

ON HOLOMORPHIC SOLUTIONS OF THE HEAT EQUATION WITH A VOLTERRA OPERATOR COEFFICIENT

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This paper is dedicated to 100 anniversary of Mark Krein.

ABSTRACT. Let A be a bounded operator on a Hilbert space and g a vector-valued function, which is holomorphic in a neighborhood of zero. The question about existence of holomorphic solutions of the Cauchy problem $\begin{cases} \frac{\partial u}{\partial t} = A \frac{\partial^2 u}{\partial x^2} \\ u(0, x) = g(x) \end{cases}$ is considered in the paper.

In 1875 Sofya Kovalevskaya showed that the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with a holomorphic initial condition can have no holomorphic solution (see [9], [11]). The researches, started by Kovalevskaya, further have been continued in numerous works (see, for example, [6], [7], [10], [12]–[14]). In the present paper we shall consider the following operator analog of the Cauchy problem for the heat equation:

$$(1) \quad \begin{cases} \frac{\partial u}{\partial t} = A \frac{\partial^2 u}{\partial x^2}, \\ u(0, x) = g(x), \end{cases}$$

where A is a bounded linear operator in a complex Hilbert space and $g(x)$ a vector-valued function, which is holomorphic in a neighborhood of zero. As a solution of the problem (1) we understand a holomorphic solution, i.e., a vector-valued function of the two complex variables t and x , which is holomorphic in a neighborhood of zero, satisfies the equation in this neighborhood and the initial condition holds in some neighborhood of the point $x_0 = 0$. The main result of the paper is a proof of local existence and uniqueness theorem on holomorphic solution of the Cauchy problem (1) in the assumption that A is a Volterra operator and the imaginary part of A is of trace class (see Theorem 2). Let us note that operators of this class were been studied in detail in the Odessa school of the operator theory (see [2], [4], [5]). The proof of Theorem 2 is based on the consideration of the formal solution of the Cauchy problem (1)

$$u(t, x) = g(x) + \sum_{n=1}^{\infty} \frac{t^n}{n!} A^n g^{(2n)}(x)$$

as a series on degrees of the “small parameter” A .

Let H be a complex Hilbert space and $A : H \rightarrow H$ a bounded linear operator.

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Lemma 1. Let $g(x) = \sum_{m=0}^{\infty} b_m x^m$ be a formal power series with coefficients in E . Then the formal power series

$$u(t, x) = \sum_{n,m=0}^{\infty} \frac{(m+2n)!}{m!n!} A^{2n} b_{m+2n} t^n x^m$$

is a unique formal solution of the problem (1). (We set $(Av)(t, x) = \sum_{n,m=0}^{\infty} Av_{n,m} t^n x^m$,

if $v(t, x) = \sum_{n,m=0}^{\infty} v_{n,m} t^n x^m$ is a formal power series with coefficients in H .)

Proof. Assume that $u(t, x) = \sum_{n,m=0}^{\infty} c_{nm} t^n x^m$, $c_{nm} \in H$, is a formal solution of the Cauchy problem (1). Then after substitution into the equation we obtain

$$\begin{aligned} \sum_{n,m=0}^{\infty} (n+1)c_{n+1m} t^n x^m &= \sum_{n,m=0}^{\infty} (m+2)(m+1)Ac_{nm+2} t^n x^m, \\ \sum_{m=0}^{\infty} c_{0m} x^m &= \sum_{m=0}^{\infty} b_m x^m. \end{aligned}$$

Hence,

$$(n+1)c_{n+1m} = (m+2)(m+1)Ac_{nm+2}$$

and

$$c_{0m} = b_m, \quad n, m \geq 0,$$

i.e.,

$$c_{n,m} = \frac{(m+2n)!}{m!n!} A^n b_{m+2n}, \quad n, m \geq 0.$$

Thus the unique formal solution is $u(t, x) = \sum_{n,m=0}^{\infty} \frac{(m+2n)!}{m!n!} A^n b_{m+2n} t^n x^m$. □

Theorem 1. Let the operator A be normal (in particular, self-adjoint). If the Cauchy problem (1) has a holomorphic solution for any vector-valued function $g(x)$, which is holomorphic in a neighborhood of zero, then $A = 0$.

Proof. Since the problem (1) has a holomorphic solution for any vector-valued function $g(x)$, the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = A \frac{\partial^2 u}{\partial x^2}, \\ u(0, x) = \frac{b}{1-x}, \end{cases}$$

has a holomorphic solution for all vector $b \in H$. Let $u(t, x)$ be a solution of this problem.

It follows from Lemma 1 that $u(t, x) = \sum_{n,m=0}^{\infty} \frac{(m+2n)!}{m!n!} A^n b t^n x^m$. Since $\sum_{m=0}^{\infty} \frac{(m+2n)!}{m!(2n)!} x^m = \frac{1}{(1-x)^{2n+1}}$, we obtain that $u(t, x) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} \frac{A^n b t^n}{(1-x)^{2n+1}}$. Therefore $\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{\frac{(2n)!}{n!} \|A^n b\|} < +\infty$. Since $\sqrt[n]{\frac{(2n)!}{n!}} \rightarrow \infty$, we obtain that $\lim_{n \rightarrow \infty} \sqrt[n]{\|A^n b\|} = 0$. Hence, $\lim_{n \rightarrow \infty} \sqrt[n]{\|A^n\|} = 0$ (see, for example, [1], Problem 6.1.10). Since A is normal, $A = 0$. □

Theorem 2. Let the operator A be quasi-nilpotent and its imaginary part $A_I := \frac{1}{2i}(A - A^*)$ be of trace class. Then the Cauchy problem (1) has a unique holomorphic solution in a neighborhood of zero.

Proof. We shall seek a solution of the Cauchy problem (1) in the form

$$(2) \quad u(t, x) = \sum_{n=0}^{\infty} A^n u_n(t, x).$$

It is easy to check that this series formally satisfies the equation $\frac{\partial u}{\partial t} = A \frac{\partial^2 u}{\partial x^2}$ if

$$\frac{\partial u_0}{\partial t} = 0 \quad \text{and} \quad \frac{\partial u_{n+1}}{\partial t} = \frac{\partial^2 u_n}{\partial x^2}, \quad n \geq 0.$$

Taking into account the initial condition, we obtain

$$u_0(t, x) = g(x) \quad \text{and} \quad u_{n+1}(t, x) = \int_0^t \frac{\partial^2 u_n}{\partial x^2}(\tau, x) d\tau, \quad n \geq 0.$$

Hence,

$$u_n(t, x) = g^{(2n)}(x) \frac{t^n}{n!}, \quad n \geq 0.$$

Therefore, $u(t, x) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n g^{(2n)}(x) t^n$. Now we show that there exist $T_0, R_0 > 0$ such that this series converges uniformly in $|t| < T_0, |x| < R_0$. Let $g(x) = \sum_{m=0}^{\infty} b_m x^m, |x| < R(g)$. Let $c_1, c_2, c_3, c_4 \in (0, 1)$ and $c_3 < c_1^2 c_2 (1 - c_4)^2$ (for example $c_1 = 3/4, c_2 = 9/10, c_3 = 1/8, c_4 = 1/2$). If $0 < r_1 = c_1 R(g)$, then there exists a constant $M_1 > 0$ such that $\|b_m\| \leq \frac{M_1}{r_1^m}, m = 0, 1, \dots$. Hence, for $|x| < r_1$ and $n = 0, 1, \dots$, we obtain

$$\begin{aligned} \|g^{(2n)}(x)\| &= \left\| \sum_{m=0}^{\infty} \frac{(m+2n)!}{m!} b_{m+2n} x^m \right\| \leq \sum_{m=0}^{\infty} \frac{(m+2n)!}{m!} \|b_{m+2n}\| |x|^m \\ &\leq \frac{M_1}{r_1^{2n}} \sum_{m=0}^{\infty} \frac{(m+2n)!}{m!} \left(\frac{|x|}{r_1}\right)^m = \frac{M_1 (2n)!}{r_1^{2n} (1 - \frac{|x|}{r_1})^{2n+1}}. \end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} \left\| \frac{1}{n!} A^n g^{(2n)}(x) t^n \right\| \leq M_1 \sum_{n=0}^{\infty} \frac{(2n)!}{n!} \|A^n\| \frac{\left(\frac{|t|}{r_1}\right)^n}{\left(1 - \frac{|x|}{r_1}\right)^{2n+1}}.$$

Let us consider the Fredholm resolvent $F_A(z) := (1 - zA)^{-1}$ of A . Since A is quasi-nilpotent, $F_A(z)$ is an entire function. Moreover, A is quasi-nilpotent and A_I is compact. Hence A is compact (see [5], Ch. 1, Th. 5.4). So, A is a Volterra operator. Now from ([5], Ch. 4, Rem. 8.3 and Ch. 5, Th. 5.2), we obtain that $F_A(z)$ is of exponential type, i.e., $\overline{\lim}_{z \rightarrow \infty} \frac{\ln \|F_A(z)\|}{|z|} < +\infty$. Hence, $\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{\|A^n\|} < +\infty$ (see [3], Ch. 1, Problem 22). It

follows from the Stirling formula that $\frac{1}{R_A} := \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{\frac{(2n)!}{n!} \|A^n\|} < +\infty$. Hence, the series $\sum_{n=0}^{\infty} \frac{(2n)!}{n!} \|A^n\| r_2^n$ converges, where $r_2 = c_2 R_A$. Therefore, there exists a constant $M_2 > 0$ such that $\frac{(2n)!}{n!} \|A^n\| \leq \frac{M_2}{r_2^n}, n = 0, 1, \dots$. Hence, for $|x| < r_1$ and $n = 0, 1, \dots$, we obtain

$$M_1 \frac{(2n)!}{n!} \|A^n\| \frac{\left(\frac{|t|}{r_1}\right)^n}{\left(1 - \frac{|x|}{r_1}\right)^{2n+1}} \leq \frac{M_1 M_2}{r_1^{2n} r_2^n} \frac{|t|^n}{\left(1 - \frac{|x|}{r_1}\right)^{2n+1}}.$$

Let now $T_0 = c_3 R_A R(g)^2, R_0 = c_1 c_4 R(g)$. If $|t| < T_0$ and $|x| < R_0$, then

$$\frac{|t|}{r_1^2 r_2 (1 - \frac{|x|}{r_1})^2} < \frac{c_3 R_A R(g)^2}{c_2 R_A c_1^2 R(g)^2 (1 - \frac{R_0}{r_1})^2} = \frac{c_3}{c_1^2 c_2 (1 - c_4)^2} < 1,$$

i.e., the series $\sum_{n=0}^{\infty} \frac{1}{n!} A^n g^{(2n)}(x) t^n$ converges uniformly in the dicylinder $|t| < T_0$, $|x| < R_0$. Hence, $u(t, x) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n g^{(2n)}(x) t^n$ is holomorphic in this dicylinder and $u(t, x)$ is a solution of the problem (1). The uniqueness follows from Lemma 1. \square

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