Methods of Functional Analysis and Topology
Vol. 13 (2007), no. 4, pp. 333–337

GENERALIZED SELFADJOINTNESS OF DIFFERENTIATION OPERATOR ON WEIGHT HILBERT SPACE

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Abstract. We consider examples of operators that act in some Hilbert rigging from positive Hilbert space into the negative one. For the first derivative operator we investigate a "generalized" selfadjointness in the sense of weight Hilbert riggings of the spaces $L^2([0,1])$ and $L^2(\mathbb{R})$. We will show that an example of the operator $i\frac{d}{dt}$ in some rigging scales, which is selfadjoint in usual case and not generalized selfadjoint, can not be constructed.

1. Introduction

In the work of Yu. M. Berezansky and J. Brasche [2], an example of an operator which is "good" (i.e., in particular, selfadjoint) in the sense of H_0 space is considered but it's not selfadjoint with respect to a Hilbert rigging of H_0 in the general sense ([2], example 3.4). But it's easy to see that its construction is not correct (noticed by Berezansky). In view of this further we will show that such example can't be constructed in Hilbert weight riggings of the spaces $L^2([0,1])$ and $L^2(\mathbb{R})$ for the first order differentiation operator. Moreover, we will show that in this cases generalized selfadjointness equivalent to usual.

2. Basic definitions

Consider complex Hilbert space with a rigging,

$$H_- \supset H_0 \supset H_+, \tag{1}$$

defined in the usual way ([3], Ch. 14, §1), $H_- = (H_+)'$. Let $\mathbb{I} : H_- \to H_+$, $\mathbb{J} : H_- \to H_0$, $J : H_0 \to H_+$ be standard isometric operators connected with (1) such that the identity $\mathbb{I} = J\mathbb{J}$ takes place.

Let $A : H_+ \to H_-$ be a linear operator with domain $D(A)$ dense in H_+. For A it's easy to define an adjoint operator, $A^+ : H_+ \to H_-$. So, let $\psi \in H_+$ be such that the functional $\varphi \to (A\varphi, \psi)_{H_0} \in \mathbb{C}$, defined on $D(A)$, is continuous and therefore has the representation $(A\varphi, \psi)_{H_0} = (\varphi, \psi^+)_{H_0}, \psi^+ \in H_-$. Such ψ form $D(A^+)$ of the operator A^+, and $A^+\psi := \psi^+$. If $H_+ = H_0$, than we have the usual definition of the adjoint operator.

Remark 1. *For any A, the operator A^+ is closed. If $A \subset B$, than $A^+ \supset B^+$.*

$A : H_+ \to H_-$ is called generalized selfadjoint if $A^+ = A$ and generalized essentially selfadjoint if $A^+ = \widetilde{A}$.

Let $A : H_+ \to H_-$. Construct $\mathbb{I}A : H_+ \to H_+$. In the work [2], it was proved that the next property of generalized selfadjointness ([2], Proposition 2.1) holds true.

Proposition 1. *Properties of the above-mentioned operator A are equivalent to the corresponding classical properties of the operator $\mathbb{I}A$ as an operator from H_+. Also, $\mathbb{I}A^+ = (\mathbb{I}A)^*$, where $*$ denotes taking the adjoint in H_+ .*

1991 *Mathematics Subject Classification.* 47B25, 47A70.

Key words and phrases. Selfadjoint operators, generalized selfadjoint operators, Hilbert space rigging.

3. Criteria of generalized essential selfadjointness for weight rigging $L_2[0,1]$

Further we will consider a first order differential operator in weight Hilbert rigging and in this concrete case we will established necessary and sufficient conditions for its selfadjointness.

Consider the rigging

$$L_2([0,1],p^{-1}dt)\supset L_2([0,1],dt)\supset L_2([0,1],pdt), \tag{2}$$

where $L_2([0,1],pdt)$ is the Lebesgue space with a weight p such that

$$p\in C^1(0,1],\quad p(t)\geq 1,\quad t\in[0,1],\quad p(t)\to+\infty,\quad t\to 0. \tag{3}$$

It follows that $p^{-\frac{1}{2}}(t)\in C^1(0,1]$, $p^{-\frac{1}{2}}(t)>0\,\forall t\in(0,1]$ and $p^{-\frac{1}{2}}(t)\in C[0,1]$ $(p^{-\frac{1}{2}}(0):=0)$. Further we will also use the notations H_-,H_0,H_+ for the rigging (2). In this case, the isometric operators connected with (2) have the representation $(\mathbb{I}g)(t)=p^{-1}(t)g(t)$, $(\mathbb{J}g)(t)=p^{-\frac{1}{2}}(t)g(t)$, $(Jf)(t)=p^{-\frac{1}{2}}(t)f(t)$, where $g\in H_-$, $f\in H_0$.

Consider the operator $A\colon H_0\to H_0$ of the form $(Af)(t)=i\frac{df}{dt}$ with domain $D(A)$, dense in H_0, such that $D(A)=\{x\in AC[0,1]\mid x^{'}\in L_2(0,1),x(0)=x(1)\}$. Here $AC[0,1]$ is the set of absolutely continuous functions on the line segment $[0,1]$.

Remark 2. *Considering the operator A as an operator that acts as $A\colon H_0\to H_0$ is sufficient for defining it on $D(A)$. But since $D(A)\not\subseteq H_+$ in general, to define the operator as the map $A\colon H_+\to H_-$, it's necessary to construct its domain in H_+.*

So in the future, the operator A is understood as $A\restriction D_+(A)$, where $D_+(A)=D(A)\cap H_+$.

Theorem 1. *Let p satisfy (3). The operator $A\colon H_0\to H_0$ with the domain $D_+(A)$ is selfadjoint iff*

$$\int_0^1 p(t)\,dt<\infty. \tag{4}$$

Proof. **Necessity.** Let us suppose the opposite, that is,

$$\int_0^1 p(t)\,dt=\infty. \tag{5}$$

We will show that $D_+(A)\subset\{x\in AC[0,1]\mid x^{'}\in L_2(0,1),x(0)=x(1)=0\}=:D_1$.

If not, $\exists\,x\in D_+(A)$ such that $x(0)\neq 0$. Since $x\in AC[0,1]\subset C[0,1]$, there exist $c>0$ and $\delta>0$ such that $\forall\,t\in[0,\delta]\ \mid x(t)\mid>c$. As (5) takes place and the weight p satisfies (3), we see that $\int_0^\delta p(t)\,dt=\infty$. Therefore, $x\overline{\in}D_+(A)$, because $\|x\|_+=\int_0^1\mid x(t)\mid^2p(t)\,dt\geq\int_0^\delta\mid x(t)\mid^2p(t)\,dt>c^2\int_0^\delta p(t)\,dt=\infty$. We obtain a contradiction.

It is well known that the operator $A_1=i\frac{d}{dt}\colon D_1\to H_0$ is Hermitian but not selfadjoint ([1], Ch. 4, §55 or [5] Ch. 8, §2) and $D(A_1^*)=\{x\in AC[0,1]\mid x^{'}\in L_2(0,1)\}$. Form this and Remark 1 it follows that $A\subset A_1\subset A_1^*\subset A^*$ and A is not selfadjoint, because the inclusion $A_1\subset A_1^*$ is strict. We obtain a contradiction.

Sufficiency. It's easy to see that under the condition (4), $D(A)\subset H_+$ and $D_+(A)=D(A)$. So, as well known ([1], Ch. 4, §55), the operator $A\colon D(A)\to H_0$ is selfadjoint. □

Consider the operator $A\colon H_+\to H_-$ with domain $D_+(A)$ dense in H_+. The generalized selfadjointness of an operator $A\colon H_+\to H_-$, according to Prop. 1, is equivalent to selfadjointness of $\mathbb{I}A\colon H_+\to H_+$. So the generalized selfadjointness is equivalent to the usual selfadjointness of the operator $S:=\mathbb{J}AJ=J^{-1}\mathbb{I}AJ\colon H_0\to H_0$, where $D(S)=\{x\in H_0\big|Jx\in D_+(A)\}$. So further we will investigate the operator $S=ip^{-\frac{1}{2}}(\cdot)\frac{d}{dt}p^{-\frac{1}{2}}(\cdot)\colon L_2[0,1]\to L_2[0,1]$ with the domain $D(S)=\{x\in L_2[0,1]\mid p^{-\frac{1}{2}}x\in$

$AC[0,1], (p^{-\frac{1}{2}}x)' \in L_2(0,1), (p^{-\frac{1}{2}}x)(0) = (p^{-\frac{1}{2}}x)(1)\}$. Selfadjointness of the operator S depends on properties of the weight p. We will formulate results which give necessary and sufficient selfadjointness conditions for the operator S, and the initial operator A. The operator S is Hermitian as A because, for all $x, y \in D(S)$, we have $(J^{-1}\mathbb{I}AJx, y) = (\mathbb{I}AJx, Jy) = (Jx, \mathbb{I}AJy) = (x, J^{-1}\mathbb{I}AJy)$, since $D(S) = \{x \mid Jx \in D(A)\}$.

Theorem 2. *Let p satisfy (3). Then the operator $A\colon H_+ \to H_-$ with domain $D_+(A)$ is generalized essentially selfadjoint iff p satisfies (4).*

Proof. **Necessity.** Let's suppose to the contrary that (5) takes place. At first consider $D(S)$. Note that $\forall x \in D(S) \quad p^{-\frac{1}{2}}x \in AC[0,1] \subset C[0,1]$. We will prove that $(p^{-\frac{1}{2}}x)(0) = 0$. Let, by contradiction, $(p^{-\frac{1}{2}}x)(0) = c \neq 0$. Then $x \sim c\,p^{\frac{1}{2}}, t \to 0$. But since $\int\limits_0^1 p(t)\,dt = \infty$, we see that $\int\limits_0^1 x^2(t)\,dt = \infty$. From this it follows that $x \overline{\in} L_2[0,1]$. We obtain a contradiction. So from (5), it follows that $D(S) = \{x \in L_2[0,1] \mid p^{-\frac{1}{2}}x \in AC[0,1], (p^{-\frac{1}{2}}x)' \in L_2(0,1), (p^{-\frac{1}{2}}x)(0) = (p^{-\frac{1}{2}}x)(1) = 0\}$.

Let's show that $D(S^*) \supseteq \{y \in L_2[0,1] \mid p^{-\frac{1}{2}}(t)y(t) \in AC[0,1], (p^{-\frac{1}{2}}(t)y(t))' \in L_2[0,1]\} =: D_2$.

Consider $x, y \in D(S)$. Then

$$\begin{aligned}(Sx,y) &= \int\limits_0^1 ip^{-\frac{1}{2}}(t)(p^{-\frac{1}{2}}(t)x(t))'\overline{y(t)}\,dt\\ &= i[p^{-\frac{1}{2}}(1)x(1)\overline{(p^{-\frac{1}{2}}(1)y(1))} - p^{-\frac{1}{2}}(0)x(0)\overline{(p^{-\frac{1}{2}}(0)y(0))}]\\ &+ \int\limits_0^1 x(t)\overline{(ip^{-\frac{1}{2}}(t)(p^{-\frac{1}{2}}(t)y(t))')}\,dt\end{aligned}$$

so $(Sx, y) = (x, Sy)$, because the expression outside the integral sign equals zero. But it is zero also in case where $x \in D(S)$ and $y \in D_2$. So any function $y \in D_2$ belongs to $D(S^*)$ and $(S^*y)(t) = p^{-\frac{1}{2}}(t)(p^{-\frac{1}{2}}(t)y(t))'$.

Let's show that S is not essentially selfadjoint. It's sufficient to show that $\operatorname{Ker}(S^* \pm i) \neq \{0\}$. Let $y \in \operatorname{Ker}(S^* \pm i)$. Then $\forall x \in D(S)$, $0 = (y, (S \mp i)x) = -i\int\limits_0^1 y(t)(p^{-1}(t)x'(t) + (p^{-\frac{1}{2}}(t))'p^{-\frac{1}{2}}(t)x(t) \mp x(t))\,dt$ or $\int\limits_0^1 p^{-1}(t)y(t)x'(t)\,dt = \int\limits_0^1 (-(p^{-\frac{1}{2}}(t))'p^{-\frac{1}{2}}(t) \pm 1)y(t)x(t)\,dt$. From the form of $D(S)$ and it's density in H_0, we obtain the equation $-(p^{-1}(t)y(t))' + ((p^{-\frac{1}{2}}(t))'p^{-\frac{1}{2}}(t) \mp 1)y(t) = 0$ integrating left-hand side of the last equality by parts. Solutions of this equation are functions of the form $y_\pm(t) = c_\pm p^{\frac{1}{2}}(t)\exp\{\pm\int\limits_1^t p(s)\,ds\}$. It's easy to see that $y_+ \in L_2[0,1]$ and $y_- \overline{\in} L_2[0,1]$. Indeed,

$$\|y_+\|_{L_2[0,1]} = \int\limits_0^1 p(t)\exp\{-2\int\limits_t^1 p(s)\,ds\}dt = 1/2\int\limits_0^1 d(\exp\{-2\int\limits_t^1 p(s)\,ds\}) = 1/2.$$

Also, $y_+ \in D(S^*)$ because $y_+ \in D_2 \subset D(S^*)$. Therefore, S is not essentially selfadjoint.

Sufficiency. Let the weight p be such that (4) holds. Then $\exists x \in D(S)$ such that $(p^{-\frac{1}{2}}x)(0) = (p^{-\frac{1}{2}}x)(1) = c \neq 0$. Indeed, take $x := p^{1/2}$. It's easy to see that $x \in D(S)$. But $(p^{-\frac{1}{2}}x)(0) = 1$.

Let $y \in D(S^*)$ $S^*y = y^*$. It's easy to see that $(S^*y)(t) = ip^{-\frac{1}{2}}(t)\big(p^{-\frac{1}{2}}(t)y(t)\big)'$. Then for any $x \in D(S)$, we have

$$\int_0^1 ip^{-\frac{1}{2}}(t)(p^{-\frac{1}{2}}(t)x(t))'\overline{y(t)}\,dt = (Sx, y) = (x, y^*)$$

$$= \int_0^1 x(t)\overline{ip^{-\frac{1}{2}}(t)(p^{-\frac{1}{2}}(t)y(t))'}dt = -ip^{-\frac{1}{2}}(t)x(t)\,p^{-\frac{1}{2}}(t)y(t)\Big|_0^1$$

$$+ \int_0^1 ip^{-\frac{1}{2}}(t)(p^{-\frac{1}{2}}(t)x(t))'\overline{y(t)}\,dt.$$

So, for any $y \in D(S^*)$, $(p^{-\frac{1}{2}}y)(0) = (p^{-\frac{1}{2}}y)(1)$.

Lets us show that S is essentially selfadjoint. Let $y \in \operatorname{Ker}(S^* \pm i)$. Then y satisfies the equation $-(p^{-1}(t)y(t))' + ((p^{-\frac{1}{2}}(t))'p^{-\frac{1}{2}}(t) \mp 1)y(t) = 0$, which is proved in the same way as for the necessity part. The solutions are $y_\pm(t) = c_\pm p^{\frac{1}{2}}(t)\exp\{\pm\int_1^t p(s)\,ds\}$. But $y_\pm(t)$ doesn't satisfy the necessary conditions, because, since $0 < \int_0^1 p(t)\,dt < \infty$, we have $(p^{-1/2}y_\pm)(0) \neq 0$, $(p^{-1/2}y_\pm)(1) = 0$.

So, $\operatorname{Ker}(S^* \pm i) = \{0\}$ and S is essentially selfadjoint. □

From the above we have the following. The operator $A\colon H_+ \to H_-$ with the domain $D_+(A)$ is generalized essentially selfadjoint iff it is selfadjoint in the usual sense considered as an operator on H_0.

4. Generalized essential selfadjointness for weight rigging $L_2(\mathbb{R})$

Consider a complex Hilbert space rigging,

$$L_2(\mathbb{R}, p^{-1}dt) \supset L_2(\mathbb{R}, dt) \supset L_2(\mathbb{R}, pdt), \tag{6}$$

where $L_2(\mathbb{R}, pdt)$ is the Lebesgue space with a weight p such that $p \in C^\infty(\mathbb{R})$; $p\colon \mathbb{R} \to [1, +\infty)$. So we have that $p^{-\frac{1}{2}}(t) \in C^\infty(\mathbb{R})$, $p^{-\frac{1}{2}}\colon \mathbb{R} \to (0, 1]$. Isometric operators connected with (6) have the same representation as in previous case.

Consider an operator $A\colon H_+ \to H_-$ defined by $(Af)(t) = i\frac{df}{dt}$ with domain $D(A) = C_0^\infty(\mathbb{R})$ dense in H_+. Here $C_0^\infty(\mathbb{R})$ is the set of finite and infinitely differentiable functions on $\overline{\mathbb{R}}$. As well known ([4], p. 160), the operator $A\colon H_0 \to H_0$ with domain $D(A)$ is essentially selfadjoint. Instead of investigating the operator $A\colon H_+ \to H_-$, we consider $S := \mathbb{J}AJ = J^{-1}\mathbb{I}AJ\colon H_0 \to H_0$, where $D(S) = \{x \in H_0 \big| Jx \in D(A)\}$. Since $p^{-\frac{1}{2}} \in C^\infty(\mathbb{R})$ and $\forall t \in \mathbb{R}\; p^{-\frac{1}{2}}(t) > 0$, we have $D(S) = D(A) = C_0^\infty(\mathbb{R})$.

Proposition 2. *The operator S is essentially selfadjoint in $L_2(\mathbb{R})$.*

Proof. It's sufficient to show that $\operatorname{Ker}(S^* \pm i) = \{0\}$. Let $y \in \operatorname{Ker}(S^* \pm i)$. Then y satisfies the equation $-(p^{-1}(t)y(t))' + ((p^{-\frac{1}{2}}(t))'p^{-\frac{1}{2}}(t) \mp 1)y(t) = 0$, solutions of which have the form $y_\pm(t) = c_\pm p^{\frac{1}{2}}(t)\exp\{\pm\int_0^t p(s)\,ds\}$. But $y_\pm(t) \overline{\in} L_2(\mathbb{R})$, since

$$\begin{aligned}\int_{-\infty}^{+\infty} y_\pm^2(t)\,dt &= \int_{-\infty}^{+\infty} p(t)\exp\{\pm 2\int_0^t p(s)\,ds\}dt = 1/2\int_{-\infty}^{+\infty} \pm d\Big(\exp\{\pm 2\int_0^t p(s)\,ds\}\Big) \\ &= \pm 1/2\Big(\exp\{\pm 2\int_0^{+\infty} p(s)\,ds\} - \exp\{\pm 2\int_0^{-\infty} p(s)\,ds\}\Big) = +\infty.\end{aligned}$$

□

So the operator $A\colon H_+ \to H_-$ in the rigging sense (6) with the domain $D(A) = C_0^\infty(\mathbb{R})$ is generalized essentially selfadjoint and, at the same time, $A\colon H_0 \to H_0$ on $D(A)$ is essential selfadjoint in the usual sense.

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Received 04/04/2007; Revised 21/06/2007