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ON RANK ONE PERTURBATION OF CONTINUOUS SPECTRUM WHICH GENERATES PRESCRIBED FINITE POINT SPECTRUM

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Dedicated to the 100th anniversary of Mark Krein.

Abstract. The perturbations of Nevanlinna type functions which preserve the set of zeros of this function or add to this set new points are discussed.

1. Statement of the problem

The point spectrum in the case of rank one perturbation of purely continuous spectrum may be very rich. In general, this spectrum contains the eigenvalues as well the spectral singularities. We will not give a review of the references (we only indicate [1] and [2]).

Let us consider the case where non-perturbed continuous spectrum coincides with the half line $[0,\infty)$. The eigenvalues of the perturbed operator is obtained as a set of zeros of the function

$$\delta(\zeta) = 1 + \int_0^\infty \frac{h(\tau)}{\tau-\zeta}d\tau, \quad \zeta \notin [0,\infty) \tag{1.1}$$

(called the "denominator" of the resolvent). Some properties of such function one can find in [4]. Spectral singularities of the perturbed operator coincide with zeros of the functions

$$\delta_\pm(\sigma) = \lim_{\varepsilon\to\pm 0} \delta(\sigma+i\varepsilon), \quad \sigma > 0. \tag{1.2}$$

We discuss following questions: how to describe a) perturbations of the function $h(\tau)$ such that the perturbed function $\delta(\zeta)$ has new prescribed zeros without changing the other zeros; b) all perturbations of the function $h(\tau)$, which does not change the set of zeros of the function $\delta(\zeta)$.

The set of zeros of the function $\delta_+(\sigma)$ or $\delta_-(\sigma)$, $\sigma > 0$, are considered in the same way.

2. The construction of the function $\delta(\zeta)$ without zeros or with a prescribed finite set of zeros

Let us consider, in the space $L^2(0,\infty)$, the operator generated by the expression

$$\begin{cases} ly = -y'', \quad x > 0 \\ y(0) + (y,\eta)_{L^2(0,\infty)} = 0 \end{cases} \tag{2.1}$$

with a non-local boundary condition. The function to study (the "denominator" of the resolvent) is

$$\delta(\zeta) = 1 + \int_0^\infty \frac{\gamma(\tau)}{\tau-\zeta}\sqrt{\tau}d\tau, \quad \zeta \notin [0,\infty), \tag{2.2}$$

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where $\pi\gamma(\tau)$ denotes the sin-Fourier transformation of the function $\eta(x)$. Note that $\eta \in L^2(0,\infty)$, $\gamma \in L^2_\rho(0,\infty)$, $\rho(\tau) = \frac{1}{\pi}\sqrt{\tau}$ and that the function (2.2) is of the form (1.1).

Lemma 2.1. *Let $a_i \in \mathbb{C}$, $\tau_i > 0$, $i = 1,\dots,n$, be arbitrary numbers, then the function*

$$\delta(\zeta) = \frac{(\sqrt{\zeta}+ia_1)\dots(\sqrt{\zeta}+ia_n)}{(\sqrt{\zeta}+i\sqrt{\tau_1})\dots(\sqrt{\zeta}+i\sqrt{\tau_n})}, \quad \operatorname{Im}\sqrt{\zeta} \geqslant 0, \tag{2.3}$$

admits the representation (2.2).

Proof. The expression (2.3) is a rational function of $\sqrt{\zeta}$, so, the decomposition into simple fractions gives

$$\delta(\zeta) = 1 + \frac{A_1}{\sqrt{\zeta}+i\sqrt{\tau_1}} + \dots + \frac{A_n}{\sqrt{\zeta}+i\sqrt{\tau_n}}, \quad A_k = \text{const}.$$

Using the identity

$$\left(\frac{1}{\tau-\zeta}, \frac{1}{\tau+\tau_0}\right)_{L^2_\rho} = \frac{i}{\sqrt{\zeta}+i\sqrt{\tau_0}}, \quad \operatorname{Im}\sqrt{\zeta} > 0, \quad \tau_0 > 0, \tag{2.4}$$

we obtain the representation (2.2) where

$$\gamma(\tau) = -\frac{A_1 i}{\tau+\tau_1} - \dots - \frac{A_n i}{\tau+\tau_n}, \quad \tau > 0.$$

Lemma is proved. □

If $\operatorname{Im}\sqrt{\sigma+i\varepsilon} > 0$ then $\lim_{\varepsilon\to\pm 0}\sqrt{\sigma+i\varepsilon} = \pm\sqrt{\sigma}$. So, due to (1.2) and (2.3) we have

$$\delta_\pm(\sigma) = \frac{(\sqrt{\sigma}\pm ia_1)\dots(\sqrt{\sigma}\pm ia_n)}{(\sqrt{\sigma}\pm i\sqrt{\tau_1})\dots(\sqrt{\sigma}\pm i\sqrt{\tau_n})}, \quad \sigma > 0. \tag{2.5}$$

The representation (2.3)–(2.5) prove the following proposition.

Proposition 2.2. *Let the representation* (2.3) *hold. Then for $k = 1,\dots,n$, $\zeta \notin [0,\infty)$ and $\sigma > 0$, we have the following*:

1) *if $a_k > 0$ then $\delta(\zeta) \neq 0$, $\delta_+(\sigma) \neq 0$, $\delta_-(\sigma) \neq 0$,*
2) *if $a_k < 0$ then $\delta(-a_k^2) = 0$, $\delta_+(\sigma) \neq 0$, $\delta_-(\sigma) \neq 0$,*
3) *if $a_k = i\alpha_k$, $\alpha_k > 0$ then $\delta(\zeta) \neq 0$, $\delta_+(\alpha_k^2) = 0$, $\delta_-(\sigma) \neq 0$.*

As a consequence one can give examples of the function $\gamma(\tau)$ such that the functions $\delta(\zeta)$, $\delta_+(\sigma)$ and $\delta_-(\sigma)$ have a prescribed set of zeros.

In view of the condition $\tau_i > 0$, $i = 1,\dots,n$, the function $\delta(\zeta)$ is bounded,

$$\sup_{\zeta\notin[0,\infty)} |\delta(\zeta)| < \infty$$

(see (2.3)). We denote the relation (1.1) by $\delta(\zeta) \sim h(\tau)$ and the Hilbert transformation by

$$\mathcal{H}h(\sigma) = \text{V. p.}\int_0^\infty \frac{h(\tau)}{\tau-\sigma}d\tau, \quad \sigma > 0.$$

Theorem 2.3. *Let $\delta_1(\zeta) \sim h_1(\tau)$ and $\delta_2(\zeta) \sim h_2(\tau)$. Suppose that $h_{1,2} \in L^2(0,\infty) \cap C^1[0,\infty)$ and that the function $\delta_2(\zeta)$ is bounded in the domain $\zeta \notin [0,\infty)$.*

Let $\delta(\zeta) \sim h(\tau)$ where

$$h(\tau) = h_1(\tau) + h_2(\tau) + h_1(\tau)\mathcal{H}h_2(\tau) + h_2(\tau)\mathcal{H}h_1(\tau). \tag{2.6}$$

Then $\delta(\zeta) = \delta_1(\zeta)\delta_2(\zeta)$.

Proof. Denote $\widetilde{\delta}(\zeta) = \delta_1(\zeta)\delta_2(\zeta)$, then we must prove that $\widetilde{\delta}(\zeta) = \delta(\zeta)$. According to the definition (2.2) we have

$$\widetilde{\delta}(\zeta) = \delta_2(\zeta) + \delta_2(\zeta)\int_0^\infty \frac{h_1(\tau)}{\tau-\zeta}d\tau = 1 + \int_0^\infty \frac{h_2(\tau)}{\tau-\zeta}d\tau + \delta_2(\zeta)\int_0^\infty \frac{h_1(\tau)}{\tau-\zeta}d\tau. \tag{2.7}$$

Since $h_{1,2} \in L^2(0,\infty)$, the integrals $\displaystyle\int_0^\infty \frac{h_{1,2}(\tau)}{\tau-\zeta}d\tau$ belong to a Hardy space in the domain $\mathrm{Im}\zeta > 0$ and $\mathrm{Im}\zeta < 0$. The function $\widetilde{\delta}(\zeta)-1$ belongs to this Hardy space, too. Therefore, there exists a function $\widetilde{h} \in L^2(-\infty,\infty)$ such that

$$\widetilde{\delta}(\zeta) - 1 = \int_0^\infty \frac{\widetilde{h}(\tau)}{\tau-\zeta}d\tau.$$

Here, due to (2.7), $\widetilde{h}(\sigma) = \dfrac{1}{2\pi i}\left(\widetilde{\delta}_+(\sigma) - \widetilde{\delta}_-(\sigma)\right) = 0$, $\sigma < 0$. In view of $h_{1,2} \in C^1[0,\infty)$, the limit-values (like (1.2)) exist as continuous functions, for example,

$$\delta_{1,\pm}(\sigma) = 1 \pm \pi i h_1(\sigma) + \mathcal{H}h_1(\sigma), \quad \sigma > 0.$$

So, for $\sigma > 0$ we have (see (2.6)–(2.7))

$$\begin{aligned}
2\pi i\widetilde{h}(\sigma) &= \widetilde{\delta}_+(\sigma) - \widetilde{\delta}_-(\sigma) \\
&= 2\pi i h_2(\sigma) + +(1 + \pi i h_2(\sigma) + \mathcal{H}h_2(\sigma))(\pi i h_1(\sigma) + \mathcal{H}h_1(\sigma)) \\
&- (1 - \pi i h_2(\sigma) + \mathcal{H}h_2(\sigma))(-\pi i h_1(\sigma) + \mathcal{H}h_1(\sigma)) \\
&= 2\pi i h(\sigma),
\end{aligned}$$

i.e., $\widetilde{h}(\sigma) = h(\sigma)$, $\sigma > 0$. Therefore,

$$\widetilde{\delta}(\zeta) = 1 + \int_{-\infty}^{+\infty} \frac{\widetilde{h}(\tau)}{\tau-\zeta}d\tau = 1 + \int_0^{+\infty} \frac{h(\tau)}{\tau-\zeta}d\tau = \delta(\zeta).$$

Theorem is proved. □

Corollary 2.4. *To add a finite set of zeros to the function $\delta_1(\zeta) \sim h_1(\tau)$ one can use the function $\delta_2(\zeta) \sim h_2(\tau)$ with this set of zeros and replace the function $h_1(\tau)$ by the function $h(\tau)$ (see* (2.6)*).*

3. The perturbation which preserves the set of zeros

We consider the perturbation $\Delta\gamma(\tau)$ of the function $\gamma(\tau)$ in the representation

$$\delta(\zeta) = 1 + \int_0^\infty \frac{\gamma(\tau)}{\tau-\zeta}\sqrt{\tau}d\tau, \quad \zeta \notin [0,\infty). \tag{3.1}$$

To simplify the calculations, we will study the function $\delta_+(\sigma)$, $\sigma > 0$, only (the functions $\delta_+(\sigma)$, $\sigma > 0$ and $\delta(\zeta)$, $\zeta \notin [0,\infty)$ are considered similarly). We are looking for a function $\Delta\gamma(\tau)$ such that replacing $\gamma(\tau)$ with $\gamma(\tau) + \Delta\gamma(\tau)$ in (3.1) we obtain the same set of zeros for the function $\delta_+(\sigma)$, $\sigma > 0$.

Let $0 < \sigma_1 < \cdots < \sigma_n$ be the set of all zeros of the function $\delta_+(\sigma)$, $\sigma > 0$. We suppose that $\gamma \in C^1[0,\infty]$ and that the multiplicities of the zeros are equal to 1.

We need the notations

$$\begin{cases} \mathcal{R}_\sigma\gamma(\tau) = \dfrac{\gamma(\tau)-\gamma(\sigma)}{\tau-\sigma}, \\[2ex] \mathcal{R}\gamma(\tau) = \gamma(\tau) + \sum_{j=1}^n A_j\mathcal{R}_{\sigma_j}\gamma(\tau), \end{cases} \tag{3.2}$$

where the coefficients $A_k = \mathrm{const}$ are defined below.

Lemma 3.1. *Suppose that $\sigma_j > 0$, $j = 1, \dots, n$, are arbitrary numbers. Let $p(\tau) = \prod_{j=1}^{n}(\tau - \sigma_j)$, $q(\tau) = p(\tau)/(1+\tau)^n$. Then there exists the coefficients $A_k = \mathrm{const}$ and polynomials $f_j(\tau)$ of degree $n-1$, $f_j(\sigma_j) = 1$, $f_j(\sigma_k) = 1$, $k \neq j$, $k, j = 1, \dots, n$, such that following decomposition holds*:

$$\gamma(\tau) = \sum_{j=1}^{n} \gamma(\sigma_j) f_j(\tau) + q(\tau)\mathcal{R}\gamma(\tau). \tag{3.3}$$

Proof. We define the values A_j as coefficients of the decomposition into simple fractions,

$$\frac{1}{q(\tau)} = 1 + \sum_{j=0}^{n} \frac{A_j}{\tau - \sigma_j}.$$

The decomposition (3.3) follows from the identity

$$\frac{\gamma(\tau)}{q(\tau)} = \sum_{j=1}^{n} \gamma(\sigma_j) \frac{A_j}{\tau - \sigma_j} + \left[\gamma(\tau) + \sum_{j=1}^{n} A_j \frac{\gamma(\tau) - \gamma(\sigma_j)}{\tau - \sigma_j}\right],$$

then $f_j(\tau) = A_j \dfrac{q(\tau)}{\tau - \sigma_j}$.

Lemma is proved. □

Note that the set of numbers $\gamma(\sigma_1), \dots, \gamma(\sigma_n)$ and the function $\Gamma(\tau) = \mathcal{R}\gamma(\tau)$ are independents composants of the decomposition (3.3) of the function $\gamma(\tau)$.

Obviously, the function $f_j(\tau)$ admits a decomposition in the form

$$f_j(\tau) = \sum_{l=1}^{n} \frac{M_{lj}}{(\tau+1)^l}, \quad M_{lj} = \mathrm{const}. \tag{3.4}$$

The functions $f_1(\tau), \dots, f_n(\tau)$ are linearly independents, so

$$\det(M_{lj}) \neq 0. \tag{3.5}$$

In view of (3.1) and the relation $\gamma(\tau) = \sum_{j=1}^{n} \gamma(\sigma_j) f_j(\tau) + q(\tau)\Gamma(\tau)$ we have for $\sigma > 0$ that

$$\delta_+(\sigma) = 1 + \sum_{i=1}^{n} \gamma(\sigma_i) l_i^+(\sigma) + \left(\int_0^\infty \frac{q(\tau)}{\tau - \sigma} \Gamma(\tau)\sqrt{\tau} d\tau\right)_+, \quad \Gamma = \mathcal{R}\gamma, \tag{3.6}$$

where

$$l_i^+(\sigma) = \left(\int_0^\infty \frac{f_i(\tau)}{\tau - \sigma} \sqrt{\tau} d\tau\right)_+, \quad \sigma > 0. \tag{3.7}$$

Lemma 3.2. *Let $\mathcal{L} = (l_i^+(\sigma_j))$. Then* $\det \mathcal{L} \neq 0$.

Proof. Taking the derivative of the equality (2.4) in the case $\sqrt{\zeta} \to \sqrt{\sigma}$ with respect to τ_0 and substituting $\tau_0 = 1$ we obtain for $l = 2, 3, \dots$

$$\left(\frac{1}{\tau - \sigma}, \frac{1}{(\tau+1)^l}\right)_+ = \sum_{\alpha=1}^{l} P_{\alpha,l}^+ u(\sigma)^\alpha, \quad u(\sigma) = \frac{i}{\sqrt{\sigma} + i}, \quad P_{l,l}^+ \neq 0.$$

In view of (3.4), (3.7),

$$\frac{1}{\pi} l_i^+(\sigma) = \left(\frac{1}{\tau - \sigma}, f_i\right)_+ = \sum_{l=1}^{n} M_{li} \left(\frac{1}{\tau - \sigma}, \frac{1}{(\tau+1)^l}\right)_+ = \sum_{\alpha=1}^{n} L_{i,\alpha}^+ u(\sigma)^\alpha$$

where

$$L_{i,\alpha}^+ = \sum_{l=\alpha}^{n} M_{li} P_{\alpha l}^+.$$

Obviously, the matrix

$$\begin{pmatrix} L^+_{11} & \dots & L^+_{1n} \\ \dots & \dots & \dots \\ L^+_{n1} & \dots & L^+_{nn} \end{pmatrix} = \begin{pmatrix} \dots & \dots & \dots & \dots \\ M_{1i} & M_{2i} & \dots & M_{ni} \\ \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \ddots & & 0 \\ & P^+_{\alpha,\alpha} & \\ & \vdots & \ddots & \\ & P_{\alpha,n} & & P^+_{n,n} \end{pmatrix}$$

is nondegenerate (see (3.5)).

So, the column $\begin{pmatrix} l^+_1(\sigma) \\ \vdots \\ l^+_n(\sigma) \end{pmatrix}$ is a nondegenerate stable (with respect to σ) transformation of the column $\begin{pmatrix} u(\sigma) \\ \vdots \\ u(\sigma)^n \end{pmatrix}$. Since $u(\sigma_i) \neq u(\sigma_j)$, $i \neq j$, we have $\det \mathcal{L} \neq 0$ because

$$\begin{vmatrix} u(\sigma_1) & \dots & u(\sigma_n) \\ \vdots & & \vdots \\ u(\sigma_1)^n & \dots & u(\sigma_n)^n \end{vmatrix} \neq 0.$$

Lemma is proved. $\square$

We consider two functions

$$\delta(\zeta) = 1 + \int_0^\infty \frac{\gamma(\tau)}{\tau - \zeta}\sqrt{\tau}d\tau, \quad \delta^0(\zeta) = 1 + \int_0^\infty \frac{\gamma^0(\tau)}{\tau - \zeta}\sqrt{\tau}d\tau, \quad \zeta \notin [0, \infty).$$

Obviously, the set $\{\sigma_j\}$ is a set of zeros of the function $\delta_+(\sigma)$ iff

$$0 < \left|\frac{\delta_+(\sigma)}{q(\sigma)}\right|, \quad \sigma > 0.$$

Lemma 3.3. *Let $\{\sigma_j\}$ be a set of zeros of $\delta^0_+(\sigma)$. Then $\{\sigma_j\}$ is a set of zeros of $\delta_+(\sigma)$ iff*

$$\left|\frac{\delta_+(\sigma)}{q(\sigma)}\right| > 0, \quad \sigma > 0, \tag{3.8}$$

$$\delta_+(\sigma) - \delta^0_+(\sigma) = 0, \quad j = 1, \dots, n. \tag{3.9}$$

Note that due the identity

$$\frac{1}{q(\sigma)}\delta_+(\sigma) = \frac{1}{q(\sigma)}\delta^0_+(\sigma) + \frac{1}{q(\sigma)}\Big(\delta_+(\sigma) - \delta^0_+(\sigma)\Big) \tag{3.10}$$

the condition $\inf_{(0,\infty)} \left|\frac{\delta^0_+(\sigma)}{q(\sigma)}\right| > 0$ follows from the condition

$$\inf_{(0,\infty)} \left|\frac{\delta_+(\sigma)}{q(\sigma)}\right| > \frac{1}{2} \inf_{(0,\infty)} \left|\frac{\delta^0_+(\sigma)}{q(\sigma)}\right|$$

if

$$\sup_{(0,\infty)} \left|\frac{1}{q(\sigma)}\Big(\delta_+(\sigma) - \delta^0_+(\sigma)\Big)\right| < \frac{1}{2} \inf_{(0,\infty)} \left|\frac{\delta^0_+(\sigma)}{q(\sigma)}\right|. \tag{3.11}$$

Lemma 3.4. 1) *If* $\gamma(\tau)$, $\gamma^0(\tau)$ *are arbitrary functions, then*

$$\begin{aligned}\delta_+(\sigma_j) - \delta^0_+(\sigma_j) = \sum_{i=1}^{n} \left(\gamma(\sigma_i) - \gamma^0(\sigma_i)\right) l^+_i(\sigma_j) \\ + \int_0^\infty \mathcal{R}_{\sigma_j} q(\tau) \left(\Gamma(\tau) - \Gamma^0(\tau)\right) \sqrt{\tau} d\tau,\end{aligned} \tag{3.12}$$

where $\Gamma = \mathcal{R}\gamma$, $\Gamma^0 = \mathcal{R}\gamma^0$.

2) *If* $\delta_+(\sigma_j) = \delta^0_+(\sigma_j) = 0$, *then*

$$\begin{aligned}\frac{1}{q(\sigma)} \left(\delta_+(\sigma) - \delta^0_+(\sigma)\right) = \sum_{i=1}^{n} \left(\gamma(\sigma_i) - \gamma^0(\sigma_i)\right) \mathcal{R} l^+_i(\sigma) \\ + \int_0^\infty \mathcal{R}\left(\mathcal{R}_\sigma q(\tau)\right) \left(\Gamma(\tau) - \Gamma^0(\tau)\right) \sqrt{\tau} d\tau + \left(\int_0^\infty \frac{\Gamma(\tau) - \Gamma^0(\tau)}{\tau - \sigma} \sqrt{\tau} d\tau\right)_+.\end{aligned} \tag{3.13}$$

Proof. 1) The decomposition (3.6) for the functions $\delta_+(\sigma)$, $\delta^0_+(\sigma)$ follow from the equality (3.12).

2) The equality $\delta_+(\sigma_i) = 0$ signifies (see (3.6))

$$1 + \sum_{i=1}^{n} \gamma(\sigma_i) l^+_i(\sigma_j) + \int_0^\infty \mathcal{R}_{\sigma_j} q(\tau) \Gamma(\tau) \sqrt{\tau} d\tau = 0, \quad i = 1, \ldots, n. \tag{3.14}$$

Like (3.3) we introduce the decompositions

$$l^+_i(\sigma) = \sum_{j=1}^{n} l^+_i(\sigma_j) f_j(\sigma) + q(\sigma) \mathcal{R} l^+_i(\sigma),$$

$$1 = \sum_{j=1}^{n} f_j(\sigma) + q(\sigma),$$

$$\mathcal{R}_\sigma q(\tau) = \sum_{j=1}^{\infty} \mathcal{R}_{\sigma_j} q(\tau) f_j(\sigma) + q(\sigma) \mathcal{R}\left(\mathcal{R}_\sigma q(\tau)\right).$$

Subtracting from the equation (see (3.6))

$$\delta_+(\sigma) = 1 + \sum_{i=1}^{n} \gamma(\sigma_i) l^+_i(\sigma) + \int_0^\infty \mathcal{R}_\sigma q(\tau) \Gamma(\tau) \sqrt{\tau} d\tau + q(\sigma) \left(\int_0^\infty \frac{\Gamma(\tau)\sqrt{\tau}}{\tau - \sigma} d\tau\right)_+$$

the equations (3.14) which are multiplied by $f_j(\sigma)$ we obtain

$$\frac{\delta_+(\sigma)}{q(\sigma)} = 1 + \sum_{i=1}^{n} \gamma(\sigma_i) \mathcal{R} l^+_i(\sigma) + \int_0^\infty \mathcal{R}\left(\mathcal{R}_\sigma q(\tau)\right) \Gamma(\tau) \sqrt{\tau} d\tau + \left(\int_0^\infty \frac{\Gamma(\tau)\sqrt{\tau}}{\tau - \sigma} d\tau\right)_+. \tag{3.15}$$

The same decomposition for the function $\dfrac{\delta^0_+(\sigma)}{q(\sigma)}$ gives (3.13).

Lemma is proved. □

Since $\det(l^+_i(\sigma_j)) \neq 0$, the relation

$$\sum_{i=1}^{n} \left(\gamma(\sigma_i) - \gamma^0(\sigma_i)\right) l^+_i(\sigma_j) + \int_0^\infty \mathcal{R}_{\sigma_j} q(\tau) \left(\Gamma(\tau) - \Gamma^0(\tau)\right) \sqrt{\tau} d\tau = 0 \tag{3.16}$$

and (3.13) define uniquely a linear operator N^+ such that

$$\frac{1}{q(\sigma)} \left(\delta_+(\sigma) - \delta^0_+(\sigma)\right) = (N_+(\Gamma - \Gamma^0))(\sigma). \tag{3.17}$$

Let

$$\| \Gamma \|_1 = \sup_{[0;\infty)} \left((1+\tau)|\Gamma(\tau)|\right) + \sup_{[0;\infty)} \left(\left(1+\tau^{3/2}\right)|\Gamma'(\tau)|\right).$$

Theorem 3.5. *Suppose that* $\| \Gamma \|_1 < \infty$. *If* $f(\tau) = \Gamma(\tau)\sqrt{\tau}$, *then*

$$F(\sigma) \equiv \mathrm{V.\,p.} \int_0^\infty \frac{f(\tau)}{\tau-\sigma} d\tau \to 0, \quad \tau \to \infty,$$

and

$$\sup_{[0;\infty)} |F(\sigma)| \leqslant C \| \Gamma \|_1, \quad C = \mathrm{const.}$$

Proof. We use the estimates

$$\sup_{[0;\infty)} |\sqrt{\tau} f(\tau)| \leqslant \| \Gamma \|_1, \quad \sup_{[0;\infty)} |\tau f'(\tau)| \leqslant 2 \| \Gamma \|_1$$

for the following expressions.

1) Let $\sigma > 4$ and

$$F(\sigma) = \left(\int_0^{\sigma-\sqrt{\sigma}} + \int_{\sigma-\sqrt{\sigma}}^{\sigma+\sqrt{\sigma}} + \int_{\sigma+\sqrt{\sigma}}^{\infty}\right) \frac{f(\tau)}{\tau-\sigma} d\tau \equiv F_1(\sigma) + F_2(\sigma) + F_3(\sigma).$$

Then

$$a) \quad |F_1(\sigma)| = \left| \int_0^{\sigma-\sqrt{\sigma}} \frac{f(\tau)\sqrt{\tau}}{\sqrt{\tau}(\tau-\sigma)} d\tau \right| \leqslant \sup_{[0;\infty)} |\sqrt{\tau} f(\tau)| \int_0^{\sigma-\sqrt{\sigma}} \frac{d\tau}{\sqrt{\tau}|\tau-\sigma|} \to 0, \quad \sigma \to \infty.$$

$$b) \quad |F_2(\sigma)| = \left| \mathrm{V.\,p.} \int_{\sigma-\sqrt{\sigma}}^{\sigma+\sqrt{\sigma}} \frac{f(\tau)}{\tau-\sigma} d\tau \right| = \left| \int_{\sigma-\sqrt{\sigma}}^{\sigma+\sqrt{\sigma}} \frac{f(\tau)-f(\sigma)}{\tau-\sigma} d\tau \right| = \left| \int_{\sigma-\sqrt{\sigma}}^{\sigma+\sqrt{\sigma}} f'(c) d\tau \right|$$

$$\leqslant \sup_{[2;\infty)} |\tau f'(\tau)| \int_{\sigma-\sqrt{\sigma}}^{\sigma+\sqrt{\sigma}} \frac{d\tau}{\sigma-\sqrt{\sigma}} \to 0, \quad \sigma \to \infty,$$

here $\sigma - \sqrt{\sigma} < c < \sigma + \sqrt{\sigma}$.

$$c) \quad |F_3(\sigma)| = \left| \int_{\sigma+\sqrt{\sigma}}^{\infty} \frac{f(\tau)}{\tau-\sigma} d\tau \right| \leqslant \sup_{[0;\infty)} |\sqrt{\tau} f(\tau)| \int_{\sigma+\sqrt{\sigma}}^{\infty} \frac{d\tau}{\sqrt{\tau}(\tau-\sigma)} \to 0, \quad \sigma \to \infty.$$

2) Let $0 < \sigma < 4$ and

$$F(\sigma) = \left(\int_0^5 + \int_5^\infty\right) \frac{f(\tau)}{\tau-\sigma} d\tau \equiv G_1(\sigma) + G_2(\sigma).$$

Then

$$a) \quad G_1(\sigma) = \mathrm{V.\,p.} \int_0^5 \frac{\sqrt{\tau}\Gamma(\tau)}{\tau-\sigma} d\tau = \int_0^5 \sqrt{\tau} \frac{\Gamma(\tau)-\Gamma(\sigma)}{\tau-\sigma} d\tau$$

$$+ \Gamma(\sigma) \mathrm{V.\,p.} \int_0^5 \frac{\sqrt{\tau}}{\tau-\sigma} d\tau = O(1), \quad \sigma \to 0$$

and

$$|G_1(\sigma)| \leqslant C(\sup_{[0;5]} |\Gamma'(\tau)| + \sup_{[0;5]} |\Gamma(\tau)|) \leqslant C \| \Gamma \|_1 .$$

$$b) \quad |G_2(\sigma)| = \left| \int_5^\infty \frac{f(\tau)}{\tau-\sigma} d\tau \right| \leqslant \sup_{[5;\infty)} |\sqrt{\tau} f(\tau)| \int_5^\infty \frac{d\tau}{\sqrt{\tau}(\tau-\sigma)} \leq C \| \Gamma \|_1 .$$

Theorem is proved. □

Corollary 3.6. *If* $\| \Gamma \|_1 < \infty$ *then* (*see* (3.17)) $(N_+\Gamma)(\sigma) \to 0, \quad \sigma \to \infty$, *and*

$$\sup_{[0;\infty)} |(N_+\Gamma)(\sigma)| \leqslant K \| \Gamma \|_1, \quad K = \mathrm{const.}$$

Proof. The function $q(\tau)$ is a linear combination of the functions $\dfrac{1}{(\tau+1)^k}$, $\quad k=1,2\dots$ and

$$\mathcal{R}_\sigma\left(\frac{1}{(\tau+1)^k}\right)=\frac{1}{\tau-\sigma}\left[\frac{1}{(\tau+1)^k}-\frac{1}{(\sigma+1)^k}\right]=\frac{r(\tau,\sigma)}{(\tau+1)^k(\sigma+1)^k}$$

where $r(\tau,\sigma)$ is a polynomial in τ and σ of degree $k-1$.

The same form holds for $\mathcal{R}(\mathcal{R}_\sigma q(r))$. Due to Theorem 3.5, the estimate

$$|\mathcal{R}_{\sigma_j}q(\tau)\Gamma(\tau)|\leqslant|\mathcal{R}_{\sigma_j}q(\tau)|\frac{\|\Gamma\|_1}{1+\tau}$$

and the presentation

$$\left(\int_0^\infty\frac{\Gamma(\tau)\sqrt{\tau}}{\tau-\sigma}d\tau\right)_+=\pi i\Gamma(\sigma)\sqrt{\sigma}+\mathrm{V.\,p.}\int_0^\infty\frac{\Gamma(\tau)\sqrt{\tau}}{\tau-\sigma}d\tau$$

prove the Corollary 3.6. □

Corollary 3.7. *If $\delta_+(0)\neq 0$ then the value*

$$m(\gamma)\equiv\inf_{[0;\infty)}\left|\frac{\delta_+(\sigma)}{q(\sigma)}\right|$$

is non-zero, i.e., $m(\gamma)>0$.

Proof. The continuous function $\dfrac{\delta_+(\sigma)}{q(\sigma)}$, $\sigma\in[0;\infty)$, does not have zeros and, by Corollary 3.6, for $\Gamma=\gamma$ and (3.15), $\dfrac{\delta_+(\sigma)}{q(\sigma)}\to 1$, $\sigma\to\infty$. Therefore, $m(\gamma)>0$. □

Let γ^0, γ be two functions, then we denote $\Delta\gamma=\gamma-\gamma^0$, $\Delta\Gamma=\Gamma-\Gamma^0$, $\Gamma=\mathcal{R}\gamma$, $\Gamma^0=\mathcal{R}\gamma^0$.

We denote by $\Omega(\gamma)$ the set of zeros of the function $\delta_+(\sigma)$, $\sigma>0$.

Theorem 3.8. *Let the function γ^0 be such that $m(\gamma^0)>0$. We define the function $\Delta\gamma(\tau)$ by its decomposition*

$$\Delta\gamma(\tau)=\sum_{i=1}^n\Delta\gamma(\sigma_i)f_i(\tau)+q(\tau)\Delta\Gamma(\tau),\quad\Gamma=\mathcal{R}\gamma,\tag{3.18}$$

where $\Delta\Gamma(\tau)$ is an arbitrary function such that (*see Corollary* 3.6)

$$K\,\|\Delta\Gamma\|_1<\frac{1}{2}m(\gamma^0)\tag{3.19}$$

and the numbers $\Delta\gamma(\sigma_i)$ are defined by $\Delta\Gamma(\tau)$ from the system (3.16)*. So, if*

$$\gamma=\gamma^0+\Delta\gamma$$

then $\Omega(\gamma)=\Omega(\gamma^0)$, i.e., the sets of zeros of the functions $\delta_+(\sigma)$ and $\delta_+^0(\sigma)$ coincide.

Proof. Let $\Omega(\gamma^0)=\{\sigma_1,\dots\sigma_n\}$. Suppose that some function $\Delta\Gamma$ satisfies the condition (3.19). The system (3.16) where $\Gamma(\tau)-\Gamma^0(\tau)=\Delta\Gamma(\tau)$ defines the numbers $\gamma(\sigma_j)-\gamma^0(\sigma_j)=\Delta\gamma(\sigma_j)$. The relation (3.18) gives the function $\Delta\gamma(\tau)$, so, we obtain $\gamma(\tau)=\gamma^0(\tau)+\Delta\gamma(\tau)$. Using the known function $\gamma(\tau)$ we define the function $\delta(\zeta)$ by (3.5). The system (3.16) signifies that $\delta_+(\sigma_j)-\delta_+^0(\sigma_j)=0$, i.e., $\delta_+(\sigma_j)=0$. Therefore,

$\Omega(\gamma) \subset \Omega(\gamma^0)$. It remains to prove that $\delta_+(\sigma) \neq 0$ if $\sigma \neq \sigma_j$, $j = 1, \dots, n$. We use the identity (3.10). According to Corollary 3.6 and (3.19), we have

$$\sup_{[0;\infty)} \left| \frac{1}{q(\tau)} (\delta_+(\sigma) - \delta_+^0(\sigma)) \right| = \sup_{[0;\infty)} |N_+(\Delta\Gamma)(\sigma)| \leqslant K \parallel \Delta\Gamma \parallel_1 < \frac{1}{2} m(\gamma^0)$$
$$= \frac{1}{2} \inf_{[0;\infty)} \left| \frac{1}{q(\sigma)} \delta_+^0(\sigma) \right|.$$

Therefore,

$$\inf_{[0;\infty)} \left| \frac{1}{q(\sigma)} \delta_+(\sigma) \right| > \frac{1}{2} \inf_{[0;\infty)} \left| \frac{1}{q(\sigma)} \delta_+^0(\sigma) \right| > 0,$$

i.e., $\Omega(\gamma) = \Omega(\gamma^0)$.

Theorem is proved. □

4. Application to non-local Sturm-Liouville operator with trivial potential

Let $ly = -y''$, we denote by B and A the operators in the space $L^2(0, \infty)$ generated by the same differential expression ly and the corresponding boundary condition $y(0) = 0$ and $y(0) + (y, \eta)_{L^2(0,\infty)} = 0$.

We need the Fourier transformation which diagonalizes the nonperturbed operator B, namely $\mathcal{F}$: $L^2(0, \infty) \to L^2_\rho(0, \infty)$, $\rho(\tau) = \frac{1}{\pi}\sqrt{\pi}$, where

$$\mathcal{F}y(\tau) = \int_0^\infty y(x) \frac{\sin x\sqrt{\tau}}{\sqrt{\tau}} dx, \quad \tau > 0,$$

and we need the relation

$$\mathcal{F}(e^{i\sqrt{\zeta}x})(\tau) = \frac{1}{\tau - \zeta}, \quad \tau > 0, \quad \operatorname{Im}\sqrt{\zeta} > 0. \tag{4.1}$$

We denote $R_\zeta(B) = (B - \zeta)^{-1}$, $R_\zeta(A) = (A - \zeta)^{-1}$ and

$$\delta(\zeta) = 1 + \int_0^\infty \frac{\overline{\mathcal{F}\eta(\tau)}\rho(\tau)}{\tau - \zeta} d\tau, \quad \rho(\tau) = \frac{1}{\pi}\sqrt{\tau}, \quad \zeta \notin [0, \infty), \tag{4.2}$$

where the function $\eta(x)$ defines the boundary condition for the perturbed operator A.

Theorem 4.1. *Let $\zeta \notin [0, \infty)$, then $\zeta \in \rho(A)$ iff $\delta(\zeta) \neq 0$, in this case,*

$$R_\zeta(A)f = R_\zeta(B)f - \frac{1}{\delta(\zeta)} (R_\zeta(B)f, \eta)_{L^2(0,\infty)} e_\zeta \tag{4.3}$$

where $e_\zeta(x) = \exp(i\sqrt{\zeta}x)$, $\operatorname{Im}\sqrt{\zeta} > 0$. A value $\zeta \notin [0, \infty)$ is an eigen-value of the operator A iff $\delta(\zeta) = 0$.

Proof. Let $e \in L^2(0, \infty)$ be an arbitrary function such that $le \in L^2(0, \infty)$, $e(0) = 1$, and $(e, \eta)_{L^2(0,\infty)} = -2$.

Every element $z \in L^2(0, \infty)$ admits the representation

$$z = y + (y, \eta)_{L^2(0,\infty)} e \tag{4.4}$$

where $y = z + (z, \eta)_{L^2(0,\infty)} e$. If $z \in D(A)$ then $z(0) + (z, \eta)_{L^2(0,\infty)} = 0$, so $y(0) = 0$, i.e., $z \in D(B)$.

Applying the operation $l - \zeta$ to both sides of the equality (4.4) we obtain

$$(A - \zeta)z = (B - \zeta)y + (y, \eta)_{L^2(0,\infty)} (l - \zeta)e.$$

The equation $(A - \zeta)z = f$ becomes

$$y + (y, \eta)_{L^2(0,\infty)} R_\zeta(B)(l - \zeta)e = R_\zeta(B)f. \tag{4.5}$$

The identity $\mathcal{F}le(\tau) - \tau\mathcal{F}e(\tau) \equiv -e(0) = -1$ gives

$$\frac{1}{\tau-\zeta}\mathcal{F}(l-\zeta)e(\tau) - \mathcal{F}e(\tau) = -\frac{1}{\tau-\zeta}$$

or

$$\mathcal{F}R_\zeta(B)(l-\zeta)e(\tau) - \mathcal{F}e(\tau) = -\mathcal{F}e_\zeta(\tau),\ e_\zeta(x) = \exp(i\sqrt{\zeta}x)$$

from which $R_\zeta(B)(l-\zeta)e = e - e_\zeta$. Substituting into (4.5) we obtain

$$y + (y,\eta)_{L^2(0,\infty)}(e - e_\zeta) = R_\zeta(B)f. \tag{4.6}$$

The multiplication by η gives

$$(y,\eta)_{L^2(0,\infty)}\left[1 + (-2 - (e_\zeta,\eta)_{L^2(0,\infty)})\right] = (R_\zeta(B)f,\eta)_{L^2(0,\infty)},$$

therefore,

$$(y,\eta)_{L^2(0,\infty)} = -\frac{1}{\delta(\zeta)}(R_\zeta(B)f,\eta)_{L^2(0,\infty)}.$$

In view of (4.4), the equality (4.6) becomes (4.3). If $\delta(\zeta) \neq 0$ then the operator $R_\zeta(A)$ is bounded, so $\zeta \in \rho(A)$. Other statements are simple to prove.

Theorem is proved. □

We will consider the functions that are more general than (4.2), namely,

$$\delta(\zeta) = a + \int_0^\infty \frac{h(\tau)}{\tau-\zeta}d\tau, \tag{4.7}$$

where $a \in \mathbb{C}$, $h \in L^2(0,\infty)$. We denote $\delta(\zeta) \sim (a,h)$ and assume that $h(\tau)$ is a function such that the limit values $\delta_\pm(\cdot)$ of the function (4.7) are continuous on $(0,\infty)$.

Let us introduce the normed space

$$U = \{u = (a,h) : a \in \mathbb{C}, h \in L^2(0,\infty), \delta_\pm(\cdot) \in C[0,\infty)\},$$

where the norm is

$$||u||_1 = \sup_{\sigma>0}|\delta_\pm(\sigma)| + ||\delta_+ - \delta_-||_{L^2(0,\infty)}. \tag{4.8}$$

The definition (4.8) is correct, since the transformation $(a,h) \to \delta(\zeta)$ is invertible,

$$\begin{cases} a = \lim_{\sigma\to-\infty}\delta(\sigma), \\ h(\tau) = \frac{1}{2\pi i}\left(\delta_+(\tau) - \delta_-(\tau)\right). \end{cases} \tag{4.9}$$

Lemma 4.2. *Let some function $\delta(\zeta)$ be holomorphic in the domain $\zeta \notin [0,\infty)$, have continuous on $[0,\infty)$ limit-values $\delta_\pm(\cdot)$ and a finite value of the right side of (4.8) and a finite value of the limit $a = \lim_{\sigma\to-\infty}\delta(\sigma)$. Then the function $\delta(\zeta)$ admits the representation (4.7), $\delta(\zeta) \sim (a,h)$, where $(a,h) \in U$.*

Proof. It is sufficient to consider the difference (see (4.9))

$$\delta(\zeta) - \left[a + \int_0^\infty \frac{h(\tau)}{\tau-\zeta}d\tau\right]$$

and use the principle of symmetry known as a property of analytic functions. □

Lemma 4.3. *The space U is closed.*

Proof. Let us rewrite the norm (4.8) in the form

$$||u||_1 = \sup_{\sigma>0}|\delta_\pm(\sigma)| + 2\pi||h||_{L^2(0,\infty)}. \tag{4.10}$$

Let $u_n = (a_n,h_n) \in U$ be a fundamental sequence, in view of (4.10) the sequence $\{h_n\}$ is fundamental in $L^2(0,\infty)$. We denote $h = \lim h_n$. Since $|\delta_{n\pm}(\sigma) - \delta_{m\pm}(\sigma)| \leqslant ||u_n - u_m||_1$, the sequence $\{\delta_{n\pm}(\sigma)\}$ is fundamental too in the space of continuous function $C[\sigma_1,\sigma_2]$ for every $0 < \sigma_1 < \sigma_2$.

So, from the equality

$$\delta_{n+}(\sigma) - \delta_{m+}(\sigma) = a_n - a_m + \left(\int_0^\infty \frac{h_n(\tau) - h_m(\tau)}{\tau - \sigma} d\tau \right)_+$$

taking into account the boundedness of the Hilbert transform in the space $L^2(\sigma_1, \sigma_2)$ it follows that the sequence $\{a_n\}$ is fundamental. Let $a = \lim_{n\to\infty} a_n$. Obviously, $(a, h) = \lim_{n\to\infty}(a_n, h_n)$ in the sense of the norm $||\cdot||_1$. The functions $\delta_\pm(\cdot)$ corresponding to the pair (a, h) (see (4.7)) are continuous on $[0, \infty)$, being uniform limits of the sequence of the continuous functions $\{\delta_{n\pm}(\sigma)\}$. So, $(a, h) \in U$, therefore the space U is closed.

Lemma is proved. □

According to Lemma 4.3, the space U is a Banach space.

We introduce the product of the pairs $u_1 = (a_1, h_1)$ and $u_2 = (a_2, h_2)$ by the relation

$$u_1 * u_2 \sim \delta_1(\zeta)\delta_2(\zeta)$$

(see (4.7)).

Lemma 4.4. *If $u_{1,2} \in U$, then $u_1 * u_2 \in U$ and*

$$||u_1 * u_2||_1 \leqslant ||u_1||_1 \cdot ||u_2||_1.$$

Proof. The function $\delta(\zeta) = \delta_1(\zeta)\delta_2(\zeta)$ is holomorphic in the domain $\zeta \notin [0, \infty)$, has continuous on $[0, \infty)$ limit-values $\delta_\pm(\sigma)$ and has the finite limit $\lim_{\sigma\to-\infty} \delta(\sigma) = a_1 a_2$,

$$\begin{aligned} ||u_1 * u_2||_1 &= \sup_{\sigma>0} |\delta_{1\pm}(\sigma)\delta_{2\pm}(\sigma)| + ||\delta_{1+}\delta_{2+} - \delta_{1-}\delta_{2-}||_{L^2(0,\infty)} \\ &\leqslant \sup_{\sigma>0} |\delta_{1\pm}(\sigma)| \sup_{\sigma>0} |\delta_{2\pm}(\sigma)| + \sup_{\sigma>0} |\delta_{2+}(\sigma)| \cdot ||\delta_{1+} - \delta_{1-}||_{L^2(0,\infty)} \\ &+ \sup_{\sigma>0} |\delta_{1-}(\sigma)| \cdot ||\delta_{2+} - \delta_{2-}||_{L^2(0,\infty)} \leqslant ||u_1||_1 \cdot ||u_2||_1 \end{aligned}$$

According to Lemma 4.2, the function $\delta(\zeta)$ admits the representation (4.7), $\delta_1(\zeta)\delta_2(\zeta) \sim u$, where $u \in U$. Therefore, $u_1 * u_2 \in U$.

Lemma is proved. □

According to Lemma 4.4, the Banach space U is a normed ring.

Let us come back to Theorem 2.3. If $\delta_2(\zeta) \neq 0$, $\zeta \in [0, \infty)$, then the relation (2.6) gives a sufficiency condition for such a transformation of the function $h_1(\tau)$ which preserves the roots of the function $= \delta_1(\zeta)$. We will show that this condition is closed to a necessary condition.

Lemma 4.5. *Let $\delta(\zeta)$ be an arbitrary function in the form* (4.7) *and $\delta_1(\zeta)$ be a function in the form* (2.3) *with the same roots as the function $\delta(\zeta)$. Then*

$$\delta(\zeta) = \delta_1(\zeta)\delta_2(\zeta)$$

where the function (which has no roots) admits the representation (4.7).

Proof results from Lemma 4.2. □

We will give some sufficient condition for the function $\delta(\zeta)$ to not have roots (compare (4.2) and (4.11)).

Theorem 4.6. *Let*

$$\delta(\zeta) = 1 + \int_0^\infty \frac{h(\tau)}{\tau - \zeta} d\tau, \quad \zeta \notin [0, \infty). \tag{4.11}$$

Suppose that the function $h(\tau)$ and its Hilbert transform $\mathcal{H}(\sigma) = \mathrm{v.\,p.} \int_0^\infty \frac{h(\tau)}{\tau-\sigma} d\tau$ are continuous on $[0, \infty)$.

Then, if

$$\sup_{\sigma>0} | \pm \pi i h(\sigma) + \mathcal{H}h(\sigma)| + 2\pi ||h||_{L^2(0,\infty)} < 1,$$

then the function $\delta(\zeta)$, $\zeta \notin [0,\infty)$, *has no roots.*

Proof results from the statement that an element u of the ring U is invertible if $||1-u||_1 < 1$ where 1 denotes the pair $(1,0)$. □

Theorem 4.7. *Let* $\Omega = \{\zeta : \mathrm{dist}(\zeta, [\tau_0,\infty)) < \varepsilon\}$ *for some* $\tau_0 > 0$, $\varepsilon > 0$. *Suppose that the function* $h(\tau)$ *admits an analytic continuation, which belongs to the Hardy space* $H^2(\Omega)$.

Suppose

$$\sup_{\sigma>0} | \pm \pi i h(\sigma) + \mathcal{H}h(\sigma)| < 1. \tag{4.12}$$

Then the function $\delta(\zeta)$, $\zeta \notin [0,\infty)$, *has no roots.*

Proof. We have the inequality

$$\left| \int_0^\infty \frac{h(\tau)}{\tau - \zeta} d\tau \right| < 1, \quad \zeta \notin [0,\infty)$$

using Phragmén-Lindelöf theorem and the estimate (4.12). □

Note that Theorem 4.6 and 4.7 complete the Theorem 3.8 and that one can easily rewrite all results in terms of the problem (2.1).

5. Conclusion

The case where the function $\delta(\zeta)$ has zeros with multiplicities > 1 is considered in the same way.

The traditional inverse problem requires to find a unique operator such that its spectrum coincides with a given set in the complex plane. But the problem to describe all the operators such that only a part of their spectrum coincides with a given set has a sense too. The question of how to choose a perturbation as to obtain a given change is interesting. One can compare such a problem with the construction of the well-known transform of the potential which adds to the spectrum of the Sturm-Liouvill operator one new point only (see, e.g. [3]). The decomposition (3.3) is useful for the Friedrichs' model (see [5]).

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