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BOUNDARY PROBLEMS FOR FULLY NONLINEAR PARABOLIC EQUATIONS WITH LÉVY LAPLACIAN

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Abstract. We suggest a method to solve boundary and initial-boundary value problems for a class of nonlinear parabolic equations with the infinite dimensional Lévy Laplacian Δ_L

$$f\Big(U(t,x), \frac{\partial U(t,x)}{\partial t}, \Delta_L U(t,x)\Big) = 0$$

in fundamental domains of a Hilbert space.

1. Introduction

In the paper by S. Albeverio, Ya. I. Belopolskaya, M. N. Feller [1] we have constructed a solution of the Cauchy problem for a fully nonlinear parabolic equation with the Lévy Laplacian

$$f\Big(U(t,x), \frac{\partial U(t,x)}{\partial t}, \Delta_L U(t,x)\Big) = 0, \quad U(0,x) = U_0(x),$$

where $f(\xi,\eta,\zeta)$ is a function on R^3.

In the present paper we continue the investigation started in [1] . We develop a method to solve the boundary value problem

$$f\Big(U(t,x), \frac{\partial U(t,x)}{\partial t}, \Delta_L U(t,x)\Big) = 0, \quad U(t,x) = G(t,x) \quad \text{on} \quad \Gamma,$$

and the initial-boundary value problem

$$f\Big(U(t,x), \frac{\partial U(t,x)}{\partial t}, \Delta_L U(t,x)\Big) = 0, \quad U(0,x) = U_0(x), \quad U(t,x) = G(t,x) \quad \text{on} \quad \Gamma,$$

in fundamental domains $\Omega \cup \Gamma$ of a Hilbert space.

2. Preliminaries

Let H be a separable real Hilbert space, $F(x)$ be a scalar function defined on H.

An infinite dimensional Laplacian was introduced by P. Lévy [2] through the formula

$$\Delta_L F(x) = 2 \lim_{\rho \to 0} \frac{\mathfrak{M} F(x+\rho y) - F(x)}{\rho^2},$$

where $\mathfrak{M}\Phi$ is the mean value of the function $\Phi(y)$ over the sphere $\|y\|_H^2 = 1$.

If $F(x)$ is a twice strongly differentiable function at the point x_0, then the Lévy Laplacian is defined (when it exists) by the formula

$$\Delta_L F(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (F''(x_0) f_k, f_k)_H, \tag{1}$$

where $F''(x)$ is the Hessian of the function $F(x)$, $\{f_k\}_1^\infty$ is an orthonormal basis in H.

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In the sequel we need a property of the Lévy Laplacian that has been established in [2], (see as well [3]) which we describe now.

Let

$$F(x) = f(U_1(x), \dots, U_m(x)),$$

$f(u_1, \dots, u_m)$ be a twice continuously differentiable function with m arguments defined on the domain $\{U_1(x), \dots, U_m(x)\} \subset R^m$, where $(U_1(x), \dots, U_m(x))$ is a vector of values of the functions $U_1(x), \dots, U_m(x)$. Assume that $U_j(x)$ are uniformly continuous in a bounded domain $\Omega \subset H$ and twice strongly differentiable functions and $\Delta_L U_j(x)$ exist $(j = 1, \dots, m)$. Then $\Delta_L F(x)$ exists and

$$\Delta_L F(x) = \sum_{j=1}^{m} \frac{\partial f}{\partial u_j}\Big|_{u_j = U_j(x)} \Delta_L U_j(x). \tag{2}$$

Actually the second differential of the function $F(x)$ at the point x in the direction $h \in H$ has the form

$$d^2 F(x; h) = (F''(x)h, h)_H = \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial u_i \partial u_j}\Big|_{u_l = U_l(x)} (U_i'(x), h)_H (U_j'(x), h)_H$$
$$+ \sum_{j=1}^{m} \frac{\partial f}{\partial u_j}\Big|_{u_j = U_j(x)} (U_j''(x)h, h)_H.$$

By (1),

$$\Delta_L F(x) = \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial u_i \partial u_j}\Bigg|_{u_l = U_l(x)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (U_i'(x), f_k)_H (U_j'(x), f_k)_H$$
$$+ \sum_{j=1}^{m} \frac{\partial f}{\partial u_j}\Bigg|_{u_j = U_j(x)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (U_j''(x) f_k, f_k)_H.$$

But

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (U_i'(x), f_k)_H (U_j'(x), f_k)_H = 0,$$

(since $(U_l'(x), f_k)_H \to 0$ as $k \to \infty$), and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (U_j''(x) f_k, f_k)_H = \Delta_L U_j(x).$$

From this we obtain

$$\Delta_L F(x) = \sum_{j=1}^{m} \frac{\partial f}{\partial u_j}\Big|_{u_j = U_j(x)} \Delta_L U_j(x).$$

Let Ω be a bounded domain in the Hilbert space H, that is, a bounded open set in H, and let $\overline{\Omega} = \Omega \bigcup \Gamma$ be the corresponding domain in H with boundary Γ.

In the space H we define a domain Ω with the surface Γ as follows:

$$\Omega = \{x \in H : 0 \le Q(x) < R^2\}, \quad \Gamma = \{x \in H : Q(x) = R^2\},$$

where $Q(x)$ is a twice continuously differentiable function such that $\Delta_L Q(x) = \gamma$ for a nonzero positive constant γ. Domains of this type are called fundamental domains.

Let us give some examples of fundamental domains.

1) The ball $\overline{\Omega} = \{x \in H : \|x\|_H^2 \le R^2\}$.

2) The ellipsoid $\overline{\Omega} = \{x \in H : (Bx, x)_H \le R^2\}$, where $B = \gamma E + S(x)$, E is the identity operator and $S(x)$ is a compact operator on H.

Let us introduce a function $T(x) = \frac{R^2 - Q(x)}{\gamma}$. This function possesses the following properties:

$$0 < T(x) \le \frac{R^2}{\gamma}, \quad \Delta_L T(x) = -1 \quad \text{if} \quad x \in \Omega,$$
$$T(x) = 0 \quad \text{if} \quad x \in \Gamma.$$

3. THE PROBLEM WITHOUT INITIAL AND BOUNDARY CONDITIONS

Consider the nonlinear equation

$$f\Big(U(t,x), \frac{\partial U(t,x)}{\partial t}, \Delta_L U(t,x)\Big) = 0, \tag{3}$$

where $U(t,x)$ is a function on $[0,\mathcal{T}] \times H$, $f(\xi,\eta,\zeta)$ is a given function with three arguments.

Theorem 1. 1. *Let $f(\xi,\eta,\zeta)$ be a continuous twice differentiable function with three arguments taking values in the domain $\{U(t,x), \frac{\partial U(t,x)}{\partial t}, \Delta_L U(t,x)\}$ in R^3 .*

2. *Assume that one can solve the equation $f(\xi,\eta,c\eta) = 0$ with respect to η, $\eta = \phi(\xi,c)$ (although the original equation $f(\xi,\eta,\zeta) = 0$, in general, might have no solution η) and the solution admits separation of the variables ξ and c, i.e., $\phi(\xi,c) = \alpha(c)\beta(\xi)$ (for some functions $\alpha(c)$, $\beta(\xi)$ on R^1, $\beta(\xi) \ne 0$).*

Then a solution of (3) can be given in an implicit form

$$\varphi(U(t,x)) = \alpha(\Psi(x))t + \delta(\Psi(x))\frac{\|x\|_H^2}{2} + \Phi(x), \tag{4}$$

where $\varphi(\xi) = \int \frac{d\xi}{\beta(\xi)}$, $\delta(c) = c\alpha(c)$, and $\Psi(x)$, $\Phi(x)$ are arbitrary harmonic functions on H.

Proof. We deduce from (4), using (2) and the relation $\delta(\Psi(x)) = \Psi(x)\alpha(\Psi(x))$, that

$$\begin{aligned}
\varphi'_\xi(U(t,x))\frac{\partial U(t,x)}{\partial t} &= \alpha(\Psi(x)),\\
\varphi'_\xi(U(t,x))\Delta_L U(t,x) &= \alpha'_c(\Psi(x))\Delta_L\Psi(x)t + \Psi(x)\alpha'_c(\Psi(x))\Delta_L\Psi(x)\frac{\|x\|_H^2}{2}\\
&\quad + \Delta_L\Psi(x)\alpha(\Psi(x))\frac{\|x\|_H^2}{2} + \Psi(x)\alpha(\Psi(x))\frac{1}{2}\Delta_L\|x\|_H^2 + \Delta_L\Phi(x)\\
&= \Psi(x)\alpha(\Psi(x))
\end{aligned}$$

(since $\Delta_L\Psi(x) = \Delta_L\Phi(x) = 0$ by harmonicity and $\Delta_L\|x\|_H^2 = 2$ according to (1)).

Since $\varphi'_\xi = \frac{1}{\beta(\xi)}$ this implies that

$$\begin{aligned}
\frac{\partial U(t,x)}{\partial t} &= \alpha(\Psi(x))\beta(U(t,x)) = \phi(U(t,x),\Psi(x)),\\
\Delta_L U(t,x) &= \delta(\Psi(x))\beta(U(t,x)) = \Psi(x)\phi(U(t,x),\Psi(x)).
\end{aligned}$$

Substituting these relations into (3) we obtain

$$f\Big(U(t,x), \phi(U(t,x),\Psi(x)), \Psi(x)\phi(U(t,x),\Psi(x))\Big) = 0.$$

Due to condition 2) in the statement of the theorem this yields the identity

$$\phi(U(t,x),\Psi(x)) = \phi(U(t,x),\Psi(x)).$$

$\square$

4. Boundary problem

Consider the boundary problem

$$f\Big(U(t,x), \frac{\partial U(t,x)}{\partial t}, \Delta_L U(t,x)\Big) = 0 \quad \text{in} \quad \Omega, \tag{5}$$

$$U(t,x) = G(t,x) \quad \text{on} \quad \Gamma, \tag{6}$$

where $U(t,x)$ is a function on $[0,\mathcal{T}] \times H$, $f(\xi,\eta,\zeta)$ is a given function having three arguments and $G(t,x)$ is a given function.

Theorem 2. 1. *Let $f(\xi,\eta,\zeta)$ be a continuous twice differentiable function with three arguments defined in the domain $\{U(t,x), \frac{\partial U(t,x)}{\partial t}, \Delta_L U(t,x)\}$ in R^3.*

2. *Assume that one can solve the equation $f(\xi,\eta,c\eta) = 0$ with respect to η, $\eta = \phi(\xi,c)$, and the solution admits the separation of variables ξ and c that is $\phi(\xi,c) = \alpha(c)\beta(\xi)$ (for some functions $\alpha(c)$, $\beta(\xi)$ on R^1, $\beta(\xi) \neq 0$).*

3. *Assume that there exist a primitive $\varphi(\xi) = \int \frac{d\xi}{\beta(\xi)}$ and its inverse function φ^{-1}.*

4. *Assume that the domain $\overline{\Omega}$ is fundamental.*

5. *Assume that in some functional space there exists a solution $V(\tau,x)$ of the boundary problem for the heat equation*

$$\frac{\partial V(\tau,x)}{\partial \tau} = \Delta_L V(\tau,x) \quad \text{in} \quad \Omega, \quad V(\tau,x)\Big|_{\Gamma} = G(\tau,x). \tag{7}$$

6. *Consider the equation*

$$\alpha'_c\Bigg(\alpha^{-1}\bigg(\frac{\frac{\partial V(\tau,x)}{\partial \tau}\Big|_{\tau=X+T(x)}}{\beta(V(X+T(x),x))}\bigg)\Bigg)[t-X] - \delta'_c\Bigg(\alpha^{-1}\bigg(\frac{\frac{\partial V(\tau,x)}{\partial \tau}\Big|_{\tau=X+T(x)}}{\beta(V(X+T(x),x))}\bigg)\Bigg)T(x) = 0, \tag{8}$$

where $\delta(c) = c\alpha(c)$, and assume that it can be solved with respect to $X = \chi(t,x)$, and $\chi(t,x)\Big|_{\Gamma} = t$.

Then the solution of the boundary problem (5), (6) *in the same functional space is given by the formula*

$$\varphi(U(t,x)) = \alpha(\psi(\chi(t,x)))[t-\chi(t,x)] - \delta(\psi(\chi(t,x))T(x) + \varphi(V(\chi(t,x)+T(x),x))), \tag{9}$$

where

$$\psi(\chi(t,x)) = \alpha^{-1}\left(\frac{\frac{\partial V(\tau,x)}{\partial \tau}\Big|_{\tau=\chi(t,x)+T(x)}}{\beta(V(\chi(t,x)+T(x),x))}\right) \tag{10}$$

($\psi(z)$ *is a function on* R^1).

Proof. Since $\varphi'_\xi = \frac{1}{\beta(\xi)}$ we deduce from (9)

$$\begin{aligned}
\varphi'_\xi(U(t,x))\frac{\partial U(t,x)}{\partial t} &= \frac{1}{\beta(U(t,x))}\frac{\partial U(t,x)}{\partial t}\\
&= \alpha(\psi(\chi(t,x))) - \alpha(\psi(\chi(t,x)))\frac{\partial \chi(t,x)}{\partial t}\\
&+ \alpha'_c(\psi(\chi(t,x)))\psi'_z(\chi(t,x))\frac{\partial \chi(t,x)}{\partial t}[t-\chi(t,x)]\\
&- \delta'_c(\psi(\chi(t,x)))\psi'_z(\chi(t,x))\frac{\partial \chi(t,x)}{\partial t}T(x)\\
&+ \frac{\frac{\partial V(\tau,x)}{\partial \tau}\Big|_{\tau=\chi(t,x)+T(x)}}{\beta(V(\chi(t,x)+T(x),x))}\frac{\partial \chi(t,x)}{\partial t}\\
&= \alpha(\psi(\chi(t,x))) + \Big\{\alpha'_c(\psi(\chi(t,x)))[t-\chi(t,x)]\\
&- \delta'_c(\psi(\chi(t,x)))T(x)\Big\}\psi'_z(\chi(t,x))\frac{\partial \chi(t,x)}{\partial t}\\
&- \left[\alpha(\psi(\chi(t,x))) - \frac{\frac{\partial V(\tau,x)}{\partial \tau}\Big|_{\tau=\chi(t,x)+T(x)}}{\beta(V(\chi(t,x)+T(x),x))}\right]\frac{\partial \chi(t,x)}{\partial t}.
\end{aligned}$$

Note that $\chi(t,x)$ solves (8) and due to (10) we have

$$\frac{\partial U(t,x)}{\partial t} = \alpha(\psi(\chi(t,x)))\beta(U(t,x)). \tag{11}$$

Since $\varphi'_\xi = \frac{1}{\beta(\xi)}$, and $\Delta_L T(x) = -1$, applying (2) we deduce from (9)

$$\begin{aligned}
&\varphi'_\xi(U(t,x))\Delta_L U(t,x) = \frac{1}{\beta(U(t,x))}\Delta_L U(t,x)\\
&= -\alpha(\psi(\chi(t,x)))\Delta_L\chi(t,x) + \alpha'_c(\psi(\chi(t,x)))\psi'_z(\chi(t,x))\Delta_L\chi(t,x)[t-\chi(t,x)]\\
&- \delta'_c(\psi(\chi(t,x)))\psi'_z(\chi(t,x))\Delta_L\chi(t,x)T(x) - \delta(\psi(\chi(t,x)))\Delta_L T(x)\\
&+ \frac{\frac{\partial V(\tau,x)}{\partial \tau}\Big|_{\tau=\chi(t,x)+T(x)}}{\beta(V(\chi(t,x)+T(x),x))}\Big[\Delta_L\chi(t,x) + \Delta_L T(x)\Big] + \frac{\Delta_L V(\tau,x)\Big|_{\tau=\chi(t,x)+T(x)}}{\beta(V(\chi(t,x)+T(x),x))}\\
&= \delta(\psi(\chi(t,x))) + \Big\{\alpha'_c(\psi(\chi(t,x)))[t-\chi(t,x)]\\
&- \delta'_c(\psi(\chi(t,x))))T(x)\Big\}\,\psi'_z(\chi(t,x))\Delta_L\chi(t,x)\\
&- \left[\alpha(\psi(\chi(t,x))) - \frac{\frac{\partial V(\tau,x)}{\partial \tau}\Big|_{\tau=\chi(t,x)+T(x)}}{\beta(V(\chi(t,x)+T(x),x))}\right]\Delta_L\chi(t,x)\\
&- \frac{\Big[\frac{\partial V(\tau,x)}{\partial \tau} - \Delta_L V(\tau,x)\Big]\Big|_{\tau=\chi(t,x)+T(x)}}{\beta(V(\chi(t,x)+T(x),x))}.
\end{aligned}$$

Recall that $\chi(t,x)$ solves (8), $\frac{\partial V(\tau,x)}{\partial \tau} = \Delta_L V(\tau,x)$, then taking into account (10) we obtain

$$\Delta_L U(t,x) = \delta(\psi(\chi(t,x)))\beta(U(t,x)). \tag{12}$$

Substituting (11) and (12) into (5) we derive

$$f\Big(U(t,x), \alpha(\psi(\chi(t,x)))\beta(U(t,x)), \psi(\chi(t,x))\alpha(\psi(\chi(t,x)))\beta(U(t,x))\Big) = 0. \tag{13}$$

By condition 2) in the statement of the theorem the identity

$$\alpha(\psi(\chi(t,x)))\beta(U(t,x)) = \alpha(\psi(\chi(t,x)))\beta(U(t,x))$$

can be deduced from (13).

At the surface Γ we have $T(x) = 0$, and $\chi(t,x) = t$. Setting $T(x) = 0$, $\chi(t,x) = t$ in (9) and keeping in mind that $V(t,x)\Big|_\Gamma = G(t,x)$ we obtain

$$\varphi\Big(U(t,x)\Big|_\Gamma\Big) = \varphi\Big(V(t,x)\Big|_\Gamma\Big) = \varphi(G(t,x)) \quad \text{and} \quad U(t,x)\Big|_\Gamma = G(t,x).$$

□

5. Initial-boundary value problem

Consider an initial-boundary value problem with uniform initial data

$$f\Big(U(t,x), \frac{\partial U(t,x)}{\partial t}, \Delta_L U(t,x)\Big) = 0 \quad \text{in} \quad \Omega, \tag{14}$$

$$U(0,x) = 0, \tag{15}$$

$$U(t,x) = G(t,x) \quad \text{on} \quad \Gamma, \tag{16}$$

where $U(t,x)$ is a function on $[0,\mathcal{T}] \times H$, $f(\xi,\eta,\zeta)$ is a given function with three arguments, $G(t,x)$ is a given function.

Theorem 3. 1. *Let $f(\xi,\eta,\zeta)$ be a continuous twice differentiable function with three arguments defined in the domain $\{U(t,x), \frac{\partial U(t,x)}{\partial t}, \Delta_L U(t,x)\}$ in R^3.*

2. *Assume that one can solve the equation $f(\xi,\eta,c\eta) = 0$ with respect to η, $\eta = \phi(\xi,c)$, and the solution admits the separation of variables ξ and c that is $\phi(\xi,c) = \alpha(c)\beta(\xi)$ (for some functions $\alpha(c), \beta(\xi)$ on $\mathrm{R}^1, \beta(\xi) \neq 0$).*

3. *Assume that there exist a primitive $\varphi(\xi) = \int \frac{d\xi}{\beta(\xi)}$ and its inverse function φ^{-1}.*

4. *Assume that the domain $\overline{\Omega}$ is fundamental .*

5. *Assume that in some functional space there exists a solution $V(\tau,x)$ of the initial-boundary value problem for the heat equation*

$$\frac{\partial V(\tau,x)}{\partial \tau} = \Delta_L V(\tau,x)) \quad \text{in} \quad \Omega, \quad V(0,x) = 0, \quad V(\tau,x)\Big|_\Gamma = G(t,x). \tag{17}$$

6. *Assume that the equation*

$$\alpha'_c\left(\alpha^{-1}\left(\frac{\frac{\partial V(\tau,x)}{\partial \tau}\Big|_{\tau=X+T(x)}}{\beta(V(X+T(x),x))}\right)\right)[t-X] - \delta'_c\left(\alpha^{-1}\left(\frac{\frac{\partial V(\tau,x)}{\partial \tau}\Big|_{\tau=X+T(x)}}{\beta(V(X+T(x),x)}\right)\right)T(x) = 0, \tag{18}$$

where $\delta(c) = c\alpha(c)$, can be solved with respect to $X = \chi(t,x)$, and $\chi(t,x)\Big|_\Gamma = t$, $\chi(0,x) < r$.

7. *Assume in addition that the function $G(t,x)$ is uniformly continuous in $\overline{\Omega}$ for each $t \in [0,\mathcal{T}]$, and the mean value $\mathfrak{M}G(t, x + \sqrt{2T(x)}y)$ for all $t \in [0,\mathcal{T}]$ and $G(t,x) = 0$, $G'_t(t,x) = 0$ for $t \leq r$ $(r > 0)$ exists .*

Then the solution of initial-boundary value problem (14)–(16) *in this functional space is given by*

$$\varphi(U(t,x)) = \alpha(\psi(\chi(t,x)))[t-\chi(t,x)] - \delta(\psi(\chi(t,x)))T(x) + \varphi(V(\chi(t,x)+T(x),x)), \tag{19}$$

where

$$\psi(\chi(t,x)) = \alpha^{-1}\left(\frac{\frac{\partial V(\tau,x)}{\partial \tau}\Big|_{\tau=\chi(t,x)+T(x)}}{\beta(V(\chi(t,x)+T(x),x))}\right) \tag{20}$$

($\psi(z)$ *is a function on* R^1).

Proof. We can prove that the function given by (19) satisfies (14) in Ω and at the surface Γ $U(t,x) = G(t,x)$, in the similar to the proof of theorem 2.

Let us show that $U(0,x) = 0$.

First we prove that if $G(\tau,x) = 0$ for $\tau \le 0$, then under the conditions of the theorem (namely condition (7)) the solution of the problem

$$\frac{\partial V(t,x)}{\partial t} = \Delta_L V(t,x) \quad \text{in} \quad \Omega, \quad V(0,x) = 0, \quad V(t,x)\Big|_{\Gamma} = G(t,x)$$

can be written in the form

$$V(t,x) = \mathfrak{M}G(t - T(x), x + \sqrt{2T(x)}y), \tag{21}$$

where $\mathfrak{M}\Phi$ is a mean value of the function $\Phi(y)$ over the sphere $\|y\|_H^2 = 1$.

Eventually, at one hand

$$\frac{\partial V(t,x)}{\partial t} = \frac{\partial \mathfrak{M}G(t - T(x), x + \sqrt{2T(x)}y)}{\partial \tau}. \tag{22}$$

At the other hand using (2) we derive

$$\begin{aligned}&\Delta_L V(t,x)\\ &\quad = -\frac{\partial \mathfrak{M}G(t - T(x), x + \sqrt{2T(x)}y)}{\partial \tau}\Delta_L T(x) + \Delta_L \mathfrak{M}G(\tau, x + \sqrt{2T(x)}y)\Big|_{\tau = t - T(x)}.\end{aligned}$$

It was shown in the paper [4] by E. M. Polischuk that if the function $F(x)$ is uniformly continuous in Ω and has a mean value $\mathfrak{M}F(x + \sqrt{2T(x)}y)$, then this mean value is a harmonic function in Ω, that is $\Delta_L \mathfrak{M}F(x + \sqrt{2T(x)}y) = 0$ $(x \in \Omega)$.

Hence taking into account that $\Delta_L T(x) = -1$, we obtain

$$\Delta_L V(t,x) = \frac{\partial \mathfrak{M}G(t - T(x), x + \sqrt{2T(x)}y)}{\partial \tau}. \tag{23}$$

Substituting (22) and (23) into the equation $\frac{\partial V(t,x)}{\partial t} = \Delta_L V(t,x)$ we get the identity

$$\frac{\partial \mathfrak{M}G(t - T(x), x + \sqrt{2T(x)}y)}{\partial \tau} = \frac{\partial \mathfrak{M}G(t - T(x), x + \sqrt{2T(x)}y)}{\partial \tau}.$$

Setting $t = 0$ in (21) we obtain $V(0,x) = \mathfrak{M}G(-T(x), x + \sqrt{2T(x)}y) = 0$, since by conditions of the theorem we have $G(\tau,x) = 0$ for $\tau \le 0$.

At the surface Γ $T(x) = 0$, and (21) yields $V(t,x)\Big|_{\Gamma} = \mathfrak{M}G(t,x) = G(t,x)$.

It results from (21) that

$$V(\chi(t,x) + T(x), x) = \mathfrak{M}G(\chi(t,x), x + \sqrt{2T(x)}y)$$

and thus

$$V(\chi(0,x) + T(x), x) = \mathfrak{M}G(\chi(0,x), x + \sqrt{2T(x)}y) = 0 \tag{24}$$

(since by theorem conditions $\chi(0,x) \le r$, and $G(\tau,x) = 0$ for $\tau \le r$).

The relation

$$\alpha(\psi(\chi(t,x))) = \frac{\frac{\partial V(\tau,x)}{\partial \tau}\Big|_{\tau = \chi(t,x) + T(x)}}{\beta(V(\chi(t,x)) + T(x), x))} = \frac{\mathfrak{M}G'_\tau(\chi(t,x), x + \sqrt{2T(x)}y)}{\beta(V(\chi(t,x)) + T(x), x))}$$

follows from (20) and leads to $\alpha(\psi(\chi(0,x))) = 0$ (since by theorem conditions $\chi(0,x) < r$, and $G'(\tau,x) = 0$ for $\tau \le r$).

Setting $t = 0$ in (19) and taking into account (24) and the existence of φ^{-1}, we deduce $\varphi(U(0,x)) = \varphi(V(\chi(0,t) + T(x), x)) = \varphi(0)$, that yields $U(0,x) = 0$. $\square$

Remark. If there exist unique solutions η, c and $\chi(t,x)$ to the equations $f(\xi,\eta,c\eta)=0$, $\alpha(c)=z$ and (18) respectively, and if the initial-boundary value problem (17) for the heat equation has a unique solution in a certain functional class then the solution of the initial-boundary value problem (14)–(16) is unique in this functional space.

Example. Let us solve the initial-boundary value problem in a ball of the Hilbert space H: $\overline{\Omega}=\{x\in H:\ \|x\|_H^2\le R^2\}$

$$\Big(\frac{\partial U(t,x)}{\partial t}\Big)^3 - U(t,x)\Big(\frac{\partial U(t,x)}{\partial t}\Big)^2 + \Big(\frac{\partial U(t,x)}{\partial t}\Big)\Delta_L U(t,x) = U(t,x)\Delta_L U(t,x) \quad \text{in} \quad \Omega, \tag{25}$$

$$U(0,x)=0, \tag{26}$$

$$U(t,x)\Big|_{\|x\|_H^2=R^2} = g\Big(t-\frac{1}{2}\|x\|_H^2\Big), \tag{27}$$

where $g(\lambda)=\lambda^2$ if $\lambda\ge 0$, $g(\lambda)=0$ if $\lambda\le 0$.

To apply theorem 3 we note that the function f in (25) has the form

$$f(\xi,\eta,\zeta)=\eta^3-\xi\eta^2+\eta\zeta-\xi\zeta.$$

Hence the equation that appears in the condition 2) of theorem 3 has the form

$$\eta^3-\xi\eta^2+c\eta^2-c\xi\eta=0. \tag{28}$$

The solutions of the equation (28) have the form $\eta=-c$, $\eta=\xi$ and $\eta=0$.

Let us take the solution $\eta=-c$. In this case $\eta=\phi(\xi,c)=-c$. Hence $\alpha(c)=-c$, $\beta(\xi)=1$ which yields $\delta(c)=-c^2$, $\varphi(\xi)=\xi$.

A solution of the initial-boundary value problem for the heat equation

$$\frac{\partial V(\tau,x)}{\partial\tau}=\Delta_L V(\tau,x) \quad \text{in} \quad \Omega, \quad V(0,x)=0, \quad V(\tau,x)\Big|_{\|x\|_H^2=R^2} = g\Big(\tau-\frac{1}{2}\|x\|_H^2\Big)$$

is given by the formula

$$V(\tau,x)=g\Big(\tau+\frac{1}{2}\|x\|_H^2-R^2\Big).$$

It results

$$V(\tau,x)\Big|_{\tau=X+T} = g\Big(X-\frac{R^2}{2}\Big), \quad \frac{\partial V(\tau,x)}{\partial\tau}\Bigg|_{\tau=X+T} = 2\sqrt{g\big(X-R^2/2\big)}.$$

Now the equation (18) (in the condition 6) of theorem 3) takes the form

$$t-(1-4T(x))X-2R^2T(x)=0.$$

A solution of this equation is given by

$$X=\chi(t,x)=\frac{t-2R^2T(x)}{1-4T(x)},$$

and in addition we have $\chi(t,x)\Big|_{\Gamma}=t$.

Since for such a form of $\chi(t,x)$

$$V(\chi(t,x)+T(x),x)=g\Big(\chi(t,x)-\frac{R^2}{2}\Big)=\frac{g(t-\frac{R^2}{2})}{(1-4T(x))^2},$$

$$\frac{\partial V(\tau,x)}{\partial\tau}\Bigg|_{\tau=\chi(t,x)+T(x)} = \frac{2\sqrt{g\big(t-R^2/2\big)}}{1-4T(x)},$$

we deduce

$$\alpha(\psi(\chi(t,x))) = \frac{2\sqrt{g(t-R^2/2)}}{1-4T(x)}, \quad \delta(\psi(\chi(t,x))) = -\frac{4g(t-\frac{R^2}{2})}{(1-4T(x))^2}.$$

It results from (19) in theorem 3 that the solution of the problem (25)–(27) has the form

$$U(t,x) = \frac{g\Big(t-\frac{R^2}{2}\Big)}{1-2(R^2-\|x\|_H^2)}.$$

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