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ON AN EXTENDED STOCHASTIC INTEGRAL AND THE WICK CALCULUS ON THE CONNECTED WITH THE GENERALIZED MEIXNER MEASURE KONDRATIEV-TYPE SPACES

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Abstract. We introduce an extended stochastic integral and construct elements of the Wick calculus on the Kondratiev-type spaces of regular and nonregular generalized functions, study the interconnection between the extended stochastic integration and the Wick calculus, and consider examples of stochastic equations with Wick-type nonlinearity. Our researches are based on the general approach that covers the Gaussian, Poissonian, Gamma, Pascal and Meixner analyses.

0. Introduction

In 1934 J. Meixner ([30]) proved that there exist exactly five types of orthogonal polynomials on $\mathbb{R}$ with the generating function $\gamma(\lambda)e^{x\alpha(\lambda)}$ (the polynomials with such a generating function are called the *Schefer polynomials* or, in another terminology, the *generalized Appell polynomials*): the Hermite, Charlier, Laguerre, Meixner and Meixner-Pollaczek polynomials, which are orthogonal with respect to the Gaussian, Poissonian, Gamma, Pascal and Meixner measures correspondingly. In an infinite-dimensional analysis the situation is more complicated, but all mentioned measures and polynomials have the corresponding counterparts, and the orthogonality preserves. Nevertheless, if for the Gaussian and Poissonian measures the orthogonality of the (infinite-dimensional) Hermite and Charlier polynomials correspondingly is quite simple, and therefore it is respectively easy to construct corresponding analyses; then for the Gamma, Pascal and Meixner measures the orthogonality of the corresponding polynomials is more tricky (see Theorem 1.2 in Section 1), and therefore the situation is much more complicated. As a result, the connected with the Gamma, Pascal and Meixner measures infinite-dimensional analyses are considerably more 'poor' than the analyses that are connected with the Gaussian and Poissonian measures. (Note that the question of orthogonality of polynomials is connected with the so-called Chaotic Representation Property (CRP) of the measure. The CRP is very important in the stochastic integration theory. Between five mentioned above measures only the Gaussian and Poissonian ones have CRP.)

In the papers [28, 29] E. W. Lytvynov made first (as far as it is known to the author) attempt to generalize the results of [30] to the infinite-dimensional case and to construct elements of the corresponding analysis with "stochastic applications" (in this connection we have to remember also the paper of Yu. M. Berezansky [4]). In the paper [31] I. V. Rodionova constructed the analysis that is based on generalization of results [29]; and for the first time considered connected with the Gaussian, Poissonian, Gamma, Pascal and Meixner measures infinite-dimensional analyses from a common point of view. However,

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in these papers an extended stochastic integral and a stochastic derivative were not discussed. (Nevertheless, studied in [29, 31] operators ∂_x (actually this is the generalized Hida derivative at the point x) and $\partial_x^\dagger$ are closely connected with the extended stochastic integral, see Section 3 for more details.)

The main aim of this paper is to introduce the extended stochastic integral and to construct elements of the Wick calculus on the Kondratiev-type spaces of regular and nonregular generalized functions, to consider the interconnection between the stochastic integration and the Wick calculus, and to illustrate these considerations by simple examples (by simple stochastic equations with Wick-type nonlinearity). Our researches are based on the proposed in [31] general approach that covers the Gaussian, Poissonian, Gamma, Pascal and Meixner analyses. The construction of the stochastic derivative is affected only (in the connection with studying of properties of the extended stochastic integral), the detailed study will be given in forthcoming papers.

The paper is based on the results of I. V. Rodionova [31] and of the author [21, 17], and can be considered as a natural development and generalization of results [21, 17].

Finally we mention that in this paper (in the same way as in [31]) the base probability measure (see Definition 1.1 in Section 1) is *centered*, but all results hold true for the case of the described in [29] *noncentered* measure (in this connection see also Remarks 1.5 and 3.2 below).

The paper is organized in the following manner. In the first section we recall necessary definitions and results, and prove several important for our considerations statements (note that some definitions in this section seem "artificial" because they are based on calculations of [31], the interested reader can find the detailed explanations in the mentioned paper). In the second section we introduce the Kondratiev-type spaces of test and (regular and nonregular) generalized functions and construct natural bases in these spaces. In the third section we introduce the extended stochastic integral and study its properties (in particular, its interconnection with the generalized Hida derivative). The fourth section is devoted to the Wick calculus and its interconnection with the extended stochastic integration; in the end of the section we consider examples.

1. Preliminaries

Let σ be a measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ (here and below $\mathcal{B}$ denotes the Borel σ-algebra) satisfying the following assumptions:

1) σ is absolutely continuous with respect to the Lebesgue measure and the density is an infinite differentiable function on $\mathbb{R}_+$;
2) σ is nondegenerate measure, i.e., for each nonempty open set $O \subset \mathbb{R}_+$ $\sigma(O) > 0$.

Remark 1.1. Note that these assumptions are the "simplest *sufficient* ones" for our considerations; actually it is possible to consider much more general σ. Moreover, one can use the space $\mathbb{R}$ instead of $\mathbb{R}_+$; but in this case it is necessary either to introduce the integration stochastic process on $\mathbb{R}$ (such situation is unnatural) or to overcome unjustified technical problems.

By $\mathcal{D}$ denote the set of all real-valued infinite differentiable functions on $\mathbb{R}_+$ with compact supports. This set can be naturally endowed with a (projective limit) topology of a nuclear space (by analogy with, e.g., [8]): $\mathcal{D} = \mathrm{pr}\lim_{\tau\in T} \mathcal{H}_\tau$, where T is the set of all pairs $\tau = (\tau_1, \tau_2)$, $\tau_1 \in \mathbb{N}$, τ_2 is an infinite differentiable function on $\mathbb{R}_+$ such that $\tau_2(t) \geq 1$ $\forall t \in \mathbb{R}_+$; $\mathcal{H}_\tau = \mathcal{H}_{(\tau_1,\tau_2)}$ is the Sobolev space on $\mathbb{R}_+$ of order τ_1 weighted by the function τ_2, i.e., the denoted by $(\cdot,\cdot)_\tau$ scalar product in $\mathcal{H}_\tau$ is given by the formula

$$(f, g)_\tau := \int_{\mathbb{R}_+} (f(t)g(t) + \sum_{k=1}^{\tau_1} f^{(k)}(t)g^{(k)}(t))\tau_2(t)\sigma(dt).$$

Hence in what follows, we understand $\mathcal{D}$ as the corresponding *topological space*.

Remark 1.2. By analogy with [28, 29, 31] one can consider a complete, connected, oriented C^∞ non-compact Riemannian manifold X instead of $\mathbb{R}_+$ and work with $\mathcal{D}(X)$ etc. But such a generalization is not essential for our considerations (and leads to a technical complication), therefore we shall restrict ourselves, in this paper, to the case $X=\mathbb{R}_+$.

Let $\mathcal{H}_{\tau,\mathbb{C}} := \mathcal{H}_\tau \oplus i\mathcal{H}_\tau$ be the complexification of $\mathcal{H}_\tau$ (here and below by the subindex $\mathbb{C}$ denote complexifications of spaces). By $|\cdot|_\tau$ denote the corresponding to the scalar product $(\cdot,\cdot)_\tau$ norm in $\mathcal{H}_{\tau,\mathbb{C}}$, i.e., $|f|_\tau^2=(f,\overline{f})_\tau$. In the forthcoming statement we describe the important property of $\mathcal{H}_{\tau,\mathbb{C}}$.

Lemma 1.1. *The space $\mathcal{H}_{\tau,\mathbb{C}}$ $(\tau\in T)$ is a Banach algebra with respect to the usual (pointwise) multiplication of functions, i.e., for each $\tau\in T$ there exists a constant $c_\tau>0$ such that*

$$|fg|_\tau \le c_\tau |f|_\tau |g|_\tau \quad \forall f,g\in\mathcal{H}_{\tau,\mathbb{C}}.$$

The proof is completely analogous to the proof of Theorem 7.1 in [6]. □

Let us consider the (nuclear) chain (the rigging of $L^2(\mathbb{R}_+,\sigma)$)

$$\mathcal{D}'=\operatorname*{ind\,lim}_{\tau'\in T}\mathcal{H}_{-\tau'}\supset\mathcal{H}_{-\tau}\supset L^2(\mathbb{R}_+,\sigma)=:\mathcal{H}\supset\mathcal{H}_\tau\supset\operatorname*{pr\,lim}_{\tau'\in T}\mathcal{H}_{\tau'}=\mathcal{D}, \tag{1.1}$$

where $\mathcal{H}_{-\tau}$, $\mathcal{D}'$ are the dual to $\mathcal{H}_\tau$, $\mathcal{D}$ with respect to $\mathcal{H}$ spaces correspondingly. By $|\cdot|_{-\tau}$ and $|\cdot|_0$ denote the norms in $\mathcal{H}_{-\tau}$ and $\mathcal{H}$. Let $\langle\cdot,\cdot\rangle$ be the generated by the scalar product in $\mathcal{H}$ dual pairing between elements of $\mathcal{D}'$ and $\mathcal{D}$ (and also $\mathcal{H}_{-\tau}$ and $\mathcal{H}_\tau$). The notation $|\cdot|_\tau$, $|\cdot|_0$, $|\cdot|_{-\tau}$, $(\cdot,\cdot)_\tau$, and $\langle\cdot,\cdot\rangle$ will be preserved for tensor powers and complexifications of spaces.

Remark 1.3. Note that *all scalar products and pairings in this paper are real*, i.e., they are *bilinear* functionals. In particular, $\langle\cdot,\cdot\rangle$ is a *real* pairing in complexifications of spaces.

Let us fix arbitrary functions $\alpha,\beta:\mathbb{R}_+\to\mathbb{C}$ that are smooth and satisfy

$$\theta:=-\alpha-\beta\in\mathbb{R},\quad \eta:=\alpha\beta\in\mathbb{R}_+, \tag{1.2}$$

θ and η are bounded on $\mathbb{R}_+$. Further, let $\widetilde{\upsilon}(\alpha,\beta,ds)$ be a probability measure on $\mathbb{R}$ that is defined by its Fourier transform

$$\int_{\mathbb{R}} e^{ius}\widetilde{\upsilon}(\alpha,\beta,ds)$$
$$=\exp\left\{-iu(\alpha+\beta)+2\alpha\beta\sum_{m=1}^{\infty}\frac{(\alpha\beta)^{m-1}}{m}\left[\sum_{n=2}^{\infty}\frac{(-iu)^n}{n!}(\beta^{n-2}+\beta^{n-3}\alpha+\cdots+\alpha^{n-2})\right]^m\right\},$$

$\upsilon(\alpha,\beta,ds):=\frac{1}{s^2}\widetilde{\upsilon}(\alpha,\beta,ds)$.

Definition 1.1. We say that the probability measure μ on the measurable space $(\mathcal{D}',\mathcal{F})$ (here and below $\mathcal{F}$ is the generated by cylinder sets σ-algebra on $\mathcal{D}'$) with the Fourier transform

$$\int_{\mathcal{D}'} e^{i\langle x,\xi\rangle}\mu(dx)=\exp\left\{\int_{\mathbb{R}_+}\sigma(dt)\int_{\mathbb{R}}\upsilon(\alpha(t),\beta(t),ds)(e^{is\xi(t)}-1-is\xi(t))\right\}$$

(here $\xi\in\mathcal{D}$) is called the *generalized Meixner measure*.

Theorem 1.1. [31] *The generalized Meixner measure μ is a generalized stochastic process with independent values in the sense of* [11]. *The Laplace transform of μ is given in a* *neighborhood of zero* $\mathcal{U}_0 \subset \mathcal{D}_{\mathbb{C}}$ *by the following formula:*

$$l_\mu(\lambda)=\int_{\mathcal{D}'} e^{\langle x,\lambda\rangle}\mu(dx)=\exp\left\{\int_{\mathbb{R}_+}\sum_{m=1}^{\infty}\frac{(\alpha(t)\beta(t))^{m-1}}{m}\right.$$
$$\left.\times\left(\sum_{n=2}^{\infty}\frac{(-\lambda)^n}{n!}(\beta(t)^{n-2}+\beta(t)^{n-3}\alpha(t)+\cdots+\alpha(t)^{n-2})\right)^m\sigma(dt)\right\},\quad \lambda\in\mathcal{U}_0. \tag{1.3}$$

Remark 1.4. Accordingly to the classical classification [30] (see also [29, 31]) for $\alpha=\beta=0$ (here and below all such equalities we understand σ-a.e.) μ is the Gaussian measure; for $\alpha\neq 0$ (here and below $a(\cdot)\neq b(\cdot)$ means that $a-b\neq 0$ on some measurable set M such that $\sigma(M)>0$), $\beta=0$ μ is the centered Poissonian measure; for $\alpha=\beta\neq 0$ μ is the centered Gamma measure; for $\alpha\neq\beta$, $\alpha\beta\neq 0$, $\alpha,\beta:\mathbb{R}_+\to\mathbb{R}$ μ is the centered Pascal measure; for $\alpha=\overline{\beta}$, $Im(\alpha)\neq 0$ μ is the centered Meixner measure.

Remark 1.5. The introduced above generalized Meixner measure μ is *centered* by construction; but this is not essential for our considerations because actually our constructions do not depend on a "centrality" of μ (the only difference is described in Remark 3.2 in Section 3 below). For example, one can use the introduced in [29] *noncentered* generalized Meixner measure (the particular case of this measure is the *noncentered* Gamma measure that was introduced in [25] and studied in [23], see also [21, 17]).

It is known that the Gaussian measure, the Poissonian measure and the Gamma measure are concentrated on a "pre-limit" space $\mathcal{H}_{-\tau}$ (for some $\tau\in T$). Let us prove that this result holds true for the generalized Meixner measure μ.

Lemma 1.2. *There exists* $\widetilde{\tau}\in T$ *such that the generalized Meixner measure is concentrated on* $\mathcal{H}_{-\widetilde{\tau}}$, *i.e.,* $\mu(\mathcal{H}_{-\widetilde{\tau}})=1$.

Proof. It follows from (1.3) and Lemma 1.1 (see also [31]) that l_μ is continuous at zero in the topology of $\mathcal{H}_{\tau,\mathbb{C}}$ for all $\tau\in T$. Let us fix $\tau'\in T$. Since $\mathcal{D}'$ is a nuclear space, there exists $\widetilde{\tau}\in T$ such that the embedding $\mathcal{H}_{\widetilde{\tau}}\hookrightarrow\mathcal{H}_{\tau'}$ is of Hilbert-Schmidt type, therefore by the Minlos-Sazonov theorem μ is concentrated on $\mathcal{H}_{-\widetilde{\tau}}$. □

Remark 1.6. *In what follows, we assume that* μ *is concentrated on* $\mathcal{H}_{-\tau}$ *for all* $\tau\in T$. In fact, it is sufficient to exclude from T the indexes τ such that μ is not concentrated on $\mathcal{H}_{-\tau}$.

Now by $(L^2)=L^2(\mathcal{D}',\mu)$ denote the space of square integrable with respect to μ complex-valued functions on $\mathcal{D}'$. Let us construct orthogonal polynomials on (L^2).

Definition 1.2. We define a so-called *Wick exponential* (a generating function of the orthogonal polynomials) by setting

$$:\exp(x;\lambda): \stackrel{\text{def}}{=} \exp\left\{-\int_{\mathbb{R}_+}\left(\frac{\lambda(t)^2}{2}+\sum_{n=3}^{\infty}\frac{\lambda(t)^n}{n}(\alpha(t)^{n-2}+\alpha(t)^{n-3}\beta(t)+\cdots+\beta(t)^{n-2})\right)\sigma(dt)\right.$$
$$\left.+\left\langle x,\lambda+\sum_{n=2}^{\infty}\frac{\lambda^n}{n}(\alpha^{n-1}+\alpha^{n-2}\beta+\cdots+\beta^{n-1})\right\rangle\right\}, \tag{1.4}$$

where $\lambda\in\mathcal{U}_0\subset\mathcal{D}_{\mathbb{C}}$, $x\in\mathcal{D}'$, $\mathcal{U}_0$ is some neighborhood of $0\in\mathcal{D}_{\mathbb{C}}$.

Remark 1.7. It was proved in [31] that

$$:\exp(x;\lambda):=\frac{e^{\langle x,\Psi(\lambda)\rangle}}{l_\mu(\Psi(\lambda))}$$

with $\Psi(\lambda)=\lambda+\sum_{n=2}^{\infty}\frac{\lambda^n}{n}(\alpha^{n-1}+\alpha^{n-2}\beta+\dots+\beta^{n-1})$, therefore $:\exp(x;\cdot):$ is a generating function of the so-called *Schefer polynomials* (or the *generalized Appell polynomials* in another terminology). This fact gives us the possibility to use well-known results of the so-called "biorthogonal analysis" (see, e.g., [2, 1, 27, 18, 19, 24, 3, 7] and references therein) in order to construct the connected with the generalized Meixner measure μ analysis; but actually this is not very important for our considerations in this paper.

It is clear (see also [31]) that $:\exp(x;\cdot):$ is a holomorphic at zero function on $\mathcal{D}_{\mathbb{C}}$ for each $x\in\mathcal{D}'$. Therefore using the Cauchy inequalities (see, e.g., [10]) and the kernel theorem (see, e.g., [8]) one can obtain the representation

$$:\exp(x;\lambda):=\sum_{n=0}^{\infty}\frac{1}{n!}\langle P_n(x),\lambda^{\otimes n}\rangle,\quad P_n(x)\in{\mathcal{D}'_{\mathbb{C}}}^{\widehat{\otimes}n},\quad x\in\mathcal{D}',\quad \lambda\in\mathcal{D}_{\mathbb{C}}.$$

Here (and below) $\widehat{\otimes}$ denotes the symmetric tensor product, $\lambda^{\otimes 0}=1$ even for $\lambda\equiv 0$.

Remark 1.8. It follows from the given in [31] recurrence formula for $P_n(x)$ that actually $P_n(x)\in{\mathcal{D}'}^{\widehat{\otimes}n}$ for $x\in\mathcal{D}'$. Moreover, if $\tau\in T$ is such that the Dirac delta-function $\delta_0\in\mathcal{H}_{-\tau}$ (it means that $\delta_s\in\mathcal{H}_{-\tau}$ $\forall s\in\mathbb{R}_+$, see, e.g., [8]) then *for $x\in\mathcal{H}_{-\tau}$ we have $P_n(x)\in\mathcal{H}_{-\tau}^{\widehat{\otimes}n}$*.

In what follows, we assume that this statement holds true for all $\tau\in T$. In fact, by analogy with Remark 1.6 it is sufficient to exclude from T the indexes τ such that $\delta_0\notin\mathcal{H}_{-\tau}$.

Definition 1.3. We say that the polynomials $\langle P_n,f^{(n)}\rangle$, $f^{(n)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$ are called the *generalized Meixner polynomials*.

Remark 1.9. Depending on α and β in (1.4) the generalized Meixner polynomials can be the generalized Hermite polynomials ($\alpha=\beta=0$); the generalized Charlier polynomials ($\alpha\neq 0,\beta=0$); the generalized Laguerre polynomials ($\alpha=\beta\neq 0$); the Meixner polynomials ($\alpha\neq\beta$, $\alpha\beta\neq 0$, $\alpha,\beta:\mathbb{R}_+\to\mathbb{R}$); the Meixner-Pollaczek polynomials ($\alpha=\overline{\beta}$, $Im(\alpha)\neq 0$). (See also Remark 1.4.)

In order to formulate a statement on an orthogonality of the generalized Meixner polynomials we need the following

Definition 1.4. We define the scalar product $\langle\cdot,\cdot\rangle_{\mathrm{ext}}$ on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$ ($n\in\mathbb{N}$) by the formula
(1.5)

$$\begin{aligned}\langle f^{(n)},g^{(n)}\rangle_{\mathrm{ext}}:=&\sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n}}\frac{n!}{l_1^{s_1}\dots l_k^{s_k}s_1!\dots s_k!}\\ &\times\int_{\mathbb{R}_+^{s_1+\dots+s_k}}f^{(n)}(\underbrace{t_1,\dots,t_1}_{l_1},\dots,\underbrace{t_{s_1},\dots,t_{s_1}}_{l_1},\dots,\underbrace{t_{s_1+\dots+s_k},\dots,t_{s_1+\dots+s_k}}_{l_k})\\ &\times g^{(n)}(\underbrace{t_1,\dots,t_1}_{l_1},\dots,\underbrace{t_{s_1},\dots,t_{s_1}}_{l_1},\dots,\underbrace{t_{s_1+\dots+s_k},\dots,t_{s_1+\dots+s_k}}_{l_k})\eta(t_1)^{l_1-1}\dots\eta(t_{s_1})^{l_1-1}\\ &\times\eta(t_{s_1+1})^{l_2-1}\dots\eta(t_{s_1+s_2})^{l_2-1}\dots\eta(t_{s_1+\dots+s_{k-1}+1})^{l_k-1}\dots\eta(t_{s_1+\dots+s_k})^{l_k-1}\\ &\times\sigma(dt_1)\dots\sigma(dt_{s_1+\dots+s_k}).\end{aligned}$$

Denote by $|\cdot|_{\mathrm{ext}}$ the corresponding norm, i.e., $|f^{(n)}|_{\mathrm{ext}}^2=\langle f^{(n)},\overline{f^{(n)}}\rangle_{\mathrm{ext}}$. For $n=0$ $\langle f^{(0)},g^{(0)}\rangle_{\mathrm{ext}}:=f^{(0)}g^{(0)}$, $|f^{(0)}|_{\mathrm{ext}}=|f^{(0)}|$.

Example 1.1. It is easy to see that for $n=1$

$$\langle f^{(1)},g^{(1)}\rangle_{\mathrm{ext}}=\langle f^{(1)},g^{(1)}\rangle=\int_{\mathbb{R}_+}f^{(1)}(t)g^{(1)}(t)\sigma(dt).$$

Further, for $n = 2$

$$\langle f^{(2)}, g^{(2)}\rangle_{\rm ext} = \langle f^{(2)}, g^{(2)}\rangle + \int_{\mathbb{R}_+} f^{(2)}(t,t)g^{(2)}(t,t)\eta(t)\sigma(dt).$$

If $\eta = 0$ (this corresponds to the Gaussian and Poissonian measures μ, see Remark 1.4) then $\langle f^{(n)}, g^{(n)}\rangle_{\rm ext} = \langle f^{(n)}, g^{(n)}\rangle$ for all $n \in \mathbb{Z}_+$, in a general case $\langle f^{(n)}, g^{(n)}\rangle_{\rm ext} = \langle f^{(n)}, g^{(n)}\rangle + \cdots$.

Theorem 1.2. [31] *The generalized Meixner polynomials are orthogonal in* (L^2) *in the sense that*

$$\int_{\mathcal{D}'} \langle P_n(x), f^{(n)}\rangle \langle P_m(x), g^{(m)}\rangle \mu(dx) = \delta_{mn} n! \langle f^{(n)}, g^{(n)}\rangle_{\rm ext}. \tag{1.6}$$

By $\mathcal{H}^{(n)}_{\rm ext}$ ($n \in \mathbb{N}$) denote the closure of $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$ with respect to the connected with scalar product (1.5) norm $|\cdot|_{\rm ext}$, $\mathcal{H}^{(0)}_{\rm ext} := \mathbb{C}$.

Remark 1.10. It is not difficult to prove by analogy with [5] that the space $\mathcal{H}^{(n)}_{\rm ext}$ is, generally speaking, the orthogonal sum of $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n} \equiv L^2(\mathbb{R}_+, \sigma)_{\mathbb{C}}^{\widehat{\otimes} n}$ and some another Hilbert spaces (as a "limit case" one can consider $\eta = 0$, in this case $\mathcal{H}^{(n)}_{\rm ext} = \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$). In this sense $\mathcal{H}^{(n)}_{\rm ext}$ *is an extension of* $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$.

One can give another explanation of the fact that $\mathcal{H}^{(n)}_{\rm ext}$ is a more wide space than $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$. Namely, let $f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ ($f^{(n)}$ is an equivalence class in $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$). We select a representative (a function) $\widetilde{f}^{(n)} \in f^{(n)}$ with a "zero diagonal", i.e., $\widetilde{f}^{(n)}$ is such that $\widetilde{f}^{(n)}(t_1, \dots, t_n) = 0$ if $\exists i, j \in \{1, \dots, n\}$, $i \neq j$ such that $t_i = t_j$. This function generates an equivalence class $\widehat{f}^{(n)}$ in $\mathcal{H}^{(n)}_{\rm ext}$ because $|\widetilde{f}^{(n)}|_{\rm ext} = |\widetilde{f}^{(n)}|_0$. If $\widetilde{f}_1^{(n)} \in f^{(n)}$ is another function with the "zero diagonal" then $\widetilde{f}_1^{(n)} \in \widehat{f}^{(n)}$ because $\widetilde{f}^{(n)} - \widetilde{f}_1^{(n)}$ has the "zero diagonal" and therefore $|\widetilde{f}^{(n)} - \widetilde{f}_1^{(n)}|_{\rm ext} = |\widetilde{f}^{(n)} - \widetilde{f}_1^{(n)}|_0 = 0$. Further, let $f^{(n)}, g^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ be *different* elements of $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ and $\widehat{f}^{(n)}, \widehat{g}^{(n)} \in \mathcal{H}^{(n)}_{\rm ext}$ be the corresponding (constructed above) elements of $\mathcal{H}^{(n)}_{\rm ext}$. Since $|\widehat{f}^{(n)} - \widehat{g}^{(n)}|_{\rm ext} = |\widetilde{f}^{(n)} - \widetilde{g}^{(n)}|_{\rm ext} = |\widetilde{f}^{(n)} - \widetilde{g}^{(n)}|_0 > 0$, $\widehat{f}^{(n)}$ and $\widehat{g}^{(n)}$ are different elements of $\mathcal{H}^{(n)}_{\rm ext}$. Thus there exists the injective isometric mapping $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n} \ni f^{(n)} \mapsto \widehat{f}^{(n)} \in \mathcal{H}^{(n)}_{\rm ext}$ that can be accepted as a *generalized* embedding of $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ in $\mathcal{H}^{(n)}_{\rm ext}$.

Definition 1.5. For $f^{(n)} \in \mathcal{H}^{(n)}_{\rm ext}$ ($n \in \mathbb{Z}_+$) we define $\langle P_n, f^{(n)}\rangle \in (L^2)$ as an (L^2)-limit

$$\langle P_n, f^{(n)}\rangle := \lim_{k\to\infty} \langle P_n, f_k^{(n)}\rangle, \tag{1.7}$$

where $(f_k^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n})_{k=1}^\infty$ is a sequence of "smooth" functions such that $f_k^{(n)} \to f^{(n)}$ (as $k \to \infty$) in $\mathcal{H}^{(n)}_{\rm ext}$.

Let us prove the correctness of this definition. Let $(f_k^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n})_{k=1}^\infty$ be such that $f_k^{(n)} \to f^{(n)}$ (as $k \to \infty$) in $\mathcal{H}^{(n)}_{\rm ext}$. Since $\|\langle P_n, f_k^{(n)}\rangle - \langle P_n, f_l^{(n)}\rangle\|^2_{(L^2)} = \|\langle P_n, f_k^{(n)} - f_l^{(n)}\rangle\|^2_{(L^2)} = \int_{\mathcal{D}'} \langle P_n(x), f_k^{(n)} - f_l^{(n)}\rangle \overline{\langle P_n(x), f_k^{(n)} - f_l^{(n)}\rangle} \mu(dx) = n! |f_k^{(n)} - f_l^{(n)}|^2_{\rm ext} \to 0$ as $k, l \to \infty$ (see (1.6)), the sequence $(\langle P_n, f_k^{(n)}\rangle)_{k=1}^\infty$ is a Cauchy one in (L^2) and therefore there exists the (L^2)-limit $\langle P_n, f^{(n)}\rangle := \lim_{k\to\infty} \langle P_n, f_k^{(n)}\rangle \in (L^2)$. Let $(g_k^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n})_{k=1}^\infty$ be another sequence of "smooth" functions such that $g_k^{(n)} \to f^{(n)}$ (as $k \to \infty$) in $\mathcal{H}^{(n)}_{\rm ext}$, $\langle P_n, f^{(n)}\rangle_g := \lim_{k\to\infty} \langle P_n, g_k^{(n)}\rangle$ in (L^2). We consider the "mixed" sequence $(f_1^{(n)}, g_1^{(n)}, f_2^{(n)}, g_2^{(n)}, \dots)$ and by $h_k^{(n)}$ denote the k-th element of this sequence. Let $\langle P_n, f^{(n)}\rangle_h := \lim_{k\to\infty} \langle P_n, h_k^{(n)}\rangle$ in (L^2). It is obvious that $\langle P_n, f^{(n)}\rangle_h = \langle P_n, f^{(n)}\rangle_g$ and

$\langle P_n, f^{(n)}\rangle_h = \langle P_n, f^{(n)}\rangle$, thus $\langle P_n, f^{(n)}\rangle$ is well-defined as an element of (L^2) and does not depend on the choice of an "approximating sequence". □

Remark 1.11. It is easy to see that for "smooth" $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$ limit (1.7) is a generalized Meixner polynomial, therefore the accepted in Definition 1.5 notation is natural.

The following statement from results of [31] follows.

Theorem 1.3. *A function $f \in (L^2)$ if and only if there exists a sequence of kernels $(f^{(n)} \in \mathcal{H}_{\rm ext}^{(n)})_{n=0}^{\infty}$ such that f can be presented in the form*

$$f = \sum_{n=0}^{\infty} \langle P_n, f^{(n)}\rangle, \tag{1.8}$$

where the series converges in (L^2), i.e., the (L^2)-norm of f

$$\|f\|_{(L^2)}^2 = \sum_{n=0}^{\infty} n! |f^{(n)}|_{\rm ext}^2 < \infty. \tag{1.9}$$

Furthermore, the system $\{\langle P_n, f^{(n)}\rangle,\ f^{(n)} \in \mathcal{H}_{\rm ext}^{(n)},\ n \in \mathbb{Z}_+\}$ plays a role of an orthogonal basis in (L^2) in the sense that for $f, g \in (L^2)$

$$(f, g)_{(L^2)} = \sum_{n=0}^{\infty} n! \langle f^{(n)}, g^{(n)}\rangle_{\rm ext},$$

where $f^{(n)}, g^{(n)}$ are the kernels from decompositions (1.8) *for f, g (in particular,* (1.6) *for $f^{(n)} \in \mathcal{H}_{\rm ext}^{(n)}$, $g^{(m)} \in \mathcal{H}_{\rm ext}^{(m)}$ holds true).*

2. Kondratiev-type spaces

In this section we introduce the Kondratiev-type spaces of test and (regular and non-regular) generalized functions (see, e.g., [2, 1, 27, 24, 12, 19]) and construct natural (orthogonal in the spaces that are Hilbert ones) bases in these spaces. Note that the term "Kondratiev spaces" is connected with the fact that for the first time such spaces were introduced by Yu.G. Kondratiev in [22] (in the Gaussian analysis).

In the classical Gaussian and Poissonian analysis the Kondratiev-type spaces are "based" on the tensor powers of complexification of chain (1.1)

$$\mathcal{D}_{\mathbb{C}}'^{\widehat{\otimes} n} \supset \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n} \supset \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n} \supset \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \supset \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}, \quad \tau \in T. \tag{2.1}$$

But in view of orthogonality relation (1.6) now it will be more natural to use $\mathcal{H}_{\rm ext}^{(n)}$ as "central spaces" (by analogy with the Gamma analysis, see, e.g., [21]). In order to construct corresponding chains we need

Proposition 2.1. *There exists $\widetilde{\tau} \in T$ such that for each $n \in \mathbb{N}$ $\mathcal{H}_{\widetilde{\tau},\mathbb{C}}^{\widehat{\otimes} n}$ is densely and continuously embedded in $\mathcal{H}_{\rm ext}^{(n)}$ and, moreover, for all $f^{(n)} \in \mathcal{H}_{\widetilde{\tau},\mathbb{C}}^{\widehat{\otimes} n}$ the estimate*

$$|f^{(n)}|_{\rm ext}^2 \le n! c^n |f^{(n)}|_{\widetilde{\tau}}^2 \tag{2.2}$$

with some $c > 0$ is valid.

Proof. First we prove that $\exists \widetilde{\tau} \in T$ such that $\forall f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$ estimate (2.2) is valid. Since the Laplace transform of μ l_μ is a holomorphic at $0 \in \mathcal{D}_{\mathbb{C}}$ function (see (1.3), and [31] for more details), it follows from results of [26] that there exist $\tau' \in T$ and $\varepsilon > 0$ such that $k_1 := \int_{\mathcal{H}_{-\tau'}} e^{\varepsilon |x|_{-\tau'}} \mu(dx) < \infty$ (we remind that T is modified in accordance with Remarks 1.6, 1.8, hence by Lemma 1.2 we can integrate by $\mathcal{H}_{-\tau'}$ instead of $\mathcal{D}'$). Further, since $\mathcal{D}$ is a nuclear space, there exists $\widetilde{\tau} \in T$ such that $\mathcal{H}_{\widetilde{\tau}} \hookrightarrow \mathcal{H}_{\tau'}$ and this embedding is of Hilbert-Schmidt type. In accordance with, e.g., [18] $|P_n(x)|_{-\widetilde{\tau}} \le n! c_1^n e^{k_2 |x|_{-\tau'}}$, where on can select $k_2 \in (0, \varepsilon/2]$ (the result from [18] can be used because the generalized Meixner polynomials are the generalized Appell polynomials, see Remark 1.7). Therefore $\int_{\mathcal{H}_{-\tau'}} |P_n(x)|^2_{-\widetilde{\tau}}\mu(dx) \le (n!)^2 c_1^{2n} \int_{\mathcal{H}_{-\tau'}} e^{2k_2|x|_{-\tau'}}\mu(dx) \le (n!)^2 c^n$, where $c \ge c_1^2 \max(1, k_1)$. Let $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$. Using (1.6) we can estimate as follows:

$$
\begin{aligned}
n!|f^{(n)}|^2_{\mathrm{ext}} &= \int_{\mathcal{D}'} \langle P_n(x), f^{(n)}\rangle \langle P_n(x), \overline{f^{(n)}}\rangle \mu(dx) = \int_{\mathcal{H}_{-\tau'}} \langle P_n(x), f^{(n)}\rangle \overline{\langle P_n(x), f^{(n)}\rangle}\mu(dx) \\
&= \int_{\mathcal{H}_{-\tau'}} |\langle P_n(x), f^{(n)}\rangle|^2 \mu(dx) \le \int_{\mathcal{H}_{-\tau'}} |P_n(x)|^2_{-\widetilde{\tau}}\mu(dx) |f^{(n)}|^2_{\widetilde{\tau}} \le (n!)^2 c^n |f^{(n)}|^2_{\widetilde{\tau}},
\end{aligned}
$$

from where for $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$ (2.2) follows.

Further, let a sequence $(f_k^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n})_{k=1}^{\infty}$ be a Cauchy one in $\mathcal{H}_{\widetilde{\tau},\mathbb{C}}^{\widehat{\otimes} n}$ and $\lim_{k\to\infty} f_k^{(n)} = 0$ in $\mathcal{H}_{\mathrm{ext}}^{(n)}$. In order to prove that $\mathcal{H}_{\widetilde{\tau},\mathbb{C}}^{\widehat{\otimes} n}$ is embedded in $\mathcal{H}_{\mathrm{ext}}^{(n)}$ we have to show that $\lim_{k\to\infty} f_k^{(n)} = 0$ in $\mathcal{H}_{\widetilde{\tau},\mathbb{C}}^{\widehat{\otimes} n}$ (see, e.g., [8]). In fact, since $|\cdot|_0 \le |\cdot|_{\mathrm{ext}}$ (see Definition 1.4), $\lim_{k\to\infty} f_k^{(n)} = 0$ in $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$. But $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n} \hookrightarrow \mathcal{H}_{\widetilde{\tau},\mathbb{C}}^{\widehat{\otimes} n} \hookrightarrow \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$, therefore $\lim_{k\to\infty} f_k^{(n)} = 0$ in $\mathcal{H}_{\widetilde{\tau},\mathbb{C}}^{\widehat{\otimes} n}$. Estimate (2.2) for general $f^{(n)} \in \mathcal{H}_{\widetilde{\tau},\mathbb{C}}^{\widehat{\otimes} n}$ can be obtained from the corresponding estimate for $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$ by passing to a limit. Finally, the embedding $\mathcal{H}_{\widetilde{\tau},\mathbb{C}}^{\widehat{\otimes} n} \subset \mathcal{H}_{\mathrm{ext}}^{(n)}$ is dense because $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n} \hookrightarrow \mathcal{H}_{\widetilde{\tau},\mathbb{C}}^{\widehat{\otimes} n}$ and $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$ is a dense set in $\mathcal{H}_{\mathrm{ext}}^{(n)}$; and the continuity of this embedding from estimate (2.2) follows. □

Remark 2.1. Let $\mathcal{H}_\tau$ be continuously embedded in $\mathcal{H}_{\widetilde{\tau}}$ ($\tau, \widetilde{\tau} \in T$, $\widetilde{\tau}$ from Proposition 2.1). Then it easily follows from Proposition 2.1 that for each $n \in \mathbb{N}$ $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ is densely and continuously embedded in $\mathcal{H}_{\mathrm{ext}}^{(n)}$. Moreover, since now there exists $k(\tau) > 0$ such that $|\cdot|_{\widetilde{\tau}} \le \sqrt{k(\tau)}|\cdot|_\tau$, it follows from (2.2) that

$$|f^{(n)}|^2_{\mathrm{ext}} \le n! c^n k(\tau) |f^{(n)}|^2_\tau$$

with the same c (of course, one can easily obtain from here or by direct calculation the estimate $|f^{(n)}|^2_{\mathrm{ext}} \le n! c(\tau)^n |f^{(n)}|^2_\tau$, which is the full formal analog of (2.2)).

Therefore by analogy with Remarks 1.6, 1.8 one can exclude from T indexes τ such that there is no a continuous embedding $\mathcal{H}_\tau$ in $\mathcal{H}_{\widetilde{\tau}}$, and *assume in what follows, that the results of Proposition 2.1 hold true for all* $\tau \in T$.

Finally we note that since $|\cdot|_{-\tau} \le \sqrt{k(\tau)}|\cdot|_{-\widetilde{\tau}}$, for each $\tau \in T$

$$
\begin{aligned}
\| |P_n(\cdot)|_{-\tau} \|_{(L^2)} &= \sqrt{\int_{\mathcal{H}_{-\tau'}} |P_n(x)|^2_{-\tau}\mu(dx)} \le \sqrt{k(\tau) \int_{\mathcal{H}_{-\tau'}} |P_n(x)|^2_{-\widetilde{\tau}}\mu(dx)} \\
&\le n! c^{n/2} \sqrt{k(\tau)}
\end{aligned}
\tag{2.3}
$$

(here c does not depend on τ).

Now we can consider the chains

$$\mathcal{D}_{\mathbb{C}}^{\prime\,(n)} \supset \mathcal{H}_{-\tau,\mathbb{C}}^{(n)} \supset \mathcal{H}_{\mathrm{ext}}^{(n)} \supset \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \supset \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}, \tag{2.4}$$

where $\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$, $\mathcal{D}_{\mathbb{C}}^{\prime\,(n)} = \operatorname{ind\,lim}_{\tau\in T} \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ are the dual to $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$, $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$ with respect to $\mathcal{H}_{\mathrm{ext}}^{(n)}$ spaces correspondingly. For the generated by the scalar product in $\mathcal{H}_{\mathrm{ext}}^{(n)}$ (real) dual pairings between elements of $\mathcal{D}_{\mathbb{C}}^{\prime\,(n)}$ and $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$ (in the same way as $\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ and $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$) we preserve the notation $\langle\cdot,\cdot\rangle_{\mathrm{ext}}$.

Of course, for $n = 1$ chain (2.4) has the form

$$\mathcal{D}'_{\mathbb{C}} \supset \mathcal{H}_{-\tau,\mathbb{C}} \supset \mathcal{H}_{\mathrm{ext}}^{(1)} = \mathcal{H}_{\mathbb{C}} \supset \mathcal{H}_{\tau,\mathbb{C}} \supset \mathcal{D}_{\mathbb{C}},$$

i.e., this chain coincides with the complexification of chain (1.1). But for $n>1$ and $\eta\neq 0$ chain (2.4) *is not a tensor power of chain of type* (1.1). Nevertheless, there exists the natural interconnection between chains (2.1) and (2.4). In fact, since ${\mathcal{D}'_{\mathbb{C}}}^{(n)}$ in the same way as ${\mathcal{D}'_{\mathbb{C}}}^{\widehat{\otimes}n}$ ($n\in\mathbb{Z}_+$) are the sets of linear continuous functionals on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$, there exist linear bijective operators $U_n:{\mathcal{D}'_{\mathbb{C}}}^{(n)}\to{\mathcal{D}'_{\mathbb{C}}}^{\widehat{\otimes}n}$ such that $\forall F_{\rm ext}^{(n)}\in{\mathcal{D}'_{\mathbb{C}}}^{(n)}$, $\forall f^{(n)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$

$$\langle U_nF_{\rm ext}^{(n)},f^{(n)}\rangle=\langle F_{\rm ext}^{(n)},f^{(n)}\rangle_{\rm ext}. \tag{2.5}$$

By analogy, since for all $\tau\in T$ $\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ and $\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n}$ are the sets of linear continuous functionals on $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}$, there exist linear isometric operators $U_{n,\tau}:\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}\to\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n}$ such that $\forall F_{\rm ext}^{(n)}\in\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$, $\forall f^{(n)}\in\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}$ $\langle U_{n,\tau}F_{\rm ext}^{(n)},f^{(n)}\rangle=\langle F_{\rm ext}^{(n)},f^{(n)}\rangle_{\rm ext}$.

Proposition 2.2. *For each $\tau\in T$ and each $n\in\mathbb{Z}_+$ the restriction of the operator U_n on $\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ coincides with $U_{n,\tau}$.*

Proof. Let $F_{\rm ext}^{(n)}\in\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}\subset{\mathcal{D}'_{\mathbb{C}}}^{(n)}$. For each $f^{(n)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$ we have $\langle U_nF_{\rm ext}^{(n)},f^{(n)}\rangle=\langle F_{\rm ext}^{(n)},f^{(n)}\rangle_{\rm ext}=\langle U_{n,\tau}F_{\rm ext}^{(n)},f^{(n)}\rangle$, therefore $U_nF_{\rm ext}^{(n)}=U_{n,\tau}F_{\rm ext}^{(n)}$ as an element of ${\mathcal{D}'_{\mathbb{C}}}^{\widehat{\otimes}n}$. But $U_{n,\tau}F_{\rm ext}^{(n)}\in\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n}$ by definition, and $\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n}\subset{\mathcal{D}'_{\mathbb{C}}}^{\widehat{\otimes}n}$; so the proposition is proved. □

Corollary. *Let $\tau,\tau'\in T$ be such that $\mathcal{H}_\tau\subset\mathcal{H}_{\tau'}$. Then for each $n\in\mathbb{N}$ $\mathcal{H}_{-\tau',\mathbb{C}}^{(n)}\subset\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ and the restriction of $U_{n,\tau}$ on $\mathcal{H}_{-\tau',\mathbb{C}}^{(n)}$ coincides with $U_{n,\tau'}$.*

Taking into account Proposition 2.2 and its Corollary, in what follows, we omit a subindex τ for operators $U_{n,\tau}$, i.e., we'll write always U_n for such operators.

Remark 2.2. We note that for $n=0$ and $n=1$ $U_n=id$; but for $n>1$ and $\eta\neq 0$ $U_n\mathcal{H}_{\rm ext}^{(n)}\neq\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$, i.e., the restriction of U_n on $\mathcal{H}_{\rm ext}^{(n)}$ is not an isomorphism between $\mathcal{H}_{\rm ext}^{(n)}$ and $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$. This fact was proved in [21] for $\eta\equiv 1$ (in the Gamma analysis), the proof in the general case can be constructed by analogy.

Let $\mathcal{P}$ be the set of all continuous polynomials on $\mathcal{D}'$. It follows from results of [18, 24] that any element of $\mathcal{P}$ can be presented in the form

$$f=\sum_{n=0}^{N_f}\langle P_n,f^{(n)}\rangle,\quad f^{(n)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n},\quad N_f\in\mathbb{Z}_+. \tag{2.6}$$

We define on $\mathcal{P}$ a family of scalar products by setting for $f,g\in\mathcal{P}$, $\tau\in T$, $q\in\mathbb{N}$

$$(f,g)_{\tau,q}:=\sum_{n=0}^{\min(N_f,N_g)}(n!)^2 2^{qn}(f^{(n)},g^{(n)})_\tau,$$

where $f^{(n)}$, $g^{(n)}$ are the kernels from decompositions (2.6) for f and g respectively. By $\|\cdot\|_{\tau,q}$ denote the corresponding norm, i.e., for $f\in\mathcal{P}$ of form (2.6) we have

$$\|f\|_{\tau,q}^2=(f,\overline{f})_{\tau,q}=\sum_{n=0}^{N_f}(n!)^2 2^{qn}|f^{(n)}|_\tau^2.$$

Definition 2.1. By $(\mathcal{H}_\tau)_q$ denote a Hilbert space that is the closure of $\mathcal{P}$ with respect to the norm $\|\cdot\|_{\tau,q}$. Let also $(\mathcal{H}_\tau):=\mathrm{pr}\lim_{q\in\mathbb{N}}(\mathcal{H}_\tau)_q$, $(\mathcal{D}):=\mathrm{pr}\lim_{\tau\in T,q\in\mathbb{N}}(\mathcal{H}_\tau)_q$. The spaces $(\mathcal{H}_\tau)_q$, $(\mathcal{H}_\tau)$, $(\mathcal{D})$ are called the *Kondratiev-type test functions spaces.*

It is clear that $f \in (\mathcal{H}_\tau)_q$ if and only if f can be presented in form (1.8) with $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$, and the series converges in the sense that

$$\|f\|_{\tau,q}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{qn} |f^{(n)}|_\tau^2 < \infty. \tag{2.7}$$

Further, $f \in (\mathcal{H}_\tau)$ if and only if f has form (1.8) and norm (2.7) is finite for all $q \in \mathbb{N}$; and $f \in (\mathcal{D})$ if and only if norm (2.7) for f is finite for all $\tau \in T$ and $q \in \mathbb{N}$ (in this case, of course, the kernels from decomposition (1.8) $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$).

Remark 2.3. Elements of the Kondratiev-type test functions spaces $(\mathcal{H}_\tau)_q$, as distinguished from elements of $\mathcal{P}$, are *not functions with arguments from $\mathcal{D}'$*. But under some conditions on τ and q elements of $(\mathcal{H}_\tau)_q$ can be considered as functions on $\mathcal{H}_{-\tau}$ (moreover, these functions are continuous). A more detailed discussion of this question is given in, e.g., [7].

Remark 2.4. Let $f, g \in (\mathcal{H}_\tau)_q$. Then

$$(f, g)_{\tau,q} = \sum_{n=0}^{\infty} (n!)^2 2^{qn} (f^{(n)}, g^{(n)})_\tau,$$

where $f^{(n)}, g^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ are the kernels from decompositions (1.8) for f and g respectively; therefore the system of the generalized Meixner polynomials plays a role of an orthogonal basis in $(\mathcal{H}_\tau)_q$.

In order to define the Kondratiev-type spaces of generalized functions we need the following

Proposition 2.3. *There exists $q_0 \in \mathbb{N}$ such that for all natural $q \geq q_0$ and for all $\tau \in T$ the dense and continuous embedding $(\mathcal{H}_\tau)_q \hookrightarrow (L^2)$ takes place (we remind that T is modified in accordance with Remarks 1.6, 1.8, 2.1).*

Proof. Let $f \in (\mathcal{H}_\tau)_q$ ($\tau \in T$, $q \in \mathbb{N}$), and $\{f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}\}_{n=0}^{\infty}$ be the kernels from decomposition (1.8) for f. Using estimate (2.3) we can evaluate as follows:

$$\begin{aligned}\|f\|_{(L^2)} &\leq \sum_{n=0}^{\infty} \|\langle P_n(\cdot), f^{(n)}\rangle\|_{(L^2)} \leq \sum_{n=0}^{\infty} \||P_n(\cdot)|_{-\tau}\|_{(L^2)} |f^{(n)}|_\tau \leq \sqrt{k(\tau)} \sum_{n=0}^{\infty} n! c^{n/2} |f^{(n)}|_\tau \\ &= \sqrt{k(\tau)} \sum_{n=0}^{\infty} [n! 2^{qn/2} |f^{(n)}|_\tau][2^{-q}c]^{n/2} \leq \sqrt{k(\tau)} \sqrt{\sum_{n=0}^{\infty} (n!)^2 2^{qn} |f^{(n)}|_\tau^2} \sqrt{\sum_{n=0}^{\infty} [2^{-q}c]^n} \\ &= K \|f\|_{\tau,q} < \infty,\end{aligned}$$

where $K := \sqrt{k(\tau)}\sqrt{\sum_{n=0}^{\infty}[2^{-q}c]^n} < \infty$ if q is so large that $2^q > c$.

Let q_0 be the minimal natural number such that $2^{q_0} > c$. Now in order to prove that for all natural $q \geq q_0$ $(\mathcal{H}_\tau)_q$ is continuously embedded in (L^2), we have to prove that any Cauchy sequence $(f_k \in \mathcal{P})_{k=1}^{\infty}$ in $(\mathcal{H}_\tau)_q$ with $\lim_{k\to\infty} f_k = 0$ in (L^2) tends to zero in $(\mathcal{H}_\tau)_q$ (see, e.g., [8]). Let $f = \lim_{k\to\infty} f_k$ in $(\mathcal{H}_\tau)_q$. Then

$$\|f\|_{(L^2)} = \|f - f_k + f_k\|_{(L^2)} \leq \|f - f_k\|_{(L^2)} + \|f_k\|_{(L^2)} \leq K\|f - f_k\|_{\tau,q} + \|f_k\|_{(L^2)} \underset{k\to\infty}{\to} 0,$$

therefore $\|f\|_{(L^2)} = 0$. Let $\{f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}\}_{n=0}^{\infty}$ be the kernels from decomposition (1.8) for $f \in (\mathcal{H}_\tau)_q$. By Theorem 1.3 $0 = \|f\|_{(L^2)}^2 = \sum_{n=0}^{\infty} n! |f^{(n)}|_{\mathrm{ext}}^2$ (we remind that by Proposition 2.1 for each $n \in \mathbb{Z}_+$ $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \hookrightarrow \mathcal{H}_{\mathrm{ext}}^{(n)}$, therefore the norms $|f^{(n)}|_{\mathrm{ext}}$ are well-defined), hence $\forall n \in \mathbb{Z}_+$ $|f^{(n)}|_{\mathrm{ext}} = 0$ and (again in view of the embedding of $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ in

$\mathcal{H}_{\rm ext}^{(n)}$) $|f^{(n)}|_\tau = 0$. But it means that $\|f\|_{\tau,q}^2 = \sum_{n=0}^{\infty}(n!)^2 2^{qn}|f^{(n)}|_\tau^2 = 0$, thus $f = 0$ as an element of $(\mathcal{H}_\tau)_q$ and the continuous embedding of $(\mathcal{H}_\tau)_q$ in (L^2) is proved. The density of this embedding from the density of the embedding $\mathcal{P} \hookrightarrow (L^2)$ follows (the last fact is a consequence of a holomorphy at zero of the Laplace transform of μ, see, e.g., [32]). □

Remark 2.5. Let $\mathbb{N}_{q_0} := \{q_0, q_0+1, \dots\} \subseteq \mathbb{N}$. Then we can reformulate Proposition 2.3 as follows: for all $q \in \mathbb{N}_{q_0}$ and for all $\tau \in T$ the dense and continuous embedding $(\mathcal{H}_\tau)_q \hookrightarrow (L^2)$ is valid.

Now one can consider the chain

$$(2.8) \quad (\mathcal{D}')' \supset (\mathcal{H}_{-\tau}) \supset (\mathcal{H}_{-\tau})_{-q} \supset (L^2) \supset (\mathcal{H}_\tau)_q \supset (\mathcal{H}_\tau) \supset (\mathcal{D}), \quad q \in \mathbb{N}_{q_0}, \quad \tau \in T,$$

where $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau}) = \text{ind}\lim_{q\in\mathbb{N}_{q_0}}(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{D}')' = \text{ind}\lim_{q\in\mathbb{N}_{q_0},\tau\in T}(\mathcal{H}_{-\tau})_{-q}$ are the dual to $(\mathcal{H}_\tau)_q$, $(\mathcal{H}_\tau)$, $(\mathcal{D})$ with respect to (L^2) spaces correspondingly.

Definition 2.2. The spaces $(\mathcal{H}_{-\tau})_{-q}$ ($q \in \mathbb{N}_{q_0}$, $\tau \in T$), $(\mathcal{H}_{-\tau})$, $(\mathcal{D}')'$ are called the *Kondratiev-type spaces of nonregular generalized functions*.

The generated by the scalar product in (L^2) (real) dual pairing between elements of $(\mathcal{H}_{-\tau})_{-q}$ and $(\mathcal{H}_\tau)_q$ (in the same way as $(\mathcal{H}_{-\tau})$ and $(\mathcal{H}_\tau)$, $(\mathcal{D}')'$ and $(\mathcal{D})$) will be denoted by $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ (for example, if $f, g \in (L^2)$ then $\langle\!\langle f, g\rangle\!\rangle = (f,g)_{(L^2)} = \int_{\mathcal{D}'} f(x)g(x)\mu(dx)$).

One can construct orthogonal bases in the spaces $(\mathcal{H}_{-\tau})_{-q}$ by different ways. The simplest solution of this problem consists in using of results of the so-called biorthogonal analysis (see, e.g., [18, 3, 19, 24, 7]); but in this case elements of orthogonal bases in $(\mathcal{H}_{-\tau})_{-q}$ are some generalized functions that are "not concordant" with the special orthogonality of the generalized Meixner polynomials in (L^2), this is very inconvenient for our consequent considerations. Another way is based on the well-known result of the general duality theory: since the generalized Meixner polynomials are orthogonal in $(\mathcal{H}_\tau)_q$ and in (L^2), these polynomials are orthogonal in $(\mathcal{H}_{-\tau})_{-q}$. This fact is not sufficient in order to accept the set $\{\langle P_n, f^{(n)}\rangle\colon\ f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}\}$ as an orthogonal basis in $(\mathcal{H}_{-\tau})_{-q}$ (the "coefficients" of a basis must be distributions), but the corresponding basis can be constructed "on the base of the generalized Meixner polynomials". An example of such a construction is given in [21] for the connected with the Gamma measure generalized Laguerre polynomials.

Now we construct the connected with the generalized Meixner polynomials orthogonal bases in $(\mathcal{H}_{-\tau})_{-q}$ and establish the interconnection of these bases with ones that are given by the "biorthogonal approach". Note that our approach differs from the offered in [21] one (here we give more general and independent presentation). We begin from two statements that are simple generalizations of well-known results for Fock spaces.

Lemma 2.1. *Let $H^0, H^1, \dots$ be (complex) Hilbert spaces, $v = (v_n > 0)_{n=0}^{\infty}$ be a numerical sequence. The space*

$$\mathcal{H}_v := \bigoplus_{n=0}^{\infty} H^n v_n \equiv \left\{ f = (f^{(0)}, f^{(1)}, \dots) :\ f^{(n)} \in H^n,\ n \in \mathbb{Z}_+,\ \|f\|_{\mathcal{H}_v}^2 := \sum_{n=0}^{\infty} |f^{(n)}|_{H^n}^2 v_n < \infty \right\}$$

is a Hilbert one with the (real) scalar product

$$(f,g)_{\mathcal{H}_v} = \sum_{n=0}^{\infty} (f^{(n)}, g^{(n)})_{H^n} v_n$$

(here $|\cdot|_{H^n}$ and $(\cdot,\cdot)_{H^n}$ are the norms and the (real) scalar products in H^n correspondingly, $n \in \mathbb{Z}_+$).

This result follows from the well-known fact that a direct sum of Hilbert spaces is a Hilbert space. □

Using Lemma 2.1 and the theory of rigged Hilbert spaces (see, e.g., [8]), one can easily prove the following

Lemma 2.2. *Let* $(H_-^n \supseteq H_0^n \supseteq H_+^n)_{n=0}^\infty$ *be a sequence of chains of Hilbert spaces (riggings of* H_0^n*),* $(v_n)_{n=0}^\infty$*,* $(w_n)_{n=0}^\infty$ *(*$0 < w_n \leq v_n$ $\forall n \in \mathbb{Z}_+$*) be numerical sequences. Then* $\mathop{\oplus}\limits_{n=0}^{\infty} H_+^n v_n$ *is densely and continuously embedded in* $\mathop{\oplus}\limits_{n=0}^{\infty} H_0^n w_n$*, and* $\mathop{\oplus}\limits_{n=0}^{\infty} H_-^n \frac{w_n^2}{v_n}$ *is the space that is dual to* $\mathop{\oplus}\limits_{n=0}^{\infty} H_+^n v_n$ *with respect to the zero space* $\mathop{\oplus}\limits_{n=0}^{\infty} H_0^n w_n$*, i.e.,*

$$\mathop{\oplus}\limits_{n=0}^{\infty} H_-^n \frac{w_n^2}{v_n} \supseteq \mathop{\oplus}\limits_{n=0}^{\infty} H_0^n w_n \supseteq \mathop{\oplus}\limits_{n=0}^{\infty} H_+^n v_n \tag{2.9}$$

is a chain of Hilbert spaces.

By construction the space $(\mathcal{H}_\tau)_q$ ($\tau \in T$, $q \in \mathbb{N}_{q_0}$ (see Remark 2.5)) is isometrically isomorphic to the space $\mathop{\oplus}\limits_{n=0}^{\infty} \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}(n!)^2 2^{qn}$

$$(\mathcal{H}_\tau)_q \ni f = \sum_{n=0}^{\infty} \langle P_n, f^{(n)} \rangle \leftrightarrow \widetilde{f} = (f^{(0)}, f^{(1)}, \dots) \in \mathop{\oplus}\limits_{n=0}^{\infty} \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}(n!)^2 2^{qn},$$

$\|f\|_{\tau,q} = \|\widetilde{f}\|_{\mathop{\oplus}\limits_{n=0}^{\infty} \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}(n!)^2 2^{qn}}$ (see (2.7) and Lemma 2.1). Therefore there exists the isometric isomorphism between the space $(\mathcal{H}_{-\tau})_{-q}$ and the space $[\mathop{\oplus}\limits_{n=0}^{\infty} \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}(n!)^2 2^{qn}]'$ of linear continuous functionals on $\mathop{\oplus}\limits_{n=0}^{\infty} \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}(n!)^2 2^{qn}$. Different representations of the space $[\mathop{\oplus}\limits_{n=0}^{\infty} \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}(n!)^2 2^{qn}]'$ can be associated with different orthogonal bases in $(\mathcal{H}_{-\tau})_{-q}$. We consider two representations of $[\mathop{\oplus}\limits_{n=0}^{\infty} \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}(n!)^2 2^{qn}]'$ (and, correspondingly, two bases in $(\mathcal{H}_{-\tau})_{-q}$) that are connected with chains (2.4), (2.1).

Let

$$\mathop{\oplus}\limits_{n=0}^{\infty} \mathcal{H}_{-\tau,\mathbb{C}}^{(n)} 2^{-qn} \supset \mathop{\oplus}\limits_{n=0}^{\infty} \mathcal{H}_{\mathrm{ext}}^{(n)} n! \supset \mathop{\oplus}\limits_{n=0}^{\infty} \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}(n!)^2 2^{qn} \tag{2.10}$$

be chain (2.9) that is based on chains (2.4). Now $[\mathop{\oplus}\limits_{n=0}^{\infty} \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}(n!)^2 2^{qn}]'$ is represented by the Hilbert space $\mathop{\oplus}\limits_{n=0}^{\infty} \mathcal{H}_{-\tau,\mathbb{C}}^{(n)} 2^{-qn}$, therefore each element $F \in (\mathcal{H}_{-\tau})_{-q}$ can be identified with $\widetilde{F} = (F_{\mathrm{ext}}^{(0)}, F_{\mathrm{ext}}^{(1)}, \dots) \in \mathop{\oplus}\limits_{n=0}^{\infty} \mathcal{H}_{-\tau,\mathbb{C}}^{(n)} 2^{-qn}$, i.e., there exists the orthogonal basis in $(\mathcal{H}_{-\tau})_{-q}$ that has a form of a family of generalized functions $\{\langle \widetilde{P}_n, F_{\mathrm{ext}}^{(n)} \rangle \in (\mathcal{H}_{-\tau})_{-q}$: $F_{\mathrm{ext}}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$, $n \in \mathbb{Z}_+\}$ such that

$$F = \sum_{n=0}^{\infty} \langle \widetilde{P}_n, F_{\mathrm{ext}}^{(n)} \rangle \tag{2.11}$$

and this formal series converges in the sense that

$$\|F\|_{-\tau,-q}^2 := \|F\|_{(\mathcal{H}_{-\tau})_{-q}}^2 = \|\widetilde{F}\|^2_{\mathop{\oplus}\limits_{n=0}^{\infty} \mathcal{H}_{-\tau,\mathbb{C}}^{(n)} 2^{-qn}} = \sum_{n=0}^{\infty} 2^{-qn} |F_{\mathrm{ext}}^{(n)}|_{-\tau,\mathrm{ext}}^2 < \infty \tag{2.12}$$

(here and below by $|\cdot|_{-\tau,\mathrm{ext}}$ denote the norms in $\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$). Note that, as is easily seen, for all $n \in \mathbb{Z}_+$, $c_1, c_2 \in \mathbb{C}$, $F_{\mathrm{ext}}^{(n)}, G_{\mathrm{ext}}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ $\langle \widetilde{P}_n, c_1 F_{\mathrm{ext}}^{(n)} + c_2 G_{\mathrm{ext}}^{(n)} \rangle = c_1 \langle \widetilde{P}_n, F_{\mathrm{ext}}^{(n)} \rangle +$

$c_2\langle\widetilde{P}_n, G^{(n)}_{\rm ext}\rangle$. Further, for each $f \in (\mathcal{H}_\tau)_q$ that corresponds to $\widetilde{f} = (f^{(0)}, f^{(1)}, \dots) \in \mathop{\oplus}\limits_{n=0}^{\infty} \mathcal{H}^{\widehat{\otimes}n}_{\tau,\mathbb{C}}(n!)^2 2^{qn}$ (here $\{f^{(n)}\}_{n=0}^{\infty}$ are the kernels from decomposition (1.8) for f) we have

$$\langle\!\langle F, f\rangle\!\rangle = (\widetilde{F}, \widetilde{f})_{\mathop{\oplus}\limits_{n=0}^{\infty}\mathcal{H}^{(n)}_{\rm ext}n!} = \sum_{n=0}^{\infty} n!\langle F^{(n)}_{\rm ext}, f^{(n)}\rangle_{\rm ext}. \tag{2.13}$$

In particular,

$$\langle\!\langle\langle\widetilde{P}_n, F^{(n)}_{\rm ext}\rangle, \langle P_m, f^{(m)}\rangle\rangle\!\rangle = \delta_{nm} n!\langle F^{(n)}_{\rm ext}, f^{(n)}\rangle_{\rm ext}.$$

Therefore taking into account (1.6) and a density in (L^2) of the set of polynomials, one can conclude that *for smooth* $F^{(n)}_{\rm ext} \in \mathcal{H}^{\widehat{\otimes}n}_{\tau,\mathbb{C}}$ $\langle\widetilde{P}_n, F^{(n)}_{\rm ext}\rangle$ *is a generalized Meixner polynomial*, hence it is natural to accept the notation

$$\langle\widetilde{P}_n, F^{(n)}_{\rm ext}\rangle = \langle P_n, F^{(n)}_{\rm ext}\rangle \quad \forall F^{(n)}_{\rm ext} \in \mathcal{H}^{(n)}_{-\tau,\mathbb{C}}.$$

Remark 2.6. One can understand $\langle\widetilde{P}_n, F^{(n)}_{\rm ext}\rangle$ with $F^{(n)}_{\rm ext} \in \mathcal{H}^{(n)}_{-\tau,\mathbb{C}}$ as a limit in $(\mathcal{H}_{-\tau})_{-q}$ (for *any* $q \in \mathbb{N}_{q_0}$) of a sequence of generalized Meixner polynomials (cf. $\langle P_n, f^{(n)}\rangle$, $f^{(n)} \in \mathcal{H}^{(n)}_{\rm ext}$): if $\mathcal{H}^{\widehat{\otimes}n}_{\tau,\mathbb{C}} \ni F^{(n)}_k \underset{k\to\infty}{\to} F^{(n)}_{\rm ext} \in \mathcal{H}^{(n)}_{-\tau,\mathbb{C}}$ in the topology of $\mathcal{H}^{(n)}_{-\tau,\mathbb{C}}$ then

$$\begin{aligned}\|\langle\widetilde{P}_n, F^{(n)}_{\rm ext}\rangle - \langle P_n, F^{(n)}_k\rangle\|_{-\tau,-q} &= \|\langle\widetilde{P}_n, F^{(n)}_{\rm ext}\rangle - \langle\widetilde{P}_n, F^{(n)}_k\rangle\|_{-\tau,-q}\\ &= \|\langle\widetilde{P}_n, F^{(n)}_{\rm ext} - F^{(n)}_k\rangle\|_{-\tau,-q} = 2^{-qn/2}|F^{(n)}_{\rm ext} - F^{(n)}_k|_{-\tau,\rm ext} \underset{k\to\infty}{\to} 0.\end{aligned}$$

Note that it is possible to *define* $\langle\widetilde{P}_n, F^{(n)}_{\rm ext}\rangle$ as a limit of a sequence of the generalized Meixner polynomials in $(\mathcal{H}_{-\tau})_{-q}$, and then to prove that such limits form an orthogonal basis in this space (for example, with using of Lemma 2.2).

Remark 2.7. Actually the fact that $\langle\widetilde{P}_n, F^{(n)}_{\rm ext}\rangle$ ($n \in \mathbb{Z}_+$) are direct generalizations of the generalized Meixner polynomials is connected with the result of Theorem 1.3: the "central space" of chain (2.10) is isometrically isomorphic to (L^2), i.e.,

$$(L^2) \ni f = \sum_{n=0}^{\infty}\langle P_n, f^{(n)}\rangle \leftrightarrow \widetilde{f} = (f^{(0)}, f^{(1)}, \dots) \in \mathop{\oplus}\limits_{n=0}^{\infty}\mathcal{H}^{(n)}_{\rm ext}n!$$

(and for $f^{(n)} \in \mathcal{H}^{\widehat{\otimes}n}_{\tau,\mathbb{C}}$ $\langle P_n, f^{(n)}\rangle$ ($n \in \mathbb{Z}_+$) here are the generalized Meixner polynomials).

Let now

$$\mathop{\oplus}\limits_{n=0}^{\infty}\mathcal{H}^{\widehat{\otimes}n}_{-\tau,\mathbb{C}}2^{-qn} \supset \mathop{\oplus}\limits_{n=0}^{\infty}\mathcal{H}^{\widehat{\otimes}n}_{\mathbb{C}}n! \supset \mathop{\oplus}\limits_{n=0}^{\infty}\mathcal{H}^{\widehat{\otimes}n}_{\tau,\mathbb{C}}(n!)^2 2^{qn}$$

be chain (2.9) that is based on chains (2.1) (note that all spaces in this chain are weighted Fock ones). The space $[\mathop{\oplus}\limits_{n=0}^{\infty}\mathcal{H}^{\widehat{\otimes}n}_{\tau,\mathbb{C}}(n!)^2 2^{qn}]'$ is represented by $\mathop{\oplus}\limits_{n=0}^{\infty}\mathcal{H}^{\widehat{\otimes}n}_{-\tau,\mathbb{C}}2^{-qn}$, therefore each element $F \in (\mathcal{H}_{-\tau})_{-q}$ can be identified with $\breve{F} = (F^{(0)}, F^{(1)}, \dots) \in \mathop{\oplus}\limits_{n=0}^{\infty}\mathcal{H}^{\widehat{\otimes}n}_{-\tau,\mathbb{C}}2^{-qn}$, i.e., there exists the orthogonal basis in $(\mathcal{H}_{-\tau})_{-q}$ that has a form of a family of generalized functions $\{Q_n(F^{(n)}; \cdot) \equiv Q_n(F^{(n)}) \in (\mathcal{H}_{-\tau})_{-q}$: $F^{(n)} \in \mathcal{H}^{\widehat{\otimes}n}_{-\tau,\mathbb{C}}$, $n \in \mathbb{Z}_+\}$ such that

$$F = \sum_{n=0}^{\infty} Q_n(F^{(n)}) \tag{2.14}$$

and this formal series converges in the sense that

$$\|F\|^2_{-\tau,-q} = \|\breve{F}\|^2_{\mathop{\oplus}\limits_{n=0}^{\infty}\mathcal{H}^{\widehat{\otimes}n}_{-\tau,\mathbb{C}}2^{-qn}} = \sum_{n=0}^{\infty} 2^{-qn}|F^{(n)}|^2_{-\tau} < \infty \tag{2.15}$$

(and it is easy to see that for all $n \in \mathbb{Z}_+$, $c_1, c_2 \in \mathbb{C}$, $F^{(n)}, G^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n}$ $Q_n(c_1 F^{(n)} + c_2 G^{(n)}) = c_1 Q_n(F^{(n)}) + c_2 Q_n(G^{(n)})$). Note that, of course, for the same $F \in (\mathcal{H}_{-\tau})_{-q}$ the sums of series in the right hand sides of (2.12) and (2.15) are coincide (in fact, $\widetilde{F}$ and $\breve{F}$ are different representations of the same linear continuous functional l_F on $\bigoplus_{n=0}^{\infty} \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}(n!)^2 2^{qn}$, therefore $\|\widetilde{F}\|_{\bigoplus_{n=0}^{\infty} \mathcal{H}_{-\tau,\mathbb{C}}^{(n)} 2^{-qn}} = \|l_F\| = \|\breve{F}\|_{\bigoplus_{n=0}^{\infty} \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n} 2^{-qn}}$). Further, for each $f \in (\mathcal{H}_\tau)_q$ that corresponds to $\widetilde{f} = (f^{(0)}, f^{(1)}, \dots) \in \bigoplus_{n=0}^{\infty} \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}(n!)^2 2^{qn}$ (here $\{f^{(n)}\}_{n=0}^{\infty}$ are the kernels from decomposition (1.8) for f) we have

$$\langle\langle F, f\rangle\rangle = (\breve{F}, \widetilde{f})_{\bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n} n!} = \sum_{n=0}^{\infty} n! \langle F^{(n)}, f^{(n)}\rangle. \tag{2.16}$$

In particular,

$$\langle\langle Q_n(F^{(n)}), \langle P_m, f^{(m)}\rangle\rangle\rangle = \delta_{nm} n! \langle F^{(n)}, f^{(n)}\rangle$$

(this is the so-called *biorthogonality relation*, cf. [27, 18, 3, 24, 7]).

In view of (2.5) the representatives $\widetilde{F} = (F_{\mathrm{ext}}^{(0)}, F_{\mathrm{ext}}^{(1)}, \dots) \in \bigoplus_{n=0}^{\infty} \mathcal{H}_{-\tau,\mathbb{C}}^{(n)} 2^{-qn}$ and $\breve{F} = (F^{(0)}, F^{(1)}, \dots) \in \bigoplus_{n=0}^{\infty} \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n} 2^{-qn}$ of the same functional l_F are connected by the formulas $F^{(n)} = U_n F_{\mathrm{ext}}^{(n)}$, $n \in \mathbb{Z}_+$ (and also it follows from (2.4), (2.1) and (2.5) that $|\cdot|_{-\tau,\mathrm{ext}} = |U_n \cdot|_{-\tau}$, i.e., for each $F_{\mathrm{ext}}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ $|F_{\mathrm{ext}}^{(n)}|_{-\tau,\mathrm{ext}} = |U_n F_{\mathrm{ext}}^{(n)}|_{-\tau}$). Therefore

$$\langle P_n, F_{\mathrm{ext}}^{(n)}\rangle = Q_n(U_n F_{\mathrm{ext}}^{(n)}) \tag{2.17}$$

(or, which is the same, $Q_n(F^{(n)}) = \langle P_n, U_n^{-1} F^{(n)}\rangle$).

Let us formulate all connected with orthogonal bases in $(\mathcal{H}_{-\tau})_{-q}$ obtained above results as a theorem.

Theorem 2.1. *A generalized function $F \in (\mathcal{H}_{-\tau})_{-q}$ ($\tau \in T$, $q \in \mathbb{N}_{q_0}$) if and only if there exists a sequence*

$$(F_{\mathrm{ext}}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)})_{n=0}^{\infty} \tag{2.18}$$

such that F can be presented in form (2.11)*, where the formal series converges in $(\mathcal{H}_{-\tau})_{-q}$, i.e., norm* (2.12) *is finite. Furthermore, the system $\{\langle P_n, F_{\mathrm{ext}}^{(n)}\rangle\colon F_{\mathrm{ext}}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)},\ n \in \mathbb{Z}_+\}$ plays a role of an orthogonal basis in $(\mathcal{H}_{-\tau})_{-q}$ in the sense that for $F, G \in (\mathcal{H}_{-\tau})_{-q}$*

$$(F, G)_{(\mathcal{H}_{-\tau})_{-q}} = \sum_{n=0}^{\infty} 2^{-qn} (F_{\mathrm{ext}}^{(n)}, G_{\mathrm{ext}}^{(n)})_{-\tau,\mathrm{ext}},$$

where $F_{\mathrm{ext}}^{(n)}, G_{\mathrm{ext}}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ are the kernels from decompositions (2.11) *for F and G correspondingly, $(\cdot,\cdot)_{-\tau,\mathrm{ext}}$ is the scalar product in $\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$.*

Alternatively, instead of sequence (2.18) *one can use the sequence*

$$(F^{(n)} = U_n F_{\mathrm{ext}}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n})_{n=0}^{\infty} \tag{2.19}$$

(see (2.5)*), in this case F has form* (2.14)*, norm* (2.15) *must be finite; and the system $\{Q_n(F^{(n)})\colon F^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n},\ n \in \mathbb{Z}_+\}$ plays a role of an orthogonal basis in $(\mathcal{H}_{-\tau})_{-q}$ in the sense that for $F, G \in (\mathcal{H}_{-\tau})_{-q}$*

$$(F, G)_{(\mathcal{H}_{-\tau})_{-q}} = \sum_{n=0}^{\infty} 2^{-qn} (F^{(n)}, G^{(n)})_{-\tau},$$

where $F^{(n)}, G^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n}$ are the kernels from decompositions (2.14) *for F and G correspondingly, $(\cdot,\cdot)_{-\tau}$ is the scalar product in $\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n}$ (we note that $(F_{\rm ext}^{(n)}, G_{\rm ext}^{(n)})_{-\tau,\rm ext} = (U_n F_{\rm ext}^{(n)}, U_n G_{\rm ext}^{(n)})_{-\tau}$, $\|F\|_{-\tau,-q} = \sqrt{(F,\overline{F})_{(\mathcal{H}_{-\tau})_{-q}}}$).*

The generalized functions $\langle P_n, F_{\rm ext}^{(n)}\rangle$ and $Q_n(F^{(n)})$ are connected by formula (2.17)*. The (generated by the scalar product in (L^2)) real dual pairing between elements of $(\mathcal{H}_{-\tau})_{-q}$ and $(\mathcal{H}_\tau)_q$ is given by* (2.13) *or, equivalently, by* (2.16)*.*

Remark 2.8. It is easy to see that $F \in (\mathcal{H}_{-\tau})$ (correspondingly $F \in (\mathcal{D}')'$) if and only if there exists sequence (2.18) such that F can be presented in form (2.11) with finite norm (2.12) for some $q \in \mathbb{N}$ (correspondingly for some $q \in \mathbb{N}$ and some $\tau \in T$). Alternatively, one can use sequence (2.19), representation (2.14) and norm (2.15).

Remark 2.9. One can construct the generalized functions $Q_n(F^{(n)})$ and $\langle P_n, F_{\rm ext}^{(n)}\rangle$, and to prove that they play the role of orthogonal bases in $(\mathcal{H}_{-\tau})_{-q}$ ($\tau \in T$, $q \in \mathbb{N}_{q_0}$) by another way (by analogy with [21]). Namely, on "monomials" $\langle P_m, f^{(m)}\rangle$, $f^{(m)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m}$ we define a generalized differential operator $\langle F^{(n)}, : D :^{\widehat{\otimes} n}\rangle$, $F^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n}$ by setting

$$\langle F^{(n)}, : D :^{\widehat{\otimes} n}\rangle \langle P_m, f^{(m)}\rangle := 1_{\{m\geq n\}} \frac{m!}{(m-n)!} \langle P_{m-n} \widehat{\otimes} F^{(n)}, f^{(m)}\rangle$$

(here and below 1_A denotes the indicator of A). One can prove (see, e.g., [18]) that this operator can be extended to a linear continuous operator acting in $(\mathcal{H}_\tau)_q$ (we preserve for this extension the previous notation). Let $\langle F^{(n)}, : D :^{\widehat{\otimes} n}\rangle^* : (\mathcal{H}_{-\tau})_{-q} \to (\mathcal{H}_{-\tau})_{-q}$ be the dual to $\langle F^{(n)}, : D :^{\widehat{\otimes} n}\rangle$ with respect to (L^2) operator, i.e.,

$$\langle\!\langle \langle F^{(n)}, : D :^{\widehat{\otimes} n}\rangle^* F, f\rangle\!\rangle = \langle\!\langle F, \langle F^{(n)}, : D :^{\widehat{\otimes} n}\rangle f\rangle\!\rangle \ \forall F \in (\mathcal{H}_{-\tau})_{-q}, \quad \forall f \in (\mathcal{H}_\tau)_q.$$

We *define* $\widetilde{Q}_n(F^{(n)}) := \langle F^{(n)}, : D :^{\widehat{\otimes} n}\rangle^* 1$. It is easy to show that for all $n, m \in \mathbb{Z}_+$, $F^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n}$ and $f^{(m)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m}$

$$\langle\!\langle \widetilde{Q}_n(F^{(n)}), \langle P_m, f^{(m)}\rangle\rangle\!\rangle = \delta_{nm} n! \langle F^{(n)}, f^{(n)}\rangle,$$

therefore $\widetilde{Q}_n(F^{(n)}) = Q_n(F^{(n)})$. It follows from general results of non-Gaussian infinite-dimensional analysis (see, e.g., [18, 24, 19]) that the system $\{\widetilde{Q}_n(F^{(n)})\colon F^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+\}$ plays a role of an orthogonal basis in $(\mathcal{H}_{-\tau})_{-q}$ in the sense of Theorem 2.1. The generalized functions $\{\langle P_n, F_{\rm ext}^{(n)}\rangle\colon F_{\rm ext}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}, n \in \mathbb{Z}_+\}$ can be *defined* now by (2.17), their properties from properties of the "Q-system" follow.

Now let us introduce the Kondratiev-type spaces of *regular* test and generalized functions (cf. [12, 17, 16]). First we consider the set $\widetilde{\mathcal{P}} := \{f = \sum_{n=0}^{N_f} \langle P_n, f^{(n)}\rangle,\ f^{(n)} \in \mathcal{H}_{\rm ext}^{(n)},\ N_f \in \mathbb{Z}_+\} \subset (L^2)$ of polynomials and $\forall q \in \mathbb{N}$ introduce on this set the scalar product $(\cdot,\cdot)_q$ by setting for $f = \sum_{n=0}^{N_f} \langle P_n, f^{(n)}\rangle$, $g = \sum_{n=0}^{N_g} \langle P_n, g^{(n)}\rangle$

$$(f,g)_q := \sum_{n=0}^{\min(N_f,N_g)} (n!)^2 2^{qn} \langle f^{(n)}, g^{(n)}\rangle_{\rm ext}.$$

Let $\|\cdot\|_q$ be the corresponding norm: $\|f\|_q = \sqrt{(f,\overline{f})_q} = \sqrt{\sum_{n=0}^{N_f} (n!)^2 2^{qn} |f^{(n)}|_{\rm ext}^2}$.

Definition 2.3. We define the *Kondratiev-type spaces of ("regular") test functions* $(L^2)_q^1$ ($q \in \mathbb{N}$) as the closures of $\widetilde{\mathcal{P}}$ with respect to the norms $\|\cdot\|_q$, $(L^2)^1 := \mathrm{pr}\lim_{q\in\mathbb{N}} (L^2)_q^1$.

It is not difficult to see that $f \in (L^2)^1_q$ if and only if f can be presented in form (1.8) with

$$\|f\|^2_q = \sum_{n=0}^{\infty} (n!)^2 2^{qn} |f^{(n)}|^2_{\mathrm{ext}} < \infty, \tag{2.20}$$

and for $f, g \in (L^2)^1_q$ $(f,g)_q := (f,g)_{(L^2)^1_q} = \sum_{n=0}^{\infty} (n!)^2 2^{qn} \langle f^{(n)}, g^{(n)} \rangle_{\mathrm{ext}}$, where $f^{(n)}, g^{(n)} \in \mathcal{H}^{(n)}_{\mathrm{ext}}$ are the kernels from decompositions (1.8) for f and g correspondingly. Therefore the generalized Meixner polynomials play a role of an orthogonal basis in $(L^2)^1_q$.

It is obvious that for each $q \in \mathbb{N}$ $\|\cdot\|_{(L^2)} \le \|\cdot\|_q$. Further, let a sequence $(f_k \in \widetilde{\mathcal{P}})_{k=1}^{\infty}$ be a Cauchy one in $(L^2)^1_q$ and tends to zero in (L^2), and let $f := \lim_{k\to\infty} f_k$ in $(L^2)^1_q$. We have $\|f\|_{(L^2)} = \|f - f_k + f_k\|_{(L^2)} \le \|f - f_k\|_{(L^2)} + \|f_k\|_{(L^2)} \le \|f - f_k\|_q + \|f_k\|_{(L^2)} \to 0$ as $k \to \infty$, so, $\|f\|_{(L^2)} = 0$. But it follows from here that for all kernels $f^{(n)} \in \mathcal{H}^{(n)}_{\mathrm{ext}}$ from decomposition (1.8) for f $|f^{(n)}|_{\mathrm{ext}} = 0$ (see (1.9)) whence $\|f\|_q = 0$. Therefore $f_k \to 0$ (as $k \to \infty$) in $(L^2)^1_q$. Thus (see, e.g., [8]) $(L^2)^1_q$ is continuously embedded in (L^2). Moreover, it is obvious that this embedding is dense. Therefore one can consider the chain

$$(L^2)^{-1} = \operatorname*{ind\,lim}_{\widetilde{q}\in\mathbb{N}} (L^2)^{-1}_{-\widetilde{q}} \supset (L^2)^{-1}_{-q} \supset (L^2) \supset (L^2)^1_q \supset (L^2)^1, \tag{2.21}$$

where $(L^2)^{-1}_{-q}$, $(L^2)^{-1}$ are the dual to $(L^2)^1_q$, $(L^2)^1$ with respect to (L^2) spaces correspondingly.

Definition 2.4. The spaces $(L^2)^{-1}_{-q}$, $(L^2)^{-1}$ are called the *Kondratiev-type spaces of regular generalized functions.*

Let us construct the natural orthogonal bases in $(L^2)^{-1}_{-q}$. By construction the space $(L^2)^1_q$ ($q \in \mathbb{N}$) is isometrically isomorphic to the space $\bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}_{\mathrm{ext}} (n!)^2 2^{qn}$:

$$(L^2)^1_q \ni f = \sum_{n=0}^{\infty} \langle P_n, f^{(n)} \rangle \leftrightarrow \widetilde{f} = (f^{(0)}, f^{(1)}, \dots) \in \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}_{\mathrm{ext}} (n!)^2 2^{qn},$$

$\|f\|_q = \|\widetilde{f}\|_{\bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}_{\mathrm{ext}} (n!)^2 2^{qn}}$ (see (2.20) and Lemma 2.1). Therefore there exists the isometric isomorphism between the space $(L^2)^{-1}_{-q}$ and the space $[\bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}_{\mathrm{ext}} (n!)^2 2^{qn}]'$ of linear continuous functionals on $\bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}_{\mathrm{ext}} (n!)^2 2^{qn}$. Let

$$\bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}_{\mathrm{ext}} 2^{-qn} \supset \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}_{\mathrm{ext}} n! \supset \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}_{\mathrm{ext}} (n!)^2 2^{qn}$$

be chain (2.9) that is based on the (degenerated) chains $\mathcal{H}^{(n)}_{\mathrm{ext}} \supseteq \mathcal{H}^{(n)}_{\mathrm{ext}} \supseteq \mathcal{H}^{(n)}_{\mathrm{ext}}$. Now the space $[\bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}_{\mathrm{ext}} (n!)^2 2^{qn}]'$ is represented by the Hilbert space $\bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}_{\mathrm{ext}} 2^{-qn}$, therefore each element $F \in (L^2)^{-1}_{-q}$ can be identified with $\widehat{F} = (F^{(0)}_{\mathrm{ext}}, F^{(1)}_{\mathrm{ext}}, \dots) \in \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}_{\mathrm{ext}} 2^{-qn}$, i.e., there exists the orthogonal basis in $(L^2)^{-1}_{-q}$ that has a form of a family of regular generalized functions $\{\langle \widehat{P}_n, F^{(n)}_{\mathrm{ext}} \rangle \in (L^2)^{-1}_{-q}\colon F^{(n)}_{\mathrm{ext}} \in \mathcal{H}^{(n)}_{\mathrm{ext}}, n \in \mathbb{Z}_+\}$ such that

$$F = \sum_{n=0}^{\infty} \langle \widehat{P}_n, F^{(n)}_{\mathrm{ext}} \rangle \tag{2.22}$$

and this formal series converges in the sense that

$$\|F\|^2_{-q} := \|F\|^2_{(L^2)^{-1}_{-q}} = \|\widehat{F}\|^2_{\bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}_{\mathrm{ext}} 2^{-qn}} = \sum_{n=0}^{\infty} 2^{-qn} |F^{(n)}_{\mathrm{ext}}|^2_{\mathrm{ext}} < \infty. \tag{2.23}$$

Note that, as is easily seen, for all $n \in \mathbb{Z}_+$, $c_1, c_2 \in \mathbb{C}$, $F^{(n)}_{\mathrm{ext}}, G^{(n)}_{\mathrm{ext}} \in \mathcal{H}^{(n)}_{\mathrm{ext}}$ $\langle \widehat{P}_n, c_1 F^{(n)}_{\mathrm{ext}} + c_2 G^{(n)}_{\mathrm{ext}} \rangle = c_1 \langle \widehat{P}_n, F^{(n)}_{\mathrm{ext}} \rangle + c_2 \langle \widehat{P}_n, G^{(n)}_{\mathrm{ext}} \rangle$. Further, for each $f \in (L^2)^1_q$ that corresponds to $\widetilde{f} = (f^{(0)}, f^{(1)}, \dots) \in \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}_{\mathrm{ext}} (n!)^2 2^{qn}$ (here $\{f^{(n)}\}_{n=0}^{\infty}$ are the kernels from decomposition (1.8) for f) we have

$$\langle\!\langle F, f \rangle\!\rangle = (\widehat{F}, \widetilde{f})_{\bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}_{\mathrm{ext}} n!} = \sum_{n=0}^{\infty} n! \langle F^{(n)}_{\mathrm{ext}}, f^{(n)} \rangle_{\mathrm{ext}}.$$

In particular,

$$\langle\!\langle \langle \widehat{P}_n, F^{(n)}_{ext} \rangle, \langle P_m, f^{(m)} \rangle \rangle\!\rangle = \delta_{nm} n! \langle F^{(n)}_{\mathrm{ext}}, f^{(n)} \rangle_{\mathrm{ext}}.$$

Therefore taking into account (1.6) and a density in (L^2) of the set of polynomials, one can conclude that *for smooth* $F^{(n)}_{\mathrm{ext}} \in \mathcal{D}^{\widehat{\otimes} n}_{\mathbb{C}}$ $\langle \widehat{P}_n, F^{(n)}_{\mathrm{ext}} \rangle$ *is a generalized Meixner polynomial*; and for a general $F^{(n)}_{\mathrm{ext}} \in \mathcal{H}^{(n)}_{\mathrm{ext}}$ $\langle \widehat{P}_n, F^{(n)}_{\mathrm{ext}} \rangle$ is an (L^2)-limit of the corresponding sequence of the generalized Meixner polynomials (see Definition 1.5). Note that this result is connected with the results of Theorem 1.3, cf. Remark 2.7. We remind that in accordance with the accepted above notation

$$\langle \widehat{P}_n, F^{(n)}_{\mathrm{ext}} \rangle = \langle P_n, F^{(n)}_{\mathrm{ext}} \rangle.$$

As above, let us sum up the obtained results in

Theorem 2.2. *A regular generalized function* $F \in (L^2)^{-1}_{-q}$ *(*$q \in \mathbb{N}$*) if and only if there exists a sequence*

$$(F^{(n)}_{\mathrm{ext}} \in \mathcal{H}^{(n)}_{\mathrm{ext}})^{\infty}_{n=0} \tag{2.24}$$

such that F *can be presented in form* (2.22)*, where the formal series converges in* $(L^2)^{-1}_{-q}$*, i.e., norm* (2.23) *is finite. Furthermore, the system* $\{\langle P_n, F^{(n)}_{\mathrm{ext}} \rangle\colon F^{(n)}_{\mathrm{ext}} \in \mathcal{H}^{(n)}_{\mathrm{ext}},\ n \in \mathbb{Z}_+\}$ *plays a role of an orthogonal basis in* $(L^2)^{-1}_{-q}$ *in the sense that for* $F, G \in (L^2)^{-1}_{-q}$

$$(F, G)_{(L^2)^{-1}_{-q}} = \sum_{n=0}^{\infty} 2^{-qn} \langle F^{(n)}_{\mathrm{ext}}, G^{(n)}_{\mathrm{ext}} \rangle_{\mathrm{ext}},$$

where $F^{(n)}_{\mathrm{ext}}, G^{(n)}_{\mathrm{ext}} \in \mathcal{H}^{(n)}_{\mathrm{ext}}$ *are the kernels from decompositions* (2.22) *for* F *and* G *correspondingly. The (generated by the scalar product in* (L^2)*) real dual pairing between elements of* $(L^2)^{-1}_{-q}$ *and* $(L^2)^1_q$ *is given by* (2.13).

Remark 2.10. It is easy to see that $F \in (L^2)^{-1}$ if and only if there exists a sequence (2.24) such that F can be presented in form (2.22) with finite norm (2.23) for some $q \in \mathbb{N}$.

Remark 2.11. Note that one can introduce the spaces $(\mathcal{D})_q := \mathrm{pr} \lim_{\tau \in T} (\mathcal{H}_\tau)_q$ $(q \in \mathbb{N})$ and the corresponding dual ones; but all results for these spaces are similar to the results for $(\mathcal{H}_\tau)_q$ and $(\mathcal{H}_{-\tau})_{-q}$.

Finally we note that in order to construct and study the extended stochastic integral and elements of the Wick calculus on the spaces of *regular* generalized functions it is not necessary to use the result of Lemma 1.2.

3. An extended stochastic integral

In this section we introduce and study a natural extended stochastic integral that is connected with the generalized Meixner measure μ. Note that our construction is similar to the construction of the connected with the Gamma measure extended stochastic integral, see [21, 17].

First, let us recall the classical definition of the extended stochastic integral. Let γ be the Gaussian measure on $\mathcal{D}'$, i.e., the probability measure with the Laplace transform

$$l_\gamma(\lambda) = \int_{\mathcal{D}'} \exp\{\langle x, \lambda\rangle\}\gamma(dx) = \exp\Big\{\frac{1}{2}\langle\lambda,\lambda\rangle\Big\}.$$

By the Wiener-Itô chaos decomposition theorem (see, e.g., [14]) the Gaussian measure has the so-called Chaotic Representation Property (CRP), i.e., we can write any function $f \in L^2(\mathcal{D}',\gamma)$ in the form

$$f = \sum_{n=0}^{\infty} n! \int_0^\infty \int_0^{u_n} \dots \int_0^{u_2} f^{(n)}(u_1,\dots,u_n)\, dW_{u_1}\dots dW_{u_n}, \tag{3.1}$$

where for each $n \in \mathbb{N}$ $f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n} = L^2(\mathbb{R}_+,\sigma)_{\mathbb{C}}^{\widehat{\otimes} n}$, the term with $n = 0$ in (3.1) is just a constant $f^{(0)} \in \mathbb{C}$, and $W_\cdot$ is a standard Wiener process.

Let now $f \in L^2(\mathcal{D}',\gamma)\otimes\mathcal{H}_{\mathbb{C}}$. It follows from (3.1) that

$$f(\cdot) = \sum_{n=0}^{\infty} n! \int_0^\infty \int_0^{u_n} \dots \int_0^{u_2} f_\cdot^{(n)}(u_1,\dots,u_n)\, dW_{u_1}\dots dW_{u_n}, \tag{3.2}$$

where for each $n \in \mathbb{Z}_+$ $f_\cdot^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}\otimes\mathcal{H}_{\mathbb{C}}$. We assume in addition that f *is adapted with respect to the flow of (full by definition)* σ*-algebras* $\{\mathcal{F}_s = \sigma(W_u, u \le s)\}_{s\ge 0}$ *that is generated by the Wiener process* (more exactly, there is a *representative* (a function) $f \ni \dot f : \mathbb{R}_+ \to L^2(\mathcal{D}',\gamma)$ such that $\dot f(s)$ is $\mathcal{F}_s$-measurable $\forall s \in \mathbb{R}_+$). As is well known (see, e.g., [13]), f is integrable by Itô and the corresponding Itô integral has the form

$$\int_0^\infty f(s)\, dW_s = \sum_{n=0}^{\infty} n! \int_0^\infty \int_0^s \int_0^{u_n} \dots \int_0^{u_2} f_s^{(n)}(u_1,\dots,u_n)\, dW_{u_1}\dots dW_{u_n} dW_s. \tag{3.3}$$

In fact, the "adaptiveness" of f means that, roughly speaking, if for some $k \in \{1,\dots,n\}$ $u_k > s$ then $f_s^{(n)}(u_1,\dots,u_n) = 0$ (more exactly, such a function belongs to the corresponding equivalence class $f_\cdot^{(n)}$). Therefore (3.2) can be rewritten in the form

$$f(\cdot) = \sum_{n=0}^{\infty} n! \int_0^\cdot \int_0^{u_n} \dots \int_0^{u_2} f_\cdot^{(n)}(u_1,\dots,u_n)\, dW_{u_1}\dots dW_{u_n}, \tag{3.4}$$

and integrating this series term by term we obtain (3.3) (the correctness of such integration follows from the estimate

$$\begin{aligned}\Big\| \int_0^\infty f(s)\, dW_s\Big\|^2_{L^2(\mathcal{D}',\gamma)} &\equiv \mathbf{E}\Big[\Big|\int_0^\infty f(s)\, dW_s\Big|^2\Big] \\ &= \mathbf{E}\Big[\int_0^\infty |f(s)|^2\sigma(ds)\Big] \equiv \|f\|^2_{L^2(\mathcal{D}',\gamma)\otimes\mathcal{H}_{\mathbb{C}}} < \infty,\end{aligned}$$

here and below $\mathbf{E}$ denotes the expectation).

Let $\widehat{f}^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n+1}$ be the symmetrization of $f_\cdot^{(n)}$ with respect to $n+1$ variables (more exactly, the projection of $f_\cdot^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}\otimes\mathcal{H}_{\mathbb{C}}$ on $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n+1}$). It is easy to see that (3.3) can be rewritten in the form

(3.5)
$$\int_0^\infty f(s)\,dW_s=\sum_{n=0}^\infty (n+1)!\int_0^\infty\int_0^s\int_0^{u_n}\dots\int_0^{u_2}\widehat{f}^{(n)}(u_1,\dots,u_n,s)\,dW_{u_1}\dots dW_{u_n}dW_s.$$

If f is *not adapted* with respect to $\{\mathcal{F}_s\}_{s\geq 0}$ then the Itô integral $\int_0^\infty f(s)\,dW_s$ has no sense (note that now f can not be presented in form (3.4), and at least one term in series (3.2) is not integrable by Itô, therefore term by term integration of (3.2) is impossible). Nevertheless, one can consider symmetrizations $\widehat{f}^{(n)}$ of kernels $f_\cdot^{(n)}$ and write out *formal* series (3.5) that can diverge in $L^2(\mathcal{D}',\gamma)$, generally speaking. Let us *assume* that $f\in L^2(\mathcal{D}',\gamma)\otimes\mathcal{H}_{\mathbb{C}}$ is such that series (3.5) converges in $L^2(\mathcal{D}',\gamma)$, i.e.,

$$\Big\|\sum_{n=0}^\infty (n+1)!\int_0^\infty\int_0^s\int_0^{u_n}\dots\int_0^{u_2}\widehat{f}^{(n)}(u_1,\dots,u_n,s)\,dW_{u_1}\dots dW_{u_n}dW_s\Big\|^2_{L^2(\mathcal{D}',\gamma)}$$
$$=\sum_{n=0}^\infty (n+1)!|\widehat{f}^{(n)}|_0^2<\infty$$

(remind that *any* adapted with respect to $\{\mathcal{F}_s\}_{s\geq 0}$ $f\in L^2(\mathcal{D}',\gamma)\otimes\mathcal{H}_{\mathbb{C}}$ satisfies this assumption). Then the sum of series (3.5) is called the *extended (Skorohod) stochastic integral*, we denote this integral by $\int_0^\infty f(s)\,\widehat{d}W_s$.

So, the extended stochastic integral $\int_0^\infty \circ(s)\,\widehat{d}W_s$ is an extension of the Itô stochastic integral $\int_0^\infty \circ(s)\,dW_s$ in the sense that the domain of the Itô integral is embedded in the domain of the extended stochastic integral, there are integrable "in the extended sense" and not integrable by Itô functions, and if f is integrable by Itô then $\int_0^\infty f(s)\,\widehat{d}W_s=\int_0^\infty f(s)\,dW_s$.

Remark 3.1. Note that for not adapted f sums of series (3.3) and (3.5) are *different* (in this case the right hand side of (3.3) is the Itô integral from the element of $L^2(\mathcal{D}',\gamma)\otimes\mathcal{H}_{\mathbb{C}}$ that is *defined* by the right hand side of (3.4)). In a sense one can understand (3.3) as a *not Skorohod* extended stochastic integral.

On the other hand, it is well known (see, e.g., [13]) that for each $n\in\mathbb{Z}_+$ one can identify the multiple stochastic integral with the corresponding generalized Hermite polynomial, i.e.,

(3.6)
$$n!\int_0^\infty\int_0^{u_n}\dots\int_0^{u_2} f_\cdot^{(n)}(u_1,\dots,u_n)\,dW_{u_1}\dots dW_{u_n}=\langle H_n,f_\cdot^{(n)}\rangle\quad \forall f_\cdot^{(n)}\in\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}\otimes\mathcal{H}_{\mathbb{C}},$$

where $H_n(x)\in\mathcal{D}'^{\widehat{\otimes}n}$ is the kernel of the Hermite polynomial of power n from the decomposition

$$\exp\Big\{\langle x,\lambda\rangle-\frac{1}{2}\langle\lambda,\lambda\rangle\Big\}=\sum_{n=0}^\infty\frac{1}{n!}\langle H_n(x),\lambda^{\otimes n}\rangle, \tag{3.7}$$

λ belongs to some neighborhood of zero in $\mathcal{D}_{\mathbb{C}}$. Thus one can rewrite the integrand f and the stochastic integral $\int_0^\infty f(s)\,\widehat{d}W_s$ (formulas (3.2) and (3.5)) in the form

$$f(\cdot)=\sum_{n=0}^\infty\langle H_n,f_\cdot^{(n)}\rangle$$

and

$$\int_0^\infty f(s)\,\widehat{d}W_s=\sum_{n=0}^\infty\langle H_{n+1},\widehat{f}^{(n)}\rangle$$

correspondingly (note that the Wiener process can be written out in this notation in the form $W_\cdot = \langle H_1, 1_{[0,\cdot)}\rangle$).

If instead of the space $L^2(\mathcal{D}', \gamma)$ we want to use the space $(L^2) = L^2(\mathcal{D}', \mu)$ with the generalized Meixner measure μ then the full analog of the recalled above construction of the extended stochastic integral can not be obtained. In the first place, μ has no the CRP if $\eta \neq 0$ (see (1.2) for a definition of η), i.e., we can not present *any* element $f \in (L^2)$ in form (3.1) with the corresponding stochastic process (actually this follows from Theorem 1.3 and the construction of the Itô stochastic integral; one can found the proof for the Gamma measure (corresponding to $\eta = 1$) in, e.g., [9]). In the second place, an attempt to "go around" the absence of the CRP leads to use $\mathcal{H}_{\text{ext}}^{(n)}$ instead of $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$. But since for $\eta \neq 0$ and $n > 1$ $\mathcal{H}_{\text{ext}}^{(n)}$ are not tensor powers of Hilbert spaces, it is impossible to use only the symmetrization in order to construct the kernels for the Meixner-analog of decomposition (3.5), i.e., it is impossible to proceed by analogy with the Gaussian case. So, in order to construct the *natural* extended stochastic integral that is connected with the generalized Meixner measure, we need a modification of the described above classical scheme. The idea of this modification is the same as in the Gamma analysis (see [21, 17]) and very simple: in order to construct $\widehat{f}^{(n)} \in \mathcal{H}_{\text{ext}}^{(n+1)}$ starting from $f^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ we "exclude a diagonal of $f^{(n)}$", i.e., non-strictly speaking, we symmetrize the "function"

$$\widetilde{f}^{(n)}(t_1, \dots, t_n, s) := \begin{cases} f_s^{(n)}(t_1, \dots, t_n), & \text{if } s \neq t_1, \dots, s \neq t_n \\ 0, & \text{in other cases} \end{cases} \tag{3.8}$$

(cf. [21, 17]).

Now let us pass to the construction of stochastic integrals in the Meixner analysis. By analogy with the classical Gaussian analysis one can consider the Meixner process $M_\cdot$ on the probability space $(\mathcal{D}', \mathcal{F}, \mu)$: for each $s \in \mathbb{R}_+$ $M_s := \langle P_1, 1_{[0,s)}\rangle \in (L^2)$.

Remark 3.2. Since now the measure μ is centered, $P_1(x) = x$. For a noncentered measure μ (such measures were considered in, e.g., [23, 21, 15, 29, 17]) $\langle P_1, 1_{[0,s)}\rangle = \langle \cdot, 1_{[0,s)}\rangle - \mathbf{E}\langle \cdot, 1_{[0,s)}\rangle$. In both cases $\mathbf{E}\langle P_1, 1_{[0,s)}\rangle = 0$, i.e., the random process $M_\cdot = \langle P_1, 1_{[0,\cdot)}\rangle$ is a compensated one.

The process $M_\cdot$ has orthogonal increments (see (1.6) and Theorem 1.3), therefore $M_\cdot$ is a martingale with respect to the flow of σ-algebras $\{\mathcal{F}_s := \sigma(M_u : u \leq s)\}_{s \geq 0}$ (as above, all $\mathcal{F}_s$ are *full* by definition). Further, this martingale is (locally) square integrable: on each interval $[0, S]$, $S \in \mathbb{R}_+$

$$\begin{aligned} \sup_{s \leq S} \mathbf{E} M_s^2 &= \sup_{s \leq S} \langle\!\langle \langle P_1, 1_{[0,s)}\rangle^2, 1 \rangle\!\rangle = \sup_{s \leq S} \int_{\mathcal{D}'} \langle P_1(x), 1_{[0,s)}\rangle^2 \mu(dx) \\ &= \sup_{s \leq S} \langle 1_{[0,s)}, 1_{[0,s)}\rangle = \sup_{s \leq S} \sigma\{[0, s)\} = \sigma\{[0, S)\} < \infty, \end{aligned}$$

therefore $M_\cdot$ has the Doob-Meyer decomposition $M_\cdot^2 = m_\cdot + A_\cdot$, where $m_\cdot$ is an $\mathcal{F}_s$-martingale and $A_\cdot$ is a natural increasing process. Finally, since by Theorem 1.1 the generalized stochastic process (the Meixner white noise) $\{M'_s = \langle P_1, \delta_s\rangle \in (\mathcal{H}_{-\tau})_{-q}\}_{s \geq 0}$ ($\tau \in T$, $q \in \mathbb{N}_{q_0}$) has independent values, *$M_\cdot$ has independent increments*, therefore *$A_\cdot$ is nonrandom*, so, *$M_\cdot$ is a (square integrable) normal $\mathcal{F}_s$-martingale*. Hence one can consider the Itô stochastic integral with respect to $M_\cdot$, and any adapted with respect to $\{\mathcal{F}_s\}_{s \geq 0}$ $f \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ is integrable by Itô on $\mathbb{R}_+$ (and, therefore, on $[0, t)$ $\forall t \in [0, +\infty]$) in the sense of the so-called L^2-theory. We denote the Itô integral of f on $[0, t)$ by $\int_0^t f(s)\, dM_s$.

Now let us pass to the construction of the extended stochastic integral. Let $f \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$. It follows from Theorem 1.3 that f can be presented in the form

$$f(\cdot) = \sum_{n=0}^{\infty} \langle P_n, f_{\cdot}^{(n)} \rangle, \quad f_{\cdot}^{(n)} \in \mathcal{H}_{\rm ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}} \tag{3.9}$$

with

$$\|f\|^2_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} = \sum_{n=0}^{\infty} n! |f_{\cdot}^{(n)}|^2_{\mathcal{H}_{\rm ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}} < \infty.$$

Remark 3.3. Generally speaking, (3.9) is not a "Meixner analog" of (3.2) because the terms in series (3.9) can not be presented by analogy with (3.6) as repeated Itô integrals with respect to $M_\cdot$. Nevertheless, *for* $f \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ *with the kernels from decomposition* (3.9) $f_{\cdot}^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\mathbb{C}}$ (now $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ $(n > 1)$ are considered as subspaces of $\mathcal{H}_{\rm ext}^{(n)}$, see Remark 1.10) *representation* (3.9) *is a "Meixner analog" of* (3.2)

$$\langle P_n, f_{\cdot}^{(n)} \rangle = n! \int_0^{\infty} \int_0^{u_n} \dots \int_0^{u_2} f_{\cdot}^{(n)}(u_1, \dots, u_n) \, dM_{u_1} \dots dM_{u_n} \quad \forall n \in \mathbb{Z}_+ \tag{3.10}$$

(cf. (3.6)). In fact, for $n = 0$ (3.10) is obvious. Let $n \in \mathbb{N}$ and $\Delta_1, \dots, \Delta_n \subset \mathbb{R}_+$ be disjoint measurable sets. Then, as is easily seen,

$$n! \int_0^{\infty} \int_0^{u_n} \dots \int_0^{u_2} (1_{\Delta_1} \widehat{\otimes} \dots \widehat{\otimes} 1_{\Delta_n})(u_1, \dots, u_n) \, dM_{u_1} \dots dM_{u_n} = \langle P_1, 1_{\Delta_1} \rangle \dots \langle P_1, 1_{\Delta_n} \rangle. \tag{3.11}$$

On the other hand, it follows from results of [31] that for each $m \in \mathbb{Z}_+$

$$\begin{aligned} \langle P_m, f^{(m)} \rangle \langle P_1, g^{(1)} \rangle &= \langle P_{m+1}, f^{(m)} \widehat{\otimes} g^{(1)} \rangle + m \langle P_m, Pr(\theta(\cdot) g^{(1)}(\cdot) f^{(m)}(\cdot, \cdot_2, \dots, \cdot_m)) \rangle \\ &+ m \langle P_{m-1}, \langle f^{(m)}, g^{(1)} \rangle \rangle + m(m-1) \langle P_{m-1}, Pr(\eta(\cdot) g^{(1)}(\cdot) f^{(m)}(\cdot, \cdot, \cdot_3 \dots, \cdot_m)) \rangle \end{aligned} \tag{3.12}$$

(see (1.2) for a definition of θ and η), where $f^{(m)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} m}$, $g^{(1)} \in \mathcal{D}_{\mathbb{C}}$, Pr is the symmetrization operator, $\langle f^{(m)}, g^{(1)} \rangle := \int_{\mathbb{R}_+} f^{(m)}(s, \cdot_2, \dots, \cdot_m) g^{(1)}(s) \sigma(ds) \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} m-1}$. Let $(e_k^l \in \mathcal{D})_{l=1}^{\infty})$, $k \in \{1, \dots, n\}$ be sequences of "smooth" functions such that $e_k^l \to 1_{\Delta_k}$ as $l \to \infty$ in $\mathcal{H}$ (it is clear that one can select e_k^l that satisfy estimates $|e_k^l| \le c(k)$ $\forall l \in \mathbb{N}$). It is not difficult to show that since $\Delta_1, \dots, \Delta_n$ are disjoint sets, $e_1^l \widehat{\otimes} \dots \widehat{\otimes} e_n^l \to 1_{\Delta_1} \widehat{\otimes} \dots \widehat{\otimes} 1_{\Delta_n}$ as $l \to \infty$ in $\mathcal{H}_{\rm ext}^{(n)}$ (here we understand $1_{\Delta_1} \widehat{\otimes} \dots \widehat{\otimes} 1_{\Delta_n} \in \mathcal{H}^{\widehat{\otimes} n}$ as an element of $\mathcal{H}_{\rm ext}^{(n)}$, see Remark 1.10); $Pr(\theta(\cdot) e_n^l(\cdot)(e_1^l \widehat{\otimes} \dots \widehat{\otimes} e_{n-1}^l)(\cdot, \cdot_2, \dots, \cdot_{n-1})) \to 0$ as $l \to \infty$ in $\mathcal{H}_{\rm ext}^{(n-1)}$; $\langle e_1^l \widehat{\otimes} \dots \widehat{\otimes} e_{n-1}^l, e_n^l \rangle \to 0$ as $l \to \infty$ and $Pr(\eta(\cdot) e_n^l(\cdot)(e_1^l \widehat{\otimes} \dots \widehat{\otimes} e_{n-1}^l)(\cdot, \cdot, \cdot_3, \dots, \cdot_{n-1})) \to 0$ as $l \to \infty$ in $\mathcal{H}_{\rm ext}^{(n-2)}$. Therefore substituting in (3.12) $m = n-1$, $f^{(n-1)} = e_1^l \widehat{\otimes} \dots \widehat{\otimes} e_{n-1}^l$, $g^{(1)} = e_n^l$ and passing to the limit as $l \to \infty$ we obtain

$$\langle P_{n-1}, 1_{\Delta_1} \widehat{\otimes} \dots \widehat{\otimes} 1_{\Delta_{n-1}} \rangle \langle P_1, 1_{\Delta_n} \rangle = \langle P_n, 1_{\Delta_1} \widehat{\otimes} \dots \widehat{\otimes} 1_{\Delta_n} \rangle.$$

By analogy $\langle P_{n-2}, 1_{\Delta_1} \widehat{\otimes} \dots \widehat{\otimes} 1_{\Delta_{n-2}} \rangle \langle P_1, 1_{\Delta_{n-1}} \rangle = \langle P_{n-1}, 1_{\Delta_1} \widehat{\otimes} \dots \widehat{\otimes} 1_{\Delta_{n-1}} \rangle$ (if $n > 1$) etc. So, $\langle P_1, 1_{\Delta_1} \rangle \dots \langle P_1, 1_{\Delta_n} \rangle = \langle P_n, 1_{\Delta_1} \widehat{\otimes} \dots \widehat{\otimes} 1_{\Delta_n} \rangle$. Substituting this result in (3.11) and taking into consideration that $\{1_{\Delta_1} \widehat{\otimes} \dots \widehat{\otimes} 1_{\Delta_n}\}$ with disjoint $\Delta_1, \dots, \Delta_n \subset \mathbb{R}_+$ is a total set in $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$, we obtain

$$\langle P_n, f^{(n)} \rangle = n! \int_0^{\infty} \int_0^{u_n} \dots \int_0^{u_2} f^{(n)}(u_1, \dots, u_n) \, dM_{u_1} \dots dM_{u_n} \ \forall f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}, \ \forall n \in \mathbb{Z}_+,$$

from where (3.10) follows (cf. [23]).

In order to give a definition of the extended stochastic integral we need

Lemma 3.1. *For given* $f_\cdot^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ *and* $t \in [0, +\infty]$ *we construct the element* $\widehat{f}_{[0,t)}^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n+1)}$ *by the following way. Let* $\dot{f}_\cdot^{(n)} \in f_\cdot^{(n)}$ *be some representative (function) from the equivalence class* $f_\cdot^{(n)}$. *We set*

$$\widetilde{\dot{f}}_{[0,t)}^{(n)}(u_1,\dots,u_n,u) := \begin{cases} \dot{f}_u^{(n)}(u_1,\dots,u_n)1_{[0,t)}(u), \text{ if } u \neq u_1,\dots,u \neq u_n, \\ 0, \text{ in other cases} \end{cases},$$

$\widehat{\dot{f}}_{[0,t)}^{(n)} := Pr\widetilde{\dot{f}}_{[0,t)}^{(n)}$, *where* Pr *is the symmetrization operator. Let* $\widehat{f}_{[0,t)}^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n+1)}$ *be the generated by* $\widehat{\dot{f}}_{[0,t)}^{(n)}$ *equivalence class in* $\mathcal{H}_{\mathrm{ext}}^{(n+1)}$. *This class is well-defined, does not depend on the representative* $\dot{f}_\cdot^{(n)}$, *and the estimate*

$$|\widehat{f}_{[0,t)}^{(n)}|_{\mathrm{ext}} \leq |f_\cdot^{(n)}1_{[0,t)}(\cdot)|_{\mathcal{H}_{\mathrm{ext}}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}} \leq |f_\cdot^{(n)}|_{\mathcal{H}_{\mathrm{ext}}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}} \tag{3.13}$$

is valid.

Proof. By direct calculation that is based on formula (1.5), properties of $\widetilde{\dot{f}}_{[0,t)}^{(n)}$, well-know estimate $|\sum_{l=1}^p a_l|^2 \leq p\sum_{l=1}^p |a_l|^2$ and nonatomicity of the measure σ, by analogy with the calculation in the proof of Lemma 2.1 in [21] one can obtain the estimate

$$|\widehat{\dot{f}}_{[0,t)}^{(n)}|_{\mathrm{ext}} \leq |\dot{f}_\cdot^{(n)}1_{[0,t)}(\cdot)|_{\mathcal{H}_{\mathrm{ext}}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}} \leq |\dot{f}_\cdot^{(n)}|_{\mathcal{H}_{\mathrm{ext}}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}},$$

therefore $\widehat{f}_{[0,t)}^{(n)}$ is well-defined as an element of $\mathcal{H}_{\mathrm{ext}}^{(n+1)}$ and estimate (3.13) is valid. Let $\dot{g}_\cdot^{(n)} \in f_\cdot^{(n)}$ be another representative of $f_\cdot^{(n)}$, $\widehat{g}_{[0,t)}^{(n)}$ be the corresponding element of $\mathcal{H}_{\mathrm{ext}}^{(n+1)}$. We have $|\widehat{f}_{[0,t)}^{(n)} - \widehat{g}_{[0,t)}^{(n)}|_{\mathrm{ext}} = |\widehat{\dot{f}}_{[0,t)}^{(n)} - \widehat{\dot{g}}_{[0,t)}^{(n)}|_{\mathrm{ext}} \leq |\dot{f}_\cdot^{(n)} - \dot{g}_\cdot^{(n)}|_{\mathcal{H}_{\mathrm{ext}}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}} = 0$ (it is obvious that if $\dot{h}_\cdot^{(n)} = \dot{f}_\cdot^{(n)} - \dot{g}_\cdot^{(n)}$ then $\widehat{\dot{h}}_{[0,t)}^{(n)} = \widehat{\dot{f}}_{[0,t)}^{(n)} - \widehat{\dot{g}}_{[0,t)}^{(n)}$). Therefore $\widehat{f}_{[0,t)}^{(n)}$ does not depend on a choice of $\dot{f}_\cdot^{(n)} \in f_\cdot^{(n)}$ and the Lemma is proved. □

Definition 3.1. Let $t \in [0, +\infty]$ and $f \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ be such that

$$\sum_{n=0}^{\infty}(n+1)!|\widehat{f}_{[0,t)}^{(n)}|_{\mathrm{ext}}^2 < \infty, \tag{3.14}$$

where $\widehat{f}_{[0,t)}^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n+1)}$ ($n \in \mathbb{Z}_+$) are constructed in Lemma 3.1 starting from the kernels $f_\cdot^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ from decomposition (3.9) for f. We define the extended stochastic integral $\int_0^t f(s)\,\widehat{d}M_s \in (L^2)$ by setting

$$\int_0^t f(s)\,\widehat{d}M_s := \sum_{n=0}^{\infty}\langle P_{n+1}, \widehat{f}_{[0,t)}^{(n)}\rangle. \tag{3.15}$$

Since $\|\int_0^t f(s)\,\widehat{d}M_s\|_{(L^2)}^2 = \sum_{n=0}^{\infty}(n+1)!|\widehat{f}_{[0,t)}^{(n)}|_{\mathrm{ext}}^2 < \infty$, this definition is correct. □

Remark 3.4. Note that for the Gaussian or Poissonian measure $\eta = 0$ and $\widehat{f}_{[0,t)}^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n+1}$ is the symmetrization of $f^{(n)}(\cdot_1,\dots,\cdot_n;\cdot)1_{[0,t)}(\cdot) \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\mathbb{C}}$ with respect to $n+1$ "arguments" (more exactly, $\widehat{f}_{[0,t)}^{(n)}$ is the projection of $f_\cdot^{(n)}1_{[0,t)}(\cdot)$ on $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n+1}$). For the general Meixner measure we have to "exclude a diagonal of $f_\cdot^{(n)}1_{[0,t)}(\cdot)$ before symmetrization", i.e., (non-strictly speaking) we symmetrize "function" (3.8).

It is easy to show (by analogy with the classical Gaussian case) that if $f\in(L^2)\otimes\mathcal{H}_{\mathbb{C}}$ is integrable by Itô (with respect to the martingale $M_\cdot$) and satisfies the addition condition of Remark 3.3 then $\int_0^t f(s)\,\widehat{d}M_s=\int_0^t f(s)\,dM_s$. For a general $f\in(L^2)\otimes\mathcal{H}_{\mathbb{C}}$ the "classical" proof can not be adapted; nevertheless, the result holds true (and therefore Definition 3.1 is natural). More exactly, we have the following

Theorem 3.1. *Let $f\in(L^2)\otimes\mathcal{H}_{\mathbb{C}}$ be adapted with respect to the generated by $M_\cdot$ flow of σ-algebras. Then $\forall t\in[0,+\infty]$ f is integrable on $[0,t)$ by Itô and in the extended sense, and $\int_0^t f(s)\,\widehat{d}M_s=\int_0^t f(s)\,dM_s$.*

Proof. Since $f\in(L^2)\otimes\mathcal{H}_{\mathbb{C}}$, $\mathbf{E}\int_0^\infty|f(s)|^2\sigma(ds)=\|f\|^2_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}<\infty$, therefore f is integrable by Itô. Further, let us prove that now
(3.16)

$$\|f\|^2_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}=\sum_{n=0}^{\infty}n!|f_\cdot^{(n)}|^2_{\mathcal{H}^{(n)}_{\text{ext}}\otimes\mathcal{H}_{\mathbb{C}}}=\sum_{n=0}^{\infty}(n+1)!|\widehat{f}^{(n)}_{[0,+\infty)}|^2_{\text{ext}}=\Big\|\int_0^{+\infty}f(s)\,\widehat{d}M_s\Big\|^2_{(L^2)}$$

(here the kernels $f_\cdot^{(n)}\in\mathcal{H}^{(n)}_{\text{ext}}\otimes\mathcal{H}_{\mathbb{C}}$, $n\in\mathbb{Z}_+$ are from decomposition (3.9) for f, $\widehat{f}^{(n)}_{[0,+\infty)}\in\mathcal{H}^{(n+1)}_{\text{ext}}$, $n\in\mathbb{Z}_+$ are constructed in Lemma 3.1 starting from $f_\cdot^{(n)}$). For this purpose we will show that for each $n\in\mathbb{Z}_+$ $|\widehat{f}^{(n)}_{[0,+\infty)}|^2_{\text{ext}}=\frac{1}{n+1}|f_\cdot^{(n)}|^2_{\mathcal{H}^{(n)}_{\text{ext}}\otimes\mathcal{H}_{\mathbb{C}}}$. Using the notation of Lemma 3.1 and (1.5) we can write

$$
\begin{aligned}
|\widehat{f}^{(n)}_{[0,+\infty)}|^2_{\text{ext}}=&\sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n+1}}\frac{(n+1)!}{l_1^{s_1}\dots l_k^{s_k}s_1!\dots s_k!}\\
&\times\int_{\mathbb{R}_+^{s_1+\dots+s_k}}|\widehat{\dot{f}}^{(n)}_{[0,+\infty)}(\underbrace{t_1,\dots,t_1}_{l_1},\dots,\underbrace{t_{s_1+\dots+s_k},\dots,t_{s_1+\dots+s_k}}_{l_k})|^2\\
&\times\eta(t_1)^{l_1-1}\dots\eta(t_{s_1+\dots+s_k})^{l_k-1}\sigma(dt_1)\dots\sigma(dt_{s_1+\dots+s_k})\\
=&\sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n+1}}\frac{(n+1)!}{l_1^{s_1}\dots l_k^{s_k}s_1!\dots s_k!(n+1)^2}\\
&\times\int_{\mathbb{R}_+^{s_1+\dots+s_k}}[|\tilde{\dot{f}}^{(n)}_{[0,+\infty)}(\underbrace{t_1,\dots,t_1}_{l_1},\dots,\underbrace{t_{s_1+\dots+s_k},\dots,t_{s_1+\dots+s_k}}_{l_k})|^2\\
&+|\tilde{\dot{f}}^{(n)}_{[0,+\infty)}(t_{s_1+\dots+s_k},\underbrace{t_1,\dots,t_1}_{l_1},\dots,\underbrace{t_{s_1+\dots+s_k},\dots,t_{s_1+\dots+s_k}}_{l_k-1})|^2\\
&+\dots+|\tilde{\dot{f}}^{(n)}_{[0,+\infty)}(\underbrace{t_1,\dots,t_1}_{l_1-1},\dots,\underbrace{t_{s_1+\dots+s_k},\dots,t_{s_1+\dots+s_k}}_{l_k},t_1)|^2]\\
&\times\eta(t_1)^{l_1-1}\dots\eta(t_{s_1+\dots+s_k})^{l_k-1}\sigma(dt_1)\dots\sigma(dt_{s_1+\dots+s_k}).
\end{aligned}
\tag{3.17}
$$

Here we used the equality $\tilde{\dot{f}}^{(n)}_{[0,+\infty)}(t_{\tau(1)},\dots,t_{\tau(n+1)})\cdot\tilde{\dot{f}}^{(n)}_{[0,+\infty)}(t_{\tau'(1)},\dots,t_{\tau'(n+1)})=0$, where τ and τ' are *different* permutations of numbers $1,2,\dots,n+1$. In fact, either in the first multiplier $t_{\tau(n+1)}$ less then one of previous arguments, therefore the first multiplier is equal to zero (since f is adapted with respect to the generated by $M_\cdot$ flow of σ-algebras, we can accept $\dot{f}^{(n)}_u(u_1,\dots,u_n)=0$ if $\exists j\in\{1,\dots,n\}$ such that $u_j>u$, this follows from (3.12): the generalized Meixner polynomial of power n is a "measurable combination" of polynomials of power 1), or in the second multiplier $t_{\tau'(n+1)}$ less then one of previous arguments, therefore the second multiplier is equal to zero, or in the first multiplier $t_{\tau(n+1)}$

coincides with one of previous arguments, in this case $\overset{\sim}{\dot{f}}{}^{(n)}_{[0,+\infty)}(t_{\tau(1)},\dots,t_{\tau(n+1)})=0$ by construction.

The integrals in the right hand side of (3.17) can be not equal to zero if and only if $l_k=1$ (by construction of $\overset{\sim}{\dot{f}}{}^{(n)}_{[0,+\infty)}$). In these cases (for fixed $k,l.,s.$, $l_k=1$), obviously, all the nonzero integrals of the connected with symmetrization summands are equal, and the quantity of such integrals is s_k. Hence we can continue (3.17) as follows:

$$
\begin{aligned}
|\widehat{f}^{(n)}_{[0,+\infty)}|^2_{\mathrm{ext}} &= \frac{1}{n+1}\sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_{k-1}>1,\\ l_1s_1+\dots+l_{k-1}s_{k-1}+(s_k-1)=n}}\frac{n!}{l_1^{s_1}\dots l_{k-1}^{s_{k-1}}s_1!\dots s_{k-1}!(s_k-1)!}\\
&\times\int_{\mathbb{R}_+^{s_1+\dots+s_k}}|\overset{\sim}{\dot{f}}{}^{(n)}_{[0,+\infty)}(\underbrace{t_1,\dots,t_1}_{l_1},\dots,t_{s_1+\dots+s_k})|^2\\
&\times\eta(t_1)^{l_1-1}\dots\eta(t_{s_1+\dots+s_{k-1}})^{l_{k-1}-1}\sigma(dt_1)\dots\sigma(dt_{s_1+\dots+s_k})\\
&=\frac{1}{n+1}|f_{\cdot}^{(n)}|^2_{\mathcal{H}^{(n)}_{\mathrm{ext}}\otimes\mathcal{H}_{\mathbb{C}}}
\end{aligned}
$$

(here we used the nonatomicity of σ).

It follows from (3.16) that 1) f is integrable on $\mathbb{R}_+$ (and therefore on $[0,t)$ for each $t\in[0,+\infty]$) in the extended sense; 2) if a sequence of adapted step-functions converges to f in $(L^2)\otimes\mathcal{H}_{\mathbb{C}}$ then the sequence of extended stochastic integrals on $[0,t)$ ($t\in[0,+\infty]$) of these step-functions converges to $\int_0^t f(s)\,\widehat{d}M_s$ in (L^2).

Therefore it remains to prove that for each $t\in[0,+\infty]$ $\int_0^t f(s)\widehat{d}M_s=\int_0^t f(s)dM_s$ for an adapted step-function $f\in(L^2)\otimes\mathcal{H}_{\mathbb{C}}$. Since stochastic integrals are linear operators, we can consider without loss of generality $f=g\cdot 1_{[u_1,u_2)}(\cdot)\in(L^2)\otimes\mathcal{H}_{\mathbb{C}}$, where $g=\sum_{n=0}^\infty\langle P_n,g^{(n)}\rangle\in(L^2)$ is $\mathcal{F}_{u_1}$-measurable (we remind that $\mathcal{F}_{u_1}=\sigma(M_u:u\le u_1)$), $g^{(n)}\in\mathcal{H}^{(n)}_{\mathrm{ext}}$.

Let $t\ge u_2$. We have $\int_0^t f(s)dM_s=g(M_{u_2}-M_{u_1})=\sum_{n=0}^\infty\langle P_n,g^{(n)}\rangle\langle P_1,1_{[u_1,u_2)}\rangle$, $\int_0^t f(s)\widehat{d}M_s=\sum_{n=0}^\infty\langle P_{n+1},\widehat{g}^{(n)}_{[0,t)}\rangle$, where $\widehat{g}^{(n)}_{[0,t)}\in\mathcal{H}^{(n+1)}_{\mathrm{ext}}$, $n\in\mathbb{Z}_+$ are constructed in Lemma 3.1 starting from $g^{(n)}\cdot 1_{[u_1,u_2)}(\cdot)\in\mathcal{H}^{(n)}_{\mathrm{ext}}\otimes\mathcal{H}_{\mathbb{C}}$. Thus it is sufficient to prove that for each $n\in\mathbb{Z}_+$ $\langle P_n,g^{(n)}\rangle\langle P_1,1_{[u_1,u_2)}\rangle=\langle P_{n+1},\widehat{g}^{(n)}_{[0,t)}\rangle$. But this result easily follows from (3.12): we can consider sequences $D_{\mathbb{C}}^{\widehat{\otimes}n}\ni g_k^{(n)}\to g^{(n)}$ as $k\to\infty$ in $\mathcal{H}^{(n)}_{\mathrm{ext}}$, $D_{\mathbb{C}}\ni h_k\to 1_{[u_1,u_2)}$ as $k\to\infty$ in $\mathcal{H}_{\mathbb{C}}$ (both sequences must be uniformly bounded outside of $\operatorname{supp} g^{(n)}$ and $[u_1,u_2]$ correspondingly), to substitute in (3.12) $g_k^{(n)}$ instead of $f^{(n)}$, h_k instead of $g^{(1)}$, and pass to the limit as $k\to\infty$, taking into account $\mathcal{F}_{u_1}$-measurability of g (roughly speaking, the last means that for each $n\in\mathbb{N}$ $g^{(n)}$ vanishes outside of $[0,u_1]^n$), cf. [15].

In the case $t\in[u_1,u_2)$ it is convenient to consider $f=g\cdot 1_{[u_1,t)}(\cdot)$; the case $t<u_1$ is trivial. □

Corollary. *If f satisfies assumptions of Theorem 3.1 then for all $t\in[0,+\infty]$ estimate (3.14) is fulfilled.*

This statement from (3.16) follows. □

Remark 3.5. Let $f \in (L^2)^1_{q+1} \otimes \mathcal{H}_{\mathbb{C}} \subset (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ ($q \in \mathbb{N}$), $t \in [0, +\infty]$. Then using (3.13) we obtain

$$\Big\| \int_0^t f(s) \widehat{d}M_s \Big\|_q^2 = \sum_{n=0}^{\infty} ((n+1)!)^2 2^{q(n+1)} |\widehat{f}^{(n)}_{[0,t)}|^2_{\rm ext} \le \sum_{n=0}^{\infty} ((n+1)!)^2 2^{q(n+1)} |f^{(n)}_{\cdot}|^2_{\mathcal{H}^{(n)}_{\rm ext} \otimes \mathcal{H}_{\mathbb{C}}}$$
$$= 2^q \sum_{n=0}^{\infty} [(n+1)^2 2^{-n}] (n!)^2 2^{(q+1)n} |f^{(n)}_{\cdot}|^2_{\mathcal{H}^{(n)}_{\rm ext} \otimes \mathcal{H}_{\mathbb{C}}} \le 9 \cdot 2^{q-2} \|f\|^2_{(L^2)^1_{q+1} \otimes \mathcal{H}_{\mathbb{C}}} < \infty$$

(because $\max_{n \in \mathbb{Z}_+} [(n+1)^2 2^{-n}] = 9/4$), therefore in this case $\int_0^t f(s) \widehat{d}M_s \in (L^2)^1_q$. If $f \in (\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_{\tau,\mathbb{C}}$ ($\tau \in T$, $q \in \mathbb{N}$) then again $\int_0^t f(s) \widehat{d}M_s \in (L^2)^1_q$ because $(\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_{\tau,\mathbb{C}} \subset (L^2)^1_{q+1} \otimes \mathcal{H}_{\mathbb{C}}$. In order to obey the inclusion $\int_0^t f(s) \widehat{d}M_s \in (\mathcal{H}_\tau)_q$ that seems "natural", f must satisfy addition conditions (cf. Remark 3.3). This question is connected with the generalized stochastic derivative and will be discussed in details in forthcoming papers.

As is well known (see, e.g., [13]), the extended stochastic integral in the Gaussian analysis can be constructed as the operator that is dual to the stochastic derivative (or, equivalently, as "the integral" of the operator that is dual to the "Hida derivative at a point", or the Malliavin derivative in another notation). In the Meixner analysis such an approach also is possible, let us explain this in details.

By analogy with the Gaussian analysis we define the "Hida derivative" $\partial_\cdot$ by setting $\partial_\cdot := \langle \delta_\cdot, : D : \rangle$ (see Remark 2.9), where δ is the Dirac delta-function. Since for each $\tau \in T$ $\delta_s \in \mathcal{H}_{-\tau}$ $\forall s \in \mathbb{R}_+$ (see Remark 1.8), ∂_s is a linear continuous operator acting in $(\mathcal{H}_\tau)_q$, $\tau \in T$, $q \in \mathbb{N}$. Moreover, $\partial_\cdot$ is a linear continuous operator acting from $(\mathcal{H}_\tau)_q$ to $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$. In fact, if for $f \in (\mathcal{H}_\tau)_q$ $f^{(n)} \in \mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}}$ ($n \in \mathbb{Z}_+$) are the kernels from decomposition (1.8) then

$$\partial_\cdot f = \sum_{n=0}^{\infty} (n+1) \langle P_n, f^{(n+1)}(\cdot) \rangle,$$
$$\|\partial_\cdot f\|^2_{(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}} = \sum_{n=0}^{\infty} (n!)^2 2^{qn} (n+1)^2 |f^{(n+1)}(\cdot)|^2_{\mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}} \otimes \mathcal{H}_{\tau,\mathbb{C}}} \tag{3.18}$$
$$= 2^{-q} \sum_{n=0}^{\infty} ((n+1)!)^2 2^{q(n+1)} |f^{(n+1)}|^2_\tau \le 2^{-q} \|f\|^2_{\tau,q},$$

where $f^{(n+1)}(\cdot) \in \mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}} \otimes \mathcal{H}_{\tau,\mathbb{C}}$ are obtained from $f^{(n+1)} \in \mathcal{H}^{\widehat{\otimes} n+1}_{\tau,\mathbb{C}}$ by "separating" of one argument (since $f^{(n+1)}$ are symmetric functions, actually $f^{(n+1)}(\cdot) \in \mathcal{H}^{\widehat{\otimes} n+1}_{\tau,\mathbb{C}} \subset \mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}} \otimes \mathcal{H}_{\tau,\mathbb{C}}$ and $|f^{(n+1)}(\cdot)|_{\mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}} \otimes \mathcal{H}_{\tau,\mathbb{C}}} = |f^{(n+1)}|_\tau$).

Unfortunately, the introduced in Remark 2.9 generalized differential operator $\langle \delta_\cdot, : D : \rangle$ can not be continued by continuity on (L^2). Therefore we have to extend the operator $\partial_\cdot$ on (L^2) "by hand" (note that the domain of this extension is not equal to (L^2)). First we need the natural generalization of $f^{(n)}(\cdot)$ ($n \in \mathbb{N}$), i.e., we have to construct for $f^{(n)} \in \mathcal{H}^{(n)}_{\rm ext}$ the element $f^{(n)}(\cdot) \in \mathcal{H}^{(n-1)}_{\rm ext} \otimes \mathcal{H}_{\mathbb{C}}$ that coincides with the introduced above $f^{(n)}(\cdot)$ if $f^{(n)} \in \mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}} \subset \mathcal{H}^{(n)}_{\rm ext}$.

Lemma 3.2. *For given $f^{(n)} \in \mathcal{H}^{(n)}_{\rm ext}$ ($n \in \mathbb{N}$) we construct the element $f^{(n)}(\cdot) \in \mathcal{H}^{(n-1)}_{\rm ext} \otimes \mathcal{H}_{\mathbb{C}}$ by the following way. Let $\dot{f}^{(n)} \in f^{(n)}$ be some representative (function) from the equivalence class $f^{(n)}$. We consider $\dot{f}^{(n)}(\cdot)$ (i.e., separate a one argument of $\dot{f}^{(n)}$). Let $f^{(n)}(\cdot) \in \mathcal{H}^{(n-1)}_{\rm ext} \otimes \mathcal{H}_{\mathbb{C}}$ be the generated by $\dot{f}^{(n)}(\cdot)$ equivalence class in $\mathcal{H}^{(n-1)}_{\rm ext} \otimes \mathcal{H}_{\mathbb{C}}$. This* *class is well-defined, does not depend on the representative $\dot f^{(n)}$, and*

$$|f^{(n)}(\cdot)|_{\mathcal{H}_{\mathrm{ext}}^{(n-1)}\otimes\mathcal{H}_{\mathbb{C}}} \le |f^{(n)}|_{\mathrm{ext}}. \tag{3.19}$$

Proof. Let $\dot f^{(n)} \in f^{(n)}$ be a representative from the equivalence class $f^{(n)}$. We fix a one argument in $\dot f^{(n)}$ and obtain the function $\dot f^{(n)}(\cdot)$. Using the definition of $|\cdot|_{\mathrm{ext}}$ (see Definition 1.4) we can write

$$\begin{aligned}
|f^{(n)}|^2_{\mathrm{ext}} = |\dot f^{(n)}|^2_{\mathrm{ext}} = &\sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n}} \frac{n!}{l_1^{s_1}\dots l_k^{s_k}s_1!\dots s_k!}\\
&\times\int_{\mathbb{R}_+^{s_1+\dots+s_k}} |\dot f^{(n)}(\underbrace{t_1,\dots,t_1}_{l_1},\dots,\underbrace{t_{s_1+\dots+s_k},\dots,t_{s_1+\dots+s_k}}_{l_k})|^2\\
&\times \eta(t_1)^{l_1-1}\dots\eta(t_{s_1+\dots+s_k})^{l_k-1}\sigma(dt_1)\dots\sigma(dt_{s_1+\dots+s_k})\\
&= \sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k>1,\\ l_1s_1+\dots+l_ks_k=n}} \frac{n!}{l_1^{s_1}\dots l_k^{s_k}s_1!\dots s_k!}\\
&\times\int_{\mathbb{R}_+^{s_1+\dots+s_k}} |\dot f^{(n)}(\underbrace{t_1,\dots,t_1}_{l_1},\dots,\underbrace{t_{s_1+\dots+s_k},\dots,t_{s_1+\dots+s_k}}_{l_k})|^2\\
&\times \eta(t_1)^{l_1-1}\dots\eta(t_{s_1+\dots+s_k})^{l_k-1}\sigma(dt_1)\dots\sigma(dt_{s_1+\dots+s_k})\\
&+ \sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k=1,\\ l_1s_1+\dots+l_{k-1}s_{k-1}+(s_k-1)=n-1}} \frac{(n-1)!n}{l_1^{s_1}\dots l_{k-1}^{s_{k-1}}s_1!\dots(s_k-1)!s_k}\\
&\times\int_{\mathbb{R}_+^{s_1+\dots+s_k}} \Big|\dot f^{(n)}(\underbrace{t_1,\dots,t_1}_{l_1},\dots,\underbrace{t_{s_1+\dots+s_{k-1}},\dots,t_{s_1+\dots+s_{k-1}}}_{l_{k-1}},\\
&\qquad t_{s_1+\dots+s_{k-1}+1},\dots,t_{s_1+\dots+s_k})\Big|^2\\
&\times \eta(t_1)^{l_1-1}\dots\eta(t_{s_1+\dots+s_{k-1}})^{l_{k-1}-1}\sigma(dt_1)\dots\sigma(dt_{s_1+\dots+s_k}) \ge |\dot f^{(n)}(\cdot)|^2_{\mathcal{H}_{\mathrm{ext}}^{(n-1)}\otimes\mathcal{H}_{\mathbb{C}}}
\end{aligned}$$

because $n\ge s_k$. Let $f^{(n)}(\cdot)\in\mathcal{H}_{\mathrm{ext}}^{(n-1)}\otimes\mathcal{H}_{\mathbb{C}}$ be the generated by $\dot f^{(n)}(\cdot)$ equivalence class in $\mathcal{H}_{\mathrm{ext}}^{(n-1)}\otimes\mathcal{H}_{\mathbb{C}}$. It is clear that this class is well-defined and (3.19) is valid. Let now $\dot f_1^{(n)}\in f^{(n)}$ be another representative from $f^{(n)}$, $f_1^{(n)}(\cdot)$ be the corresponding equivalence class in $\mathcal{H}_{\mathrm{ext}}^{(n-1)}\otimes\mathcal{H}_{\mathbb{C}}$. By analogy with the calculation above we obtain

$$|f^{(n)}(\cdot)-f_1^{(n)}(\cdot)|_{\mathcal{H}_{\mathrm{ext}}^{(n-1)}\otimes\mathcal{H}_{\mathbb{C}}} = |\dot f^{(n)}(\cdot)-\dot f_1^{(n)}(\cdot)|_{\mathcal{H}_{\mathrm{ext}}^{(n-1)}\otimes\mathcal{H}_{\mathbb{C}}} \le |\dot f^{(n)}-\dot f_1^{(n)}|_{\mathrm{ext}} = 0,$$

therefore $f^{(n)}(\cdot)\in\mathcal{H}_{\mathrm{ext}}^{(n-1)}\otimes\mathcal{H}_{\mathbb{C}}$ does not depend on the representative $\dot f^{(n)}\in f^{(n)}$. □

Corollary. *If $f^{(n)}\in\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}\subset\mathcal{H}_{\mathrm{ext}}^{(n)}$ ($\tau\in T$, $n\in\mathbb{N}$) then constructed in Lemma 3.2 $f^{(n)}(\cdot)\in\mathcal{H}_{\mathrm{ext}}^{(n-1)}\otimes\mathcal{H}_{\mathbb{C}}$ coincides with the considered above $f^{(n)}(\cdot)\in\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n-1}\otimes\mathcal{H}_{\tau,\mathbb{C}}$ (more exactly, the considered above $f^{(n)}(\cdot)\in\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n-1}\otimes\mathcal{H}_{\tau,\mathbb{C}}$ belongs to the corresponding equivalence class in $\mathcal{H}_{\mathrm{ext}}^{(n-1)}\otimes\mathcal{H}_{\mathbb{C}}$).*

Definition 3.2. Let $f\in(L^2)$ be such that

$$\sum_{n=0}^{\infty}(n+1)!(n+1)|f^{(n+1)}(\cdot)|^2_{\mathcal{H}_{\mathrm{ext}}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}}<\infty, \tag{3.20}$$

where $f^{(n+1)}(\cdot) \in \mathcal{H}_{\rm ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ ($n \in \mathbb{Z}_+$) are constructed in Lemma 3.2 starting from the kernels $f^{(n+1)} \in \mathcal{H}_{\rm ext}^{(n+1)}$ from decomposition (1.8) for f. We define the generalized Hida derivative $\partial. f \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ by formula (3.18).

Since $\|\partial. f\|^2_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} = \sum_{n=0}^{\infty}(n+1)!(n+1)|f^{(n+1)}(\cdot)|^2_{\mathcal{H}_{\rm ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}} < \infty$, this definition is correct. (Moreover, one can prove that $\partial.$ is a *closed* operator.) □

Remark 3.6. It is obvious that the restriction of the generalized Hida derivative on $(\mathcal{H}_\tau)_q$ ($\tau \in T$, $q \in \mathbb{N}$) coincides with the "Hida derivative" (3.18).

Remark 3.7. In the classical Gaussian analysis the domain of the Hida derivative $\partial.$ consists of $f \in L^2(\mathcal{D}', \gamma)$ (here γ is the Gaussian measure) such that

$$\sum_{n=0}^{\infty}(n+1)!(n+1)|f^{(n+1)}|^2_{\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n+1}} < \infty \tag{3.21}$$

(we remind that now $\eta = 0$, therefore $\mathcal{H}_{\rm ext}^{(n)} = \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ for all $n \in \mathbb{Z}_+$). Since $|f^{(n+1)}|_{\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n+1}} = |f^{(n+1)}(\cdot)|_{\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}\otimes\mathcal{H}_{\mathbb{C}}}$, (3.21) can be rewritten in the form

$$\sum_{n=0}^{\infty}(n+1)!(n+1)|f^{(n+1)}(\cdot)|^2_{\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}\otimes\mathcal{H}_{\mathbb{C}}} < \infty$$

(cf. (3.20)). At the same time in the Meixner analysis we can not accept the "natural" analog of (3.21)

$$\sum_{n=0}^{\infty}(n+1)!(n+1)|f^{(n+1)}|^2_{\rm ext} < \infty \tag{3.22}$$

because for $\eta \neq 0$ the class of elements of (L^2) satisfying (3.22) is *narrower* than the class of elements satisfying (3.20), this statement from the proof of Lemma 3.2 follows.

Theorem 3.2. *Let $t \in [0, +\infty]$, $f \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$, and $g \in (L^2)$ be such that estimate* (3.14) *for f and estimate* (3.20) *for g are fulfilled. Then*

$$\mathbf{E}\Big[g \int_0^t f(s)\,\widehat{d}M_s\Big] \equiv \langle\!\langle \int_0^t f(s)\,\widehat{d}M_s, g\rangle\!\rangle = \int_0^t \langle\!\langle f(s), \partial_s g\rangle\!\rangle\sigma(ds) \equiv \int_0^t \mathbf{E}[f(s)\partial_s g]\sigma(ds). \tag{3.23}$$

Proof. It follows from estimates (3.14) and (3.20) that all terms in (3.23) are well-defined. Further,

$$\langle\!\langle \int_0^t f(s)\,\widehat{d}M_s, g\rangle\!\rangle = \sum_{n=0}^{\infty}(n+1)!\langle \widehat{f}^{(n)}_{[0,t)}, g^{(n+1)}\rangle_{\rm ext},$$

where $\widehat{f}^{(n)}_{[0,t)} \in \mathcal{H}_{\rm ext}^{(n+1)}$, $n \in \mathbb{Z}_+$, are the kernels from decomposition (3.15) for $\int_0^t f(s)\,\widehat{d}M_s$, $\{g^{(n+1)} \in \mathcal{H}_{\rm ext}^{(n+1)}\}_{n=0}^{\infty}$ are the kernels from decomposition (1.8) for g. On the other hand, $\partial. g = \sum_{n=0}^{\infty}(n+1)\langle P_n, g^{(n+1)}(\cdot)\rangle$, so we have

$$\langle\!\langle f(\cdot), \partial. g\rangle\!\rangle = \sum_{n=0}^{\infty}(n+1)!\langle f^{(n)}_{\cdot}, g^{(n+1)}(\cdot)\rangle_{\rm ext}$$

(here $f^{(n)}_{\cdot} \in \mathcal{H}_{\rm ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$, $n \in \mathbb{Z}_+$, are the kernels from decomposition (3.9) for f, $g^{(n+1)}(\cdot) \in \mathcal{H}_{\rm ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ are constructed in Lemma 3.2. Therefore in order to finish the proof it is sufficient to show that for each $n \in \mathbb{Z}_+$

$$\langle \widehat{f}^{(n)}_{[0,t)}, g^{(n+1)}\rangle_{\rm ext} = \int_0^t \langle f^{(n)}_u, g^{(n+1)}(u)\rangle_{\rm ext}\sigma(du). \tag{3.24}$$

Using the notation of Lemma 3.1 we have (see (1.5))

$$\langle \widehat{f}^{(n)}_{[0,t)}, g^{(n+1)}\rangle_{\rm ext} = \langle \widehat{\dot{f}}^{(n)}_{[0,t)}, \dot{g}^{(n+1)}\rangle_{\rm ext} = \sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n+1}} \frac{(n+1)!}{l_1^{s_1}\dots l_k^{s_k}s_1!\dots s_k!}$$

$$\times \int_{\mathbb{R}_+^{s_1+\dots+s_k}} \widehat{\dot{f}}^{(n)}_{[0,t)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})$$

$$\times \dot{g}^{(n+1)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})$$

$$\times \eta(u_1)^{l_1-1}\dots\eta(u_{s_1+\dots+s_k})^{l_k-1}\sigma(du_1)\dots\sigma(du_{s_1+\dots+s_k})$$

$$= \sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n+1}} \frac{n!}{l_1^{s_1}\dots l_k^{s_k}s_1!\dots s_k!}$$

$$\times [\int_{\mathbb{R}_+^{s_1+\dots+s_k}} \widetilde{\dot{f}}^{(n)}_{[0,t)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})$$

$$\times \dot{g}^{(n+1)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})$$

$$\times \eta(u_1)^{l_1-1}\dots\eta(u_{s_1+\dots+s_k})^{l_k-1}\sigma(du_1)\dots\sigma(du_{s_1+\dots+s_k})$$

$$+ \int_{\mathbb{R}_+^{s_1+\dots+s_k}} \widetilde{\dot{f}}^{(n)}_{[0,t)}(u_{s_1+\dots+s_k},\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k-1})$$

$$\times \dot{g}^{(n+1)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})$$

$$\times \eta(u_1)^{l_1-1}\dots\eta(u_{s_1+\dots+s_k})^{l_k-1}\sigma(du_1)\dots\sigma(du_{s_1+\dots+s_k}) + \dots$$

$$+ \int_{\mathbb{R}_+^{s_1+\dots+s_k}} \widetilde{\dot{f}}^{(n)}_{[0,t)}(\underbrace{u_1,\dots,u_1}_{l_1-1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k},u_1)$$

$$\times \dot{g}^{(n+1)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})$$

$$\times \eta(u_1)^{l_1-1}\dots\eta(u_{s_1+\dots+s_k})^{l_k-1}\sigma(du_1)\dots\sigma(du_{s_1+\dots+s_k})]$$

$$= \sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_{k-1}>1,\\ l_1s_1+\dots+l_{k-1}s_{k-1}+(s_k-1)=n}} \frac{n!}{l_1^{s_1}\dots l_{k-1}^{s_{k-1}}s_1!\dots s_{k-1}!(s_k-1)!}$$

$$\times \int_{\mathbb{R}_+^{s_1+\dots+s_k}} \widetilde{\dot{f}}^{(n)}_{[0,t)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_{k-1}},\dots,u_{s_1+\dots+s_{k-1}}}_{l_{k-1}},$$

$$u_{s_1+\dots+s_{k-1}+1},\dots,u_{s_1+\dots+s_k-1},u)$$

$$\times \dot{g}^{(n+1)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_{k-1}},\dots,u_{s_1+\dots+s_{k-1}}}_{l_{k-1}},$$

$$u_{s_1+\dots+s_{k-1}+1},\dots,u_{s_1+\dots+s_k-1},u)$$

$$\times \eta(u_1)^{l_1-1}\dots\eta(u_{s_1+\dots+s_{k-1}})^{l_{k-1}-1}\sigma(du_1)\dots\sigma(du_{s_1+\dots+s_k-1})\sigma(du)$$

$$= \int_0^t \langle \dot{f}^{(n)}_u, \dot{g}^{(n+1)}(u)\rangle_{\rm ext}\sigma(du) = \int_0^t \langle f^{(n)}_u, g^{(n+1)}(u)\rangle_{\rm ext}\sigma(du)$$

(here a nonatomicity of σ, a symmetry of $\dot{\widetilde{f}}^{(n)}_{[0,t)}$ by first n arguments and a symmetry of $\dot{g}^{(n+1)}$ by $n+1$ arguments were used). □

Remark 3.8. If $g \in (L^2)$ satisfies the estimate

$$\sum_{n=0}^{\infty}(n+1)!(n+1)|g^{(n+1)}(\cdot)1_{[0,t)}(\cdot)|^2_{\mathcal{H}^{(n)}_{\mathrm{ext}}\otimes\mathcal{H}_{\mathbb{C}}} < \infty$$

then one can consider defined by (3.18) $\partial_{\cdot} g$ as an element of $(L^2) \otimes L^2([0,t),\sigma)_{\mathbb{C}}$. It is obvious that the result of Theorem 3.2 holds true in this case.

Since $\mathrm{dom}\,\partial_{\cdot}$ (here and below by $\mathrm{dom}\,A$ denote the domain of an operator A) is a dense set in (L^2) (see (3.20) and (3.19)), the adjoint to $\partial_{\cdot}$ operator $\partial^*_{\cdot} : (L^2)\otimes\mathcal{H}_{\mathbb{C}} \to (L^2)$ is well-defined. Since $\forall f \in \mathrm{dom}\,\partial^*_{\cdot}$, $\forall g \in \mathrm{dom}\,\partial_{\cdot}$

$$(f,\partial_{\cdot} g)_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} \equiv \int_{\mathbb{R}_+} \langle\langle f(s), \partial_s g\rangle\rangle \sigma(ds) = \langle\langle \partial^*_{\cdot} f, g\rangle\rangle, \tag{3.25}$$

it is natural to write *formally*

$$\int_{\mathbb{R}_+} \langle\langle f(s), \partial_s g\rangle\rangle \sigma(ds) = \int_{\mathbb{R}_+} \langle\langle \partial_s^{\dagger} f(s), g\rangle\rangle \sigma(ds) = \langle\langle \int_{\mathbb{R}_+} \partial_s^{\dagger} f(s)\sigma(ds), g\rangle\rangle,$$

where we accepted the notation $\int_{\mathbb{R}_+} \partial_s^{\dagger} f(s)\sigma(ds) := \partial^*_{\cdot} f$ (cf. $\partial_x^{\dagger}$ in [31]). Also we denote $\int_0^t \partial_s^{\dagger} f(s)\sigma(ds) := \int_{\mathbb{R}_+} \partial_s^{\dagger} f(s)1_{[0,t)}(s)\sigma(ds) \equiv \partial^*_{\cdot}(f(\cdot)1_{[0,t)}(\cdot))$.

Remark 3.9. *Formally* one can understand $\partial_s^{\dagger}$ ($s \in \mathbb{R}_+$) as the adjoint to ∂_s with respect to the scalar product in (L^2) operator. Strictly speaking, if we consider $\partial_{\cdot}$ on (L^2) then such a "definition" of $\partial_s^{\dagger}$ is incorrect because for $f^{(n+1)}(\cdot) \in \mathcal{H}^{(n)}_{\mathrm{ext}} \otimes \mathcal{H}_{\mathbb{C}}$ $f^{(n+1)}(s)$ is not determined and therefore ∂_s is not determined. But for the "Hida derivative" $\partial_{\cdot}$ on $(\mathcal{H}_\tau)_q$ ($\tau \in T$, $q \in \mathbb{N}_{q_0}$) ∂_s is a linear continuous operator in $(\mathcal{H}_\tau)_q$ for each $s \in \mathbb{R}_+$, therefore $\forall s \in \mathbb{R}_+$ $\partial_s^{\dagger}$ is well-defined as a linear continuous operator in $(\mathcal{H}_{-\tau})_{-q}$ (see also Theorem 3.5 below).

From Theorem 3.2 we obtain the following

Corollary. *Let $t \in [0,+\infty]$, and $f \in (L^2)\otimes\mathcal{H}_{\mathbb{C}}$ satisfy estimate* (3.14). *Then*

$$\int_0^t f(s)\,\widehat{d}M_s = \partial^*_{\cdot}(f(\cdot)1_{[0,t)}(\cdot)) = \int_0^t \partial_s^{\dagger} f(s)\sigma(ds) \tag{3.26}$$

(in particular, this means that $\int_0^t \circ(s)\,\widehat{d}M_s$ is a closed operator). This equality can be accepted as a definition of the extended stochastic integral.

Proof. We have to prove that $\mathrm{dom}\int_0^t \circ(s)\,\widehat{d}M_s = \mathrm{dom}\,\partial^*_{\cdot}(\circ(\cdot)1_{[0,t)}(\cdot))$ and (3.26) is valid. In accordance with the definitions

$\{f \in \mathrm{dom}\,\partial^*_{\cdot}(\circ(\cdot)1_{[0,t)}(\cdot))\} \Leftrightarrow \{f1_{[0,t)} \in \mathrm{dom}\,\partial^*_{\cdot}\}$

$\Leftrightarrow \{(L^2) \supset \mathrm{dom}\,\partial_{\cdot} \ni g \mapsto (f(\cdot)1_{[0,t)}(\cdot), \partial_{\cdot} g)_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}$ is a linear continuous functional$\}$.

By Riesz's theorem the last is possible if and only if $(f(\cdot)1_{[0,t)}(\cdot), \partial_{\cdot} g)_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}$ can be presented in the form $\langle\langle H, g\rangle\rangle$ with some $H \in (L^2)$. Using (3.18) and (3.24), for $f \in (L^2)\otimes\mathcal{H}_{\mathbb{C}}$ and $g \in \mathrm{dom}\,\partial_{\cdot}$ we obtain

$$\begin{aligned}(f(\cdot)1_{[0,t)}(\cdot), \partial_{\cdot} g)_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} &= \sum_{n=0}^{\infty}(n+1)!\int_0^t \langle f_s^{(n)}, g^{(n+1)}(s)\rangle_{\mathrm{ext}}\sigma(ds) \\ &= \sum_{n=0}^{\infty}(n+1)!\langle \widehat{f}^{(n)}_{[0,t)}, g^{(n+1)}\rangle_{\mathrm{ext}},\end{aligned}$$

where $f^{(n)}_{\cdot}, g^{(n+1)}(\cdot) \in \mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ are the kernels from decompositions (3.9) and (3.18) for f and $\partial_{\cdot} g$ correspondingly, $\widehat{f}^{(n)}_{[0,t)} \in \mathcal{H}_{\text{ext}}^{(n+1)}$ are constructed in Lemma 3.1 starting from $f^{(n)}$. By Theorem 1.3 the last series can be presented in the form $\langle\!\langle H, g\rangle\!\rangle$ with $H := \sum_{n=0}^{\infty} \langle P_{n+1}, \widehat{f}^{(n)}_{[0,t)}\rangle$, if, of course, this $H \in (L^2)$. Therefore $f \in \operatorname{dom} \partial^*_{\cdot}(\circ(\cdot)1_{[0,t)}(\cdot))$ if and only if estimate (3.14) is fulfilled. It means that $\operatorname{dom} \partial^*_{\cdot}(\circ(\cdot)1_{[0,t)}(\cdot)) = \operatorname{dom} \int_0^t \circ(s)\, \widehat{d}M_s$. Further, if $f \in \operatorname{dom} \int_0^t \circ(s)\, \widehat{d}M_s$ then by (3.15) $H = \int_0^t f(s)\, \widehat{d}M_s$ and by (3.25) for $g \in \operatorname{dom} \partial_{\cdot}$ $(f(\cdot)1_{[0,t)}(\cdot), \partial_{\cdot} g)_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} = \langle\!\langle \partial^*_{\cdot}(f(\cdot)1_{[0,t)}(\cdot)), g\rangle\!\rangle$. Thus $\langle\!\langle \partial^*_{\cdot}(f(\cdot)1_{[0,t)}(\cdot)), g\rangle\!\rangle = \langle\!\langle \int_0^t f(s)\, \widehat{d}M_s, g\rangle\!\rangle$ and (3.26) is valid because $\operatorname{dom} \partial_{\cdot}$ is a dense set in (L^2). □

Remark 3.10. The result of Theorem 3.2 accepts the following natural generalization. Let $t \in [0, +\infty]$, $f \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$, $g \in (L^2)$. Without additional restrictions like estimates (3.14), (3.20) $\int_0^t f(s)\, \widehat{d}M_s$ and $\partial_{\cdot} g$ are well-defined as elements of the spaces of regular generalized functions $(L^2)^{-1}_{-q}$ and $(L^2)^{-1}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ ($q \in \mathbb{N}$) correspondingly, this fact will be proved for $\int_0^t f(s)\, \widehat{d}M_s$ later and for $\partial_{\cdot} g$ in forthcoming papers. Therefore the expectations and pairings in (3.23) can be undetermined. But if either estimate (3.22) (not (3.20)!) for g is fulfilled *or* estimate (3.14) for f is fulfilled then at least one term in (3.23) is well-defined and therefore (3.23) holds true in a generalized sense. In fact, it is sufficient to prove that $\sum_{n=0}^{\infty} (n+1)! \langle \widehat{f}^{(n)}_{[0,t)}, g^{(n+1)}\rangle_{\text{ext}} < \infty$. If (3.22) for g is fulfilled then using (3.13) we can estimate as follows:

$$
\begin{aligned}
\Big| \sum_{n=0}^{\infty} (n+1)! \langle \widehat{f}^{(n)}_{[0,t)}, g^{(n+1)}\rangle_{\text{ext}} \Big| &\le \sum_{n=0}^{\infty} (n+1)! |\widehat{f}^{(n)}_{[0,t)}|_{\text{ext}} |g^{(n+1)}|_{\text{ext}} \\
&\le \sum_{n=0}^{\infty} [\sqrt{n!} |f^{(n)}_{\cdot}|_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}}][\sqrt{n!}(n+1)|g^{(n+1)}|_{\text{ext}}] \\
&\le \sqrt{\sum_{n=0}^{\infty} n! |f^{(n)}_{\cdot}|^2_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}}} \cdot \sqrt{\sum_{n=0}^{\infty} (n+1)!(n+1) |g^{(n+1)}|^2_{\text{ext}}} \\
&= \|f\|_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} \cdot \sqrt{\sum_{n=0}^{\infty} (n+1)!(n+1) |g^{(n+1)}|^2_{\text{ext}}} < \infty.
\end{aligned}
$$

If (3.14) for f is valid then

$$
\begin{aligned}
\Big| \sum_{n=0}^{\infty} (n+1)! \langle \widehat{f}^{(n)}_{[0,t)}, g^{(n+1)}\rangle_{\text{ext}} \Big| &\le \sum_{n=0}^{\infty} (n+1)! |\widehat{f}^{(n)}_{[0,t)}|_{\text{ext}} |g^{(n+1)}|_{\text{ext}} \\
&= \sum_{n=0}^{\infty} [\sqrt{(n+1)!} |\widehat{f}^{(n)}_{[0,t)}|_{\text{ext}}][\sqrt{(n+1)!} |g^{(n+1)}|_{\text{ext}}] \\
&\le \sqrt{\sum_{n=0}^{\infty} (n+1)! |\widehat{f}^{(n)}_{[0,t)}|^2_{\text{ext}}} \cdot \sqrt{\sum_{n=0}^{\infty} (n+1)! |g^{(n+1)}|^2_{\text{ext}}} \\
&\le \sqrt{\sum_{n=0}^{\infty} (n+1)! |\widehat{f}^{(n)}_{[0,t)}|^2_{\text{ext}}} \cdot \|g\|_{(L^2)} < \infty.
\end{aligned}
$$

Now let us pass to constructing of the extended stochastic integral on the spaces of *regular* generalized functions. Let $F \in (L^2)^{-1}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$, $q \in \mathbb{N}$. Then (see (2.22))

$$
F(\cdot) = \sum_{n=0}^{\infty} \langle P_n, F^{(n)}_{\cdot}\rangle, \quad F^{(n)}_{\cdot} \in \mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}. \tag{3.27}
$$

Definition 3.3. Let $F \in (L^2)^{-1}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$, $q \in \mathbb{N}$. For each $t \in [0, +\infty]$ we define the extended stochastic integral $\int_0^t F(s)\, \widetilde{d}M_s \in (L^2)^{-1}_{-q}$ by setting

$$\int_0^t F(s)\, \widetilde{d}M_s := \sum_{n=0}^{\infty} \langle P_{n+1}, \widehat{F}^{(n)}_{[0,t)} \rangle, \tag{3.28}$$

where the kernels $\widehat{F}^{(n)}_{[0,t)} \in \mathcal{H}^{(n+1)}_{\mathrm{ext}}$, $n \in \mathbb{Z}_+$, are constructed in Lemma 3.1 starting from the kernels $F^{(n)}_{\cdot}$ from decomposition (3.27) for F.

Since (see (3.13))

$$\begin{aligned}\Big\| \int_0^t F(s)\, \widetilde{d}M_s \Big\|^2_{-q} &= \sum_{n=0}^{\infty} 2^{-q(n+1)} |\widehat{F}^{(n)}_{[0,t)}|^2_{\mathrm{ext}} \le 2^{-q} \sum_{n=0}^{\infty} 2^{-qn} |F^{(n)}_{\cdot}|^2_{\mathcal{H}^{(n)}_{\mathrm{ext}} \otimes \mathcal{H}_{\mathbb{C}}} \\ &= 2^{-q} \|F\|^2_{(L^2)^{-1}_{-q} \otimes \mathcal{H}_{\mathbb{C}}} < \infty,\end{aligned}$$

this definition is correct, and, moreover, as distinguished from the integration on $(L^2) \otimes \mathcal{H}_{\mathbb{C}}$, $\int_0^t \circ(s)\, \widetilde{d}M_s$ is a linear *continuous* operator acting from $(L^2)^{-1}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ to $(L^2)^{-1}_{-q}$. □

Comparing (3.28) and (3.15) we obtain

Theorem 3.3. *The restriction of* $\int_0^t \circ(s)\, \widetilde{d}M_s$ *on* dom $\int_0^t \circ(s)\, \widehat{d}M_s \subset (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ *(here* $t \in [0, +\infty]$*) coincides with* $\int_0^t \circ(s)\, \widehat{d}M_s$.

It follows from this statement that it is natural to accept for integral (3.28) the notation $\int_0^t F(s)\, \widehat{d}M_s$.

Now let us obtain the analog of Theorem 3.2 and its Corollary. First we note that if $f \in (L^2)^1_q$ ($q \in \mathbb{N}$) then f satisfies estimate (3.20) because $\forall n \in \mathbb{Z}_+$ $(n+1)!(n+1)|f^{(n+1)}(\cdot)|^2_{\mathcal{H}^{(n)}_{\mathrm{ext}} \otimes \mathcal{H}_{\mathbb{C}}} \le ((n+1)!)^2 2^{q(n+1)} |f^{(n+1)}|^2_{\mathrm{ext}}$, see (3.19). Therefore the generalized Hida derivative is well-defined for each $f \in (L^2)^1_q$. Moreover, since (see (3.18), (3.19))

$$\begin{aligned}\|\partial_{\cdot} f\|^2_{(L^2)^1_q \otimes \mathcal{H}_{\mathbb{C}}} &= \sum_{n=0}^{\infty} (n!)^2 2^{qn} (n+1)^2 |f^{(n+1)}(\cdot)|^2_{\mathcal{H}^{(n)}_{\mathrm{ext}} \otimes \mathcal{H}_{\mathbb{C}}} \\ &\le 2^{-q} \sum_{n=0}^{\infty} ((n+1)!)^2 2^{q(n+1)} |f^{(n+1)}|^2_{\mathrm{ext}} \le 2^{-q} \|f\|^2_q < \infty,\end{aligned}$$

the restriction of $\partial_{\cdot}$ on $(L^2)^1_q$ is a linear *continuous* operator acting from $(L^2)^1_q$ to $(L^2)^1_q \otimes \mathcal{H}_{\mathbb{C}}$. Therefore the defined by (3.25) adjoint to $\partial_{\cdot}$ operator $\partial^*_{\cdot}$ is a linear continuous one acting from $(L^2)^{-1}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ to $(L^2)^{-1}_{-q}$ (here $q \in \mathbb{N}$).

Theorem 3.4. *Let* $t \in [0, +\infty]$, $F \in (L^2)^{-1}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$, $f \in (L^2)^1_q$ *(*$q \in \mathbb{N}$*). Then*

$$\mathbf{E}\Big[f \int_0^t F(s)\, \widehat{d}M_s\Big] \equiv \langle\!\langle \int_0^t F(s)\, \widehat{d}M_s, f \rangle\!\rangle = \int_0^t \langle\!\langle F(s), \partial_s f \rangle\!\rangle \sigma(ds) \equiv \int_0^t \mathbf{E}[F(s) \partial_s f] \sigma(ds) \tag{3.29}$$

and

$$\int_0^t F(s)\, \widehat{d}M_s = \partial^*_{\cdot}(F(\cdot) 1_{[0,t)}(\cdot)) \equiv \int_0^t \partial^{\dagger}_s F(s) \sigma(ds). \tag{3.30}$$

Moreover, (3.30) *can be accepted as a definition of the extended stochastic integral on* $(L^2)^{-1}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$.

Proof. Since now the extended stochastic integral and the generalized Hida derivative are linear *continuous* operators, dom $\partial_{\cdot} = (L^2)^1_q$ and dom $\int_0^t \circ(s)\, \widehat{d}M_s = \mathrm{dom}\, \partial^*_{\cdot}(\circ(\cdot) 1_{[0,t)}(\cdot))$ $= (L^2)^{-1}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$. The equalities (3.29) and (3.30) can be obtained by direct calculation as in the proofs of Theorem 3.2 and its Corollary. □

Remark 3.11. Note that for $F \in (L^2)^{-1} \otimes \mathcal{H}_{\mathbb{C}}$ the defined by (3.28) integral $\int_0^t F(s)\,\widehat{d}M_s$ is well-defined as an element of $(L^2)^{-1}$ and, moreover, $\int_0^t \circ(s)\,\widehat{d}M_s$ is a linear continuous operator acting from $(L^2)^{-1} \otimes \mathcal{H}_{\mathbb{C}}$ to $(L^2)^{-1}$. Of course, the results of Theorem 3.3 hold true for $F \in (L^2)^{-1} \otimes \mathcal{H}_{\mathbb{C}}$ and $f \in (L^2)^1$.

Finally, we consider the extended stochastic integral on the spaces of *nonregular* generalized functions. In the paper [20] such an integral "on the language of the so-called Q-system in the biorthogonal analysis" was constructed. Now we recall the definition from [20] and prove that the restriction on $(L^2)^{-1}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ ($q \in \mathbb{N}$) of the integral "from [20]" coincides with the integral that is given by Definition 3.3.

Let $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ ($\tau \in T$, $q \in \mathbb{N}_{q_0}$). It follows from Theorem 2.1 that F can be presented in the form (see (2.14))

$$F(\cdot) = \sum_{n=0}^{\infty} Q_n(F_{\cdot}^{(n)}), \quad F_{\cdot}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\mathbb{C}}. \tag{3.31}$$

Definition 3.4. (cf. [20]) Let $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}}$, $\tau \in T$, $q \in \mathbb{N}_{q_0}$. For each $t \in [0, +\infty]$ we define the extended stochastic integral $\int_0^t F(s)\,\overline{d}M_s \in (\mathcal{H}_{-\tau})_{-q}$ by setting

$$\int_0^t F(s)\,\overline{d}M_s := \sum_{n=0}^{\infty} Q_{n+1}(\widehat{F}_{[0,t)}^{(n)}), \tag{3.32}$$

where the kernels $\widehat{F}_{[0,t)}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n+1}$, $n \in \mathbb{Z}_+$, are the symmetrizations (the projections on $\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n+1}$) of $F_{\cdot}^{(n)} 1_{[0,t)}(\cdot)$, $\{F_{\cdot}^{(n)}\}_{n=0}^{\infty}$ are from decomposition (3.31) for F.

Since

$$\begin{aligned} \Big\| \int_0^t F(s)\,\overline{d}M_s \Big\|^2_{-\tau,-q} &= \sum_{n=0}^{\infty} 2^{-q(n+1)} |\widehat{F}_{[0,t)}^{(n)}|^2_{-\tau} \\ &\le 2^{-q} c \sum_{n=0}^{\infty} 2^{-qn} |F_{\cdot}^{(n)}|^2_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\mathbb{C}}} = 2^{-q} c \|F\|^2_{(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}}} < \infty \end{aligned}$$

(here $c = c(\tau) > 0$ is a constant such that $|\cdot|_{-\tau} \le c|\cdot|_0$), this definition is correct and, moreover, $\int_0^t \circ(s)\overline{d}M_s$ is a linear *continuous* operator acting from $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ to $(\mathcal{H}_{-\tau})_{-q}$. □

Remark 3.12. Note that for $t = +\infty$ we can define the extended stochastic integral for $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ by formula (3.32). As is easily seen, this integral will be a linear continuous operator acting from $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ to $(\mathcal{H}_{-\tau})_{-q}$. The restriction $F \in (L^2)^{-1}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ is connected only with the necessity to multiply the integrand F by the indicator: the multiplication of an element of $\mathcal{H}_{-\tau,\mathbb{C}}$ by an element of $\mathcal{H}_{\mathbb{C}}$ is undetermined.

In according with the results of Theorem 2.1 one can rewrite formulas (3.31), (3.32) in the form (see (2.11), (2.17), (2.5); Pr is the symmetrization operator)

$$F(\cdot) = \sum_{n=0}^{\infty} \langle P_n, F_{\text{ext},\cdot}^{(n)} \rangle, \quad F_{\text{ext},\cdot}^{(n)} = U_n^{-1} F_{\cdot}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}, \tag{3.33}$$

$$\begin{aligned} &\int_0^t F(s)\,\overline{d}M_s = \sum_{n=0}^{\infty} \langle P_{n+1}, \widehat{F}_{\text{ext},[0,t)}^{(n)} \rangle, \\ &\widehat{F}_{\text{ext},[0,t)}^{(n)} = U_{n+1}^{-1} \widehat{F}_{[0,t)}^{(n)} = U_{n+1}^{-1} [Pr(U_n(F_{\text{ext},\cdot}^{(n)}) 1_{[0,t)}(\cdot))] \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n+1)}, \end{aligned} \tag{3.34}$$

this form is more natural for our considerations.

Let us study properties of integral (3.32). First we note that the defined above (see (3.18)) "Hida derivative" $\partial_\cdot : (\mathcal{H}_\tau)_q \to (\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$ ($\tau \in T$, $q \in \mathbb{N}_{q_0}$) can be considered as a linear *continuous* operator acting from $(\mathcal{H}_\tau)_q$ to $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\mathbb{C}$ because $\mathcal{H}_{\tau,\mathbb{C}}$ is *continuously* embedded in $\mathcal{H}_\mathbb{C}$. Therefore one can consider the adjoint to $\partial_\cdot$ operator $\partial_\cdot^* : (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_\mathbb{C} \to (\mathcal{H}_{-\tau})_{-q}$ (see (3.25)) that is a linear *continuous* one.

Theorem 3.5. *Let $t \in [0, +\infty]$, $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_\mathbb{C}$, $f \in (\mathcal{H}_\tau)_q$ ($\tau \in T$, $q \in \mathbb{N}_{q_0}$). Then*
(3.35)

$$\mathbf{E}\Big[f \int_0^t F(s)\, \overline{d}M_s\Big] \equiv \langle\!\langle \int_0^t F(s)\, \overline{d}M_s, f \rangle\!\rangle = \int_0^t \langle\!\langle F(s), \partial_s f \rangle\!\rangle \sigma(ds) \equiv \int_0^t \mathbf{E}[F(s)\partial_s f]\sigma(ds)$$

and

$$\int_0^t F(s)\, \overline{d}M_s = \partial_\cdot^*(F(\cdot)1_{[0,t)}(\cdot)) = \int_0^t \partial_s^\dagger F(s)\sigma(ds). \tag{3.36}$$

Formula (3.36) *can be accepted as a definition of the extended stochastic integral on $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_\mathbb{C}$.*

Proof. By analogy with the proof of Theorem 3.2, using (2.16) one can show that in order to prove (3.35) it is sufficient to establish that for each $n \in \mathbb{Z}_+$

$$\langle \widehat{F_{[0,t)}^{(n)}}, f^{(n+1)} \rangle = \int_0^t \langle F_s^{(n)}, f^{(n+1)}(s) \rangle \sigma(ds), \tag{3.37}$$

where $\widehat{F_{[0,t)}^{(n)}} \in \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n+1}$, $f^{(n+1)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n+1}$, $F_\cdot^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_\mathbb{C}$ and $f^{(n+1)}(\cdot) \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\tau,\mathbb{C}}$, $n \in \mathbb{Z}_+$, are the kernels from decompositions (3.32), (1.8), (3.31) and (3.18) correspondingly. But (3.37) is obviously true. Further, it is easy to see that $\mathrm{dom} \int_0^t \circ(s)\, \overline{d}M_s = \mathrm{dom}\, \partial_\cdot^*(F(\cdot)1_{[0,t)}(\cdot)) = (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_\mathbb{C}$. Equality (3.36) follows from (3.35) by definition of $\partial_\cdot^*$ (see (3.25)). □

Remark 3.13. For $t = +\infty$ Theorem 3.5 can be easily reformulated for the extended stochastic integral that is defined on $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ in Remark 3.12 (in this case we have to consider $\partial_\cdot^* : (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}} \to (\mathcal{H}_{-\tau})_{-q}$, this operator is a linear and *continuous* one by construction).

Comparing (3.36) and (3.30) and taking into account Remark 3.6 we obtain the following

Corollary. *If $F \in (L^2)_{-q}^{-1} \widehat{\otimes} \mathcal{H}_\mathbb{C}$ and simultaneously $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_\mathbb{C}$ then $\forall t \in [0, +\infty]$ $\int_0^t F(s)\, \overline{d}M_s, f = \int_0^t F(s)\, \widehat{d}M_s$. In particular, the restriction of $\int_0^t \circ(s)\, \overline{d}M_s$ on $(L^2) \otimes \mathcal{H}_\mathbb{C}$ coincides with $\int_0^t \circ(s)\, \widehat{d}M_s$.*

It follows from this statement that it is natural to accept for integral (3.32) the notation $\int_0^t F(s)\, \widehat{d}M_s$.

Remark 3.14. The result of the Corollary from Theorem 3.5 can be obtained by direct calculation, by analogy with the proof of Theorem 2.2 in [21]. Namely, one has to prove that

$$\langle\!\langle \int_0^t F(s)\, \overline{d}M_s, f \rangle\!\rangle = \langle\!\langle \int_0^t F(s)\, \widehat{d}M_s, f \rangle\!\rangle \quad \forall F \in (L^2) \otimes \mathcal{H}_\mathbb{C}, \quad \forall f \in (\mathcal{H}_\tau)_q. \tag{3.38}$$

Using (3.34), (1.8), (3.28) and (2.13) one can show that (3.38) is true if for each $n \in \mathbb{Z}_+$

$$\langle \widehat{F_{\mathrm{ext},[0,t)}^{(n)}}, f^{(n+1)} \rangle_{\mathrm{ext}} = \langle \widehat{F}_{[0,t)}^{(n)}, f^{(n+1)} \rangle_{\mathrm{ext}} \quad \forall f^{(n+1)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n},$$

where $\widehat{F}_{[0,t)}^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n+1)}$ are constructed in Lemma 3.1 starting from the kernels $F_{\mathrm{ext},\cdot}^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)} \otimes \mathcal{H}_\mathbb{C}$ from decomposition (3.33) for (arbitrary integrable) $F \in (L^2) \otimes \mathcal{H}_\mathbb{C} \subset$

$(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{\mathbb{C}}$, $\widehat{F^{(n)}_{\mathrm{ext},[0,t)}}\in\mathcal{H}^{(n+1)}_{-\tau,\mathbb{C}}$, $n\in\mathbb{Z}_+$, are the kernels from decomposition (3.34). Using (2.5), (3.37) and (3.24) we obtain

$$\begin{aligned}\langle\widehat{F^{(n)}_{\mathrm{ext},[0,t)}},f^{(n+1)}\rangle_{\mathrm{ext}}&=\langle\widehat{F^{(n)}_{[0,t)}},f^{(n+1)}\rangle=\int_0^t\langle F^{(n)}_s,f^{(n+1)}(s)\rangle\sigma(ds)\\&=\int_0^t\langle F^{(n)}_{\mathrm{ext},s},f^{(n+1)}(s)\rangle_{\mathrm{ext}}\sigma(ds)=\langle\widehat{F}^{(n)}_{[0,t)},f^{(n+1)}\rangle_{\mathrm{ext}}.\end{aligned}$$

Remark 3.15. As is easily seen, if we replace $(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{\mathbb{C}}$ in Definition 3.4 by $(\mathcal{H}_{-\tau})\otimes\mathcal{H}_{\mathbb{C}}$ (or by $(\mathcal{D}')'\otimes\mathcal{H}_{\mathbb{C}}$) then all described here connected with the extended stochastic integral results hold true (some necessary modifications are obvious).

4. Elements of the Wick calculus and stochastic equations

In this section we introduce a Wick product and Wick versions of holomorphic functions on the spaces $(\mathcal{D}')'$, $(\mathcal{H}_{-\tau})$ and $(L^2)^{-1}$. Then we study the interconnection of these objects with the extended stochastic integral and consider some stochastic equations with Wick type nonlinearity.

Definition 4.1. For $F\in(\mathcal{D}')'$ we define an integral S-transform $(SF)(\lambda)$ (λ belongs to some depending on F neighborhood of zero in $\mathcal{D}_{\mathbb{C}}$) by setting (see (1.4))

$$(SF)(\lambda):=\langle\!\langle F,:\exp(\cdot;\lambda):\rangle\!\rangle.$$

The S-transform is well-defined because for each $F\in(\mathcal{D}')'$ there exist $\tau\in T$ and $q\in\mathbb{N}_{q_0}$ such that $F\in(\mathcal{H}_{-\tau})_{-q}$; and for $\lambda\in\mathcal{D}_{\mathbb{C}}$ such that $2^q|\lambda|^2_\tau<1$ we have $:\exp(\cdot;\lambda):\in(\mathcal{H}_\tau)_q$. □

Remark 4.1. It is easy to see that

$$(SF)(\lambda)=\sum_{n=0}^{\infty}\langle F^{(n)},\lambda^{\otimes n}\rangle=\sum_{n=0}^{\infty}\langle F^{(n)}_{\mathrm{ext}},\lambda^{\otimes n}\rangle_{\mathrm{ext}},\tag{4.1}$$

where $F^{(n)}\in\mathcal{H}^{\widehat{\otimes}n}_{-\tau,\mathbb{C}}$, $F^{(n)}_{\mathrm{ext}}=U_n^{-1}F^{(n)}\in\mathcal{H}^{(n)}_{-\tau,\mathbb{C}}$ (see (2.5)) are the kernels correspondingly from decompositions (2.14), (2.11) for F. In particular, $(SF)(0)=F^{(0)}=F^{(0)}_{\mathrm{ext}}$, $S1\equiv 1$.

Theorem 4.1. *(*[18, 24]*) The S-transform is a topological isomorphism between the space $(\mathcal{D}')'$ and the algebra Hol_0 of germs of holomorphic at zero functions on $\mathcal{D}_{\mathbb{C}}$.*

Definition 4.2. For $F,G\in(\mathcal{D}')'$ and a holomorphic at $(SF)(0)$ function $h:\mathbb{C}\to\mathbb{C}$ we define the *Wick product* $F\Diamond G\in(\mathcal{D}')'$ and the *Wick version of* h $h^{\Diamond}(F)\in(\mathcal{D}')'$ by setting

$$F\Diamond G:=S^{-1}(SF\cdot SG),\quad h^{\Diamond}(F):=S^{-1}h(SF).$$

The correctness of this definition from Theorem 4.1 follows. □

Remark 4.2. It is easy to see that the Wick multiplication $\Diamond$ is commutative, associative and distributive (over the field $\mathbb{C}$). Further, if h from Definition 4.2 is presented in the form

$$h(u)=\sum_{n=0}^{\infty}h_n(u-(SF)(0))^n\tag{4.2}$$

then $h^{\Diamond}(F)=\sum_{n=0}^{\infty}h_n(F-(SF)(0))^{\Diamond n}$, where $F^{\Diamond n}:=\underbrace{F\Diamond\ldots\Diamond F}_{\text{n times}}$.

Let us write out the "coordinate form" of $F\Diamond G$ and $h^{\Diamond}(F)$ (this form is necessary for calculations). Using (4.1) it is easy to show that

$$F\Diamond G=\sum_{n=0}^{\infty}Q_n\Big(\sum_{k=0}^{n}F^{(k)}\widehat{\otimes}G^{(n-k)}\Big)\equiv\sum_{n=0}^{\infty}\big\langle P_n,\sum_{k=0}^{n}F_{\rm ext}^{(k)}\diamond G_{\rm ext}^{(n-k)}\big\rangle, \tag{4.3}$$

where $F^{(k)},G^{(k)}\in\mathcal{D}_{\mathbb{C}}^{'\widehat{\otimes}k}$, $F_{\rm ext}^{(k)}=U_k^{-1}F^{(k)},G_{\rm ext}^{(k)}=U_k^{-1}G^{(k)}\in\mathcal{D}_{\mathbb{C}}^{'(k)}$ (see (2.5)) are the kernels from decompositions (2.14), (2.11) for F and G correspondingly; and for $F_{\rm ext}^{(k)}\in\mathcal{D}_{\mathbb{C}}^{'(k)}$, $G_{\rm ext}^{(m)}\in\mathcal{D}_{\mathbb{C}}^{'(m)}$

$$F_{\rm ext}^{(k)}\diamond G_{\rm ext}^{(m)}:=U_{k+m}^{-1}(U_kF_{\rm ext}^{(k)}\widehat{\otimes}U_mG_{\rm ext}^{(m)})\in\mathcal{D}_{\mathbb{C}}^{'(k+m)} \tag{4.4}$$

(it is obvious that the "multiplication" $\diamond$ is commutative, associative and distributive (over the field $\mathbb{C}$)). Further, substituting (4.1) in (4.2) and applying S^{-1} one can obtain

$$\begin{aligned}h^{\Diamond}(F)&=h_0+\sum_{n=1}^{\infty}Q_n\Big(\sum_{k=1}^{n}h_k\sum_{m_1,\dots,m_k\in\mathbb{N}:m_1+\dots+m_k=n}F^{(m_1)}\widehat{\otimes}\dots\widehat{\otimes}F^{(m_k)}\Big)\\&\equiv h_0+\sum_{n=1}^{\infty}\big\langle P_n,\sum_{k=1}^{n}h_k\sum_{m_1,\dots,m_k\in\mathbb{N}:m_1+\dots+m_k=n}F_{\rm ext}^{(m_1)}\diamond\dots\diamond F_{\rm ext}^{(m_k)}\big\rangle,\end{aligned} \tag{4.5}$$

where $F^{(m)}\in\mathcal{D}_{\mathbb{C}}^{'\widehat{\otimes}m}$, $F_{\rm ext}^{(m)}=U_m^{-1}F^{(m)}\in\mathcal{D}_{\mathbb{C}}^{'(m)}$ are the kernels from decompositions correspondingly (2.14), (2.11) for F; $\{h_k\in\mathbb{C}\}_{k=0}^{\infty}$ are the coefficients from decomposition (4.2) for h. (The interested reader can find the detailed proof of (4.3) and (4.5) in [16] (see also formula (4.11) below): in this place there is no any difference in a formalism between the Meixner and Gamma cases).

Remark 4.3. It follows from (4.3) that, in particular,

$$Q_n(F^{(n)})\Diamond Q_m(G^{(m)})=Q_{n+m}(F^{(n)}\widehat{\otimes}G^{(m)})$$

or, equivalently,

$$\langle P_n,F_{\rm ext}^{(n)}\rangle\Diamond\langle P_m,G_{\rm ext}^{(m)}\rangle=\langle P_{n+m},F_{\rm ext}^{(n)}\diamond G_{\rm ext}^{(m)}\rangle.$$

Each of these formulas can be used in order to *define* the Wick product (and then the Wick version of a holomorphic function as a series) without the S-transform. Formulas (4.3) and (4.5) also can be used as definitions.

Remark 4.4. In the classical Gaussian analysis the (Gaussian) Wick exponential coincides with $\exp^{\diamond}(\langle H_1,\lambda\rangle)$ (here $H_1(x)=x$ is the kernel of the generalized Hermite polynomial, see, e.g., [27] for more details). But now (if $\eta\neq 0$)

$$\exp^{\diamond}(\langle P_1,\lambda\rangle)=\sum_{n=0}^{\infty}\frac{1}{n!}\langle P_1,\lambda\rangle^{\Diamond n}=\sum_{n=0}^{\infty}\frac{1}{n!}\langle P_n,\lambda^{\diamond n}\rangle\neq:\exp(\cdot;\lambda):=\sum_{n=0}^{\infty}\frac{1}{n!}\langle P_n,\lambda^{\otimes n}\rangle$$

and therefore the inherited from the Gaussian analysis term "Wick exponential" for $:\exp(\cdot;\lambda):$ is, strictly speaking, inaccurate (cf. [16]).

Let now $F,G\in(\mathcal{H}_{-\tau})$, $\tau\in T$. Since the space $(\mathcal{H}_{-\tau})$ is embedded in $(\mathcal{D}')'$, the Wick product $F\Diamond G$ and the Wick version of a holomorphic at $(SF)(0)$ function h $h^{\Diamond}$ are well-defined as elements of $(\mathcal{D}')'$ and "coordinate representations" (4.3), (4.5) hold true. Moreover, by Remark 2.8 and Proposition 2.2 the kernels from decompositions (4.3) and (4.5) "by Q-system" are elements of $\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n}$; and the kernels from decompositions (4.3) and (4.5) "by P-system" are elements of $\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$.

Theorem 4.2. *([19]) Let $F, G \in (\mathcal{H}_{-\tau})$, $\tau \in T$, and $h : \mathbb{C} \to \mathbb{C}$ be a holomorphic at $(SF)(0)$ function. Then the Wick product $F\lozenge G$ and the Wick version $h^{\lozenge}(F)$ are elements of $(\mathcal{H}_{-\tau})$, i.e., $F\lozenge G \in (\mathcal{H}_{-\tau})$ and $h^{\lozenge}(F) \in (\mathcal{H}_{-\tau})$. Moreover, the Wick multiplication is continuous in the topology of $(\mathcal{H}_{-\tau})$ (more exactly, for $F_1, \dots, F_m \in (\mathcal{H}_{-\tau})$, $m \in \mathbb{N}$,*

$$\|F_1\lozenge \dots \lozenge F_m\|_{-\tau,-q} \le C(m-1)\|F_1\|_{-\tau,-(q-1)} \cdots \|F_m\|_{-\tau,-(q-1)},$$

where $C(m) = \sqrt{\max_{n\in\mathbb{Z}_+}\{2^{-n}(n+1)^m\}}$, $q \in \mathbb{N}$ is such that $F_1, \dots, F_m \in (\mathcal{H}_{-\tau})_{-(q-1)}$).

Remark 4.5. Note that the proof of Theorem 4.2 consists in a direct estimation of $(\mathcal{H}_{-\tau})_{-q}$-norms of $F\lozenge G$ and $h^{\lozenge}(F)$ with using of decompositions (4.3) and (4.5).

Finally, we consider the case $F, G \in (L^2)^{-1}$. First we note that since $:\exp(\cdot;\lambda):\notin (L^2)^1_q$ $\forall q \in \mathbb{N}$ if $\lambda \neq 0$, the S-transform now can be defined on $(L^2)^{-1}$ only as a *formal* operator, i.e., by definition for $F = \sum_{n=0}^{\infty}\langle P_n, F^{(n)}_{\text{ext}}\rangle \in (L^2)^{-1}$ $(SF)(\lambda)$ is given by (4.1), where the series is a formal one (can diverge). Nevertheless, calculating *formally* $F\lozenge G$ and $h^{\lozenge}(F)$ we obtain representations (4.3) and (4.5) correspondingly. Our nearest goal is to prove that for $F^{(n)}_{\text{ext}} \in \mathcal{H}^{(n)}_{\text{ext}}$ and $G^{(m)}_{\text{ext}} \in \mathcal{H}^{(m)}_{\text{ext}}$ $F^{(n)}_{\text{ext}} \diamond G^{(m)}_{\text{ext}} \in \mathcal{H}^{(n+m)}_{\text{ext}}$ (this result can not be obtained directly from properties of the operators U_n (see (2.5)) because $U_n\mathcal{H}^{(n)}_{\text{ext}} \neq \mathcal{H}^{\widehat{\otimes}n}_{\mathbb{C}}$ if $\eta \neq 0$). Then we will show that for $F, G \in (L^2)^{-1}$ and holomorphic at $(SF)(0)$ $h : \mathbb{C} \to \mathbb{C}$ $F\lozenge G \in (L^2)^{-1}$ and $h^{\lozenge}(F) \in (L^2)^{-1}$. We begin from the following statement (in a sense this is a generalization of Lemma 3.1).

Lemma 4.1. *Let $F^{(n)} \in \mathcal{H}^{(n)}_{\text{ext}}$, $G^{(m)} \in \mathcal{H}^{(m)}_{\text{ext}}$, $n, m \in \mathbb{Z}_+$. Then defined by (4.4) $F^{(n)} \diamond G^{(m)} \in \mathcal{H}^{(n+m)}_{\text{ext}}$ and*

$$|F^{(n)} \diamond G^{(m)}|_{\text{ext}} \le |F^{(n)}|_{\text{ext}}|G^{(m)}|_{\text{ext}}. \tag{4.6}$$

More exactly, there exists $\widehat{F^{(n)}G^{(m)}} \in \mathcal{H}^{(n+m)}_{\text{ext}}$ such that $\forall q^{(n+m)} \in \mathcal{D}^{\widehat{\otimes}(n+m)}_{\mathbb{C}}$

$$\langle \widehat{F^{(n)}G^{(m)}}, q^{(n+m)}\rangle_{\text{ext}} = \langle F^{(n)} \diamond G^{(m)}, q^{(n+m)}\rangle_{\text{ext}}.$$

The element $\widehat{F^{(n)}G^{(m)}}$ has the following construction. Let $\dot{F}^{(n)} \in F^{(n)}$, $\dot{G}^{(m)} \in G^{(m)}$ be some representatives (functions) from equivalence classes $F^{(n)}$, $G^{(m)}$. Set

$$\begin{aligned}(\widetilde{F^{(n)}\dot{G}^{(m)}})(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m}) &\equiv \dot{F}^{(n)}(\widetilde{t_1, \dots, t_n})\dot{G}^{(m)}(t_{n+1}, \dots, t_{n+m}) \\ &:= \begin{cases} \dot{F}^{(n)}(t_1, \dots, t_n)\dot{G}^{(m)}(t_{n+1}, \dots, t_{n+m}), \text{ if } {}^{\forall i\in\{1,\dots,n\},}_{\forall j\in\{n+1,\dots,n+m\}}\ t_i \neq t_j, \\ 0, \text{ in other cases} \end{cases},\end{aligned} \tag{4.7}$$

$\widehat{F^{(n)}\dot{G}^{(m)}} := Pr\widetilde{F^{(n)}\dot{G}^{(m)}}$, where Pr is the symmetrization operator. Then $\widehat{F^{(n)}G^{(m)}}$ is the generated by $\widehat{F^{(n)}\dot{G}^{(m)}}$ equivalence class in $\mathcal{H}^{(n+m)}_{\text{ext}}$, this class is well-defined and does not depend on a choice of representatives $\dot{F}^{(n)}$, $\dot{G}^{(m)}$.

Remark 4.6. Note that non-strictly speaking $\widehat{F^{(n)}G^{(m)}}$ is the symmetrization of the function

$$\begin{aligned}&\widetilde{F^{(n)}G^{(m)}}(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+m}) \\ &\quad := \begin{cases} F^{(n)}(t_1, \dots, t_n)G^{(m)}(t_{n+1}, \dots, t_{n+m}), \text{ if } {}^{\forall i\in\{1,\dots,n\},}_{\forall j\in\{n+1,\dots,n+m\}}\ t_i \neq t_j, \\ 0, \text{ in other cases} \end{cases}\end{aligned}$$

with respect to $n + m$ variables.

Proof. Note that for $n=0$ or $m=0$ Lemma 4.1 is obviously true. Let $n,m\in\mathbb{N}$. By direct calculation that is based on formula (1.5), properties of function (4.7), well-know estimate $|\sum_{l=1}^{p}a_l|^2\le p\sum_{l=1}^{p}|a_l|^2$ and nonatomicity of the measure σ, by analogy with the calculation in the proof of Lemma 3.1 in [17] one can obtain the estimate

$$|\widehat{F^{(n)}\dot{G}^{(m)}}|_{\rm ext}\le|\dot{F}^{(n)}|_{\rm ext}|\dot{G}^{(m)}|_{\rm ext}, \tag{4.8}$$

therefore $\widehat{F^{(n)}G^{(m)}}$ is well-defined as an element of $\mathcal{H}_{\rm ext}^{(n+m)}$ and

$$|\widehat{F^{(n)}G^{(m)}}|_{\rm ext}\le|F^{(n)}|_{\rm ext}|G^{(m)}|_{\rm ext}. \tag{4.9}$$

Let $\dot{F}_1^{(n)}\in F^{(n)}$, $\dot{G}_1^{(m)}\in G^{(m)}$ be another representatives from $F^{(n)}$ and $G^{(m)}$. Using the obvious linearity of the operation $\widehat{\circ}$ and estimate (4.8) we obtain

$$\begin{aligned}|\widehat{F^{(n)}G^{(m)}}-\widehat{F_1^{(n)}G_1^{(m)}}|_{\rm ext}&=|\widehat{F^{(n)}\dot{G}^{(m)}}-\widehat{F_1^{(n)}\dot{G}_1^{(m)}}|_{\rm ext}\\&=|(\widehat{F^{(n)}\dot{G}^{(m)}}-\widehat{F^{(n)}\dot{G}_1^{(m)}})+(\widehat{F^{(n)}\dot{G}_1^{(m)}}-\widehat{F_1^{(n)}\dot{G}_1^{(m)}})|_{\rm ext}\\&\le|\widehat{F^{(n)}\dot{G}^{(m)}}-\widehat{F^{(n)}\dot{G}_1^{(m)}}|_{\rm ext}+|\widehat{F^{(n)}\dot{G}_1^{(m)}}-\widehat{F_1^{(n)}\dot{G}_1^{(m)}}|_{\rm ext}\\&=|\widehat{F^{(n)}(\dot{G}^{(m)}-G_1^{(m)})}|_{\rm ext}+|\widehat{(F^{(n)}-\dot{F}_1^{(n)})G_1^{(m)}}|_{\rm ext}\\&\le|\dot{F}^{(n)}|_{\rm ext}|\dot{G}^{(m)}-\dot{G}_1^{(m)}|_{\rm ext}+|\dot{F}^{(n)}-\dot{F}_1^{(n)}|_{\rm ext}|\dot{G}_1^{(m)}|_{\rm ext}=0,\end{aligned}$$

therefore $\widehat{F^{(n)}G^{(m)}}$ does not depend on a choice of representatives $\dot{F}^{(n)}$, $\dot{G}^{(m)}$.

By direct calculation that is based on formula (1.5), properties of functions (4.7), and nonatomicity of the measure σ, by analogy with the previous calculations one can show that

$$\langle\widehat{F^{(n)}G^{(m)}},\lambda^{\otimes n+m}\rangle_{\rm ext}=\langle F^{(n)},\lambda^{\otimes n}\rangle_{\rm ext}\langle G^{(m)},\lambda^{\otimes m}\rangle_{\rm ext}\quad\forall\lambda\in\mathcal{D}_{\mathbb{C}}. \tag{4.10}$$

On the other hand, by (4.4) and (2.5)

$$\begin{aligned}\langle F^{(n)}\diamond G^{(m)},\lambda^{\otimes n+m}\rangle_{\rm ext}&=\langle U_{n+m}U_{n+m}^{-1}((U_nF^{(n)})\widehat{\otimes}(U_mG^{(m)})),\lambda^{\otimes n+m}\rangle\\&=\langle U_nF^{(n)},\lambda^{\otimes n}\rangle\langle U_mG^{(m)},\lambda^{\otimes m}\rangle=\langle F^{(n)},\lambda^{\otimes n}\rangle_{\rm ext}\langle G^{(m)},\lambda^{\otimes m}\rangle_{\rm ext}.\end{aligned} \tag{4.11}$$

Comparing (4.10) and (4.11) we obtain

$$\langle\widehat{F^{(n)}G^{(m)}},\lambda^{\otimes n+m}\rangle_{\rm ext}=\langle F^{(n)}\diamond G^{(m)},\lambda^{\otimes n+m}\rangle_{\rm ext}\quad\forall\lambda\in\mathcal{D}_{\mathbb{C}}. \tag{4.12}$$

The restriction of $\widehat{F^{(n)}G^{(m)}}$ (as a linear continuous functional) on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n+m}$ is a *linear continuous functional on* $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n+m}$ that coincides (by (4.12)) with $F^{(n)}\diamond G^{(m)}\in[\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n+m}]'$ on the total in $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n+m}$ set $\{\lambda^{\otimes n+m}\colon\ \lambda\in\mathcal{D}_{\mathbb{C}}\}$. Therefore $\forall q^{(n+m)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n+m}$ $\langle\widehat{F^{(n)}G^{(m)}},q^{(n+m)}\rangle_{\rm ext}=\langle F^{(n)}\diamond G^{(m)},q^{(n+m)}\rangle_{\rm ext}$. Estimate (4.6) is equivalent to (4.9). □

Using the result of Lemma 4.1 we obtain

Theorem 4.3. *Let $F,G\in(L^2)^{-1}$ and $h:\mathbb{C}\to\mathbb{C}$ be a holomorphic at $(SF)(0)$ function. Then the Wick product $F\Diamond G$ and the Wick version $h^{\Diamond}(F)$ are elements of $(L^2)^{-1}$, i.e., $F\Diamond G\in(L^2)^{-1}$ and $h^{\Diamond}(F)\in(L^2)^{-1}$. Moreover, the Wick multiplication is continuous in the topology of $(L^2)^{-1}$ (more exactly, for $F_1,\dots,F_m\in(L^2)^{-1}$, $m\in\mathbb{N}$,*

$$\|F_1\Diamond\dots\Diamond F_m\|_{-q}\le C(m-1)\|F_1\|_{-(q-1)}\dots\|F_m\|_{-(q-1)}, \tag{4.13}$$

where $C(m)=\sqrt{\max_{n\in\mathbb{Z}_+}\{2^{-n}(n+1)^m\}}$, $q\in\mathbb{N}$ is such that $F_1,\dots,F_m\in(L^2)^{-1}_{-(q-1)}$).

Proof. The proof of this theorem is completely analogous to the proof of the corresponding statement in the Gamma analysis (see Theorem 3.2 in [17]). Therefore we shall confine ourselves to the short description of main steps.

(1) Let $F_1, \dots, F_m \in (L^2)^{-1}$. Then there exists $q \in \mathbb{N}$ such that $F_1, \dots, F_m \in (L^2)^{-1}_{-(q-1)}$. By (4.3) one can calculate the "coordinate form" of $F_1 \Diamond \dots \Diamond F_m$. Then calculating the $(L^2)^{-1}_{-q}$-norm of $F_1 \Diamond \dots \Diamond F_m$ by (2.23) and estimating this norm with using (4.6) one can obtain (4.13). In particular, it follows from (4.13) that for $F, G \in (L^2)^{-1}$ we have $F \Diamond G \in (L^2)^{-1}$.

(2) Let $F \in (L^2)^{-1}$. Then there exists $q \in \mathbb{N}$ such that $F \in (L^2)^{-1}_{-q}$. Using (4.5), (2.23) and (4.6) one can show that for a holomorphic at $(SF)(0)$ function $h : \mathbb{C} \to \mathbb{C}$

$$\|h^{\Diamond}(F)\|^2_{-q'} \le |h_0|^2 + C \sum_{s=1}^{\infty} 2^{(q-q'+2+2\widetilde{q}+2|\log_2(\|F\|_{-q})|)s-2} < \infty,$$

where $\widetilde{q} \in \mathbb{N}$ is such that $|h_n| < 2^{\widetilde{q}n}$ for all $n \in \mathbb{N}$ (here $\{h_n \in \mathbb{C}\}_{n=0}^{\infty}$ are the coefficients from decomposition (4.2) for h, such $\widetilde{q}$ exists because of holomorphy of h), $C := \frac{2^{2\widetilde{q}+2|\log_2(\|F\|_{-q})|}}{(2^{\widetilde{q}+|\log_2(\|F\|_{-q})|}-1)^2}$, $q' \in \mathbb{N}$ is sufficiently large. Therefore $h^{\Diamond}(F) \in (L^2)^{-1}_{-q'} \subset (L^2)^{-1}$.

□

Now let us consider the interconnection between the Wick calculus and the extended stochastic integration. It is well-known that in the Gaussian analysis the extended stochastic integral can be presented in the form

$$\int_0^t F(s)\,\widehat{d}W_s = \int_0^t F(s) \Diamond W'_s \sigma(ds), \tag{4.14}$$

where $W_\cdot = \langle H_1, 1_{[0,\cdot)}\rangle$ is a standard Wiener process, $W'_\cdot = \langle H_1, \delta_\cdot\rangle$ is the corresponding white noise (here H_1 is the kernel of the generalized Hermite polynomial, see (3.7)). The analog (generalization) of (4.14) in the "biorthogonal analysis" was obtained in [20]; in the Gamma analysis the corresponding results were proved in [21, 17]. Now we'll obtain the analog of (4.14) in the Meixner analysis. Note that a portion of results in this area from [20] follow, but we prefer to give a full (and more detailed than in [20]) presentation.

First we remind that $M_\cdot = \langle P_1, 1_{[0,\cdot)}\rangle$ (and correspondingly $M'_\cdot = \langle P_1, \delta_\cdot\rangle = Q_1(\delta_\cdot)$), where $M_\cdot$ is a Meixner process and $M'_\cdot$ is the corresponding Meixner white noise (this noise is the generalized stochastic process with independent values from Theorem 1.1). Let $F \in (\mathcal{D}')' \otimes \mathcal{H}_{\mathbb{C}}$. One can select a *representative* (with respect to $\mathcal{H}_{\mathbb{C}}$) $\dot{F} \in F$ (more exactly, this representative is a $(\mathcal{D}')'$-valued function on $\mathbb{R}_+$ such that $\forall g \in (\mathcal{D})$ $\langle\langle \dot{F}(\cdot), g\rangle\rangle \in \langle\langle F, g\rangle\rangle \in \mathcal{H}_{\mathbb{C}}$). It is convenient to preserve the notation F for $\dot{F}$ because we'll consider integrals with respect to σ and from this point of view different representatives are identical. Since for each $s \in \mathbb{R}_+$ $M'_s = Q_1(\delta_s) \in (\mathcal{D}')'$, we can consider

$$F(s) \Diamond M'_s = \sum_{n=0}^{\infty} Q_{n+1}(F_s^{(n)} \widehat{\otimes} \delta_s) \in (\mathcal{D}')'$$

(see Remark 4.3), where $F_s^{(n)} \in \mathcal{D}'^{\widehat{\otimes} n}_{\mathbb{C}}$ $(n \in \mathbb{Z}_+)$ are (corresponding to the representative F) representatives (at s) of the kernels $F_\cdot^{(n)} \in \mathcal{D}'^{\widehat{\otimes} n}_{\mathbb{C}} \otimes \mathcal{H}_{\mathbb{C}}$ from decomposition (3.31) for

F. For arbitrary $f \in (\mathcal{D})$ by (2.16) we have

$$\begin{aligned}\langle\!\langle F(s)\Diamond M'_s, f\rangle\!\rangle &= \sum_{n=0}^{\infty}(n+1)!\langle F_s^{(n)}\widehat{\otimes}\delta_s, f^{(n+1)}\rangle \\ &= \sum_{n=0}^{\infty}(n+1)!\langle F_s^{(n)}\otimes\delta_s, f^{(n+1)}\rangle = \sum_{n=0}^{\infty}(n+1)!\langle F_s^{(n)}, f^{(n+1)}(s)\rangle,\end{aligned}$$

where $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$, $n \in \mathbb{Z}_+$, are the kernels from decomposition (1.8) for f, and the kernels $f^{(n+1)}(s) \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$ are obtained from $f^{(n+1)}$ by substitution s as a one argument. Since there exist $\tau \in T$ and $q \in \mathbb{N}_{q_0}$ such that $F \in (\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{\mathbb{C}}$, we can estimate as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1)!\int_0^{\infty}|\langle F_s^{(n)}, f^{(n+1)}(s)\rangle|\sigma(ds) &\le \sum_{n=0}^{\infty}(n+1)!\int_0^{\infty}|F_s^{(n)}|_{-\tau}|f^{(n+1)}(s)|_{\tau}\sigma(ds) \\ &\le \sum_{n=0}^{\infty}(n+1)!\sqrt{\int_0^{\infty}|F_s^{(n)}|_{-\tau}^2\sigma(ds)}\sqrt{\int_0^{\infty}|f^{(n+1)}(s)|_{\tau}^2\sigma(ds)} \\ &= \sum_{n=0}^{\infty}(n+1)!|F_\cdot^{(n)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n}\otimes\mathcal{H}_{\mathbb{C}}}|f^{(n+1)}(\cdot)|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}\otimes\mathcal{H}_{\mathbb{C}}} \\ &\le c\sum_{n=0}^{\infty}(n+1)!|F_\cdot^{(n)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n}\otimes\mathcal{H}_{\mathbb{C}}}|f^{(n+1)}|_{\tau} \\ &= c2^{-q/2}\sum_{n=0}^{\infty}[2^{-qn/2}|F_\cdot^{(n)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n}\otimes\mathcal{H}_{\mathbb{C}}}][(n+1)!2^{q(n+1)/2}|f^{(n+1)}|_{\tau}] \\ &\le c2^{-q/2}\sqrt{\sum_{n=0}^{\infty}2^{-qn}|F_\cdot^{(n)}|^2_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n}\otimes\mathcal{H}_{\mathbb{C}}}}\sqrt{\sum_{n=0}^{\infty}((n+1)!)^2 2^{q(n+1)}|f^{(n+1)}|_{\tau}^2} \\ &\le c2^{-q/2}\|F\|_{(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{\mathbb{C}}}\|f\|_{\tau,q} < \infty,\end{aligned}$$

where $c = c(\tau) > 0$ is a constant such that $|\cdot|_{\mathcal{H}_{\mathbb{C}}} \le c|\cdot|_{\tau}$. Therefore for each $t \in [0, +\infty]$ and each $f \in (\mathcal{D})$

$$\begin{aligned}\int_0^t \langle\!\langle F(s)\Diamond M'_s, f\rangle\!\rangle\sigma(ds) &= \sum_{n=0}^{\infty}(n+1)!\int_0^t\langle F_s^{(n)}, f^{(n+1)}(s)\rangle\sigma(ds) \\ &= \sum_{n=0}^{\infty}(n+1)!\langle \widehat{F_{[0,t)}^{(n)}}, f^{(n+1)}\rangle = \langle\!\langle \int_0^t F(s)\,\widehat{d}M_s, f\rangle\!\rangle\end{aligned} \tag{4.15}$$

(see (3.37), (3.32), (2.16) and the Corollary from Theorem 3.5).

Definition 4.3. Let $F \in (\mathcal{D}')'\otimes\mathcal{H}_{\mathbb{C}}$, $t \in [0, +\infty]$. We define $\int_0^t F(s)\Diamond M'_s\sigma(ds) \in (\mathcal{D}')'$ by the formula

$$\langle\!\langle \int_0^t F(s)\Diamond M'_s\sigma(ds), f\rangle\!\rangle = \int_0^t\langle\!\langle F(s)\Diamond M'_s, f\rangle\!\rangle\sigma(ds) \quad \forall f \in (\mathcal{D}).$$

It follows from (4.15) that this definition is correct and, moreover, we have the following

Theorem 4.4. *(cf. (4.14)) For all $t \in [0, +\infty]$ and $F \in (\mathcal{D}')'\otimes\mathcal{H}_{\mathbb{C}}$*

$$\int_0^t F(s)\,\widehat{d}M_s = \int_0^t F(s)\Diamond M'_s\sigma(ds) \in (\mathcal{D}')'. \tag{4.16}$$

Remark 4.7. Of course, one can easily rewrite all corresponding to (4.16) calculations "on the language of P-system". We selected the "Q-system language" because in this case the calculations are slightly more simple from the technical point of view.

Remark 4.8. We explain that (4.16) has the following "practical" sense: in order to calculate the "action" of the extended stochastic integral $\int_0^t F(s)\,\widehat{d}M_s$ on a test function f one can

1) select a representative $F(s)$;
2) calculate the Wick product $F(s)\Diamond M_s'$;
3) calculate the pairing $\langle\!\langle F(s)\Diamond M_s', f\rangle\!\rangle =: q(s)$;
4) calculate $\int_0^t q(s)\sigma(ds)$.

To put it in another way, it is possible to interchange stochastic integration by calculation of the Wick product and Lebesgue integration.

It is easy to see that one can rewrite Definition 4.3 and Theorem 4.4 using the space $(\mathcal{H}_{-\tau})\otimes\mathcal{H}_{\mathbb{C}}$ $(\tau\in T)$ instead of $(\mathcal{D}')'\otimes\mathcal{H}_{\mathbb{C}}$, in this case $\int_0^t F(s)\Diamond M_s'\sigma(ds)\in(\mathcal{H}_{-\tau})$. But for $F\in(L^2)^{-1}\otimes\mathcal{H}_{\mathbb{C}}$ the situation is slightly more complicated. Namely, we have the following

Theorem 4.5. *For all $t\in[0,+\infty]$ and $F\in(L^2)^{-1}\otimes\mathcal{H}_{\mathbb{C}}$ $\int_0^t F_s\Diamond M_s'\sigma(ds)$ can be considered as a linear continuous functional on $(L^2)^1$ that coincides with $\int_0^t F_s\,\widehat{d}M_s$, i.e.,*

$$\int_0^t F(s)\Diamond M_s'\sigma(ds)=\int_0^t F(s)\,\widehat{d}M_s\in(L^2)^{-1}.$$

Proof. We have to prove that if $F\in(L^2)^{-1}\widehat{\otimes}\mathcal{H}_{\mathbb{C}}$ and simultaneously $F\in(\mathcal{H}_{-\tau})\widehat{\otimes}\mathcal{H}_{\mathbb{C}}$ then

$$\langle\!\langle\int_0^t F(s)\Diamond M_s'\sigma(ds),f\rangle\!\rangle=\langle\!\langle\int_0^t F(s)\widehat{d}M_s,f\rangle\!\rangle\quad\forall f\in(\mathcal{D}),$$

but this result follows from (4.16) and the Corollary from Theorem 3.5. □

Theorems 4.1–4.5 give us a possibility to consider so-called stochastic differential equations with Wick-type nonlinearity and solve such equations using the S-transform. Let us consider corresponding examples.

Example 4.1. (*a linear equation*) Let us consider the integral stochastic equation

$$X_t=X_0+\int_0^t X_s\Diamond F_1\Diamond\dots\Diamond F_n\sigma(ds)+\int_0^t X_s\Diamond G_1\Diamond\dots\Diamond G_m\widehat{d}M_s,\tag{4.17}$$

where $X_0\in(\mathcal{D}')'$ (correspondingly $(\mathcal{H}_{-\tau})$ $(\tau\in T)$, $(L^2)^{-1}$); $n,m\in\mathbb{N}$; $F_k=\langle P_1,F_k^{(1)}\rangle$, $F_k^{(1)}\in\mathcal{D}_{\mathbb{C}}'$ (correspondingly $\mathcal{H}_{-\tau,\mathbb{C}}$, $\mathcal{H}_{ext}^{(1)}=\mathcal{H}_{\mathbb{C}}$), $k\in\{1,\dots,n\}$; $G_k=\langle P_1,G_k^{(1)}\rangle$, $G_k^{(1)}\in\mathcal{D}_{\mathbb{C}}'$ (correspondingly $\mathcal{H}_{-\tau,\mathbb{C}}$, $\mathcal{H}_{ext}^{(1)}$), $k\in\{1,\dots,m\}$. Applying to (4.17) the S-transform (with regard to (4.16)), solving the obtained *algebraic* equation and applying the inverse S-transform we obtain the solution

$$X_t=X_0\Diamond\exp^{\Diamond}\{F_1\Diamond\dots\Diamond F_n\sigma([0,t))+G_1\Diamond\dots\Diamond G_m\Diamond M_t\}\in(\mathcal{D}')'$$

(correspondingly $(\mathcal{H}_{-\tau})$, $(L^2)^{-1}$).

By analogy one can solve the more general equation

$$X_t=X_0+\int_0^t X_s\Diamond F\sigma(ds)+\int_0^t X_s\Diamond G\,\widehat{d}M_s,$$

where $X_0,F,G\in(\mathcal{D}')'$ (correspondingly $(\mathcal{H}_{-\tau})$, $(L^2)^{-1}$), the solution has the form

$$X_t=X_0\Diamond\exp^{\Diamond}\{F\sigma([0,t))+G\Diamond M_t\}\in(\mathcal{D}')'$$

(correspondingly $(\mathcal{H}_{-\tau})$, $(L^2)^{-1}$).

Example 4.2. (*a Verhulst-type equation*) Let us consider the integral stochastic equation

$$X_t = X_0 + r\int_0^t X_s\lozenge(N - X_s)\sigma(ds) + \alpha\int_0^t X_s\lozenge(N - X_s)\,\widehat{d}M_s, \tag{4.18}$$

where $X_0 \in (\mathcal{D}')'$ (correspondingly $(\mathcal{H}_{-\tau})$, $(L^2)^{-1}$), $N, r, \alpha \in \mathbb{R}$, $N > 0$, $r > 0$, $(SX_0)(0) > 0$. Applying to (4.18) the S-transform (with regard to (4.16)), solving the obtained *algebraic* equation and applying the inverse S-transform, one can show by the full analogy with [20] that the solution of (4.18) has the form

$$X_t = N[1 + (NX_0^{\lozenge(-1)} - 1)\lozenge\exp^{\lozenge}\{-N(r\sigma([0,t)) + \alpha M_t)\}]^{\lozenge(-1)} \in (\mathcal{D}')'$$

(correspondingly $(\mathcal{H}_{-\tau})$, $(L^2)^{-1}$), where $Y^{\lozenge(-1)} := S^{-1}\frac{1}{SY}$.

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