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ABOUT ONE CLASS OF HILBERT SPACE UNCONDITIONAL BASES

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Abstract. Let a sequence $\{v_k\}_{-\infty}^{+\infty} \in l_2$ and a real sequence $\{\lambda_k\}_{-\infty}^{+\infty}$ such that $\left\{\lambda_k^{-1}\right\}_{-\infty}^{+\infty} \in l_2$, and an orthonormal basis $\{e_k\}_{-\infty}^{+\infty}$ of a Hilbert space be given. We describe a sequence $M = \{\mu_k\}_{-\infty}^{+\infty}$, $M \cap \mathbb{R} = \varnothing$, such that the families

$$f_k = \sum_{j\in\mathbb{Z}} v_j \left(\lambda_j - \bar{\mu}_k\right)^{-1} e_k, \quad k \in \mathbb{Z}$$

form an unconditional basis in $\mathfrak{H}$.

Let $\{\lambda_k\}_{-\infty}^{+\infty}$ be a sequence of real numbers distinct from zero, with the unique limit point ∞, and $\{v_k\}_{-\infty}^{+\infty}$ be some sequence ($v_k \neq 0,\ k \in \mathbb{Z}$) of complex numbers such that

$$\sum_{j\in\mathbb{Z}} |v_j|^2 \lambda_j^{-2} < \infty.$$

Let also $\{e_k\}_{-\infty}^{+\infty}$ be an orthonormal basis of the Hilbert space $\mathfrak{H}$. Consider in $\mathfrak{H}$ a family of vectors,

$$f_k = \sum_{j\in\mathbb{Z}} \frac{v_j \cdot e_j}{\lambda_j - \bar{\mu}_k}, \quad k \in \mathbb{Z}, \tag{1}$$

where $M := \{\mu_k\}_{-\infty}^{+\infty}$ is a some complex-valued sequence such that $M \cap \mathbb{R} = \varnothing$ and M does not have finite limit points. The problem is to describe sequences M for which family (1) forms an unconditional basis in the space $\mathfrak{H}$. Let us recall that a family $\{f_k\}_{-\infty}^{+\infty}$ of vectors, complete in $\mathfrak{H}$, is called an unconditional basis [1], if there is a constant $C > 0$ such that for an arbitrary finite sequence $\{c_k\}$,

$$C^{-1} \sum_k |c_k|^2 \|f_k\|^2 \leqslant \left\| \sum_k c_k f_k \right\|^2 \leqslant C \sum_k |c_k|^2 \|f_k\|^2.$$

This setting contains the problem of the basis property for some important families of functions. For example, if we assume, in addition, that

$$v_j = 1 \ (j \in \mathbb{Z}), \quad \inf_k |\mathrm{Im}\ \mu_k| > 0, \quad \lambda_j = j + \alpha \ (j \in \mathbb{Z})$$

where α is a some number from the interval $(0, 1)$, then we will come to the problem of unconditional basis property in the space $L_2(0, 2\pi)$ for the family of exponents $\{\exp(i\bar{\mu}_k t) : \mu_k \in M\}$. In fact, to see this, it is enough to expand every function $\exp(i\bar{\mu}_k t)$ with respect to the trigonometric system $\{\exp i(k + \alpha)t\}$, $k \in \mathbb{Z}$, in the space $L_2(0, 2\pi)$. In this paper, we consider families of the form (1), which are very differing from systems of exponents in some sense. To study their basis property, we will use the de Brange theory of spaces of entire functions [2] and results of paper [3].

Everywhere in the sequel, we assume the following:

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(1) $\{v_j\}_{-\infty}^{+\infty} \in l_2$;
(2) the sequence $\{\lambda_j\}_{-\infty}^{+\infty}$ coincides with the set of all zeros of an entire function Q of sine type [4];
(3) $\inf\limits_{k\neq j} |\lambda_k - \lambda_j| > 0$.

Besides, we assume that the condition

$$h_Q\left(\frac{\pi}{2}\right) = h_Q\left(-\frac{\pi}{2}\right) = \sigma > 0$$

holds. This is a normalization condition, (h_Q is a growth indicator for the function$Q$. In this way if we consider $Q(0) = 1$, then we have the multiplicative representation

$$Q(z) = v.p. \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{\lambda_k}\right).$$

By W^2_σ we denote the Wiener-Paley space of entire functions of exponential type g, which are not greater then σ, with the norm

$$\|g\|^2 = \int\limits_{\mathbb{R}} |g(x)|^2 dx < \infty.$$

We recall that the family of functions, $\{Q(z)/(z-\lambda_k) : k \in \mathbb{Z}\}$, forms an unconditional basis of the space W^2_σ [4] and, therefore, a unique solution of the interpolation problem

$$g(\lambda_j) = c_j, \quad j \in \mathbb{Z}, \quad \{c_j\}_{-\infty}^{+\infty} \in l_2, \tag{2}$$

in the W^2_σ-class can be represented as the Lagrange series

$$g(z) = \sum_{k\in\mathbb{Z}} c_k \frac{Q(z)}{Q'(\lambda_k)(z - \lambda_k)}.$$

Theorem 1. *If a family of vectors forms an unconditional basis in the space $\mathfrak{H}$, then the sequence of non-real numbers, M, coincides with the set of all zeros of the entire function*

$$S(z) := Q(z) + zg(z),$$

where $g \in W^2_\sigma$ and is a solution of problem (2) in which the interpolation data $\{c_k\}_{-\infty}^{+\infty}$ satisfies the additional conditions

$$c_k \neq 0, \quad \left\{c_k v_k^{-1}\right\}_{-\infty}^{+\infty} \in l_2, \quad 1 + \sum_{k\in\mathbb{Z}} c_k (Q'(\lambda_k))^{-1} \neq 0. \tag{3}$$

If we also assume that

$$\sup_{k\in\mathbb{Z}} \left\{|\lambda_k||c_k||v_k^{-1}|\right\} < \infty$$

then the elements of the sequence M can be represented as

$$\mu_k = \lambda_k + b_k, \quad k \in \mathbb{Z}, \quad \{b_k\}_{-\infty}^{+\infty} \in l_2.$$

Thus, it is sufficient to restrict the consideration to the sequences of functions from Theorem 1 when solving the basis property for family (1). Let us sketch the plan of the proof of this theorem. If family (1) forms an unconditional basis in the space $\mathfrak{H}$, then there is a bounded operator L such that

$$Lf_k = (\bar{\mu}_k)^{-1} f_k, \quad k \in \mathbb{Z}. \tag{4}$$

Introduce the operator

$$Bh := \sum_{k\in\mathbb{Z}} \lambda_k^{-1} (h, e_k) e_k, \quad h \in \mathfrak{H},$$

and find a representation for L in the form $L = B + V$. Since the vectors f_k are represented by

$$f_k = B(1 - \bar{\mu}_k B)^{-1} v, \quad v := \sum_{k\in\mathbb{Z}} v_k e_k,$$

it follows from (4) that the image of the operator V is spanned by the vector Bv, i.e.,

$$Vh = (h, u)Bv$$

for some $u \in \mathfrak{H}$. We also note that the spectrum of the operator $K := L^*$ is the closure of the sequence μ_k^{-1} and $\ker K = \ker K^* = \{0\}$. Since

$$Kh = Bh + (h, Bv)u, \quad h \in \mathfrak{H}, \tag{5}$$

it is not difficult to compute that M is the same as the set of meromorphic roots of

$$\varphi(z) = 1 - z\left(B(1 - zB)^{-1}u, v\right) = 1 - z\sum_{k\in\mathbb{Z}} \frac{u_k \bar{v}_k}{\lambda_j - z}, \tag{6}$$

where $u_k := (u, e_k),\ k \in \mathbb{Z}$, and all $u_k \neq 0$ in view of the fact that a family biorthogonal to the family (1) has a basis property. Therefore, the function $S := Q\varphi$ is represented by

$$S(z) = Q(z) + zg(z),$$

where $g \in W_\sigma^2$, and it solves the interpolation problem (2) with $c_j = Q'(\lambda)u_j\bar{v}_j,\ j \in \mathbb{Z}$. Therefore, first two conditions (3) are satisfied. The last condition (3) is represented as $1 + (u, v) \neq 0$ and is equal to $\ker K = \ker K^* = \{0\}$. The second statement of Theorem 1 follows, since S can be represented in this case as

$$S(z) = CQ(z) + g_1(z), \quad C := 1 + (u, v),$$

where $g_1 \in W_\sigma^2$.

We will consider the entire function

$$\begin{gathered} E(z) := Q(z)(1 - iz((1 - zB)^{-1}Bv, Bv)) = \gamma Q(z) + i\sum_{k\in\mathbb{Z}} \frac{Q(z)|v_k|^2}{(z - \lambda_k)}, \\ \gamma := 1 + i\sum_{k\in\mathbb{Z}} \lambda_k^{-1}|v_k|^2. \end{gathered} \tag{7}$$

It is not difficult to see that the function E satisfies the condition

$$|E(\bar{z})| < |E(z)|, \quad z \in \mathbb{C}_+,$$

and, therefore, it generates a de Brange space $\mathcal{H}(E)$ with the reproducing kernel [2]

$$k(z, \lambda) := \left(b(z)\overline{a(\lambda)} - a(z)\overline{b(\lambda)}\right)/\pi(z - \bar{\lambda}),$$

where the entire functions a, b are defined by identities

$$a(z) = \frac{1}{2}\left(\overline{E(\bar{z})} + E(z)\right), \quad b(z) = \frac{1}{zi}\left(\overline{E(\bar{z})} + E(z)\right). \tag{8}$$

The next result reduces the basis property problem for families (1) to similar problems for the families $k(z, \mu_n)$, $\mu_n \in M$, of values of the reproducing kernel in the space $\mathcal{H}(E)$. In its formulation, we assume that M is the same as the set of zeroes of some function S from Theorem 1.

Theorem 2. *There exists an isometric mapping U of space $\mathfrak{H}$ onto the de Brange space $\mathcal{H}(E)$ such that*

$$Uf_k = \sqrt{\pi}Q^{-1}(\bar{\mu}_k)k(z, \mu_k), \quad k \in \mathbb{Z}.$$

We will briefly state the idea for proving this theorem. Since M is the same as zeros of the function S, it is not difficult to build an operator K represented by (5), whose spectrum is the same as the closure of the sequence μ_k^{-1}, $\mu_k \in M$. We note that

$$\frac{1}{i}(K - K^*)h = -i(h, Bv)u + i(h, u)Bv.$$

The characteristic matrix-valued function of the operator K [5] is given by the formula

$$w(z) = E_2 - iz\Delta(z)J,$$

where E_2 is the identity matrix,

$$\Delta(z) = \begin{pmatrix} ((1-zK)^{-1}u, u) & ((1-zK)^{-1}Bv, u) \\ ((1-zK)^{-1}u, Bv) & ((1-zK)^{-1}Bv, Bv) \end{pmatrix}, \quad J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Easy computations show that w can be represented by

$$w(z) = \frac{1}{S(z)} \begin{pmatrix} d(z) & c(z) \\ -b(z) & a(z) \end{pmatrix},$$

where a, b are represented by formula (8), c, d are some entire functions taking real values on the real axis. Also it is verified that S is associated with the space $\mathcal{H}(E)$ and the matrix-valued function $w(z)$ is perfect [2]. The spectrum of the bounded in $\mathcal{H}(E)$ operator

$$(\tilde{K}f)(z) := (f(z)S(0) - S(z)f(0))/zS(0), \quad f \in \mathcal{H}(E)$$

is also the same as the closure of the sequence μ_n^{-1}, $\mu_n \in M$, moreover,

$$\frac{1}{i}(\tilde{K} - \tilde{K}^*)f = -i(f, l_2)l_1 + i(f, l_1)l_2, \quad f \in \mathcal{H}(E),$$

where the entire functions l_1, l_2 are computed by the formulas

$$l_1(z) = (a(z) - S(z))\sqrt{\pi}z, \quad l_2(z) = b(z)/\sqrt{\pi}z.$$

Using results from [6], we can verify that the characteristic matrix-valued function of the operator $\tilde{K}$ is equal to $w(z)$. Since both operators do not have nontrivial self-adjoint parts, from the theory of characteristic functions of not self-adjoint operators, it follows that K and $\tilde{K}$ are unitary equivalent, and, moreover, [5]

$$U(1 - \bar{z}K^*)^{-1}Bv = (1 - \bar{z}\tilde{K}^*)^{-1}l_2, \quad z \in \{\bar{\mu}_k^{-1}\},$$

where U is an isometry from $\mathfrak{H}$ onto $\mathcal{H}(E)$. As a result of simple computations we get the formulas

$$(1 - zK^*)^{-1}Bv = \frac{(1 - \bar{z}B)^{-1}Bv}{\overline{\varphi(\bar{z})}}, \quad (1 - \bar{z}\tilde{K}^*)^{-1}l_2 = \sqrt{\pi}\frac{k(\lambda, z)}{\overline{S(\bar{z})}},$$

where the function φ is defined by formula (6). Therefore,

$$U(1 - \bar{z}B)^{-1}Bv = \sqrt{\pi}\,(Q(\bar{z}))^{-1}\,k(\lambda, z), \quad z \notin \{\lambda_k\}$$

from which the theorem follows for $z = \mu_n \in M$.

Criteria for families $\{k(z, \mu_n) : \mu_n \in M\}$ to have the unconditional basis property in de Brange spaces are found in [3]. The next theorems are obtained applying using these criteria to the basis property problem for family (1). We recall that the non-real sequence M coincides with the roots of the entire function $S(z) = Q(z) + zg(z)$, $g \in W^2_\sigma$, and it solves some interpolation problem (2)–(3). We also recall that the function E is defined by formula (7).

Theorem 3. *If the weight $v(x) := |S(x)/E(x)|^2$ satisfies Muckenhoupt A_2-condition* [7] *on $\mathbb{R}$, then the family of vectors given by (1) forms an unconditional basis in the space $\mathfrak{H}$. Conversely, with the additional assumption that*

$$\sup_{\mu_k \in \mathbb{C}_+} |E(\bar{\mu}_k)/E(\mu_k)| < 1, \quad \sup_{\mu_k \in \mathbb{C}_-} |E(\mu_k)/E(\bar{\mu}_k)| < 1$$

it follows from the unconditional basis property of system (1) that the weight $v(x)$ satisfies the A_2-condition on the real axis.

Theorem 4. *Let the interpolation data $\{c_k\}_{-\infty}^{+\infty}$ of problem (2)–(3) satisfy the condition*

$$|c_k| \asymp |\lambda_k|^{-1}|v_k|^2, \quad k \in \mathbb{Z}.$$

If the function $S(x)/E(x)$ is uniformly continuous on $\mathbb{R}$ then the family of vectors (1) forms an unconditional basis in the space $\mathfrak{H}$.

We note that the roots γ_k of the function E lie in the lower half-plane $\mathbb{C}_-$ and can be represented as

$$\gamma_k = \lambda_k + d_k, \quad k \in \mathbb{Z}, \quad \{d_k\} \in l_2.$$

Theorem 5. *Let a sequence* $\{c_k\}_{-\infty}^{+\infty}$ *be such that* $\{\lambda_k v_k^{-1} c_k\}_{-\infty}^{+\infty} \in l_2$. *If the conditions*

$$\sup_{k\in\mathbb{Z}} \left(|\mathrm{Im}\ \gamma_k|^{-1}|\gamma_k - \mu_k|\right) < \infty, \quad \sup_{k\in\mathbb{Z}} \left(|\mathrm{Im}\ \mu_k|^{-1}|\gamma_k - \mu_k|\right) < \infty$$

are verified, the the family of vectors (1) forms an unconditional basis in the space $\mathfrak{H}$.

In the proof of this theorem we used the fact that the set of functions

$$\frac{\bar{v}_k}{\sqrt{\pi}} \frac{Q(z)}{z - \lambda_k}, \quad k \in \mathbb{Z}$$

forms an orthonormal basis in the space $\mathcal{H}(E)$.

Let us consider one simple example of unconditional bases represented by (1). Assume that the function $g \in W_\sigma^2$ is a solution of the interpolation problem

$$g(\lambda_k) = wQ'(\lambda_k)|v_k|^2, \quad k \in \mathbb{Z},$$

where w is an arbitrary non-real number. Beginning with some index, all the roots μ_k of the function $S(z) = Q(z) + zg(z)$ are simple. If all the roots μ_k are simple, then it follows from Theorem 3 that the corresponding family of vectors (1) forms an unconditional basis of the space $\mathfrak{H}$.

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