as $r \to \infty$ for each positive integer p. For $1 \le k < p(\gamma)$, we have $\inf_{r\ge 0} r^{-k}\gamma(Br) > 0$. Thus, there exist positive numbers r'_k such that

$$\frac{\gamma(Br'_k)}{(r'_k)^k} < 2\frac{\gamma(Br)}{r^k}$$

for $r > 0$ and $1 \le k < p(\gamma)$. For k in this range, we define $\alpha_k = -S(r_k;k)$, where $r_k = \inf r'_k$.

For those k, if there are any, for which $k \ge p(\gamma)$, we choose a sequence $\{r_j\}_{j=1}^{\infty}$, $\lim_{j\to\infty} r_j = \infty$, such that

$$\lim_{j\to\infty}\frac{\gamma(Br_j)}{r_j^{p(\gamma)}} = 0\,.$$

For values of k, then, such that $k \ge p(\gamma)$, we define $\alpha_k = -\lim_{j\to\infty} S(r_j;k)$. We note that the limit exists (see the proof of Proposition 13.1.14 in [5]).

Definition 3. A sequence $\{c_k(r;Z)\}$, $k=0,\pm1,\pm2,\dots$, defined by $c_0(r)=N(r)$, $c_k(r)=r^k\{\alpha_k+S(r;k)\}/2–S'(r;k)/2$ for $k=1,2,\dots$, $c_{-k}(r)=\overline{c_k(r)}$ for $k=1,2,\dots$, is said to be a sequence of Fourier coefficients associated with Z.

We remark that Definition 3 differs from Definition 13.2.3 in [5] in which $c_k(r)$ depends on a sequence α of complex numbers.

Using Definition 3 of Fourier coefficients of the sequence Z, Propositions 13.2.5, 13.2.6, and Theorems 13.4.5, 13.5.1, 13.5.2 in [5] we can formulate the following theorem.

Theorem 1. *Let $f(z)\in\mathcal{E}(\gamma)$, and let Z be the set of all nonzero roots of the function $f(z)$. Suppose that $\{c_k(r)\}=\{c_k(r;Z)\}$, $k=0,\pm1,\pm2,\dots$, is a sequence of Fourier coefficients associated with Z. Then there exists a unique entire function $\tilde I_f(z)\in\mathcal{E}(\gamma)$ with $Z(\tilde I_f)=Z$, $\tilde I_f(0)=1$, and $c_k(r;\tilde I_f)=c_k(r)$ for $k=0,\pm1,\pm2,\dots$*

Definition 4. Let a function f vanish at 0 with multiplicity $m\ge 0$. The function $I_f(z)=z^m\tilde I_f(z)$ is called the inner function of f and the function $Q_f(z)=f(z)/I_f(z)$ is called the exterior function of f.

We also note that $Q_f(z)$ belongs to $\mathcal{E}(\gamma)$.

The purpose of this note is to prove the following theorem.

Theorem 2. *Let $f=I_fQ_f\in\mathcal{E}(\gamma)$, where I_f is the inner function of f and Q_f is the exterior function of f. Then the closure of the linear manifold $f(z)\cdot$ {polynomials of z} is $I_f\cdot\mathcal{E}(\gamma)$.*

1. We denote $H(r)=\gamma(e^r)$ and suppose that $H(r)$ is a convex function on $[0,+\infty)$ such that there exists

$$\lim_{r\to+\infty}\frac{H(r)}{r}=+\infty. \tag{1}$$

Then the dual Young function $H^*(r)$ to $H(r)$ is defined, i.e.,

$$H^*(r):=\max\{ru-H(u):u\ge 0\}.$$

The function $H^*(r)$ is convex [6, §3.2], satisfies (1), and $H(r)^{**}\equiv H(r)$.

Let u_r be a point of maximum of the function $ru-H(u)$, $u\ge 0$. By definition, $H^*(r)$: $H^*(r)=ru_r-H(u_r)\le ru_r$. Using (1) for $H^*(r)$, it is easy to see that $\lim\limits_{r\to+\infty}u_r=+\infty$. Hence, for all $t\ge 0$, there exists $r_0(t)\ge 0$ such that

$$H^*(r)=\max\{ru-H(u):u\ge t\} \tag{2}$$

for all $r\ge r_0(t)$.

Using (2) in [7], Abanina proved that the function Young dual to $H_1(r)=H(r+B)+D$ is the function

$$H_1^*(r)=H^*(r)-Br-D. \tag{3}$$

Here $B,D\ge 0$ are some fixed numbers. Let $A>0$ be any fixed number. Then the function $A\gamma(r)$ also is a function of growth. The function $H_A(r)$ Young dual to $A\gamma(r)$ is convex on $[0,+\infty)$ and satisfies (1).

Let the entire function

$$f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{E}(\gamma). \tag{4}$$

Then there exist constants $A,B>0$ independent of r such that

$$M(r,f):=\max_{|z|=r}|f(z)|\le\exp(A\gamma(Br))$$

for all $r \geq 0$. Hence,

$$\ln M(r,f) \leq A\gamma(Br) = AH(\ln(Br)) = H_A(\ln B + \ln r).$$

For the maximal term of power series (4), $\mu(r,f) := \max_{n\geq 0} |a_n| r^n$, we have the following inequality:

$$\mu(r,f) \leq \ln M(r,f) \leq H_A(\ln B + \ln r).$$

By (3), the function Young dual to $H_A(\ln B + r)$ is the function $H_A^*(r) - r\ln B$. Then [6, theorem 3.2.5]

$$\ln |a_n| \leq -H_A^*(n) + n\ln B, \quad n = 0, 1, \dots \tag{5}$$

Lemma. *Let $f(z) \in \mathcal{E}(\gamma)$. Then there exists a sequence of polynomials $P_n(z)$, $n = 1, 2, \dots,$ such that $P_n(z)$ converge in $\mathcal{E}(\gamma)$ as $n \to \infty$ to $f(z)$.*

Proof. Assume that the function $f(z)$ can be written as (4). Then there exist $A, B > 0$ such that (5) holds. Hence, if (5) holds, then

$$\begin{aligned}
\sum_{k=0}^{\infty} |a_k| r^k &\leq \sum_{k=0}^{\infty} \exp\{-H_A^*(k) + k\ln B\} r^k = \sum_{k=0}^{\infty} \exp\{-H_A^*(k) + k\ln B + k\ln(2r)\}\frac{1}{2^k} \\
&\leq \max_{k\geq 0}\{\exp\{-H_A^*(k) + k\ln B + k\ln(2r)\}\} \sum_{k=0}^{\infty} \frac{1}{2^k} \\
&\leq 2\exp\{\max_{u\geq 0}\{-H_A^*(u) + u\ln(2Br)\}\} \\
&= 2\exp\{H_A^{**}(\ln(2Br)\} = 2\exp\{H_A(\ln(2Br)\} = 2\exp\{A\gamma(2Br)\}.
\end{aligned}$$

Using this inequality, we obtain that the sequence of partial sums of the series (4) converges in $\mathcal{E}(\gamma)$ as $n \to \infty$ to $f(z)$. $\square$

2. We now prove the main theorem.

a) We prove that $I_f \cdot \mathcal{E}(\gamma)$ includes the closure in $\mathcal{E}(\gamma)$ of the linear manifold $f(z)\cdot$\{polynomials of z\}. Let $g \in \mathcal{E}(\gamma)$, and let $\{P_n(z)\}$ be a sequence of polynomials such that $\{fP_n\}$ converges in $\mathcal{E}(\gamma)$ as $n \to \infty$ to $g(z)$.

Let $\{z_k\}_{k=1}^{\infty}$ be a set of zeros of the function $f(z)$. Since $\{fP_n\}$ converges to g uniformly on every compact set, we have that $\{f(z_k)P_n(z_k)\}$ converges as $n \to \infty$ to $g(z_k)$ for any $k \in \mathbb{N}$. Since we have $f(z_k)P_n(z_k) = 0$ for any $k, n \in \mathbb{N}$, $g(z_k) = 0$ follows then for any $k \in \mathbb{N}$. Then $G := g/I_f$ is an entire function. We just proved that $G \in \mathcal{E}(\gamma)$.

The following argument will be called a "standart argument". Let $C(a,\rho)$ be a disc of radius ρ about a. Let $\{C(a_n, \rho_n)\}$ be a sequence of disks. The number

$$L = \limsup_{n\to\infty} \frac{1}{r} \sum_{|a_n|\leq r} \rho_n \tag{6}$$

is called the upper density of the set $\bigcup_{n=1}^{\infty} C(a_n, \rho_n)$ [8]. Using Theorem 11 [8, Chap. 1], it is easy to see that given $f \in \mathcal{E}(\gamma)$, $0 < \eta < 1/2$, there exist $A(\eta), B > 0$ such that

$$\ln |f(re^{i\theta})| \geq -A(\eta)\gamma(Br)$$

for all $re^{i\theta} \notin C_\eta$, where C_η is a set of discs of upper density η. Using arguments of the paper [9], we can make the disks of C_η such that they are disjoint.

Thus, there exists a set C_η of disjoint discs of upper density η such that

$$\ln |G(re^{i\theta})| \leq A(\eta)\gamma(Br) \tag{7}$$

for all $re^{i\theta} \notin C_\eta$, where $A(\eta), B > 0$ are constants independent of r.

Let $re^{i\theta} \in C_\eta$ and let $C(a_n, \rho_n)$ be a disc of C_η such that $re^{i\theta} \in C(a_n, \rho_n)$. It follows from (6) that

$$\rho_n \le \frac{1+\eta}{1-\eta} r.$$

By the maximum modulus principle, relation (7) is true (probably with other constants) for all $z \in \mathbb{C}$.

b) We now prove that the closure of $f(z)\cdot\{\text{polynomials of } z\}$ includes $I_f \cdot \mathcal{E}(\gamma)$. We prove that the linear manifold $Q(z)\cdot\{\text{polynomials of } z\}$ is a set everywhere dense in the space $\mathcal{E}(\gamma)$. Let $g \in \mathcal{E}(\gamma)$. Then (using the "standard argument") $g/Q_f \in \mathcal{E}(\gamma)$. Let $\{P_n(z)\}$ be a sequence of polynomials converging in $\mathcal{E}(\gamma)$ to $g(z)/Q_f(z)$. Then the sequence $\{Q_f(z)P_n(z)\}$ converges to $g(z)$ as $n \to \infty$. The proof is complete.

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Received 02/01/2007; Revised 15/03/2007